

On the spectra and eigenspaces of the universal adjacency matrices of arbitrary lifts of graphs

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Abstract

The universal adjacency matrix U of a graph Γ , with adjacency matrix A , is a linear combination of A , the diagonal matrix D of vertex degrees, the identity matrix I , and the all-1 matrix J with real coefficients, that is, $U = c_1A + c_2D + c_3I + c_4J$, with $c_i \in \mathbb{R}$ and $c_1 \neq 0$. Thus, in particular cases, U may be the adjacency matrix, the Laplacian, the signless Laplacian, and the Seidel matrix. In this paper, we develop a method for determining the universal spectra and bases of all the corresponding eigenspaces of arbitrary lifts of graphs (regular or not). As an example, the method is applied to give an efficient algorithm to determine the characteristic polynomial of the Laplacian matrix of the symmetric squares of odd cycles, together with closed formulas for some of their eigenvalues.

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1 Introduction

For a graph Γ on n vertices, with adjacency matrix A and degree sequence d_1, \dots, d_n , the *universal adjacency matrix* is defined as

$$U = c_1A + c_2D + c_3I + c_4J, \quad \text{with } c_i \in \mathbb{R} \text{ and } c_1 \neq 0, \quad (1)$$

where $D = \text{diag}(d_1, \dots, d_n)$, I is the identity matrix, and J is the all-1 matrix. See, for instance, Haemers and Omidi [15], Ahmadi, Alinaghipour, Fallat, Fan, Meagher, and Nasserar [1], or Farrugia and Sciriha [10]. Thus, for given values of the coefficients (c_1, c_2, c_3, c_4) , the universal adjacency matrix particularizes to important matrices used in algebraic graph theory, such as the adjacency matrix $(1, 0, 0, 0)$, the Laplacian $(-1, 1, 0, 0)$, the signless Laplacian $(1, 1, 0, 0)$, and the Seidel matrix $(-2, 0, -1, 1)$.

Because of the interest in combinatorial properties, some methods were proposed to determine the spectra of graphs with some symmetries. This includes graphs with transitive groups (Lovász [17]), Cayley graphs (Babay [3]), and some lifts of voltage graphs (or covers) (Godsil and Henseland [11]). More recently, by using representation theory (see, for instance, Burrow [4], and James and Liebeck [16]), Dalfó, Fiol, and Širáň [8] considered a more general construction and derived a method for determining the spectrum of a regular lift of a ‘base’ (di)graph equipped with an ordinary voltage assignment, or, equivalently, the spectrum of a regular cover of a (di)graph. Recall, however, that by far, not all coverings are regular. A description of arbitrary graph coverings by the so-called ‘permutation voltage assignments’ was given by Gross and Tucker [13] (see the next section). In [7], the authors generalized their previous results to arbitrary lifts of graphs (regular or not). They not only gave the complete spectra of lifts but also provided bases of the corresponding eigenspaces, both in a straightforward way. In this work, we develop a method to find the spectra and corresponding eigenspaces of the universal adjacency matrix of a relative lift graph.

This paper is organized as follows. In the next subsection, we recall the basic notions of permutation voltage assignments on a graph, their associated lifts, and an equivalent description in terms of relative voltage assignments. In Subsection 2.2, we recall the fundamental concepts and results of representation theory that are used in our study. Section 3 deals with the main results, where the complete spectrum of the universal adjacency matrix of a relative lift graph is determined, together with bases of the associated eigenspaces. Our method is first illustrated by an example in Section 4. Finally, the method yields an efficient algorithm for determining the Laplacian spectra of the symmetric squares of odd cycles.

2 Preliminaries

2.1 General lifts of a graph

All the material of this section is borrowed from our previous work [7], but we include it to make the article more self-contained.

Let Γ be an undirected graph, possibly with loops and multiple edges, with a vertex set. As usual in algebraic and topological graph theory, we think of every undirected edge joining vertices u and v (not excluding the case when $u = v$) as consisting of a pair of oppositely directed arcs, denoted by a and a^- . Let $V = V(\Gamma)$ and $X = X(\Gamma)$ be the sets of vertices and arcs of Γ , respectively.

