

Abundance of attracting, repelling and elliptic periodic orbits in two-dimensional reversible maps.

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Abstract. We study dynamics and bifurcations of two-dimensional reversible maps having non-transversal heteroclinic cycles containing symmetric saddle fixed points. We consider one-parameter families of reversible maps unfolding the initial heteroclinic tangency and prove the existence of infinitely many sequences (cascades) of bifurcations and birth of asymptotically stable, unstable and elliptic periodic orbits.

*Dedicated to the memory of Leonid Pavlovich Shilnikov,
a Master whose works strongly influenced
the mathematical theory of dynamical systems.*

1. Introduction

Reversible systems have a very special status inside the realm of dynamical systems. They are often positioned “between” dissipative and conservative systems. For continuous dynamical systems, *reversibility* means that the system is invariant under the change in time-direction, $t \mapsto -t$, and a transformation in the spatial variables. In the discrete context, reversibility of a map f (a diffeomorphism) means that f and f^{-1} possess the “same” dynamics, where the term “same” can have rather different meanings. For instance, if f and f^{-1} are smoothly conjugate, that is, $f \circ h = h \circ f^{-1}$ where h is a general diffeomorphism, then f is called *weakly reversible*. Much more interesting types of reversibility appear when h possesses some structures. For example, if h is an *involution*, i.e. $h^2 = \text{Id}$ (and $h \neq \text{Id}$), then the map f is called *strongly reversible*. Since this case is the most frequent in the literature, nowadays strongly reversible maps are simply named *reversible* maps.

In contrast to conservative and dissipative systems, the study of homoclinic bifurcations in reversible systems is not so popular. Even for two-dimensional maps, only few results are known and most of them relate to “conservative and reversible” maps which form a certain codimension- ∞ subclass in the class of reversible maps. This situation is probably due to a “common belief” that conservative and dissipative phenomena of dynamics exist separately and, thus, there is no clear necessity to

study them “all together”. However, they can actually appear in this way. Indeed, a dynamical system exhibits the so-called *phenomenon of mixed dynamics* when:

- (i) it has simultaneously infinitely many hyperbolic periodic orbits of all possible types (stable, completely unstable and saddle), and
- (ii) these orbits are not *separated as a whole*, that is, the closures of sets of orbits of different types have non-empty intersections.

The property of mixed dynamics seems to be a *generic* property, i.e., it holds for residual subsets of open regions of systems. This fact is almost straightforward consequence of the Newhouse work [17], taking into account also results of Gavrilov-Shilnikov and Newhouse [6, 18] on bifurcations of homoclinic tangencies: it is sufficiently to consider tangencies associated to thick horseshoes containing saddles with Jacobians bigger and less than one. However, it is absolutely not clear where such thick horseshoes exist. In principle, it does not matter, since it was proved in [10] that such regions, *Newhouse regions with mixed dynamics*, exist near any two-dimensional diffeomorphism with a non-transversal heteroclinic cycle containing at least two saddle periodic points O_1 and O_2 satisfying that $|J(O_1)| > 1$ and $|J(O_2)| < 1$, where $J(O_i)$ is the Jacobian of the Poincaré map at the point O_i , $i = 1, 2$.

Let us recall that a *heteroclinic cycle* (contour) is a set consisting of (ordered) saddle hyperbolic periodic orbits O_1, \dots, O_n , as well as several heteroclinic orbits $\Gamma_{i,j} \subset W^u(O_i) \cap W^s(O_j)$ containing, at least, the orbits $\Gamma_{i,i+1}$, for $i = 1, \dots, n-1$, and $\Gamma_{n,1}$. In general, cycles can include also homoclinic orbits $\Gamma_{i,i} \subset W^u(O_i) \cap W^s(O_i)$. An heteroclinic cycle is called *non-transversal* (or non-rough) if at least one of the intersections $W^u(O_i) \cap W^s(O_j)$ is non-transversal.

Remind also that *Newhouse regions* are open (in C^2 -topology) domains in the space of dynamical systems in which systems with homoclinic tangencies are dense. It is known that they exist in any neighbourhood of any system having a homoclinic tangency to a saddle periodic orbit [19, 9]. Such regions appear in parameter families unfolding, generally, the initial homoclinic tangency. In the parameter space, the values of the parameters corresponding to the existence of homoclinic tangencies are dense in open domains (see, for instance, [19, 9, 20, 24] and [5] for area-preserving maps), called *Newhouse intervals* in the case of one-parameter families.

The existence of Newhouse regions near systems with non-transversal heteroclinic cycles follows from these works. Moreover, as it was proved in [10], Newhouse intervals with heteroclinic tangencies (of the same type as the initial one) exist in any general one-parameter unfolding.

It is worth mentioning that contracting-expanding heteroclinic cycles are rather usual among reversible maps, as shown in Figure 1(a). In that example the reversible map has two saddle fixed points O_1 and O_2 and two heteroclinic orbits $\Gamma_{12} \subset W^u(O_1) \cap W^s(O_2)$ and $\Gamma_{21} \subset W^u(O_2) \cap W^s(O_1)$ such that $R(O_1) = O_2$ and $R(\Gamma_{21}) = \Gamma_{21}$, $R(\Gamma_{12}) = \Gamma_{12}$. Besides, the orbit Γ_{12} is non-transversal, so that the manifolds $W^u(O_1)$ and $W^s(O_2)$ have a quadratic tangency along Γ_{12} . Since $R(O_1) = O_2$ it turns out that $J(O_1) = J^{-1}(O_2)$. If $J(O_i) \neq \pm 1$, $i = 1, 2$, then the heteroclinic cycle is contracting-expanding. This condition is robust and is perfectly compatible with reversibility.

Certainly, results of [10] can be applied to reversible maps with such heteroclinic cycles. However, reversible systems are sharply different from general ones by the fact that they can possess robust non-hyperbolic symmetric periodic orbits, e.g. *elliptic symmetric periodic points*. This leads to the idea that the phenomenon of mixed

dynamics in the case of two-dimensional reversible maps should be connected with the *coexistence of infinitely many attracting, repelling, saddle and elliptic periodic orbits*. It appears to be true that the phenomenon of mixed dynamics is universal

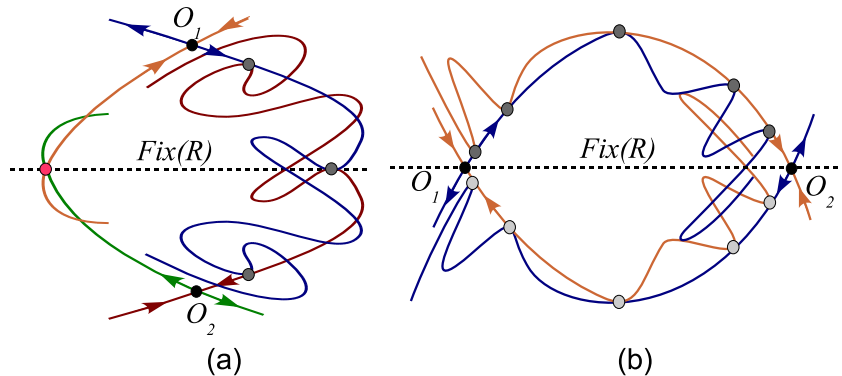


Figure 1. Two examples of planar reversible maps with symmetric non-transversal (quadratic tangency) heteroclinic cycles: (a) of the first type (there is a non-transversal symmetric heteroclinic orbit to a symmetric couple of saddle points, and (b) of the second type – there is a symmetric couple of non-transversal heteroclinic orbits to symmetric saddle points.

for reversible (two-dimensional) maps with complicated dynamics when symmetric structures (symmetric periodic, homoclinic and heteroclinic orbits) are involved. This universality can be formulated as follows:

Reversible Mixed Dynamics Conjecture *Two-dimensional reversible maps with mixed dynamics are generic in Newhouse regions where maps with symmetric homoclinic or/and heteroclinic tangencies are dense.*

We will assume, from now on, that the involution R is not trivial and leaves fixed a curve, that is, it satisfies

$$R^2 = \text{Id}, \quad \dim \text{Fix } R = 1, \quad (1.1)$$

where $\text{Fix } R = \{(x, y) \in \mathbb{R}^2 \mid R(x, y) = (x, y)\}$. We say that an object Λ is *symmetric* when $R(\Lambda) = \Lambda$. To put more emphasis, the notation self-symmetric may be used. By a *symmetric couple of objects* Λ_1, Λ_2 , we mean two different objects which are symmetric one to each other, i.e., $R(\Lambda_1) = \Lambda_2$. *Symmetric homoclinic (heteroclinic) tangencies* can be divided into two main types, namely, when: (i) there is a non-transversal symmetric heteroclinic orbit to a symmetric couple of saddle points, or (ii) there is a symmetric homoclinic tangency or a symmetric couple of non-transversal homo/heteroclinic orbits to symmetric saddle points.

The heteroclinic quadratic tangency shown in Fig. 1(a) relates to the first type. This case was studied in the paper of Lamb and Stenkin [15] where, in fact, the RMD-Conjecture was proved for general one-parameter reversible unfoldings, under the generic condition $J(O_1) = J^{-1}(O_2) \neq 1$.

Our main goal in this paper is to prove the RMD-Conjecture for general one-parameter reversible unfoldings of heteroclinic tangencies of the second type. Indeed,

let us consider a one-parameter family $\{f_\mu\}$ of R -reversible maps having, for $\mu = 0$, a map with a symmetric non-transversal heteroclinic cycle as the one in Fig. 1(b). That is, f_0 has two symmetric saddle fixed points O_1 and O_2 and a symmetric couple of heteroclinic orbits $\Gamma_{12} \subset W^u(O_1) \cap W^s(O_2)$ and $\Gamma_{21} \subset W^u(O_2) \cap W^s(O_1)$, where $R(O_1) = O_1$, $R(O_2) = O_2$ and $R(\Gamma_{12}) = \Gamma_{21}$. Since $O_1, O_2 \in \text{Fix } R$, the condition $J(O_1) = J(O_2) = 1$ holds always. Therefore, our genericity condition, the condition [C] from Section 2, relates to nonconservative properties of the map near any heteroclinic orbit Γ_{12} or Γ_{21} .

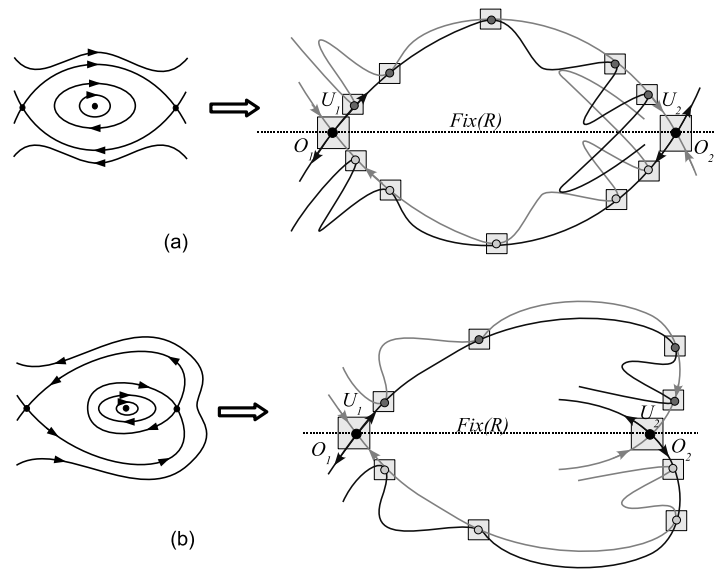


Figure 2. Examples of reversible maps with non-transversal heteroclinic cycles in the cases of (a) “inner tangency”; (b) “outer tangency”. These diffeomorphisms can be constructed as the Poincaré maps for periodically perturbed conservative planar systems (left) when the corresponding perturbations preserve the reversibility (for example, with respect to the change $x \rightarrow x, y \rightarrow -y, t \rightarrow -t$, as in the system $\dot{x} = y, \dot{y} = x - x^3 + \varepsilon \dot{x} \cos t$). Notice that the resulting reversible map cannot be area-preserving, in general

In the family $\{f_\mu\}$ we study bifurcations of *single round* periodic orbits, that is, orbits which pass just one time along a small fixed neighbourhood U of the contour $O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$. For such orbits we construct the first return (Poincaré) maps and study bifurcations of their fixed points. In Theorem 1 it is stated that infinitely many (cascade) bifurcations of *birth* and *symmetry-breaking* of single-round periodic orbits occur when varying μ near 0. The birth bifurcation is a nondegenerate reversible “fold bifurcation” under which a pair of R -symmetric saddle and elliptic fixed points appear in the corresponding first return map. One of these points undergoes next a nondegenerate reversible “pitch-fork bifurcation” which leads to the appearance of a couple of contracting-expanding fixed points. The type of this bifurcation depends on a character of the heteroclinic tangency: (a) “inner tangency” and (b) “outer tangency” (see Fig. 2). In the case of “inner tangency”, the elliptic fixed point falls into saddle, sink and source; in the case of “outer tangency”, otherwise, the saddle

fixed point falls into elliptic and two saddles with Jacobians > 1 and < 1 . Some illustration of these bifurcations is shown in Fig. 3. Let us remark that the cascade of bifurcations found is different from the well-known cascades of bifurcations observed in one-parameter unfoldings of quadratic homoclinic tangencies for general systems. In the latter case we have, first of all, the cascade of bifurcations “saddle-node plus first period doubling”, [6], which, as was posed by Shilnikov in [26] and proved by Alligood and Yorke in [32], is only the first step of the cascade of bifurcations “saddle-node plus infinite sequences of period doublings”.

The main result of this paper, contained in Theorem 2, is the existence of Newhouse intervals with mixed dynamics, which is almost an immediate consequence of Theorem 1. Both theorems are stated in Section 2. Section 3 is devoted to the proof of Theorem 1.

Section 4 contains two concrete examples of ordinary differential equations where the results in Section 2 can be applied to their associated Poincaré maps. First, in Subsection 4.1, it is shown that reversible two-dimensional maps with a priori non-conservative orbit behavior can be obtained as certain periodic perturbations of two-dimensional reversible conservative flows. As a concrete example we consider Duffing equation. In this way we can obtain a realization of a reversible map of the second type, as in Figure 2. Subsection 4.2 is devoted to the Pikovsky-Topaj model [21] of three coupled rotators in the 3-dimensional torus which gives rise to a reversible map on the two-dimensional torus exhibiting a reversible non-conservative orbit behavior.

In Section 5, we consider the so-called *cross-form* representation of reversible two-dimensional maps. It is well-known that such type of cross-forms of maps are very convenient for studying hyperbolic properties of systems with homoclinic orbits, both transversal [25] and non-transversal [6, 14]. Since L.P. Shilnikov was the first author introducing such forms and coordinates, they are often referred to as “Shilnikov cross-form” and “Shilnikov coordinates”. This cross-form appear to be pretty convenient to deal with reversible maps. And last (but not least), Section 6 contains the proofs of the lemmas needed in the proof of Theorem 1.

2. Symmetry breaking bifurcations in the case of reversible maps with non-transversal heteroclinic cycles

Let f_0 be a C^r -smooth, $r \geq 4$, two-dimensional map, reversible with respect to an involution R satisfying $\dim \text{Fix}(R) = 1$. Let us assume that f_0 satisfies the following two conditions:

- [A] f_0 has two saddle fixed points O_1 and O_2 belonging to the line $\text{Fix}(R)$ and that any point O_i has multipliers $\lambda_i, \lambda_i^{-1}$ with $0 < \lambda_i < 1$, $i = 1, 2$.
- [B] The invariant manifolds $W^u(O_1)$ and $W^s(O_2)$ have quadratic tangencies at the points of some heteroclinic orbit Γ_{12} and, therefore, by reversibility, the manifolds $W^u(O_2)$ and $W^s(O_1)$ have quadratic tangencies at the points of a heteroclinic orbit $\Gamma_{21} = R(\Gamma_{12})$.

