A NOTE ON THE TOTALLY TRANSITIVE GRAPH MAPS

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We study the totally transitive graph maps which have periodic points. In particular we show that the cofiniteness of the set of periods characterizes these maps among transitive graph maps.

1. Introduction

The aim of this paper is to generalize the main result of [Alsedà *et al.*, 1999b] to transitive graph maps. To this end we have to introduce some notation.

Let X be a topological space. A continuous map $f : X \longrightarrow X$ is called *(topologically) transitive* if for every non-empty open subsets $U, V \subset X$ there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. In a connected compact metric space a map is transitive if and only if has a dense orbit. A transitive map is called *totally transitive* if f^s is transitive for all $s \geq 1$. Also, f is said to be *topologically mixing* if for every non-empty open subsets $U, V \subset X$ there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for each $n > N$.

The study of transitive maps on compact spaces is widely motivated. For example, the asymptotic dynamics of dissipative systems are the dynamics inherent in the attractors. Here, attractor means a compact invariant set containing a dense orbit (transitivity) and whose stable set (attracting basin) has a non-empty interior or, at least, positive Lebesgue measure. Due to the dissipation, the attractors have lower dimension than the phase space has. Their dynamics often reduce to a one-dimensional one and, despite of the low dimension, interesting questions arise. One-dimensional dynamics also appear when the system has an invariant foliation of codimension one. In this way, as it has been shown in [Williams, 1977], the iteration of a map on a graph imitates the behavior of a flow in a neighborhood of a hyperbolic attractor.

It is important to know if for a transitive map there exists some iterate which is not transitive as the following example shows. Let φ be the map defined on the interval $[0,1]$ in the following way:

$$\varphi(x) = \begin{cases} 1/2 + 2x & \text{if } 0 \le x \le 1/4, \\ 3/2 - 2x & \text{if } 1/4 \le x \le 1/2, \\ 1 - x & \text{if } 1/2 \le x \le 1. \end{cases}$$

Since φ is transitive, it has an orbit which is dense in $[0,1]$ and hence, the interval is the unique invariant compact with non-empty interior. On the other hand, φ is not totally transitive, since the subintervals $[0,1/2]$ and $[1/2,1]$ are invariant under φ^2. This fact agrees with Theorem 2.1 and it may wrongly suggest that the original dynamics have two different attractors. Therefore, total transitivity is a reassuring property to discretize a continuous dynamical systems.

As we shall see the total transitivity has a strong relation with the notion below.

Definition 1.1. Let (X,d) be a compact metric space. A map $f : X \longrightarrow X$ is said to verify the *specification property* if for any $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that for any collection of points $x_1, x_2, ..., x_k$ with $k \ge 2$, for any collection of integers $a_1 \le b_1 < a_2 \le b_2 < ... < a_k \le b_k$ such that $a_i - b_{i-1} \ge M(\varepsilon)$ and for any $p \in \mathbb{N}$ such that $p \ge M(\varepsilon) + b_k - a_1$, there exists a point $y \in X$ such that $f^p(y) = y$ and $d(f^n(y), f^n(x_i)) \le \varepsilon$ for each $n, a_i \le n \le b_i$, $1 \le i \le k$. When $k = 2$ we say that f verifies the *weak specification property.*

A *(topological) graph* is a pair (G,V) where G is a connected compact Hausdorff space and V is a non-empty finite subset of G such that each connected component of $G \setminus V$ is homeomorphic to an open interval of the real line. A *tree* is a uniquely arcwise connected graph.

Let G be a graph, let $z \in G$ and let U be an open neighborhood of z (in G) whose closure is a tree. The number of connected components of $U \setminus \{z\}$ is called the *valence* of z and is denoted by $\mathrm{val}(z)$. For each $l \in \mathbb{N}$ we set

$$\nu_l = \max\left\{\min\left\{\mathrm{val}(x) : x \in A\right\} : A \in F_l\right\},$$

where F_l denotes the collection of all finite subsets of G of cardinality l.

As usual, given a map f from a topological space to itself, $\mathrm{Per}(f)$ will denote the set of periods of all periodic points of f. Also, we say that $S \subset \mathbb{N}$ is *cofinite in* $\mathbb{N}$ if $\mathbb{N} \setminus S$ is finite.

The main result of the paper is the following:

Main Theorem. *Let G be a graph and let f be a continuous transitive map from G to itself which has periodic points. Then the following statements are equivalent:*

- *(a) f is totally transitive.*
- *(b) f verifies the specification property.*
- *(c) f verifies the weak specification property.*
- *(d) f is topologically mixing.*
- *(e) $f^{(p\nu_p)!}$ is transitive, where $p \in \mathrm{Per}(f)$.*
- *(f) $\mathrm{Per}(f)$ is cofinite in $\mathbb{N}$.*

This theorem extends the main result of [Alsedà *et al.*, 1999b] to graphs except for the existence of an orbit with no division (actually this notion is not known for graphs). From [Blokh, 1987] we know that any transitive map from a graph without periodic points is conjugate to an irrational rotation of the circle. Since these maps are totally transitive the above theorem gives various characterizations of the totally transitive graph maps.

Remark 1.2. The equivalence of the statements (a), (b), (c) and (d) of the above theorem is, in fact, due to Blokh since basically it is a consequence of Theorem 1 of [Blokh, 1987].

2. Definitions and preliminary results

This section is devoted to introduce the notation and state the results which are necessary to prove the main result of the paper. The following result, which was proved in [Alsedà *et al.*, 1999a] (see also [Blokh, 1987]), plays an important role in the study of transitive maps.