The following concepts were introduced by Gross in [12]. Let G be a group. An (*ordinary*) *voltage assignment* on the graph Γ is a mapping $\alpha : X \rightarrow G$ with the property that $\alpha(a^-) = (\alpha(a))^{-1}$ for every arc $a \in X$. Thus, a voltage assigns an element $g \in G$ to each arc of the graph so that a pair of mutually reverse arcs a and a^- , forming an undirected edge, receive mutually inverse elements g and g^{-1} . The graph Γ and the voltage assignment α determine a new graph Γ^α , called the *lift* of Γ , which is defined as follows. The vertex and arc sets of the lift are simply the Cartesian products $V^\alpha = V \times G$ and $X^\alpha = X \times G$, respectively. Moreover, for every arc $a \in X$ from a vertex u to a vertex v for $u, v \in V$ (possibly, $u = v$) in Γ , and for every $g \in G$, there is an arc $(a, g) \in X^\alpha$ from the vertex $(u, g) \in V^\alpha$ to the vertex $(v, g\alpha(a)) \in V^\alpha$. Notice that, if a and a^- are a pair of mutually reverse arcs forming an undirected edge of Γ , then for every $g \in G$ the pair (a, g) and $(a^-, g\alpha(a))$ form an undirected edge of the lift Γ^α , making the lift an undirected graph. The mapping $\pi : \Gamma^\alpha \rightarrow \Gamma$ defined by erasing the second coordinate, that is, $\pi(u, g) = u$ and $\pi(a, g) = a$, for every $u \in V$, $a \in X$ and $g \in G$, is a (regular) *covering*, with its usual meaning in algebraic topology; see, for instance, Gross and Tucker [14].

To generate all (not necessarily regular) graph coverings, Gross and Tucker [13] introduced the more general concept of permutation voltage assignment. With this aim, let G be a subgroup of the symmetric group $\text{Sym}(n)$, that is, a permutation group on the set $[n] = \{1, 2, \dots, n\}$. Then, a *permutation voltage assignment* on Γ is a mapping $\alpha : X \rightarrow G$ with the same symmetric property as before. So, the lift Γ^α , with vertex set $V^\alpha = V \times [n]$ and arc set $X^\alpha = X \times [n]$, has an arc $(a, i) \in X^\alpha$ from the vertex (u, i) to the vertex (v, j) if and only if $a = (u, v) \in X$ and $j = i\alpha(a)$. Notice that we write the argument of a permutation to the left of the symbol of the permutation, with the composition of permutations being read from left to right.

The permutation voltage assignments and the corresponding lifts can, equivalently, be described as relative voltage assignments defined as follows. Let Γ be the graph considered above, K a group and H a subgroup of K of index n . Let K/H denote the set of right cosets of H in K . Furthermore, let $\beta : X \rightarrow K$ be a mapping satisfying $\beta(a^-) = (\beta(a))^{-1}$ for every arc $a \in X$. Then, β is called a *voltage assignment in K relative to H* , or simply a relative voltage assignment. The *relative lift* Γ^β has vertex set $V^\beta = V \times K/H$ and arc set $X^\beta = X \times K/H$. The incidence in the lift is given as expected: If a is an arc from a vertex u to a vertex v in Γ , then for every right coset $J \in K/H$ there is an arc (a, J) from the vertex (u, J) to the vertex $(v, J\beta(a))$ in Γ^β . We slightly abuse the notation and denote by the same symbol π the covering $\Gamma^\beta \rightarrow \Gamma$ given by suppressing the second coordinate.

Under additional and natural assumptions, there is a 1-to-1 correspondence between *connected* lifts generated by permutation and relative voltage assignments on a connected base graph Γ . To present more details, fix a vertex u in the base graph and let \mathcal{W}_u be the collection of all closed walks in the base graph that begin and end at u . If γ is either a

permutation or a relative voltage assignment on Γ in $G \leq \text{Sym}(n)$ (where ‘ \leq ’ is used for subgroup) or in a group K relative to a subgroup H , respectively, and if $W = a_1 a_2 \dots a_k$ is a sequence of arcs forming a walk in \mathcal{W}_u , we let $\gamma(W) = \gamma(a_1) \gamma(a_2) \dots \gamma(a_k)$ denote the product of the voltages taken while traversing the walk. Then, the set $\{\gamma(W); W \in \mathcal{W}_u\}$ forms a subgroup of G or K , known as the *local group*, and it will be denoted by G_u or K_u , respectively. It is well known that there is no loss of generality in the study of voltage assignments and lifts under the assumption that $G_u = G$ and $K_u H = K$, meaning that there are no ‘superfluous’ voltages in the assignments.