Hypotheses [A]-[B] define reversible maps with non-transversal symmetric heteroclinic cycles like in Figure 1(b). We ask them to satisfy one more condition. Namely, consider two points $M_1 \in W_{loc}^u(O_1)$ and $M_2 \in W_{loc}^s(O_2)$ belonging to the same heteroclinic orbit Γ_{12} and suppose $f_0^q(M_1) = M_2$ for a suitable integer q . Let some smooth local coordinates (x_i, y_i) be chosen near the points O_i in such a way that the local invariant manifolds are straightened, i.e., $W_{loc}^u(O_i)$ and $W_{loc}^s(O_i)$ have, respectively, equations

$x_i = 0$ and $y_i = 0$. Let T_{12} denote the restriction of the map f_0^q onto a small neighbourhood of the point M_1 . Then, we assume that

[C] the Jacobian of T_{12} is not constant and, moreover,

$$Q = \frac{\partial J(T_{12})}{\partial y} \Big|_{M_1} \neq 0 \quad (2.1)$$

Condition $J(T_{12}) \neq \text{constant}$ is well defined only when certain restrictions on the local coordinates hold. One possibility is when these coordinates (x_i, y_i) around O_i are chosen in such a way that $W_{loc}^u(O_i)$ and $W_{loc}^s(O_i)$ are locally straightened, i.e. have equations $\{x_i = 0\}$ and $\{y_i = 0\}$, respectively. However, the sign of $J(T_{12})$ depends also on the orientation chosen for the coordinate axes. To be precise, we choose these orientations in such a way that the heteroclinic points $M_1 \in W_{loc}^u(O_1), M_1' = R(M_1) \in W_{loc}^s(O_1), M_2 = T_{12}(M_1) \in W_{loc}^s(O_2)$ and $M_2' = R(M_2) \in W_{loc}^u(O_2)$ have positive their coordinates y_1, x_1, x_2 and y_2 , respectively (see Fig. 4, where one can put $M_1 = M_1^-, M_2 = M_2^+, M_1' = M_1^+, M_2' = M_2^-$).

The generic condition [C] implies that the (global) map defined near a heteroclinic point is neither a linear contraction (expansion) nor a conservative map. ‡

Two classes of reversible maps satisfying conditions [A]-[C] can be distinguished: those maps with “inner” (heteroclinic) tangency and those with “outer” tangency, corresponding to $J(T_{12}) > 0$ and $J(T_{12}) < 0$, respectively. Two examples of such diffeomorphisms are shown in Figure 2. Notice that in both cases the global map T_{12} preserves orientation. In the case (a) the axes x_1, y_1 and x_2, y_2 have the same orientation, whereas the orientations are different in the case (b).

Once stated the general conditions for f_0 , let us embed it into a one-parameter family $\{f_\mu\}$ of reversible maps that unfolds generally at $\mu = 0$ the initial heteroclinic tangencies at the points of Γ_{12} . Then, without loss of generality, we can take μ as the corresponding splitting parameter. By reversibility, the invariant manifolds $W^u(O_1)$ and $W^s(O_2)$ split as $W^u(O_2)$ and $W^s(O_1)$ do when μ varies. Therefore, since these heteroclinic tangencies are quadratic, only one governing parameter is needed to control this splitting.

Let U be an small enough neighbourhood of the contour $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$. It can be represented as the union of two small neighbourhoods (disks) U_1 and U_2 of the saddles O_1 and O_2 and a finite number of small disks containing those points of Γ_{12} and Γ_{21} which do not belong to U_1 and U_2 (see Figure 2). We will focus our attention on the bifurcations of the so-called *single-round periodic orbits*, that is, orbits lying entirely in U and having exactly one intersection point with every disk from the set $U \setminus (U_1 \cup U_2)$. Any point of a single-round periodic orbit is a fixed point of the corresponding *first-return map* T_{km} , that is constructed by orbits of f_μ with k and m iterations (of f_μ) in U_1 and U_2 , respectively. We will call them single-round periodic orbit of type (k, m) . However, we do not state the problem of studying the maps T_{km} for all large k and m . We suppose k and m to be large enough integers such that

$$\lambda_1^k \simeq \lambda_2^m. \quad (2.2)$$

In other words, both values of $\lambda_1^k \lambda_2^{-m}$ and $\lambda_1^{-k} \lambda_2^m$ are *uniformly* separated from 0 and bounded as $k, m \rightarrow \infty$. The values of k and m are not arbitrary but clearly dependent. The first main result is as follows:

‡ However, the property of a symmetric saddle periodic point to be a priori area-preserving is more delicate. It is well-known (see, for instance, [4]) that a symmetric reversible saddle map is “almost conservative” since its (local) analytical normal form and its C^∞ formal normal form (up to “flat terms”) are conservative.)

Theorem 1 Let $\{f_\mu\}$ be a one-parameter family of reversible diffeomorphisms that unfolds, generally, at $\mu = 0$ the initial heteroclinic tangencies. Assume that f_0 satisfies conditions [A]-[C]. Then, in any segment $[-\epsilon, \epsilon]$ with $\epsilon > 0$ small, there are infinitely many intervals (not necessarily disjoint)

$$\delta_{km} = \left[\mu_{\text{fold}}^{(k,m)}, \mu_{\text{pf}}^{(k,m)} \right] \rightarrow \{0\} \quad \text{as } m, k \rightarrow \infty,$$

such that the following assertions hold:

- (i) The value $\mu = \mu_{\text{fold}}^{(k,m)}$ corresponds to a non-degenerate reversible and self-symmetric fold bifurcation and, thus, the diffeomorphism f_μ has at $\mu \in \delta_{km}$ two symmetric, saddle and elliptic, single-round periodic orbits of type (k, m) .
- (ii) The value $\mu = \mu_{\text{pf}}^{(k,m)}$ corresponds to a symmetric (and non-degenerate if condition [C] holds) pitch-fork bifurcation depending on the type of f_0 :
 - (ii)_a In the case of “inner” tangency, a symmetric couple of single-round attracting and repelling periodic orbits of type (k, m) is born and, moreover, these orbits undergo simultaneously non-degenerate period doubling bifurcations at the value $\mu = \mu_{\text{pd}}^{(k,m)}$ (where $\mu_{\text{pd}}^{(k,m)} \rightarrow 0$ as $k, m \rightarrow \infty$).
 - (ii)_b For the “outer” tangency, a symmetric couple of contracting-expanding single-round saddle periodic orbits of type (k, m) is born. Moreover, they do not bifurcate any more (at least for $|\mu| < \epsilon$).

We refer the reader to Figure 3 for an illustration of this theorem.

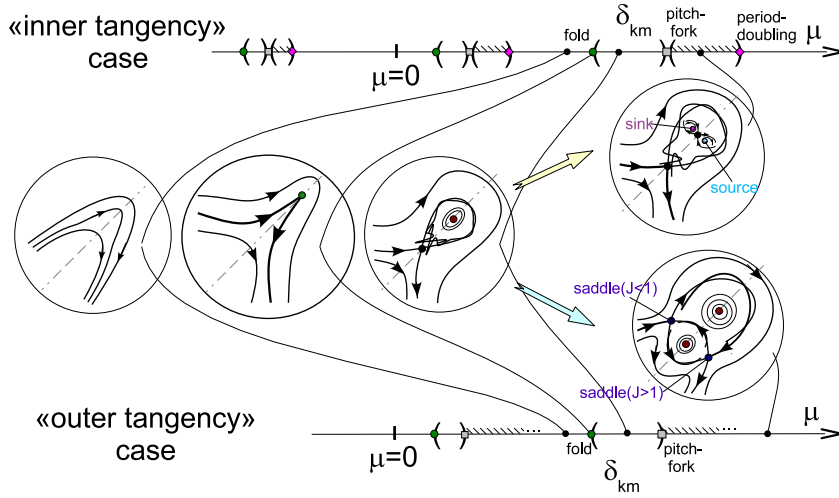


Figure 3. Shaded intervals correspond to the existence of a symmetric couple of single-round periodic orbits.

Theorem 1 and its counterpart result in [15] show that the appearance of non-conservative periodic orbits under global bifurcations can be considered as a certain general property of two-dimensional reversible maps.

Briefly, the method we use - based on a rescaling technique[§] - will allow us to prove that the first-return map T_{km} can be written asymptotically close (as $k, m \rightarrow \infty$) to an area-preserving map of the form:

$$H : \begin{cases} \bar{x} = \tilde{M} + \tilde{c}x - y^2, \\ \bar{y} = -\frac{\tilde{M}}{\tilde{c}} + \frac{1}{\tilde{c}}y + \frac{1}{\tilde{c}}(\tilde{M} + \tilde{c}x - y^2)^2, \end{cases} \quad (2.3)$$

in which the coordinates (x, y) and the parameters (\tilde{M}, \tilde{c}) can take arbitrary values except $\tilde{c} = 0$. The region $\tilde{c} < 0$ will stand for the “inner” tangency case and $\tilde{c} > 0$ for the “outer” one. Its bifurcation diagram is shown in Figure 8. The map (2.3) is, in fact, the product of two Hénon maps with Jacobian $-\tilde{c}$ and $-\tilde{c}^{-1}$. Thus, we can state (see also [31]) that map (2.3) has a complicated dynamics in the corresponding parameter intervals. **It can be seen**, in particular, that all fixed points become saddles and all of them have homoclinic and heteroclinic intersections for all values of the parameter μ including (quadratic) tangencies for dense subsets - *Newhouse phenomenon*. By Lemma 3, analogous “homoclinic tangle” is observed for the first-return map T_{km} which is reversible again but not conservative in general (see Lemma 4). It means that the map T_{km} will have four fixed points which are all saddles: two of them are self-symmetric and the other form a couple of symmetric fixed points with Jacobian greater and less than 1. It implies that periodic sinks and sources are involved into chaotic dynamics even in the “outer tangency” case, where the symmetry-breaking pitch-fork bifurcations lead directly to the appearance only of a couple of symmetric saddle periodic points. Thus, we can formulate the following result proving the RMD-Conjecture for the case under consideration.

Theorem 2 *Let $\{f_\mu\}$ be the one-parameter family of reversible maps from Theorem 1. Then, in any segment $[-\varepsilon, \varepsilon]$ of values of μ , there are Newhouse intervals with mixed dynamics connected with an abundance of attracting, repelling and elliptic periodic orbits. This is, values of parameters corresponding to maps f_μ exhibiting simultaneously infinitely many periodic orbits of all these types form a residual set (of second category) in these intervals.*

The proof of this Theorem is derived almost immediately from Theorem 1. Indeed, μ is the parameter which controls the splitting of the manifolds $W^u(O_1)$ and $W^s(O_2)$ and, by symmetry, of $W^u(O_2)$ and $W^s(O_1)$. Thus $\mu = 0$ becomes an accumulating point of homoclinic tangencies of $W^u(O_1)$ and $W^s(O_1)$ and, of course, of $W^u(O_2)$ and $W^s(O_2)$. From the work of Duarte [5], this implies that $\mu = 0$ is an accumulating point for Newhouse regions where the values of the parameter leading to quadratic homoclinic tangencies are dense. By Palis λ -Lemma, any tangency point between $W^u(O_i)$ and $W^s(O_i)$ is, in its turn, an accumulating point of pieces of both stable and unstable manifolds. This implies that, in the Newhouse intervals provided by Theorem 1, there exists a dense set of values of μ giving rise to heteroclinic tangencies between $W^u(O_1)$ and $W^s(O_2)$ and, by symmetry, between $W^u(O_2)$ and $W^s(O_1)$. Thus, Theorem 2 follows from the standard procedure of “embedded intervals” associated to the existence of periodic sinks, sources and elliptic points. We refer the reader to [10, 15, 13] for more details about this procedure.

[§] The rescaling method [29] appears to be very efficient to study homoclinic bifurcations.

3. Proof of Theorem 1

3.1. Preliminary geometric and analytic constructions

To ease the reading all the proofs of the lemmas of this section have been deferred to Sections 5, 6 and 6.3.

Let us consider first the map f_0 and let $M_1^- \in U_1$, $M_2^+ \in U_2$ be a pair of points of the orbit Γ_{12} and $M_2^- \in U_2$, $M_1^+ \in U_1$ be a pair of points of Γ_{21} . Consider $\Pi_i^+ \subset U_i$ and $\Pi_i^- \subset U_i$ small neighbourhoods of the heteroclinic points M_i^+ and M_i^- (see Figure 4). One can always take $U_i, \Pi_i^-, \Pi_i^+, i = 1, 2$ to be also R -symmetric (that is $R(U_i) = U_i, R(\Pi_i^-) = \Pi_i^-$). Let us assume that (i) the heteroclinic points are symmetric under the involution R , i.e. $M_1^- = R(M_1^+)$ and $M_2^- = R(M_2^+)$, and (ii) they are the “last” points on U_1 and U_2 , that is, $f_0(M_i^-) \notin U_i$ (and, thus, $f_0^{-1}(M_i^+) \notin U_i$). Let q be such a positive integer that $M_2^+ = f_0^q(M_1^-)$ (and, thus, $M_1^+ = f_0^q(M_2^-)$).

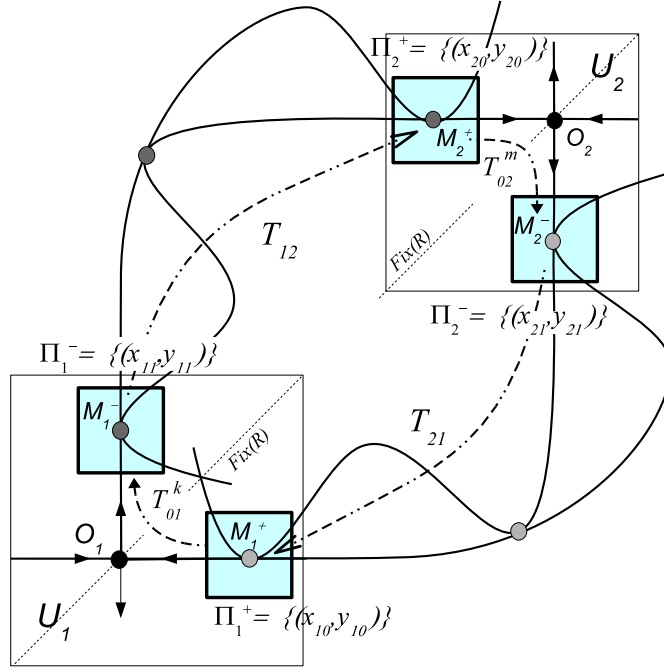


Figure 4. Schematic actions of the local (T_{01}^k and T_{02}^m) and global (T_{12} and T_{21}) maps in the neighbourhood U of the contour $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$.