Theorem 2.1. *Let X be a locally connected compact metric space and let f be a continuous transitive map from X to itself. Then exactly one of the following two statements holds:*

- *(a) f is totally transitive.*
- *(b) There exist $k > 1$ connected closed subsets $X_0, X_1, \dots, X_{k-1}$, of X with non-empty interiors such that $X = \cup_{i=0}^{k-1} X_i$, $\mathrm{Int}(X_i) \cap X_j = \emptyset$, for all $i \neq j$ and $f(X_i) = X_{i+1 \ (mod\ k)}$. Moreover, $f^k\big|_{X_i}$ is transitive for each $i = 0, 1, \dots, k-1$.*

For the case of graphs we also have (see [Alsedà *et al.*, 1999a]):

Proposition 2.2. *Let G be a graph, let $f : G \longrightarrow G$ be a continuous transitive map such that* $\mathrm{Per}(f) \neq \emptyset$ *and it is not totally transitive. Then the number k from Theorem 2.1 is smaller than or equal to $p\nu_p$ for each $p \in \mathrm{Per}(f)$.*

The following result follows from Theorem 1 of [Blokh, 1987].

Theorem 2.3. *Let G be a graph and let f be a continuous totally transitive map from G to itself which has periodic points. Then f verifies the specification property.*

3. Proof of the main result

We start by proving the following simple consequence of the weak specification property.

Proposition 3.1. *Let X be a compact metric space and let $f : X \longrightarrow X$ satisfy the weak specification property then $Per(f)$ is cofinite in $\mathbb{N}$.*

Proof. Since f satisfies the weak specification property it is easy to see that the set of periodic points of f is dense in G. Hence, there exists a periodic orbit P of f of period larger than one. Let $\varepsilon > 0$ be such that $d(x, y) > \varepsilon$ for every $x, y \in P$, $x \neq y$. Let $M(\varepsilon)$ be the natural number given in Definition 1.1. We will prove that for any $p > 5M(\varepsilon)$, f has a periodic point of period p. To prove it we take x_1 such that $d(x_1, P) > 2\varepsilon$ and $x_2 \in P$. Let $a_1 = b_1 = 0$, $a_2 = M$ and $b_2 = p - M$. Then the weak specification property implies the existence of y such that $d(y, x_1) < \varepsilon$, $d(f^n(y), P) < \varepsilon$ for $a_2 \leq n \leq b_2$ and $f^p(y) = y$. Since $f^n(y) \neq y$ for $a_2 \leq n \leq b_2$, there is no divisor p_0 of p such that $f^{p_0}(y) = y$. Otherwise, since $b_2 - a_2 > p/2$, there should exist $s \in \mathbb{N}$ such that $a_2 \leq sp_0 \leq b_2$ and $f^{sp_0}(y) = y$. Thus, y is a periodic point with period p. ∎

Proof of the Main Theorem. The implication (a) $\Rightarrow$ (b) is Theorem 2.3. (b) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) are easy to prove. Now, (c) $\Rightarrow$ (f) follows from Proposition 3.1 and (f) $\Rightarrow$ (a) follows from Theorem 2.1 (observe that in this case $\bigcup_{i,j=0, i\neq j}^{k-1} (X_i \cap X_j)$ is finite). Now we prove that (a) and (e) are equivalent. Clearly (a) implies (e). Now suppose that f is not totally transitive. By Theorem 2.1 and Proposition 2.2 there exist a proper subgraph G' of G with non-empty interior and $2 \leq k \leq p\nu_p$ such that $f^k(G') = G'$. Since k divides $(p\nu_p)!$ it follows that $f^{(p\nu_p)!}(G') = G'$. Hence, $f^{(p\nu_p)!}$ is not transitive. ∎

4. Further comments

The Main Theorem cannot be extended to any space different from a graph. So, let φ be the interval map given in the introduction, let τ be the tent map defined in the interval 0, 1], i.e.

$$\tau(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2 - 2x & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and let $\xi = \varphi \times \tau$ be the map defined in $[0, 1] \times [0, 1]$ by $\xi(x, y) = (\varphi(x), \tau(y))$. This map is transitive but not totally transitive because $[0, 1/2] \times [0, 1]$ and $[1/2, 1] \times [0, 1]$ are invariant sets by ξ^2. However, on the invariant segment $\{1/2\} \times [0, 1]$, the map ξ has periodic points of any natural period. In fact, $Per(\xi) = Per(\tau) = \mathbb{N}$. Notice also that, corresponding to the Main Theorem, τ is totally transitive and $Per(\tau)$ is cofinite, whereas φ is not totally transitive and $Per(\varphi) = \{1\} \cup \{2n \colon n \in \mathbb{N}\}$ is not cofinite. According to Sarkovskii Theorem [Sarkovskii, 1964] and the Main Theorem, an interval map is totally transitive if and only if it has some odd period different from 1.

Acknowledgements

The first author has been partially supported by the DGES grant number PB96–1153, the second author has been partially supported by the grant number PGIDT 99 PXI20704B and the third one by the DGES grant number PB98-1574.

References

Alsedà, Ll., del Río, M. A. & Rodríguez, J. A. [1999a] "A splitting theorem for transitive maps," *Journal of Math. Anal. and Appl.* **232**, 359–375.

Alsedà, Ll., del Río, M. A. & Rodríguez, J. A. [1999b] "Cofiniteness of the set of periods for totally transitive tree maps," *Int. Journal of Bifurcation and Chaos* **9**, 1877–1880.

Blokh, A. M. [1987] "The connection between entropy and transitivity for one-dimensional mappings (Russian)," *Russian Math. Surveys* **42**, 209–210.

Sarkovskii, A. N. [1964] "Coexistence of cycles of a continuous map of the line into itself (Russian)," *Ukraine Math.* **16**, 61-71.

Williams, R. F. [1977] "The structure of Lorenz attractors," *Lecture Notes in Math.* **615**, 94-112.