We are now ready to explain the correspondence between the two kinds of lifts. First, let α be a permutation voltage assignment as in the previous paragraph, taking place in a permutation group $G \leq \text{Sym}(n)$. Assume that $G = G_u$ as above and that G is transitive on $[n]$; both conditions together are equivalent to the connectivity of the lift. Then, the corresponding relative voltage group is $K = G$, with the subgroup H being the stabilizer in K of an arbitrarily chosen point in the set $[n]$. The corresponding relative assignment β is simply identical to α but acting by right multiplication on the set K/H . Observe that, in this construction, the core of H in K (that is, the intersection of all K -conjugates of H) is trivial. Conversely, let β be a voltage assignment in a group K relative to a subgroup H of index n and with a trivial core in K . Assume again that $K_u H = K$. Now, this alone guarantees the connectivity of the lift. Then, one may identify the set K/H with $[n]$ in an arbitrary way, and $\alpha(a)$ for $a \in X$ is the permutation of $[n]$ induced (in this identification) by right multiplication on the set of (right) cosets K/H by $\beta(a) \in K$.

We note that there are fine points in this correspondence to be considered on the ‘permutation’ side if G_u is intransitive or a proper subgroup of G , and on the ‘relative’ side if H is not core-free or if $K_u H$ is proper in K ; these details are, however, irrelevant for our purposes. We also point out that a covering described in terms of a permutation voltage assignment as above is regular (that is, generated by an ordinary voltage assignment) if and only if the corresponding voltage group is a regular permutation group on the set $[n]$. Of course, this is translated to the language of relative voltage assignments by the normality of H in K . In such a case, the covering admits a description in terms of ordinary voltage assignment in the factor group K/H and with voltage $H\beta(a)$ assigned to an arc $a \in X$ with original relative voltage $\beta(a)$.

2.2 Some results on group representations

In this subsection, we recall some basic results on representation theory that will be used in our study; see, for instance, the textbooks of Burrow [4], and James and Liebeck [16]. We also recall a new result from [7] on group representations, which may be of interest on its own.

For a representation ρ over \mathbb{C} of a group G , we let $d_\rho = \dim(\rho)$ denote the dimension of ρ ; as usual, $\rho(g)_{i,j}$ will be the (i, j) -th entry of the matrix $\rho(g)$ for $g \in G$. For instance, if $G = \text{Sym}(n)$, the *trivial* and *sign* representations have both dimension 1, whereas the *standard* representation is a faithful irreducible representation of dimension $n - 1$. This is obtained by the usual action of $\text{Sym}(n)$ on the basis of the $(n - 1)$ -dimensional subspace

generated by $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$, where e_i is the i -th standard unit vector. Thus, in the case of $\text{Sym}(3)$, the standard representation is

$$\begin{aligned} \iota &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (13) &\mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & (123) &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \\ (12) &\mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, & (23) &\mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & (132) &\mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \quad (2)$$

We denote by z^* the complex conjugate of a complex number z . Furthermore, let $\text{Irep}(G)$ denote a complete set of unitary irreducible representations of G . Let us also recall the Kronecker symbol $\delta_{s,t}$, for s, t in any suitable domain, which is equal to 0 if $s \neq t$ and to 1 if $s = t$.

Now let us recall a result well known as the Shurs' Great Orthogonality Theorem (see, for instance, Sengupta [19]), which is used in the proof of the main theorem.