Consider now the map f_μ . Denote $T_{0i} \equiv f_\mu|_{U_i}$, $i = 1, 2$. The μ -dependent maps T_{01} and T_{02} are called *the local maps*. We introduce also the so-called *global maps* T_{12} and T_{21} by the following relations: $T_{12} \equiv f_\mu^q : \Pi_1^- \rightarrow \Pi_2^+$ and $T_{12} \equiv f_\mu^q : \Pi_2^- \rightarrow \Pi_1^+$ (see Figure 4). Then the *first-return map* $T_{km} : \Pi_1^+ \mapsto \Pi_1^+$ is defined by the following composition of maps and neighbourhoods:

$$\Pi_1^+ \xrightarrow{T_{01}^k} \Pi_1^- \xrightarrow{T_{12}} \Pi_2^+ \xrightarrow{T_{02}^m} \Pi_2^- \xrightarrow{T_{21}} \Pi_1^+ \quad (3.1)$$

Denote local coordinates on Π_i^+ and Π_i^- as (x_{0i}, y_{0i}) and (x_{1i}, y_{1i}) , respectively. Then the chain (3.1) can be represented as

$$(x_{01}, y_{01}) \xrightarrow{T_{01}^k} (x_{11}, y_{11}) \xrightarrow{T_{12}} (x_{02}, y_{02}) \xrightarrow{T_{02}^m} (x_{12}, y_{12}) \xrightarrow{T_{21}} (\bar{x}_{01}, \bar{y}_{01}).$$

As usual, we need such local coordinates on U_1 and U_2 in which the maps T_{01} and T_{02} have their simplest form. We can not assume the maps T_{0i} are linear, since by condition [A], only C^1 -linearisation is possible here. Therefore, we consider such C^{r-1} -coordinates in which the local maps have the so-called *main normal form* or first order normal form.

Lemma 1 (Main normal form of a saddle map) *Let a C^r -smooth map T_0 be reversible with $\dim \text{Fix } T_0 = 1$. Suppose that T_0 has a saddle fixed (periodic) point O belonging to the line $\text{Fix } T_0$ and having multipliers λ and λ^{-1} , with $|\lambda| < 1$. Then there exist C^{r-1} -smooth local coordinates near O in which the map T_0 (or T_0^n , where n is the period of O) can be written in the following form:*

$$T_0 : \begin{cases} \bar{x} = \lambda x(1 + h_1(x, y)xy) \\ \bar{y} = \lambda^{-1}y(1 + h_2(x, y)xy), \end{cases} \quad (3.2)$$

where $h_1(0, 0) = -h_2(0, 0)$. The map (3.2) is reversible with respect to the standard linear involution $(x, y) \mapsto (y, x)$. In fact, it can be expressed in the so-called cross-form (see Section 5.1):

$$T_0 : \begin{cases} \bar{x} = \lambda x + \hat{h}(x, \bar{y})x^2\bar{y}, \\ y = \lambda \bar{y} + \hat{h}(\bar{y}, x)x\bar{y}^2. \end{cases} \quad (3.3)$$

The proof of this lemma has been deferred to Subsection 5.2.

In the case that T_0 is linear, i.e. of the form $\bar{x} = \lambda x$, $\bar{y} = \lambda^{-1}y$, its j -th iterates $(x_j, y_j) = T_0^j(x_0, y_0)$ are given simply by $x_j = \lambda^j x_0$, $y_j = \lambda^{-j} y_0$ in standard explicit form or by $x_j = \lambda^j x_0$, $y_0 = \lambda^j y_j$ in cross-form. If T_0 is nonlinear such cross-form expression for T_0^j exists too. Precisely, the following result holds:

Lemma 2 (Iterations of the local map) *Let T_0 be a saddle map written in the main normal form (3.2) (or (3.3)) in a small neighbourhood V of O . Let us consider points $(x_0, y_0), \dots, (x_j, y_j)$ from V such that $(x_{l+1}, y_{l+1}) = T_0(x_l, y_l)$, $l = 0, \dots, j-1$. Then one has*

$$\begin{aligned} x_j &= \lambda^j x_0 (1 + j\lambda^j h_j(x_0, y_j)), \\ y_0 &= \lambda^j y_j (1 + j\lambda^j h_j(y_j, x_0)), \end{aligned} \quad (3.4)$$

where the functions $h_j(y_j, x_0)$ are uniformly bounded with respect to j as well as all their derivatives up to order $r-2$.

Remark 1 (a) *Both lemmas 1 and 2 are true if T_0 depends on parameters.*

Moreover, if the initial T_0 is C^r with respect to coordinates and parameters, then the normal form (3.2) is C^{r-1} with respect to coordinates and C^{r-2} with respect to parameters (see [14], Lemmas 6 and 7).

(b) *Bochner Theorem (see [16]) ensures that any involution R with $\dim \text{Fix } R = 1$ is locally smoothly conjugated to its linear part around a symmetric point. It is not a loss of generality to assume that maps f_μ are reversible under an involution R with linear part given by $L(x, y) = (y, x)$. As it will be shown, this fact will be very convenient in the construction of the local maps T_{01} and T_{02} .*

(c) Similar results related to finite-smooth normal forms of saddle maps were established in [7, 11, 12, 13] for general, near-conservative and conservative maps. The proof of our lemmas are just an adapted version to the reversible setting of their proofs.

3.2. Construction of the local and global maps

By Lemma 1 one can choose in U_1 and U_2 local coordinates (x_1, y_1) and (x_2, y_2) , respectively, such that the maps T_{01} and T_{02} take the following form:

$$T_{01} : \bar{x}_1 = \lambda_1 x_1 + h_1^1(x_1, y_1) x_1^2 y_1, \quad \bar{y}_1 = \lambda_1^{-1} y_1 + h_2^1(x_1, y_1) x_1 y_1^2,$$

and

$$T_{02} : \bar{x}_2 = \lambda_2 x_2 + h_1^2(x_2, y_2) x_2^2 y_2, \quad \bar{y}_2 = \lambda_2^{-1} y_2 + h_2^2(x_2, y_2) x_2 y_2^2.$$

Furthermore, in these coordinates, the local stable and unstable invariant manifolds of both points O_1 and O_2 are straightened: $x_i = 0$ is the equation of $W_{loc}^u(O_i)$ and $y_i = 0$ is the equation of $W_{loc}^s(O_i)$, $i = 1, 2$. Then, we can write the (x, y) -coordinates of the chosen heteroclinic points as follows: $M_1^+ = (x_1^+, 0)$, $M_1^- = (0, y_1^-)$, $M_2^+ = (x_2^+, 0)$ and $M_2^- = (0, y_2^-)$. Besides, because of the reversibility, we have that

$$x_1^+ = y_1^- = \alpha_1^*, \quad x_2^+ = y_2^- = \alpha_2^* \quad (3.5)$$

We assume that $T_{0i}(\Pi_i^+) \cap \Pi_i^+ = \emptyset$ and $T_{0i}^{-1}(\Pi_i^-) \cap \Pi_i^- = \emptyset$, $i = 1, 2$. Then the domain of definition of the successor map from Π_i^+ into Π_i^- under iterations of T_{0i} consists of infinitely many non-intersecting strips σ_j^{0i} which belong to Π_i^+ and accumulate at $W_{loc}^s(O_i) \cap \Pi_i^+$ as $j \rightarrow \infty$. Analogously, the range of this map consists of infinitely many strips $\sigma_j^{1i} = T_{0i}^j(\sigma_j^{0i})$ belonging to Π_i^- and accumulating at $W_{loc}^u(O_i) \cap \Pi_i^-$ as $j \rightarrow \infty$ (see Figure 5).

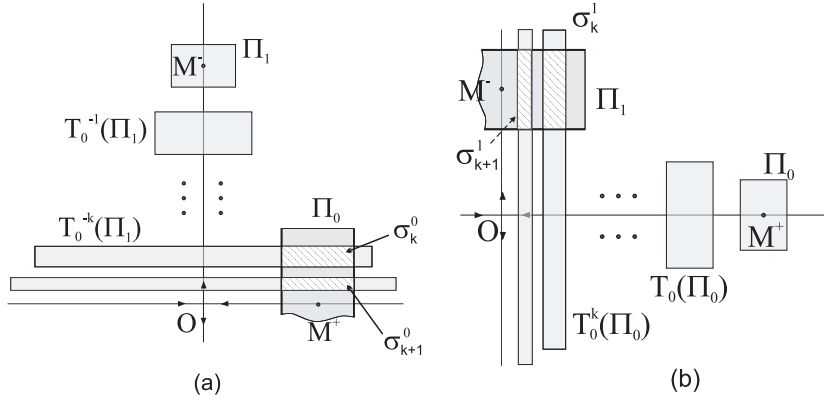


Figure 5. A geometry of creation of both domains of definition $\sigma_i^0 \subset \Pi^+$ (a) and domains of the range $\sigma_i^1 \subset \Pi^-$ (b) for the maps $T_0^i : \Pi^+ \rightarrow \Pi^-$.

It follows from Lemma 2 that the map $T_{01}^k : \sigma_k^{01} \mapsto \sigma_k^{11}$ can be written in the following form (for large enough values of k)

$$T_{01}^k : \begin{cases} x_{11} = \lambda_1^k x_{01} (1 + k \lambda_1^k h_1^1(x_{01}, y_{11})) \\ y_{01} = \lambda_1^k y_{11} (1 + k \lambda_1^k h_2^1(y_{11}, x_{01})) \end{cases} \quad (3.6)$$

and an analogous formula takes place for the map $T_{02}^m : \sigma_m^{02} \mapsto \sigma_m^{12}$:

$$T_{02}^m : \begin{cases} x_{12} = \lambda_2^m x_{02} (1 + m \lambda_2^m h_m^2(x_{02}, y_{12})) \\ y_{02} = \lambda_2^m y_{12} (1 + m \lambda_2^m h_m^2(y_{12}, x_{02})) \end{cases}$$

We write now the global map T_{12} in the following form

$$T_{12} \begin{cases} x_{02} - x_2^+ = F_{12}(x_{11}, y_{11} - y_1^-, \mu) \equiv ax_{11} + b(y_{11} - y_1^-) + \\ \quad l_{02}(y_{11} - y_1^-)^2 + \varphi_1(x_{11}, y_{11}, \mu), \\ y_{02} = G_{12}(x_{11}, y_{11} - y_1^-, \mu) \equiv \mu + cx_{11} + d(y_{11} - y_1^-)^2 + \\ \quad f_{11}x_{11}(y_{11} - y_1^-) + f_{03}(y_{11} - y_1^-)^3 + \\ \quad \varphi_2(x_{11}, y_{11}, \mu), \end{cases} \quad (3.7)$$

where $F_{12}(0) = G_{12}(0) = 0$ since $T_{12}(M_1^-) = M_2^+$ at $\mu = 0$ and

$$\begin{aligned} \varphi_1 &= O(|y_{11} - y_1^-|^3) + x_{11} O(\|(x_{11}, y_{11} - y_1^-)\|), \\ \varphi_2 &= O(|x_{11}|^2) + O(|y_{11} - y_1^-|^4) + O(x_{11}(y_{11} - y_1^-)^2). \end{aligned}$$

Since the curves $T_{12}(W_{loc}^u(O_1) : \{x_{11} = 0\})$ and $W_{loc}^s(O_2) : \{y_{02} = 0\}$ have a quadratic tangency at $\mu = 0$, it implies that

$$\frac{\partial G_{12}(0)}{\partial y_{11}} = 0, \quad \frac{\partial^2 G_{12}(0)}{\partial y_{11}^2} = 2d \neq 0.$$

The Jacobian $J(T_{12})$ has, obviously, the following form:

$$J(T_{12}) = -bc + af_{11}x_{11} + Q(y_{11} - y_1^-) + O(x_{11}^2 + (y_{11} - y_1^-)^2),$$

where

$$Q = 2ad - bf_{11} - 2cl_{02}. \quad (3.8)$$

Now condition [C] can be formulated more precisely. Namely, we require that

$$Q = \left. \frac{\partial J(T_{12})}{\partial y_{11}} \right|_{(x_{11}=0, y_{11}=y_1^-, \mu=0)} \neq 0$$

Concerning the global map T_{21} , we cannot write it now in an arbitrary form. The point is that after written a formula for the map T_{12} it is necessary to use the reversibility relations to get the one associated to it:

$$T_{21} = R T_{12}^{-1} R^{-1}, \quad T_{12} = R T_{21}^{-1} R^{-1}$$

for constructing T_{21} . Then, by (3.7), we obtain that the map $T_{21}^{-1} : \Pi_1^+ \{(x_{01}, y_{01})\} \mapsto \Pi_2^- \{(x_{12}, y_{12})\}$ must be written as follows

$$T_{21}^{-1} \begin{cases} x_{12} = G_{12}(y_{01}, x_{01} - y_1^-, \mu) = \\ \quad \mu + cy_{01} + d(x_{01} - y_1^-)^2 + f_{11}y_{01}(x_{01} - y_1^-) + \\ \quad f_{13}(x_{01} - y_1^-)^2 + \varphi_2(y_{01}, x_{01}, \mu), \\ y_{12} - x_2^+ = F_{12}(y_{01}, x_{01} - y_1^-, \mu) = \\ \quad ay_{01} + b(x_{01} - y_1^-) + l_{02}(x_{01} - y_1^-)^2 + \varphi_1(y_{01}, x_{01}, \mu) \end{cases} \quad (3.9)$$

Relation (3.9) allows to define the map $T_{21} : \Pi_2^- \{(x_{12}, y_{12})\} \mapsto \Pi_1^+ \{(x_{01}, y_{01})\}$, but in implicit form.

3.3. Construction of the first-return maps T_{km} and the Rescaling Lemma

Using relations (3.6)–(3.9), we can construct the first-return map $T_{km} = T_{21}T_{02}^mT_{12}T_{01}^k$ defined on the strip $\sigma_k^{01} \subset \Pi_1^+$. Recall that any fixed point of T_{km} corresponds to a single-round periodic orbit of type (k, m) of period $(k + m + 2q)$. We suppose k and m to be large enough integers satisfying the relation (2.2). Then the following result holds.

Lemma 3 (The rescaling lemma) *Let the map f_0 satisfy conditions [A]–[B] and $\{f_\mu\}$ be a general unfolding in the class of reversible maps. Suppose k and m are large enough integer numbers satisfying relation (2.2). Then there exist coordinates (called “rescaled coordinates”) in such a way that the first-return map $T_{km} : (x, y) \mapsto (\bar{x}, \bar{y})$ is given implicitly by*

$$\begin{aligned} M + c\bar{y} + d\bar{x}^2 + f_{11}\lambda_1^k\bar{x}\bar{y} + f_{03}\lambda_1^k\bar{x}^3 &= \\ &= b\lambda_2^m\lambda_1^{-k}y + a\lambda_2^m x + l_{02}\lambda_2^m y^2 + O(k\lambda_1^{2k}), \\ M + cx + dy^2 + f_{11}\lambda_1^k xy + f_{03}\lambda_1^k y^3 &= \\ &= b\lambda_2^m\lambda_1^{-k}\bar{x} + a\lambda_2^m\bar{y} + l_{02}\lambda_2^m\bar{x}^2 + O(k\lambda_1^{2k}), \end{aligned} \quad (3.10)$$

where

$$M = \lambda_1^{-2k} (\mu + c\lambda_1^k\alpha_1^*(1 + \dots) - \lambda_2^m\alpha_2^*(1 + \dots)) \quad (3.11)$$

and “...” stands for some coefficients tending to zero as $k, m \rightarrow \infty$. Moreover, the domain of definition of the new coordinates x and y and parameter M cover all finite values as $k, m \rightarrow \infty$.

3.4. On bifurcations of fixed points of the first-return maps T_{km}

We will approach the study of bifurcations of the first-return map T_{km} through its rescaled form (3.10). Indeed, neglecting in the latter equation all the asymptotically small terms (as $k, m \rightarrow \infty$), we obtain the following truncated form for T_{km}

$$\begin{cases} M + c\bar{y} + d\bar{x}^2 &= \beta_{km}y, \\ \beta_{km}\bar{x} &= M + cx + dy^2, \end{cases} \quad (3.12)$$

given in cross-form and where $\beta_{km} = b\lambda_1^{-k}\lambda_2^m \neq 0$ is, by hypothesis (see (2.2)), a coefficient which is uniformly bounded from zero and infinity for all admissible k and m . Introducing the following rescaled coordinates

$$x = -\frac{\beta_{km}}{d}x_{\text{new}}, \quad y = -\frac{\beta_{km}}{d}y_{\text{new}}.$$

and denoting again $(x_{\text{new}}, y_{\text{new}})$ as (x, y) , we reach the following map (in cross-form)

$$H : \begin{cases} \tilde{M} + \tilde{c}\bar{y} - \bar{x}^2 &= y, \\ \bar{x} &= \tilde{M} + \tilde{c}x - y^2, \end{cases} \quad (3.13)$$

with

$$\tilde{M} = -\frac{d}{\beta_{km}^2} M, \quad \tilde{c} = \frac{c}{\beta_{km}} = \frac{c}{b}\lambda_1^k\lambda_2^{-m} \quad (3.14)$$

Condition $c \neq 0$ implies two different scenarios for H in space of parameters (\tilde{c}, \tilde{M}) , depending on the sign of \tilde{c} .

Observe that the map H can be expressed in the explicit form (2.3) and, moreover, it can be represented as the superposition $H = \mathcal{H}_2 \circ \mathcal{H}_1$ of two Hénon maps

$$\mathcal{H}_1 : \begin{cases} \bar{x} &= y \\ \bar{y} &= \tilde{M} + \tilde{c}x - y^2, \end{cases}$$

and

$$\mathcal{H}_2 : \begin{cases} \bar{x} &= y \\ \bar{y} &= -\frac{\tilde{M}}{\tilde{c}} + \frac{1}{\tilde{c}}x + \frac{1}{\tilde{c}}y^2 \end{cases}$$

The Jacobians of these maps are constant and inverse one of each other: $J(\mathcal{H}_1) = -\tilde{c}$ and $J(\mathcal{H}_2) = -\tilde{c}^{-1}$. Therefore, the resulting map H has Jacobian equal to 1 and is, therefore, area-preserving.

The symmetric form (3.13) of H provides a simple geometric interpretation for its fixed points. Indeed, they must satisfy

$$y(1 - \tilde{c}) = \tilde{M} - x^2, \quad x(1 - \tilde{c}) = \tilde{M} - y^2, \quad (3.15)$$

which correspond, for $\tilde{c} \neq 1$, to the intersection points of a symmetric (with respect to the bisectrix $x = y$) couple of parabolas. For any fixed value of $\tilde{c} \neq 1$ varying \tilde{M} one obtain several possible intersections. For instance, in the case $\tilde{c} < 1$ (represented

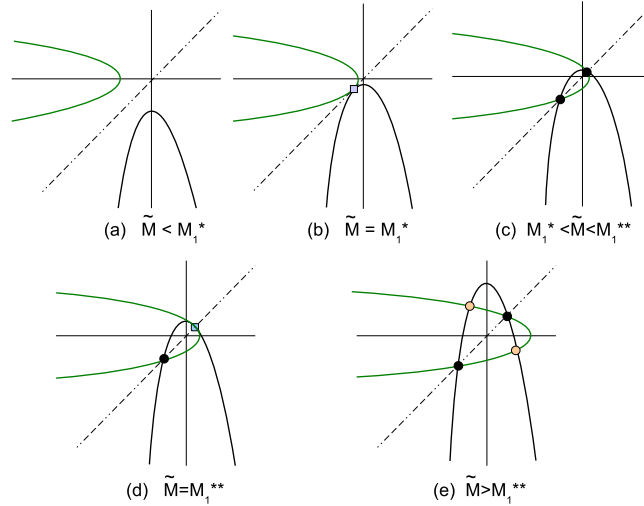


Figure 6. A geometric interpretation of the fixed points of map H in (3.13) as intersection points of a symmetric couple of parabolas (case $\tilde{c} < 1$)

in Figures 6 and 7), the following five situations are given:

- (a) the two parabolas do not intersect if $\tilde{M} < M_1^* = -\frac{1}{4}(\tilde{c} - 1)^2$;
- (b) they have a quadratic contact among them and with the bisectrix when $\tilde{M} = M_1^*$;
- (c) they have two symmetric (that is, belonging to its symmetry line $x = y$) intersection points provided $M_1^* < \tilde{M} < M_2^* = \frac{3}{4}(\tilde{c} - 1)^2$;
- (d) they present a cubic symmetric tangency when $\tilde{M} = M_2^*$ and, finally,

- (e) the two parabolas intersect in four points if $\tilde{M} > M_2^*$: two symmetric points and a symmetric couple of points.

A similar picture takes place for $\tilde{c} > 1$. However, case $\tilde{c} = 1$ is very special: equation (3.15) becomes $0 = \tilde{M} - x^2, 0 = \tilde{M} - y^2$ and, therefore a kind of "0-4"-bifurcation occurs at $\tilde{M} = 0$. There are no fixed points of H for $\tilde{M} < 0$ and four fixed points appear for $\tilde{M} > 0$.

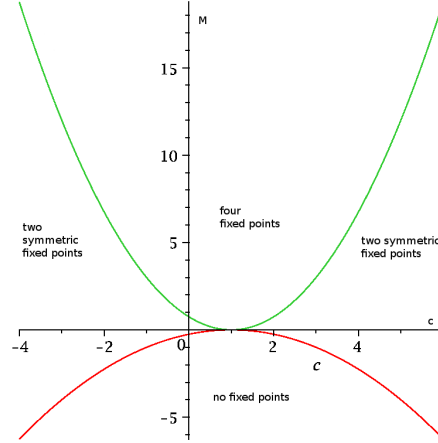


Figure 7. Regions of fixed points of map H in the space of parameters (\tilde{c}, \tilde{M})

Let us show with same detail the bifurcations one finds when varying the parameters (\tilde{c}, \tilde{M}) . We divide this study in two cases: $\tilde{c} < 0$ and $\tilde{c} > 0$ (which contains, in its turn, the cases $0 < \tilde{c} < 1$, $\tilde{c} = 1$ and $\tilde{c} > 1$). We start with the first one.

- Case $\tilde{c} < 0$:

As we have shown above, no fixed points of map H appear for a fixed $\tilde{c} < 0$ until the parameter \tilde{M} reaches the value \tilde{M}^* . At that point, a *fold bifurcation* occurs. This bifurcation is characterized by the appearance of two fixed points, saddle and elliptic ones, and has the equation

$$F_{1,2} : \quad \tilde{M} = -\frac{1}{4}(\tilde{c} - 1)^2 \quad (3.16)$$

(this curve is denoted in Figure 8 by F_1 and F_2 , associated to $\tilde{c} < 0$ and $\tilde{c} > 0$, respectively). If we call $P_+ = (p_+, p_+)$ and $P_- = (p_-, p_-)$ the (symmetric) fixed points, it is straightforward to check that they are given by

$$p_{\pm} = \frac{\tilde{c} - 1 \pm \sqrt{(\tilde{c} - 1)^2 + 4\tilde{M}_1^*}}{2}, \quad (3.17)$$

where we recall that $\tilde{M}_1^* = -\frac{1}{4}(\tilde{c} - 1)^2$.

The point P_- will remain a saddle for any value of $\tilde{M} \geq \tilde{M}^*$ while point P_+ will undergo several bifurcations. Precisely, *period-doubling bifurcations* occur when

the parameter \tilde{M} crosses the curves

$$PD^1(P_+) : \tilde{M} = 1 - \frac{1}{4}(\tilde{c} - 1)^2,$$

$$PD^3(P_+) : \tilde{M} = \frac{(\tilde{c} + 1)(3\tilde{c} - 1)}{4}.$$

In Figure 8 some “grey arrows” indicate the directions of birth of these period-2 points.

Besides these period-doubling bifurcations, a *pitch-fork bifurcation* curve exists for this point P_+ . Indeed, when crossing the curve

$$PF^1(P_+) : \tilde{M} = \frac{3}{4}(\tilde{c} - 1)^2,$$

(from region V_l to region VI_l in Figure 8) the point P_+ becomes a saddle fixed point - that we denote again by P_+ - and a symmetric couple of *elliptic* fixed points P_3 and P_4 appear. When increasing \tilde{M} , the point P_+ does not bifurcate any more whereas the points P_3 and P_4 undergo simultaneously period-doubling bifurcation at crossing the curve

$$PD(P_{3,4}) : \tilde{M} = \frac{(1 - 3\tilde{c})(3 - \tilde{c})}{4}.$$

Further variation of parameters at the region VII_l in Figure 8 will lead to a cascade of period-doubling bifurcations of symmetric couples of periodic points.

- Case $\tilde{c} > 0$:

As it has been mentioned for the case $\tilde{c} < 0$, a fold-bifurcation curve is found at the curve

$$\tilde{M} = -\frac{1}{4}(\tilde{c} - 1)^2,$$

denoted for $\tilde{c} > 0$ by F_2 in Figure 8. In this case, both $P_- = (p_-, p_-)$ and $P_+ = (p_+, p_+)$ fixed points, given by (3.17), undergo pitch-fork and period-doubling bifurcations. Namely, for P_+ , these bifurcations curves are, respectively,

$$PF^2(P_+) : \tilde{M} = \frac{3}{4}(\tilde{c} - 1)^2 \quad \text{for } 0 < \tilde{c} < 1,$$

$$PD^2(P_+) : \tilde{M} = 1 - \frac{1}{4}(\tilde{c} - 1)^2, \quad \text{for } \tilde{c} > 0.$$

Regarding the point P_- one has the following pitch-fork and period-doubling bifurcation curves:

$$PF(P_-) : \tilde{M} = \frac{3}{4}(\tilde{c} - 1)^2 \quad \text{for } \tilde{c} > 1,$$

$$PD(P_-) : \tilde{M} = \frac{(\tilde{c} + 1)(3\tilde{c} - 1)}{4} \quad \text{for } \tilde{c} > 0.$$

(3.18)

When crossing the (pitch-fork bifurcation) curve $PF^2(P_+) \cup PF(P_-)$, a symmetric couple of *saddle* fixed points P_3 and P_4 are born but they do not bifurcate any more when increasing the value of \tilde{M} .

One can expect, however, that bifurcations of the symmetric (elliptic) fixed points P_+ and P_- give rise to cascades of period-doubling bifurcations [32].

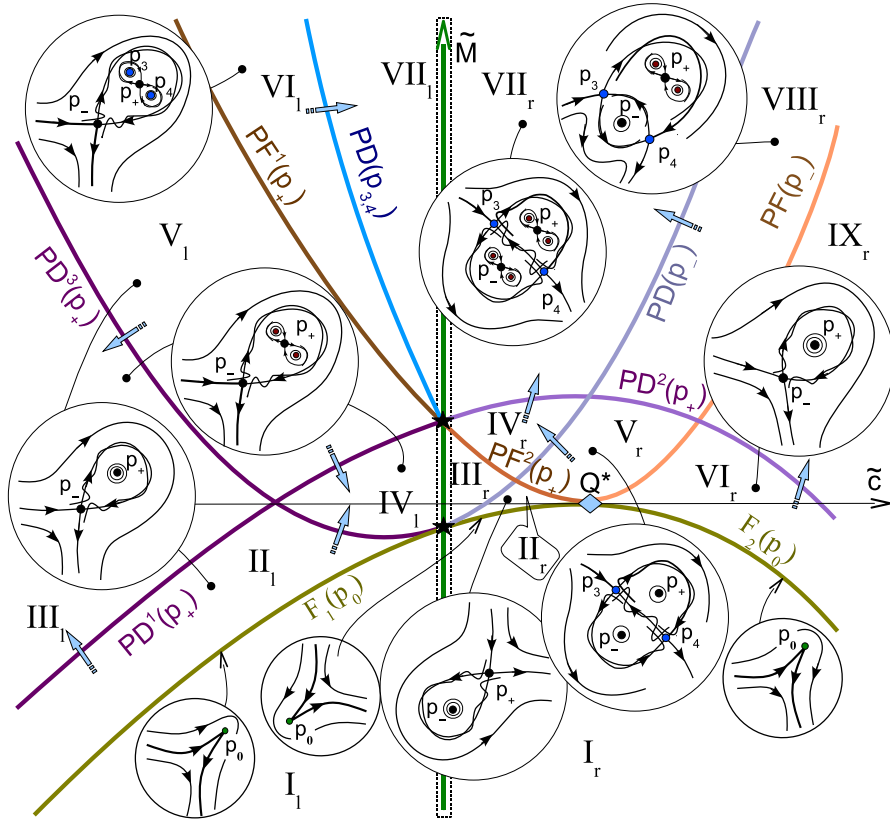


Figure 8. Elements of the bifurcation diagram for the map H .

Even though map H is a truncation of the map T_{km} , it is important to stress that its behavior around the fixed points appearing from the pitch-fork, period-doubling and fold bifurcations is not exactly the same. Indeed, pitch-fork and period-doubling bifurcations can lead in T_{km} to a slightly different behavior of the truncated area-preserving map H . Namely, pitch-fork bifurcations in T_{km} lead again to the appearance of two non-symmetric fixed points P_3 and P_4 but these points can be non-conservative. In fact, this seems to be a quite general situation as the following lemma shows.

Lemma 4 (Non-conservative fixed points) *The non-symmetric fixed points P_3 and P_4 of map (3.10), with k and m satisfying (2.2), have Jacobian J_{ns} and J_{ns}^{-1} , respectively, with*

$$J_{ns} = 1 + \frac{Q(\eta^* - \xi^*)}{bc} \lambda_1^k + o(\lambda_1^k), \quad (3.19)$$

where Q is the coefficient given by formula (3.8) and ξ^* and η^* are, respectively, the x - and y -coordinate of the fixed point (P_3 and P_4).

On the other hand, fold bifurcations of the map T_{km} lead, similarly to what happens for the map H , to the appearance of two symmetric (saddle and elliptic) fixed points

P_+ and P_- . Concerning the symmetric elliptic fixed point, we have the following result.

Lemma 5 (Symmetric elliptic fixed points) *The point P_+ is generic elliptic (i.e. KAM-stable) for open and dense sets of values of the parameters (\tilde{c}, \tilde{M}) in the domains $II_l \cup V_l$ and $IV_r \cup V_r \cup VI_r$. The same happens for the elliptic point P_- in the domain $II_r \cup V_r \cup VIII_r$.*

The proofs of lemmas 3 to 5 have been deferred to Section 6.3.