Theorem 2.1 (Great Orthogonality Theorem). *Let G be a finite group. Then, for any $\rho, \rho' \in \text{Irep}(G)$,*

$$\sum_{g \in G} \rho(g)_{i,j} \rho'(g)_{i',j'}^* = \frac{|G|}{d_\rho} \delta_{i,i'} \delta_{j,j'} \delta_{\rho,\rho'}.$$

For a subgroup H of a finite group G and for any $\rho \in \text{Irep}(G)$, we let $\rho(H) = \sum_{h \in H} \rho(h)$, that is, $\rho(H)$ is the sum of d_ρ -dimensional complex matrices $\rho(h)$ taken over all elements h of the subgroup H . We note that, in general, it may happen that $\rho(H)$ is the zero matrix, although, of course, all the matrices $\rho(h)$ for $h \in H$ are nonsingular. As usual, the symbol $\text{rank}(M)$ stands for the rank of a matrix M .

Proposition 2.2 ([7]). *For every group G and every subgroup H of G of index n ,*

$$\sum_{\rho \in \text{Irep}(G)} \dim(\rho) \cdot \text{rank}(\rho(H)) = n. \quad (3)$$

3 The spectrum of the universal adjacency matrix of a relative lift

Let Γ be a connected graph on k vertices (with loops and multiple edges allowed) and with the universal adjacency matrix U as in (1). Let β be a relative voltage assignment on the arc set X of Γ in a group G with identity element e and subgroup H of index n . As commented above, the relative lift Γ^β , associated with the pair (Γ, β) , can also be obtained from the pair (Γ, α) , where α is the (induced) permutation voltage assignment on the group of G seen as a subgroup of $\text{Sym}(n)$. Now we show that the spectrum of (the universal adjacency matrix of) the relative lift $\Gamma^\beta \cong \Gamma^\alpha$ may be computed. To this end, the key idea is to define properly the so-called base matrix as follows.

Definition 3.1. To the pair (Γ, α) as above, we assign the $k \times k$ universal base matrix $B(U)$ defined by the sum

$$B(U) = c_1 B(A) + c_2 B(D) + c_3 B(I) + c_4 B(J),$$

where the matrices $B(A), B(D), B(I)$, and $B(J)$ have entries as follows:

- $B(A)_{uv} = \alpha(a_1) + \cdots + \alpha(a_j)$ if a_1, \dots, a_j is the set of all the arcs of Γ from u to v , not excluding the case $u = v$, and $B(A)_{uv} = 0$ if $(u, v) \notin X$;
- $B(D)_{uu} = \deg(u)e$, and $B(D)_{uv} = 0$ if $u \neq v$;
- $B(I)_{uu} = e$, and $B(I)_{uv} = 0$ if $u \neq v$;
- $B(J)_{uv} = \pi^0 + \pi^1 + \cdots + \pi^{n-1}$, where $\pi(= \pi^1) = (12 \dots n) \in \text{Sym}(n)$, for any $u, v \in V$.

Recall that $e(= \pi^0)$ stands for the identity element (permutation) of G , and the sums $z_1 \pi_1 + \cdots + z_n \pi_n$, where $z_i \in \mathbb{C}$ and $\pi_i \in G$, must be seen as elements of the complex group algebra $\mathbb{C}[G]$.

Let $\rho \in \text{Irep}(G)$ be a unitary irreducible representation of G of dimension $d_\rho = \dim(\rho)$. For our graph Γ on k vertices, the assignment α in G relative to H , and the universal base matrix $B = B(U)$, let $\rho(B)$ be the $d_\rho k \times d_\rho k$ matrix obtained from B by replacing every nonzero entry $(B)_{u,v} \in \mathbb{C}[G]$ as above by the $d_\rho \times d_\rho$ matrix $\rho(B_{u,v})$. That is, each element g of the group is replaced by $\rho(g)$, and the zero entries of B are changed to all-zero $d_\rho \times d_\rho$ matrices. We refer to $\rho(B)$ as the ρ -image of the universal base matrix B .

In the following result, whose particular case of the adjacency spectrum was given by the authors in [7], we use the rank of the matrix $\rho(H) = \sum_{h \in H} \rho(h)$. For every $\rho \in \text{Irep}(G)$, we consider the ρ -image $\rho(B)$ of the universal base matrix B , and we let $\text{Sp}(\rho(B))$ denote the spectrum of $\rho(B)$, that is, the multiset of all the $d_\rho k$ eigenvalues of the matrix $\rho(B)$. Finally, the notation $\text{rank}(\rho(H)) \cdot \text{Sp}(\rho(B))$ denotes the multiset of $\text{rank}(\rho(H)) \cdot d_\rho k$ values obtained by taking each of the $d_\rho k$ entries of the spectrum $\text{Sp}(\rho(B))$ exactly $\text{rank}(\rho(H))$ times. In particular, if $\text{rank}(\rho(H)) = 0$, the set $\text{rank}(\rho(H)) \cdot \text{Sp}(\rho(B)) = \emptyset$.