3.5. End of the proof of Theorem 1.

If k and m are large enough integers satisfying (2.2) and having in mind expressions (3.11) and (3.14), the following relation between the parameters μ and \tilde{M} holds:

$$\mu = \lambda_2^m \alpha_2^*(1 + \rho_k^1) - c\lambda_1^k \alpha_1^*(1 + \rho_k^2) - \frac{b^2}{d} \tilde{M} \lambda_2^{2m} (1 + \rho_k^3),$$

where ρ_k^i , $i = 1, 2, 3$, are some small coefficients ($\rho_k^i \rightarrow 0$ as $k \rightarrow \infty$). Using formulas (3.16)–(3.18) for the bifurcation curves of the truncated map (3.13), asymptotically close to the (rescaled) first-return map (3.10), we find the following expressions for the bifurcation values of μ at the statement of Theorem 1: a value $\mu = \mu_{\text{fold}}^{(k,m)}$, which corresponds to the fold bifurcation in T_{km} ,

$$\begin{aligned} \mu_{\text{fold}}^{(k,m)} &= \lambda_2^m \alpha_2^*(1 + \rho_k^1) - c\lambda_1^k \alpha_1^*(1 + \rho_k^2) + \\ &\quad \frac{1}{4d} (b\lambda_2^m - c\lambda_1^k)^2 (1 + \rho_k^3) \end{aligned}$$

and a value $\mu = \mu_{pf}^{(k,m)}$, associated to the pitch-fork bifurcation in T_{km} (which is not conservative if we have in mind Lemma 4),

$$\begin{aligned} \mu_{pf}^{(k,m)} &= \lambda_2^m \alpha_2^*(1 + \rho_k^1) - c\lambda_1^k \alpha_1^*(1 + \rho_k^2) - \\ &\quad \frac{3}{4d} (b\lambda_2^m - c\lambda_1^k)^2 (1 + \rho_k^3). \end{aligned}$$

Finally, using formula (3.4) for the curve $PD(P_{3,4})$ of simultaneous period doubling bifurcation of the fixed points P_3 and P_4 we find that

$$\begin{aligned} \mu_{pd}^{(k,m)} &= \lambda_2^m \alpha_2^*(1 + \rho_k^1) - c\lambda_1^k \alpha_1^*(1 + \rho_k^2) - \\ &\quad \frac{1}{4d} (b\lambda_2^m - 3c\lambda_1^k)(3b\lambda_2^m - c\lambda_1^k)(1 + \rho_k^3). \end{aligned}$$

These considerations imply Theorem 1.

4. On applied reversible maps with mixed dynamics

In this section we present two concrete examples of reversible differential equations which provide a framework where Theorem 1 applies and exhibiting, therefore, mixed dynamics: a periodically perturbed Duffing equation and the Pikovsky-Topaj model [21] for coupled rotators.

4.1. A periodic perturbation of the Duffing Equation

Let us consider the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + x^3 + \varepsilon(\alpha + \beta y \sin \omega t), \end{cases} \quad (4.1)$$

where $\alpha, \beta, \omega \in \mathbb{R}$ and ε is a small perturbation parameter. The unperturbed system, for $\varepsilon = 0$, corresponds to the so-called Duffing system (also called *Anti-Duffing* for several authors). It is Hamiltonian, with

$$H(x, y) = \frac{y^2}{2} + V(x), \quad V(x) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{4}$$

and is (time)-reversible with respect to the following linear involutions $R(x, y) = (x, -y)$ and $S(x, y) = (-x, y)$. This system has three singular points: one elliptic at $(0, 0)$ and two saddles at $(\pm 1, 0)$. Moreover, these two points are connected through a symmetric couple of heteroclinic orbits Γ_h^- .

The perturbed system, for $\varepsilon \neq 0$, is still R -reversible but, in principle, non necessarily Hamiltonian. This is a particular case of a more general family of R -reversible perturbations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + x^3 + \varepsilon g(x, y, t), \end{cases}$$

satisfying that $g(x, -y, -t) = g(x, y, t)$. It is well known that two (symmetric) hyperbolic periodic orbits γ_ε^- appear close to the saddle points $(\pm 1, 0)$. Let us denote by $W^{u,s}(\gamma_\varepsilon^-)$ their corresponding unstable and stable invariant manifolds, respectively. Generically these invariant manifolds will intersect each other transversally and will remain close to the unperturbed heteroclinic connection. Its splitting is given, at first order in ε , by the celebrated Poincaré-Melnikov-Arnol'd function

$$M(t_0) = \int_{-\infty}^{+\infty} L_G F(\Gamma_h(t)) dt,$$

where $F(x, y) = (y, -x + x^3)$, $G(x, y) = \varepsilon(0, \alpha + \beta y \sin \omega t)$, $\Gamma_h(t)$ is any of both unperturbed heteroclinic connections $\Gamma_h^\pm(t)$ and where $L_G(F) = (DF)G$ corresponds to the Lie derivative of F with respect to G . Simple zeroes of $M(t_0)$ provide tangent intersections between the invariant manifolds $W^{u,s}(\gamma_\varepsilon^-)$. This system constitutes a good candidate to apply our results.

For the computation of $M(t_0)$ we consider here the positive heteroclinic orbit $\Gamma_h = \Gamma_h^+$ but, by symmetry, everything applies exactly for Γ_h^- . Thus,

$$\Gamma_h(t) = (x_h(t), y_h(t)) = (x_h(t), \dot{x}_h(t)) = \left(\tanh \frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \operatorname{sech}^2 \frac{t}{\sqrt{2}} \right)$$

and

$$M(t_0) = \int_{-\infty}^{+\infty} (DF)G|_{(x_h(t), y_h(t), t+t_0)} dt = \quad (4.2)$$

$$\begin{aligned} & \int_{-\infty}^{+\infty} y_h(t) \left(\alpha + \frac{\beta}{\sqrt{2}} \left(\operatorname{sech}^2 \frac{t}{\sqrt{2}} \right) \sin \omega(t+t_0) \right) dt = \\ & \frac{\varepsilon \alpha}{\sqrt{2}} \int_{-\infty}^{+\infty} \operatorname{sech}^2 \frac{t}{\sqrt{2}} dt + \frac{\varepsilon \beta}{2} \int_{-\infty}^{+\infty} \left(\operatorname{sech}^4 \frac{t}{\sqrt{2}} \right) \sin \omega(t+t_0) dt = \\ & \frac{\varepsilon \alpha}{\sqrt{2}} I_1 + \frac{\varepsilon \beta}{2} I_2. \end{aligned} \quad (4.3)$$

Concerning I_1 it is straightforward to check that its value is 2. Regarding I_2 , it is more convenient to compute the integral

$$\int_{-\infty}^{+\infty} \left(\operatorname{sech}^4 \frac{t}{\sqrt{2}} \right) e^{i\omega(t+t_0)} dt$$

using the method of residues. Indeed, from it we can derive that

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(\operatorname{sech}^4 \frac{t}{\sqrt{2}} \right) \sin \omega(t+t_0) dt &= \frac{2\pi}{3} \frac{\omega^2(\omega^2+2)}{\sinh \frac{\omega\pi}{\sqrt{2}}} \sin \omega t_0, \\ \int_{-\infty}^{+\infty} \left(\operatorname{sech}^4 \frac{t}{\sqrt{2}} \right) \cos \omega(t+t_0) dt &= \frac{2\pi}{3} \frac{\omega^2(\omega^2+2)}{\sinh \frac{\omega\pi}{\sqrt{2}}} \cos \omega t_0 \end{aligned}$$

and, substituting in (4.3), we get

$$\begin{aligned} M(t_0) &= \varepsilon \left(\alpha\sqrt{2} + \frac{\beta\pi}{3} \frac{\omega^2(\omega^2+2)}{\sinh \frac{\omega\pi}{\sqrt{2}}} \sin \omega t_0 \right) = \\ &= \frac{\pi\omega^2(\omega^2+2)}{3 \sinh \frac{\omega\pi}{\sqrt{2}}} \varepsilon (\alpha\mathcal{P}(\omega) + \beta \sin \omega t_0), \end{aligned}$$

provided we define

$$\mathcal{P}(\omega) = \frac{3\sqrt{2} \sinh \frac{\omega\pi}{\sqrt{2}}}{\pi\omega^2(\omega^2+2)}.$$

Therefore, for small values of ε we have: (i) if $|\beta/\alpha| > \mathcal{P}(\omega)$ then $W^u(\gamma_\varepsilon^-)$ and $W^s(\gamma_\varepsilon^+)$ intersect; (ii) if $|\beta/\alpha| < \mathcal{P}(\omega)$ they do not intersect each other and (iii) if $|\beta/\alpha| = \mathcal{P}(\omega)$ then $M(t_0)$ has zeroes which are double but not triple since $\partial M(t_0)/\partial \alpha \neq 0$; this case leads to quadratic heteroclinic tangencies.

4.2. On the Pikovsky-Topaj model [21] of coupled rotators

Let us consider the following system

$$\begin{aligned} \dot{\psi}_1 &= 1 - 2\varepsilon \sin \psi_1 + \varepsilon \sin \psi_2 \\ \dot{\psi}_2 &= 1 - 2\varepsilon \sin \psi_2 + \varepsilon \sin \psi_1 + \varepsilon \sin \psi_3 \\ \dot{\psi}_3 &= 1 - 2\varepsilon \sin \psi_3 + \varepsilon \sin \psi_2, \end{aligned} \tag{4.4}$$

where $\psi_i \in [0, 2\pi)$, $i = 1, 2, 3$, are cyclic variables. Thus, the phase space of (4.4) is the 3-dimensional torus \mathbb{T}^3 . System (4.4) is reversible with respect to the involution \mathcal{R} : $\psi_1 \rightarrow \pi - \psi_3$, $\psi_2 \rightarrow \pi - \psi_2$, $\psi_3 \rightarrow \pi - \psi_1$.

System (4.4) was suggested by Pikovsky and Topaj in the paper [21] as a simple model describing the dynamics of 4 coupled elementary rotators. By means of the coordinate change

$$\xi = \frac{\psi_1 - \psi_3}{2}, \quad \eta = \frac{\psi_1 + \psi_3 - \pi}{2}, \quad \rho = \frac{\psi_1 + \psi_3 - \pi}{2} + \psi_2 - \pi$$

and the change in time $d\tau = dt(2 + \varepsilon \cos(\rho - \eta))$ system (4.4) is led into

$$\begin{aligned} \dot{\xi} &= \frac{2\varepsilon \sin \xi \sin \eta}{2 + \varepsilon \cos(\rho - \eta)} \\ \dot{\eta} &= \frac{1 - \varepsilon \cos(\rho - \eta) - 2\varepsilon \cos \xi \cos \eta}{2 + \varepsilon \cos(\rho - \eta)} \\ \dot{\rho} &= 1 \end{aligned} \tag{4.5}$$

Then time-1 Poincaré map of system (4.5) is also reversible with respect to the same involution $R: \xi \rightarrow \xi, \eta \rightarrow -\eta$.

It was found in [21] that, for small ε , system (4.4) behaves itself as a conservative system close to integrable one and several invariant curves could be observed. However, when one increases the value of ε invariant curves break down and chaos appears (which is already noticed, for instance, at $\varepsilon \approx 0.3$). This picture looks to be quite similar to the conservative case. However, certain principal differences take place. In particular, a “strange behavior” of the invariant measure is observed. Iterations of the initial measure are convergent to some suitable limit. However, the limits $t \rightarrow +\infty$ and $t \rightarrow -\infty$ for the same initial measure are different (numerically observed, for instance, for values of $\varepsilon \approx 0.3$). This situation is impossible when the invariant measure is absolutely continuous. Therefore, it must be singular and concentrated on ”attractors and conservators” at $t \rightarrow +\infty$ or ”on repellers and conservators” at $t \rightarrow -\infty$. Here under the term “conservator” we mean the set of self-symmetric non-wandering orbits. Moreover, $+\infty$ - and $-\infty$ -invariant measures look like symmetric (with respect to the fixed line of the involution) and having non-empty intersection so there are no gaps between asymmetric and symmetric parts. This means that “visually” attractors and repellers intersect and it is an evidence of mixed dynamics in this model.

Moreover, a transition from conservative dynamics to non-conservative one can be generated by bifurcations of periodic orbits. For small enough ε periods of all such orbits are large and the corresponding resonance zones are narrow. When increasing ε , periodic orbits of no too large period appear and dissipative phenomena can become observable. For example, the map T under consideration has no points of period 1 and 2 for $\varepsilon < 0.6$ but it has, at $\varepsilon = \varepsilon^* \approx 0.445$, two period 3 orbits. Notice that these orbits are different since map T has the symmetry $\xi \rightarrow 2\pi - \xi$ that implies the appearance of 2 (in fact, an even number) different orbits. Thus, the scenario is the following: there is no fixed point for T^3 at $\varepsilon < \varepsilon^*$; at $\varepsilon = \varepsilon^*$ a symmetric couple of fixed points with double multiplier $+1$ appear in $\text{Fix } R$; at $\varepsilon > \varepsilon^*$ all these orbits fall into four orbits, a symmetric couple of elliptic orbits and a symmetric couple of contracting-expanding saddle orbits.

Bifurcations of such type are not typical in one-parameter families even in the reversible case. Here, general bifurcations are met (for symmetric fixed points) of types “ $0 \rightarrow 2$ ” or “ $1 \rightarrow 3$ ”, that is, “conservative” fold and “reversible” pitchfork, respectively. The presence of a typical bifurcation “ $0 \rightarrow 4$ ” says us about the existence of a certain additional degeneracy in the system. The “clear symmetry” $\xi \rightarrow -\xi$ is not suitable for this rôle. However, system (4.5) possesses such a “hidden symmetry” which implies that the map $T_{(\rho=0) \rightarrow (\rho=2\pi)}$ is the second power of some non-orientable map. This peculiarity is caused by the fact that the maps $T_{(\rho=\pi) \rightarrow (\rho=2\pi)}$ and $T_{(\rho=0) \rightarrow (\rho=\pi)}$ are conjugate. In particular, one can check that

$$T_{(\rho=\pi) \rightarrow (\rho=2\pi)} = S^{-1} T_{(\rho=0) \rightarrow (\rho=\pi)} S, \quad (4.6)$$

through the linear change of coordinates $\xi \rightarrow \pi - \xi, \eta \rightarrow \eta + \pi, \rho \rightarrow \rho + \pi$. Indeed, after this coordinate transformation, the right sides of system (4.5) remain the same, but the limits of integration (along orbits of system (4.5) to get the correspondence map between sections $\rho = a$ and $\rho = b$) are shifted in π . Such a property is called *time-shift symmetry*.

From (4.6) it follows that $T_{(\rho=0) \rightarrow (\rho=2\pi)} = T_{(\rho=0) \rightarrow (\rho=\pi)} S T_{(\rho=0) \rightarrow (\rho=\pi)} S^{-1}$. Since $S^2 = \text{Id}$, one has that $S = S^{-1}$ and, therefore,

$$T_{(\rho=0) \rightarrow (\rho=2\pi)} = (T_{(\rho=0) \rightarrow (\rho=\pi)} S)^2 \quad (4.7)$$

This means that the map $T_{(\rho=0)\rightarrow(\rho=2\pi)}$ considered is the second power of some map. Notice that the transformation associated to S is non-orientable and, thus, the map $T_{(\rho=0)\rightarrow(\rho=\pi)}S$ is non-orientable as well and, on its turn, our first-return map T is also the second power of some non-orientable map.

It is straightforward to check that the map $T_{(\rho=0)\rightarrow(\rho=\pi)}$ is reversible with respect to the involution $R_1(x, y) = (-x, -y)$ and that the map $T_{(\rho=0)\rightarrow(\rho=\pi)}S$ is reversible under the involution $R(x, y) = (x, -y)$. Thus, the bifurcation of map T^3 at $\varepsilon = \varepsilon^*$ can be treated as a bifurcation of a fixed point with multipliers $(+1, -1)$ in the case of a non-orientable map (in fact, the map $(T_{(\rho=0)\rightarrow(\rho=\pi)}S)^3$). So, summarising, in our case this bifurcation leads to the appearance of two elliptic points of period 2 on $\text{Fix } R$ and a symmetric couple of saddle fixed points (that is, outside $\text{Fix } R$ and symmetric one to each other). These saddle fixed points are not conservative. It can be checked numerically that the Jacobian of one point is greater than 1 and less than 1 at other point. Due to reversibility, the stable and unstable manifolds of saddles pairwise intersect and form a ‘‘heteroclinic tangle’’ zone. This zone is extremely narrow since the separatrix splitting is exponentially small. However, moving slightly away from the bifurcation moment we can find numerically heteroclinic tangencies and, hence, moments of creation of non-transversal heteroclinic cycles. Since the saddles involved are not conservative, it follows from [15] the phenomenon of mixed dynamics.

5. Cross-form type equations for reversible maps. Proof of Lemmas 1 and 2

5.1. Cross-form for reversible maps

As it will be seen along this section, the so-called Shilnikov *cross-form* variables constitute a very useful way to deal with reversible maps and a simple way to generate them. We say that a map is in *cross-form* if it is written as

$$\begin{cases} \bar{x} = h_1(x, \bar{y}), \\ y = h_2(x, \bar{y}), \end{cases}$$

where the regularity of $h_{1,2}$ varies depending on the problem. In this paper we will assume (h_1, h_2) to give rise to a C^r diffeomorphism.

On the other hand, let us consider a diffeomorphism F of the plane which is reversible with respect to a (in general, non-linear) involution R ($R^2 = \text{id}$, $R \neq \text{id}$), with the curve of points fixed under R of dimension 1. Remind that this means that $F \circ R \circ F = R$ - or, equivalently, $R \circ F \circ R = F^{-1}$ since F is a diffeo - and $\dim \text{Fix } R = 1$. Assume that the involution R reverses orientation, which is the most common situation in the literature. Our aim is to show that cross-form type can be very suitable when dealing with reversible planar maps.

We start considering the simplest setting, that is, when the reversor R is the linear (orientation-reversing) involution $L : (x, y) \mapsto (y, x)$. In this case, the following result holds:

Lemma 6 *Any diffeomorphism $F : (x, y) \mapsto (\bar{x}, \bar{y})$ defined, implicitly, by means of equations of type*

$$F : \begin{cases} \bar{x} = f(x, \bar{y}), \\ y = f(\bar{y}, x) \end{cases} \quad (5.1)$$

is always reversible with respect to $L(x, y) = (y, x)$.

Proof. Since F is a L -reversible diffeomorphism it satisfies that $L \circ F \circ L = F^{-1}$ or, equivalently, $(L \circ F \circ L)^{-1} = F$, which is the expression we will prove at the end. To do it, we first need a convenient equivalent expression for the inverse of a planar diffeomorphism. Indeed, if

$$H : \begin{cases} \bar{x} = h_1(x, y), \\ \bar{y} = h_2(x, y), \end{cases}$$

the corresponding inverse map $H^{-1} : (x, y) \mapsto (\bar{x}, \bar{y})$ can be implicitly written through the expression

$$H^{-1} : \begin{cases} x = h_1(\bar{x}, \bar{y}), \\ y = h_2(\bar{x}, \bar{y}). \end{cases}$$

This is clear since $(x, y) = H(\bar{x}, \bar{y})$ implies that $(\bar{x}, \bar{y}) = H^{-1}(x, y)$. An *algorithmic* way to get it consists just on swapping bars among the variables, that is $x \leftrightarrow \bar{x}$ and $y \leftrightarrow \bar{y}$. We apply this procedure to compute formally an expression for $(L \circ F \circ L)^{-1}$ and to check afterwards that it coincides with F . Let us do it step by step. First we have

$$F \circ L : \begin{cases} \bar{x} = f(y, \bar{y}), \\ x = f(\bar{y}, y). \end{cases}$$

To apply L onto $F \circ L$ corresponds to swap $\bar{x} \leftrightarrow \bar{y}$ in the precedent expression:

$$L \circ F \circ L : \begin{cases} \bar{y} = f(y, \bar{x}), \\ x = f(\bar{x}, y). \end{cases}$$

And finally, to get its inverse we swap bars and no-bars, that is, $x \leftrightarrow \bar{x}$ and $y \leftrightarrow \bar{y}$. Performing this change we obtain

$$(L \circ F \circ L)^{-1} : \begin{cases} y = f(\bar{y}, x), \\ \bar{x} = f(x, \bar{y}), \end{cases}$$

which is exactly the expression for F . \square

This result can be useful to provide suitable local expressions for planar reversible diffeomorphisms. Namely, we have the following lemma.

Lemma 7 *Let $F = (f_1, f_2)$ be a planar diffeomorphism, reversible with respect a general involution R , \mathcal{C}^r , $r \geq 1$, orientation reversing and with $\dim \text{Fix } R = 1$. Let us assume the origin $(0, 0)$ a fixed point of the involution R , that is $(0, 0) \in \text{Fix } R$.*

Then, if $D_{xx}f_1 + D_{yy}f_2 \neq 0$ at $(0, 0)$ there exist local coordinates, that we denote again by (x, y) , in which F admits the following implicit (normal) form

$$\begin{cases} \bar{x} = g(x, \bar{y}), \\ y = g(\bar{y}, x). \end{cases}$$

This map is reversible with respect to $L(x, y) = (y, x)$.

In the case of a saddle fixed point, the concrete type of implicit normal form that can be obtained is given in equation (3.3).

Proof. It will be proved in two steps:

- (i) First we apply Bochner Theorem [2] which allows us to conjugate, around $(0, 0)$, our involution R to its linear part $DR|_{(0,0)}$.
- (ii) Using that the partial derivatives on $(0, 0)$ do not vanish simultaneously, we apply Implicit Function Theorem to reach the final form.

We proceed as follows:

- (i) Notice that if R is a (general) involution and $p \in \text{Fix } R$ then its linear part $DR|_p$ is an involution as well. Indeed,

$$\text{Id} = R^2 \Rightarrow I = D(R^2)(p) = DR|_{R(p)} \cdot DR|_p = (DR|_p)^2.$$

Bochner Theorem ensures the existence of a C^r -diffeo ψ which conjugates, locally around p , R to $DR|_p$. We include, for completeness, a simple proof of this fact given in [23]. From the equality

$$\begin{aligned} DR|_p \circ (R + DR|_p) &= DR|_p \circ R + \text{id} = \\ \text{id} + DR|_p \circ R &= (R + DR|_p) \circ R \end{aligned}$$

it follows that $DR|_p \circ (R + DR|_p) = (R + DR|_p) \circ R$. We define $\psi = R + DR|_p \in C^r$ and check that it is a diffeomorphism in a neighbourhood of p :

$$D\psi|_p = D(R + DR|_p)|_p = DR|_p + DR|_p = 2DR|_p$$

and so $\det D\psi|_p = 2 \det DR|_p \neq 0$ since R is a diffeomorphism around p . So ψ is a C^r -diffeomorphism which conjugates R to $DR|_p$ around p .

Since R is orientation reversing its linear part around p , $DR|_p$, is also orientation reversing. Following [23] for instance, we know that there exists a transformation which conjugates $DR|_p$ to the linear involution $L(x, y) = (y, x)$, which will be the one we will consider, locally, from now on.

- (ii) Let us assume, for instance, that $D_{yy}f \neq 0$ at $(0, 0)$. Using Implicit Function Theorem, we can write from equation $\bar{y} = f_2(x, y)$ an expression for y , say $y = g(\bar{y}, x)$, for a suitable function g . Substituting it into the equations defining F we get a (locally) equivalent expression for F :

$$F : \begin{cases} \bar{x} = f_1(x, y) = f_1(x, g(\bar{y}, x)) =: h(x, \bar{y}), \\ y = g(\bar{y}, x). \end{cases}$$

As stated above, we can assume F to be locally conjugated around the origin to the linear involution $L : (x, y) \mapsto (y, x)$. So in that variables (to simplify the notation we keep the same name for the variables and the functions involved) it must satisfy that $(L \circ F \circ L)^{-1} = F$. Applying the procedure introduced in Lemma 6, one obtains that

$$F \circ L : \begin{cases} \bar{x} = h(y, \bar{y}), \\ x = g(\bar{y}, y). \end{cases}$$

We apply L (that corresponds to swapping \bar{x} and \bar{y}),

$$L \circ F \circ L : \begin{cases} \bar{y} = h(y, \bar{x}), \\ x = g(\bar{x}, y) \end{cases}$$

and, finally, we swap (x, y) for (\bar{x}, \bar{y}) ,

$$(L \circ F \circ L)^{-1} : \begin{cases} y = h(\bar{y}, x), \\ \bar{x} = g(x, \bar{y}). \end{cases}$$

Since it must coincide with F it turns out that $h(x, \bar{y}) = g(x, \bar{y})$ and so

$$F : \begin{cases} \bar{x} = g(x, \bar{y}), \\ y = g(\bar{y}, x). \end{cases}$$

□

We present now a counterpart result when the map is given in implicit form.

Lemma 8 Any map $G : (x, y) \mapsto (\bar{x}, \bar{y})$ given by

$$\begin{cases} g(x, y, \bar{x}, \bar{y}) = 0, \\ g(\bar{y}, \bar{x}, y, x) = 0, \end{cases}$$

is L -reversible, where $L : (x, y) \mapsto (y, x)$. The second equation $g(\bar{y}, \bar{x}, y, x) = 0$ is “a kind of” L -conjugate of the first equation $g(x, y, \bar{x}, \bar{y}) = 0$.

Proof. It is enough to check that $L \circ G \circ L = G^{-1}$. To do it we proceed again as in Lemma 6. First, remind that an implicit expression for G^{-1} is always obtained by swapping bars for no-bars, that is, $(x, y) \leftrightarrow (\bar{x}, \bar{y})$. So

$$G^{-1} : \begin{cases} g(\bar{x}, \bar{y}, x, y) = 0, \\ g(y, x, \bar{y}, \bar{x}) = 0. \end{cases}$$

On the other hand we compute $L \circ G \circ L$. Thus,

$$L \circ G \circ L : \begin{cases} g(y, x, \bar{x}, \bar{y}) = 0, \\ g(\bar{y}, \bar{x}, x, y) = 0, \end{cases}$$

and, swapping (\bar{x}, \bar{y}) for (\bar{y}, \bar{x}) , we get

$$L \circ G \circ L : \begin{cases} g(y, x, \bar{y}, \bar{x}) = 0, \\ g(\bar{x}, \bar{y}, x, y) = 0, \end{cases}$$

which coincides with G^{-1} . Therefore the lemma is proved. \square

The following result establishes an interesting relation between polynomial reversible and area preserving maps.

Lemma 9 ([22]) Any Taylor truncation of a planar polynomial diffeomorphism which is reversible with respect to a linear involution is area preserving. In particular, this applies to the truncation of a normal form of such diffeomorphisms.

Proof. Let $\bar{z} = G(z)$ a polynomial planar map which is reversible with respect to a linear involution S ($S^2 = \text{Id}$, $S \neq \text{Id}$). This means that $S \circ G \circ S = G^{-1}$ and, in particular, that G^{-1} is also a polynomial. Differentiating the latter expression we get

$$\begin{aligned} S DG|_{S_z} S &= D(G^{-1})|_z = (DG|_{G^{-1}(z)})^{-1} \Rightarrow \\ \det(S DG|_{S_z} S) &= \frac{1}{\det DG|_{G^{-1}(z)}}. \end{aligned}$$

Using that $\det(S DG|_{S_z} S) = (\det S)^2 \det DG|_{S_z} = \det DG|_{S_z}$ it follows that

$$(\det DG|_{S_z}) \cdot (\det DG|_{G^{-1}(z)}) = 1, \quad \forall z. \quad (5.2)$$

Since G and G^{-1} are polynomials and S linear we obtain that $\det DG|_{S_z}$ and $\det DG|_{G^{-1}(z)}$ are polynomials as well. But the product of two polynomials is a constant if and only if they are constant, that is, $\det DG|_z \equiv k = \text{constant}$. Thus, from (5.2) it follows that $k^2 = 1$ and, therefore, $\det DG|_z = \pm 1, \forall z$. \square

And last, but not least, we remark another interesting property regarding this cross-form type: any polynomial truncation of a reversible diffeomorphism written in cross-form type is also in cross-form type and, consequently, it is reversible. This means, from Lemma 9, that this truncation is always area-preserving.

5.2. Proof of Lemma 1

Let O be a fixed saddle point of a reversible map T_0 . Applying Bochner Theorem [16], we can assume the existence of local coordinates around O such that O is located at the origin and that the involution R is exactly $(x, y) \mapsto (y, x)$ in these coordinates.

Let $x = \nu(y)$ be the equation of the stable manifold. Then, by the R -reversibility, $y = \nu(x)$ is the equation of the unstable manifold. If $|d\nu/dy| < 1$, we perform the transformation $x_{new} = x - \nu(y)$, $y_{new} = y - \nu(x)$, while, if $|d\nu/dy| > 1$, the change is $x_{new} = y - \nu(x)$, $y_{new} = x - \nu(y)$. After such transformation, which commutes with R , the equations of the stable and unstable manifolds become $y = 0$ and $x = 0$, respectively. Thus, in the corresponding local coordinates, the map can be represented in the following form

$$\bar{x} = \lambda x + g_1(x, y), \quad \bar{y} = \lambda^{-1}y + g_2(x, y) \quad (5.3)$$

where $g_1(0, y) \equiv 0$, $g_2(x, 0) \equiv 0$ and $g'_i(0, 0) = 0$, $i = 1, 2$. It is very convenient to rewrite this equation in the so-called *cross-form*:

$$\bar{x} = \lambda x + \tilde{g}_1(x, \bar{y}), \quad y = \lambda \bar{y} + \tilde{g}_2(x, \bar{y}) \quad (5.4)$$

Equation (5.4) comes from (5.3) writing $y = F(x, \bar{y})$ (which exists due to the Implicit Function Theorem) and substituting it into the first equation: $\bar{x} = \lambda x + g_1(x, F(x, \bar{y}))$. The R -reversibility of (5.4) implies that $\tilde{g}_1(x, y) \equiv \tilde{g}_2(y, x)$ so we can represent map (5.4) in the form

$$\begin{aligned} \bar{x} &= \lambda x + \varphi_1(x) + \psi_1(\bar{y})x + \rho_1(x, \bar{y})x^2\bar{y} \\ y &= \lambda \bar{y} + \varphi_1(\bar{y}) + \psi_1(x)\bar{y} + \rho_1(\bar{y}, x)x\bar{y}^2 \end{aligned} \quad (5.5)$$

Performing the R -invariant change of variables

$$\xi = x + xh_1(y), \quad \eta = y + yh_1(x) \quad (5.6)$$

with $h_1(0) = 0$, it turns out the following equation for $\bar{\xi}$:

$$\begin{aligned} \bar{\xi} &= \bar{x} + \bar{x}h_1(\bar{y}) = \\ &\lambda\xi - \lambda xh_1(y) + x\psi_1(\bar{y}) + (\lambda x + \psi_1(\bar{y})x + \varphi_1(x))h_1(\bar{y}) + \\ &\varphi_1(\xi) + O(\xi^2\bar{\eta}) = \\ &\lambda\xi + \varphi_1(\xi) + O(\xi^2\bar{y}) + \\ &x[-\lambda h_1(\lambda\bar{y} + \varphi_1(\bar{y})) + \psi_1(\bar{y}) + (\lambda + \psi_1(\bar{y}))h_1(\bar{y})]. \end{aligned}$$

Since we want the expression in the square brackets to vanish identically, we ask the function $h_1(y)$ to satisfy the functional equation

$$h_1(\lambda\bar{y} + \varphi_1(\bar{y})) = h_1(\bar{y})(1 + \lambda^{-1}\psi_1(\bar{y})) + \lambda^{-1}\psi_1(\bar{y}), \quad (5.7)$$

which has solutions $h_1 = h_1(u)$ in the class of C^{r-1} -functions. Indeed, we can consider (5.7) as an equation for the strong stable invariant manifold of the following planar map

$$\begin{aligned} \bar{h}_1(\bar{u}) &= (1 + \lambda^{-1}\psi_1(u))h_1(u) + \lambda^{-1}\psi_1(u), \\ \bar{u} &= \lambda u + \varphi_1(u). \end{aligned}$$

Since $0 < |\lambda| < 1$, $\psi_1(0) = 0$ and $\varphi_1(0) = \varphi'_1(0) = 0$, this map has strong stable invariant manifold W^{ss} passing through the origin, that is, satisfying an equation $h_1 = h_1(u)$ with $h_1(0) = 0$. Therefore, after the R -invariant change (5.6), the map (5.5) takes the form

$$\begin{aligned} \bar{x} &= \lambda x + \varphi_1(x) + \rho_2(x, \bar{y})x^2\bar{y} \\ y &= \lambda \bar{y} + \varphi_1(\bar{y}) + \rho_2(\bar{y}, x)x\bar{y}^2 \end{aligned} \quad (5.8)$$

Applying a R -invariant change of variables of the form

$$\xi = x + h_2(x)x, \quad \eta = y + h_2(y)y$$

with $h_2(0) = 0$, the first equation of system (5.8) can be rewritten, in these new coordinates, as follows

$$\begin{aligned} \bar{\xi} = \lambda\xi + x[-\lambda h_2(x) + \tilde{\varphi}_1(x) + \\ h_2(\lambda x + \varphi_1(x))(\lambda + \tilde{\varphi}_1(x))] + O(\xi^2\eta), \end{aligned} \quad (5.9)$$

where we have denoted $\varphi_1(x) \equiv \tilde{\varphi}_1(x)x$. As we did above for h_1 , we seek for a function h_2 satisfying the following equation

$$h_2(\lambda x + \varphi(x)) = (1 + \lambda^{-1}\tilde{\varphi}_1(x))^{-1}(h_2(x) - \lambda^{-1}\tilde{\varphi}_1(x)), \quad (5.10)$$

which vanishes the expression inside the square brackets in (5.9). As before, equation (5.10) has solutions $h_2 = h_2(u)$ in the class of C^{r-1} -functions. Again, one can consider the expression (5.10) as an equation for the strong stable invariant manifold associated to the following planar map

$$\begin{aligned} \bar{h}_2 = (1 + \lambda^{-1}\tilde{\varphi}_1(x))^{-1}(h_2 - \lambda^{-1}\tilde{\varphi}_1(x)), \\ \bar{u} = \lambda u + \varphi_1(u). \end{aligned}$$

Having in mind that $0 < |\lambda| < 1$ and $\varphi_1(0) = \varphi_1'(0) = 0$, this map admits strong stable invariant manifold W^{ss} passing through the origin, i.e., having an equation $h_2 = h_2(u)$ with $h_2(0) = 0$. This completes the proof of the Lemma.

5.3. Proof of Lemma 2

We write the map T_0 in the following form

$$\bar{x} = \lambda x + \hat{h}(x, y), \quad \bar{y} = \gamma y + \hat{g}(x, y)$$

where we assume that $\gamma = \lambda^{-1}$ and

$$\hat{h}(x, y) \equiv x^2 y (\beta_1 + O(|x| + |y|)), \quad \hat{g}(x, y) \equiv xy^2 (\beta_2 + O(|x| + |y|)).$$

Consider the following operator $\Phi : [(x_j, y_j)]_{j=0}^k \mapsto [(\bar{x}_j, \bar{y}_j)]_{j=0}^k$:

$$\begin{aligned} \bar{x}_j &= \lambda^j x_0 + \sum_{s=0}^{j-1} \lambda^{j-s-1} \hat{h}(x_s, y_s, \mu), \\ \bar{y}_j &= \gamma^{j-k} y_k - \sum_{s=j}^{k-1} \gamma^{j-s-1} \hat{g}(x_s, y_s, \mu), \end{aligned} \quad (5.11)$$

where $j = 0, 1, \dots, k$. The operator Φ is defined on the set

$$Z(\delta) = \{z = [(x_j, y_j)]_{j=0}^k, \|z\| \leq \delta\},$$

where the norm $\|\cdot\|$ is given as the maximum of modulus of components x_j, y_j of the vector z . Notice that if $z_0 = [(x_j^0, y_j^0)]_{j=0}^k$ is a fixed point of Φ , then the following diagram takes place

$$(x_0^0, y_0^0) \xrightarrow{T_0} (x_1^0, y_1^0) \xrightarrow{T_0} \dots \xrightarrow{T_0} (x_k^0, y_k^0),$$

i.e. the fixed point of Φ gives a segment of an orbit of T_0 .

It is known [1] that, for small enough $\delta = \delta_0$ and $|x_0| \leq \delta_0/2$, $|y_k| \leq \delta_0/2$, the operator Φ maps the set $Z(\delta_0)$ into itself and it is contracting. Thus, map (5.11) has a unique fixed point $z_0 = [(x_j^0(x_0, y_k), y_j^0(x_0, y_k))]_{j=0}^k$ that is limit of iterations under Φ for any initial point from $Z(\delta_0)$. Thus, the coordinates x_j^0 and y_j^0 can be found by

applying successive approximations. As an initial approximation, we take the solution of the linear problem:

$$x_j^{0(1)} = \lambda^j x_0, \quad y_j^{0(1)} = \gamma^{j-k} y_k$$

It follows from (5.11) that the second approximation has a form

$$\begin{aligned} x_j^{0(2)} &= \lambda^j x_0 + \sum_{s=0}^{j-1} \lambda^{j-s-1} \lambda^{2s} \gamma^{s-k} x_0^2 y_k \times \\ &\quad (\beta_1 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)) = \\ &= \lambda^j x_0 + \lambda^j \gamma^{-k} \sum_{s=0}^{j-1} \lambda^{-1} \lambda^s \gamma^s x_0^2 y_k (\beta_1 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)) = \\ &= \lambda^j x_0 + (j-1) \lambda^j \gamma^{-k} \lambda^{-1} x_0^2 y_k (\beta_1 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)), \\ y_j^{0(2)} &= \gamma^{j-k} y_k + \sum_{s=j}^{k-1} \gamma^{j-s-1} \lambda^s \gamma^{2(s-k)} x_0 y_k^2 \times \\ &\quad (\beta_2 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)) = \\ &= \gamma^{j-k} y_k + \gamma^{j-2k} \sum_{s=j}^{k-1} \gamma^{-1} \lambda^s \gamma^s x_0 y_k^2 (\beta_2 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)) = \\ &= \gamma^{j-k} y_k + (k-j) \gamma^{j-2k-1} x_0 y_k^2 (\beta_2 + O(|\lambda|^s |x_0| + |\gamma|^{s-k} |y_k|)) \end{aligned}$$

Since $\gamma = \lambda^{-1}$, it follows from the precedent expression that

$$\begin{aligned} |x_j^{0(2)} - \lambda^j x_0| &\leq L_1 j \lambda^{j+k}, \\ |y_j^{0(2)} - \lambda^{k-j} y_k| &\leq L_2 (k-j) \lambda^{2k-j}, \end{aligned} \quad (5.12)$$

where L_1 and L_2 are some positive constant independent of j and k . Substituting (5.12) into (5.11) as the initial approximation, then the following ones will also satisfy estimates (5.12), with the same constants L_1 and L_2 . Thus, formula (3.5) is valid for the coordinates x_l^0 and y_0^0 of the fixed point of Φ .

The estimates for the derivatives of the functions x_l^0 and y_0^0 are deduced in the same way as done in [8] (see also modified versions of the proof in [27, 13, 14]).

6. Proofs of Lemmas 3, 4 and 5.

6.1. Proof of Lemma 3

Since coordinates (x_{01}, y_{01}) on σ_k^{01} are uniquely determined via cross-coordinates (x_{01}, y_{11}) in equations (3.6), we can express T_{km} as a map defined on points (x_{01}, y_{11}) and acting by the rule $(x_{01}, y_{11}) \mapsto (\bar{x}_{01}, \bar{y}_{11})$. As a result of this, we can express the map T_{km} in the following form

$$\begin{aligned} x_{02} - x_2^+ &= a \lambda_1^k x_{01} + b(y_{11} - y_1^-) + l_{02}(y_{11} - y_1^-)^2 + \\ &\quad \tilde{\varphi}_{1k}(x_{01}, y_{11} - y_1^-, \mu), \\ \lambda_2^m y_{12} (1 + m \lambda_2^m h_m^2(y_{12}, x_{02}, \mu)) &= \mu + c \lambda_1^k x_{01} + d(y_{11} - y_1^-)^2 + \\ &\quad f_{11} \lambda_1^k x_{01} (y_{11} - y_1^-) + f_{03} (y_{11} - y_1^-)^3 + \tilde{\varphi}_{2k}(x_{01}, y_{11} - y_1^-, \mu), \\ \lambda_2^m x_{02} (1 + m \lambda_2^m h_m^2(x_{02}, y_{12}, \mu)) &= \mu + c \lambda_1^k \bar{y}_{11} + d(\bar{x}_{01} - y_1^-)^2 + \\ &\quad f_{11} \lambda_1^k \bar{y}_{11} (\bar{x}_{01} - y_1^-) + f_{03} (\bar{x}_{01} - y_1^-)^3 + \tilde{\varphi}_{2k}(y_{11} - y_1^-, x_{01}, \mu), \\ y_{12} - x_2^+ &= a \lambda_1^k \bar{y}_{11} + b(\bar{x}_{01} - y_1^-) + l_{02}(\bar{x}_{01} - y_1^-)^2 + \\ &\quad \tilde{\varphi}_{1k}(y_{11} - y_1^-, x_{01}, \mu), \end{aligned}$$

where the coordinates x_{02} and y_{12} are “intermediate” and

$$\begin{aligned}\tilde{\varphi}_{1k}(u, v, \mu) &= O(\lambda_1^{2k}u^2 + |\lambda_1^k||uv| + |v|^3), \\ \tilde{\varphi}_{2k}(u, v, \mu) &= O(\lambda_1^{2k}(u^2 + |u|v^2) + |\lambda_1^k||u|v^2) + o(v^3).\end{aligned}\quad (6.1)$$

Now we perform the following shift in the coordinates

$$\begin{aligned}\xi_1 &= x_{01} - x_1^+ + \nu_{km}^1, & \eta_1 &= y_{11} - y_1^- + \nu_{km}^1, \\ \xi_2 &= x_{02} - x_2^+ + \nu_{km}^2, & \eta_2 &= y_{12} - y_2^- + \nu_{km}^2,\end{aligned}$$

where $\nu_{km}^i = O(\lambda_1^k)$, $i = 1, 2$, are some small coefficients which does not destroy the reversibility due to the condition (3.5). Then, for suitable ν_{km}^i , map T_{km} becomes

$$\begin{aligned}\xi_2 &= a\lambda_1^k\xi_1 + b\eta_1 + l_{02}\eta_1^2 + \tilde{\varphi}_{1k}(\xi_1, \eta_1, \mu), \\ \lambda_2^m\eta_2(1 + m\lambda_2^m h_m^2(\eta_2, \xi_2, \mu)) &= \\ \tilde{\mu} + c\lambda_1^k\xi_1 + d\eta_1^2 + f_{11}\lambda_1^k\xi_1\eta_1 + f_{03}\eta_1^3 + \tilde{\varphi}_{2k}(\xi_1, \eta_1, \mu), \\ \lambda_2^m\xi_2(1 + m\lambda_2^m h_m^2(\xi_2, \eta_2, \mu)) &= \\ \tilde{\mu} + c\lambda_1^k\bar{\eta}_1 + d\bar{\xi}_1^2 + f_{11}\lambda_1^k\bar{\eta}_1\bar{\xi}_1 + f_{03}\bar{\xi}_1^3 + \tilde{\varphi}_{2k}(\bar{\eta}_1, \bar{\xi}_1, \mu), \\ \eta_2 &= a\lambda_1^k\bar{\eta}_1 + b\bar{\xi}_1 + l_{02}\bar{\xi}_1^2 + \tilde{\varphi}_{1k}(\bar{\eta}_1, \bar{\xi}_1, \mu),\end{aligned}\quad (6.2)$$

where, since relation (3.5) holds, we have that $\tilde{\mu} = \mu - \lambda_2^m(\alpha_2^* + \dots) + c\lambda_1^k(\alpha_1^* + \dots)$ and the new functions $\tilde{\varphi}_{1k}$ and $\tilde{\varphi}_{2k}$ satisfy again conditions (6.1). One must, however, to consider coefficients a, b, \dots, f_{03} in (6.2) to be shifted by values of order $O(k\lambda_1^k)$ when comparing them with the initial coefficients in (3.7). Substituting into the second and third equations of (6.2) the expressions for ξ_2 and η_2 given by the first and the fourth precedent equations, we get an expression for T_{km} of form

$$\begin{aligned}\lambda_2^m(a\lambda_1^k\bar{\eta}_1 + b\bar{\xi}_1 + l_{02}\bar{\xi}_1^2) &= \\ \tilde{\mu} + c\lambda_1^k\xi_1 + d\eta_1^2 + f_{11}\lambda_1^k\xi_1\eta_1 + f_{03}\eta_1^3 + \tilde{\varphi}_{2k}, \\ \lambda_2^m(a\lambda_1^k\xi_1 + b\eta_1 + l_{02}\eta_1^2) &= \\ \tilde{\mu} + c\lambda_1^k\bar{\eta}_1 + d\bar{\xi}_1^2 + f_{11}\lambda_1^k\bar{\eta}_1\bar{\xi}_1 + f_{03}\bar{\xi}_1^3 + \tilde{\varphi}_{2k},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}a\lambda_2^m\bar{\eta}_1 + b\lambda_2^m\lambda_1^{-k}\bar{\xi}_1 + l_{02}\lambda_2^m\lambda_1^{-k}\bar{\xi}_1^2 &= \\ \tilde{\mu}\lambda_1^{-k} + c\xi_1 + d\lambda_1^{-k}\eta_1^2 + f_{11}\xi_1\eta_1 + f_{03}\lambda_1^{-k}\eta_1^3 + \lambda_1^{-k}\tilde{\varphi}_{2k}, \\ a\lambda_2^m\xi_1 + b\lambda_2^m\lambda_1^{-k}\eta_1 + l_{02}\lambda_2^m\lambda_1^{-k}\eta_1^2 &= \\ \tilde{\mu}\lambda_1^{-k} + c\bar{\eta}_1 + d\lambda_1^{-k}\bar{\xi}_1^2 + f_{11}\bar{\eta}_1\bar{\xi}_1 + f_{03}\lambda_1^{-k}\bar{\xi}_1^3 + \lambda_1^{-k}\tilde{\varphi}_{2k}.\end{aligned}\quad (6.3)$$

Even though they are represented by the same letter, the functions $\tilde{\varphi}_{2k}$ in the latter expression may be different of those in (6.2). We just want to stress that they still fulfil relations (6.1). Finally, rescaling coordinates,

$$\xi_1 = \lambda_1^k x, \quad \eta_1 = \lambda_1^k y,$$

system (6.3) takes the form (3.10) where the coefficients c, d, \dots, l_{02} are the “original” ones (that is, those appearing in formula (3.7)).

6.2. Proof of Lemma 4

The rescaled form (3.10) of the first-return map T_{km} is, of course, implicit one and it corresponds to a formal representation $T_{km} : F(\bar{x}, \bar{y}) \equiv G(x, y)$ which can be written

in the explicit form $(\bar{x}, \bar{y}) \equiv T_{km}(x, y) \equiv F^{-1}G(x, y)$. Then we can find the Jacobian of T_{km} using the relation

$$D(T_{km})|_{(x,y)} \equiv D(F^{-1})|_{G(x,y)} D(G(x, y)), \quad (6.4)$$

where $D(\cdot)$ is the corresponding (differential) Jacobi matrix. At the fixed point $(x = \xi^*, y = \eta^*)$ of T_{km} we can rewrite (6.4) as follows

$$D(T_{km})|_{(\xi^*, \eta^*)} \equiv \left(DF|_{(\xi^*, \eta^*)} \right)^{-1} DG|_{(\xi^*, \eta^*)}.$$

We find from (3.10) that DF and DG are of the form

$$DF = \begin{pmatrix} 2d\xi^* + f_{11}\eta^*\lambda_1^k + 3f_{03}\lambda_1^k(\xi^*)^2 & c + f_{11}\xi^*\lambda_1^k \\ b\lambda_1^{-k}\lambda_2^m + 2\lambda_2^m l_{02}\xi^* & a\lambda_2^m \end{pmatrix},$$

$$DG = \begin{pmatrix} a\lambda_2^m & b\lambda_1^{-k}\lambda_2^m + 2\lambda_2^m l_{02}\eta^* \\ c + f_{11}\eta^*\lambda_1^k & 2d\eta^* + f_{11}\xi^*\lambda_1^k + 3\lambda_1^k f_{03}(\eta^*)^2 \end{pmatrix}$$

plus terms of order $O(k\lambda_1^{2k})$. Now we compute the Jacobian as

$$J(T_{km})|_{(\xi^*, \eta^*)} = \det((DF)^{-1}DG) = \frac{\det(DG)}{\det(DF)} = \frac{-bc\lambda_1^{-k}\lambda_2^m + 2ad\lambda_2^m\eta^* - bf_{11}\lambda_2^m\eta^* - 2cl_{02}\lambda_2^m\eta^* + o(\lambda_1^k)}{-bc\lambda_1^{-k}\lambda_2^m + 2ad\lambda_1^k\lambda_2^m\xi^* - bf_{11}\lambda_2^m\xi^* - 2cl_{02}\lambda_2^m\xi^* + o(\lambda_1^k)}. \quad (6.5)$$

When the relation (2.2) is fulfilled we can rewrite (6.5) as

$$J(T_{km})|_{(\xi^*, \eta^*)} = \frac{-bc + Q\lambda_1^k \eta^* + o(\lambda_1^k)}{-bc + Q\lambda_1^k \xi^* + o(\lambda_1^k)}$$

that gives relation (3.19).

6.3. Proof of Lemma 5

It is not restrictive to prove Lemma 5 for the truncated map H , written in explicit form as

$$H : \begin{cases} \bar{x} &= \tilde{M} + \tilde{c}x - y^2, \\ \bar{y} &= -\frac{1}{\tilde{c}}\tilde{M} + \frac{1}{\tilde{c}}y + \frac{1}{\tilde{c}}\bar{x}^2. \end{cases}$$

We know that for $M > -\frac{1}{4}(\tilde{c}-1)^2$ it has a pair of symmetric fixed points $P^+ = (p_+, p_+)$ and $P^- = (p_-, p_-)$ given by formula (3.17). Denote by p either p_+ or p_- and let us assume that the corresponding fixed point P (i.e. P^+ or P^-) is elliptic. Then, \tilde{c} and \tilde{M} have to take values from the open regions in the (\tilde{c}, \tilde{M}) -space of parameters given in Figure 8.

The first step in our process is to shift the new origin of coordinates to this point $P = (p, p)$ and to perform (Jordan) linear normal form, which leads our map to the form

$$\bar{x} = \cos \psi \cdot x - \sin \psi \cdot y - \frac{2p \cos \psi}{\tilde{c} \sin \psi} y^2 + \frac{1 - 4p^2 - \tilde{c} \cos \psi}{4\tilde{c}^2 p^2 \sin \psi} (-\tilde{c} \sin \psi \cdot x + (1 - \tilde{c} \cos \psi)y + 2py^2)^2, \quad (6.6)$$

$$\bar{y} = \sin \psi \cdot x + \cos \psi \cdot y - \frac{2p}{\tilde{c}} y^2 + \frac{1}{4\tilde{c} p^2} (-\tilde{c} \sin \psi \cdot x + (1 - \tilde{c} \cos \psi)y + 2py^2)^2.$$

where x, y, \bar{x}, \bar{y} stand again for the new variables. The linear part of (6.6) is a rotation of angle ψ . Now we lead our map into Birkhoff Normal Form up to order 3, namely: $\bar{z} = e^{i\psi}z + d_{21}z^2z^* + \mathcal{O}_4$. To do it, we need to assume that $\lambda = e^{i\psi}$ is not a k th-root of unity for $k = 3, 4$ (the cases $k = 1, 2$ correspond to parabolic fixed points and, therefore, to boundaries of existence regions of elliptic fixed points in the (\tilde{c}, \tilde{M}) -parameter space). The coefficient $B_1 \equiv -id_{21}e^{-i\psi}$ is called the first Birkhoff coefficient. By the Arnol'd-Moser Twist Theorem [28], the inequality $B_1 \neq 0$ (together with the absence of strong resonances) ensures that the elliptic point is generic or, in other words, KAM-stable.

Introducing complex coordinates $z = x + iy, z^* = x - iy$, map (6.6) takes the form

$$\bar{z} = e^{i\psi}z + A_{20}z^2 + A_{11}zz^* + A_{02}(z^*)^2 + A_{21}z^2z^* + \mathcal{O}_4(z, z^*).$$

Since we are assuming $e^{i\psi}$ not to be a 3rd or 4th root of unity (and also $\psi \neq 0, \pi$), our map can be lead into BNF up to order 3 and, afterwards, provides the following formula for the first Birkhoff coefficient

$$B_1 = \frac{(\tilde{c} + 1 - 2p)(\tilde{c} + 1 + 2p)(\tilde{c} - 1 + 2p)}{32\tilde{c}^4 p \sin^3 \psi (2 \cos \psi + 1)} Q_4(\tilde{c}, p), \quad (6.7)$$

where

$$Q_4(\tilde{c}, p) = 64p^4 + 8(1 - \tilde{c})p^3 - 4(3\tilde{c}^2 + 4\tilde{c} + 3)p^2 + 2(\tilde{c} - 1)(\tilde{c} + 1)^2p - (\tilde{c} - 1)^2(\tilde{c} + 1)^2.$$

Using this formula we represent in Figure 9 curves $B_1(\tilde{c}, \tilde{M}) = 0$, where the elliptic fixed point can be, a priori, not KAM-stable. In this Figure curves related to the strong resonances are presented. For resonances 1:1 and 1:2 the equations of the corresponding curves are given in Section 3.4. The equations of the 1:3 and 1:4 curves are as follows:

$$\tilde{M} = \frac{\tilde{c}^2 + 1}{4} \pm \frac{\sqrt{\tilde{c}^2 + 1}}{2}(1 - \tilde{c}) \quad \text{for } 1 : 4 \text{ resonance}$$

and

$$\tilde{M} = \frac{\tilde{c}^2 + \tilde{c} + 1}{4} \pm \frac{\sqrt{\tilde{c}^2 + \tilde{c} + 1}}{2}(1 - \tilde{c}) \quad \text{for } 1 : 3 \text{ resonance.}$$

This completes the proof.

7. Conclusion

Generally speaking, mixed dynamics phenomenon should be considered as one more, new, kind of chaotic dynamics distinct in many principal properties from strange attractors and conservative chaos. As it follows from the work of D.Turaev [30], mixed dynamics can be observed in any dimension (in particular, Newhouse domain with mixed dynamics should exist near specific non-transversal heteroclinic cycles

|| Notice that in the particular case $\tilde{c} = -1$, map H corresponds to \mathcal{H}^2 , where $\mathcal{H} : \bar{x} = y, \bar{y} = M - x - y^2$ is the Hénon map. In this case we have that $B_1^H(\psi) = 2B_1^{\mathcal{H}(\varphi)}$ where $\psi = 2\varphi$. It is not hard to check now that, for a fixed point of \mathcal{H} with $p = -\cos \varphi$, the following relation holds

$$B_1^{\mathcal{H}^2} = B_1^{\mathcal{H}^2} = \frac{1}{4 \sin^2 \varphi} \cdot \frac{(1 + \cos \varphi)(1 + 4 \cos \varphi)}{\sin \varphi (1 + 2 \cos \varphi)},$$

which differs from the well-known formula for the Hénon map (see e.g. [3]) only by the non-zero factor $\frac{1}{4} \sin^{-2} \varphi$.

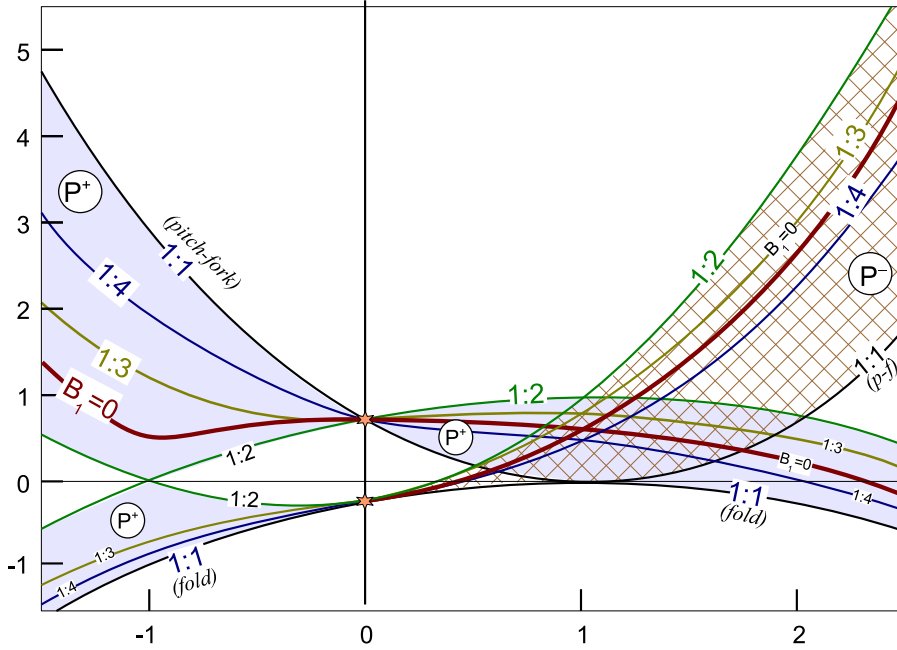


Figure 9. In the (ε, \tilde{M}) -plane, three grey and one hatching regions correspond to the existence of elliptic points: P^+ for the grey regions and P^- for the hatching one. Lines corresponding to the main resonances and vanishing the first Birkhoff coefficient for the elliptic point are shown and labelled.

proposed in [30]). However, the corresponding problem looks to be very complicated and requires quite delicate consideration.

Regarding the two dimensional reversible case, a kind of systematic program could be followed to prove the RMD-conjecture in the one parameter context and for the main cases of diffeomorphisms with symmetric homoclinic or heteroclinic tangencies.

In Figure 10 we collect some simple examples of such maps. They differ by the type of fixed points and tangencies: homoclinic or heteroclinic, quadratic or cubic¶, etc. It seems to be more important its division into two groups: a first one including those maps which are a priori non-conservative and a second one with those maps where this non-conservativity is, in some sense, hidden.

A map as in Figure 1(a) (the first one in Figure 10), containing a symmetric couple of contracting-expanding saddles, belongs to the first group since condition $J(O_{1,2}) \neq 1$ destroys certainly the conservative character. Indeed, under splitting such heteroclinic cycle, homoclinic tangencies appear both to saddles with Jacobian

¶ The existence of symmetric cubic homoclinic or heteroclinic tangencies is a codimension one bifurcation phenomenon in the class of reversible maps. Therefore, these cubic tangencies should be also considered as the main ones jointly with the pointed out quadratic ones.

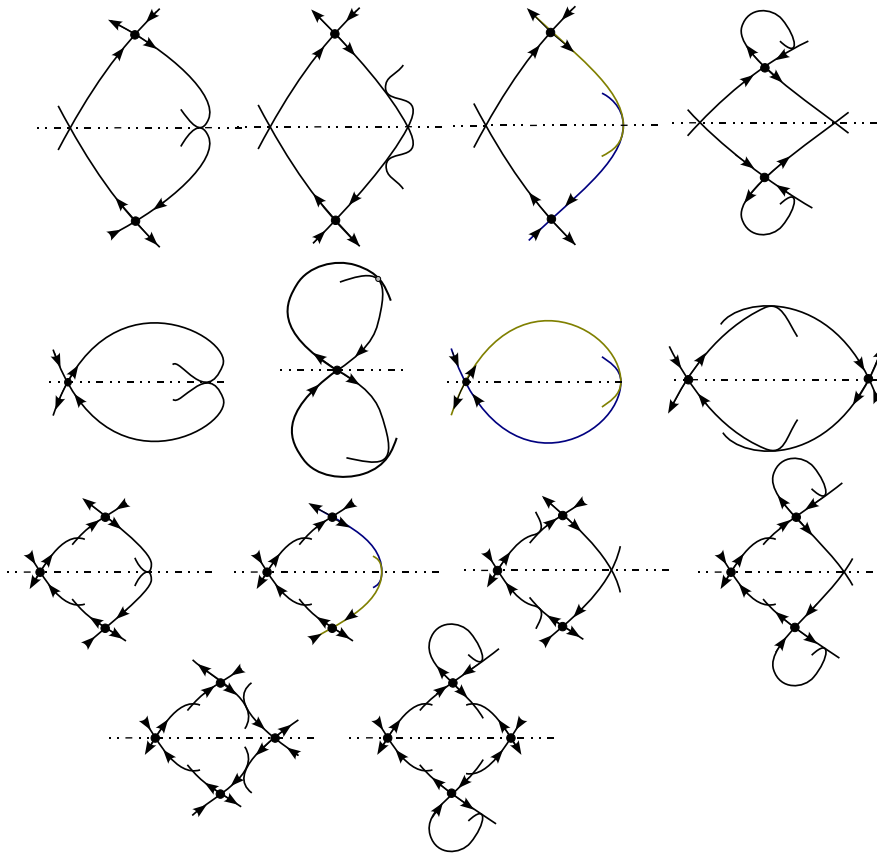


Figure 10. Some examples of reversible maps with homoclinic and heteroclinic tangencies

greater and less than 1 and, thus, attracting and repelling periodic points can be born [6]. By the same argument, all the other cases with a symmetric couple of contracting-expanding fixed (periodic) points are a priori non-conservative maps (in particular, all maps at the first, third and fourth lines in Figure 10). For maps of this type, the problem of finding symmetric elliptic periodic orbits should be considered of special relevance.

The maps at the second line of Figure 10 have only symmetric fixed points. The first, third and fourth maps can be considered as maps with “hidden non-conservativity” since, in principle, it is not clear the existence of bifurcation mechanisms leading to the appearance of attracting and repelling periodic orbits.

This is not the case for the second map (again at the second line) which has a symmetric couple of (quadratic) homoclinic tangencies to the same symmetric fixed saddle point. As in the first map at the first line, one can assume without loss of reversibility that the first-return map near a homoclinic point is not conservative, giving rise to a *symmetric couple of contracting-expanding homoclinic orbits*. As for

the “a priori conservative case”, the main problem to be considered is to prove the appearance of elliptic periodic orbits and, as a first step, to do it in the one-parameter setting.

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