Theorem 3.2. Let Γ be a base graph of order k , with universal adjacency matrix U , and universal base matrix $B = B(U)$. Let α be a voltage assignment on Γ in a group G relative to a subgroup H of index n in G . Then, the multiset of the kn eigenvalues of the universal adjacency matrix U^α for the relative lift Γ^α is

$$\text{Sp}(U^\alpha) = \bigcup_{\rho \in \text{Irep}(G)} \text{rank}(\rho(H)) \cdot \text{Sp}(\rho(B)). \quad (4)$$

Moreover, a complete base of the corresponding eigenvectors of U^α can be obtained from the eigenvectors of the matrices $\rho(B)$ for $\rho \in \text{Irep}(G)$.

Proof. Let U^α be the universal adjacency matrix of the relative lift Γ^α , where $U = c_1A + c_2D + c_3I + c_4J$, for some constants $c_i \in \mathbb{R}$. The relative lift Γ^α has vertex set $V^\alpha = V \times G/H$, where V is the vertex set of the base graph Γ , and G/H is the set of right cosets of H in G . In more detail, U^α is a $kn \times kn$ matrix indexed by ordered pairs $(u, J) \in V^\alpha$ whose entries are given as follows. If there is no arc from u to u' for $u, u' \in V$ in Γ , then there is no arc from any vertex (u, J) to any vertex (u', J') for $J, J' \in G/H$ in the lift Γ^α . So, in this case, the $((u, J), (u', J'))$ -th entry of U^α is $c_2 \deg(u) + c_3 + c_4$. In the case of existing adjacencies, if for a $g \in G$, there are exactly t arcs a_1, \dots, a_t from a vertex u to a vertex v in Γ that carry the voltage $g = \alpha(a_1) = \dots = \alpha(a_t)$, then for every right coset $J \in G/H$ the $((u, J), (v, Jg))$ -th entry of U^α is equal to $c_1t + c_2 \deg(u) + c_3 + c_4$.

From this, the steps of the proof are the following:

1. Depending on the coefficients c_i with $1 \leq i \leq 4$, consider the corresponding universal base $k \times k$ matrix $B = B(U)$ of Γ relative to H , according to Definition 3.1.
2. For each irreducible representation $\rho \in \text{Irep}(G)$, with dimension $d_\rho = d$, consider the ρ -image $\rho(B)$ of the universal base matrix.
3. Every eigenvector of $\rho(B)$ can be ‘lifted’ to at most d eigenvectors of the corresponding universal adjacency matrix U^α of the lift Γ^α . Thus, the spectrum of $\rho(B)$ is contained in the whole spectrum of U^α . Since this is a crucial step, we give more details (see Subsection 4.3 for an example). For simplicity, assume that we are dealing with an ordinary lift of a base graph Γ , with k vertices, on the group G , with n elements. Let ρ be an irreducible representation of G , as above. Let

$$(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)^\top = (u_{11}, \dots, u_{1d}, u_{21}, \dots, u_{2d}, \dots, u_{k1}, \dots, u_{kd})^\top$$

be a (right) eigenvector of $\rho(B)$, with corresponding eigenvalue μ . For each $g_h \in G$, $h = 1, \dots, n$, and $i = 1, \dots, k$, compute the vector $\rho(g_h)\mathbf{x}_i$ with components $(\rho(g_h)\mathbf{x}_i)_j$ with $j = 1, \dots, d$. Notice that, at this point, we have nk vectors, each with d components. Then, an eigenvector of U^α with corresponding eigenvalue μ is obtained when we take every j -th component of such vectors. Namely, we get the vector

$$(\rho(g_1)\mathbf{x}_1)_j, \dots, (\rho(g_n)\mathbf{x}_1)_j, (\rho(g_1)\mathbf{x}_2)_j, \dots, (\rho(g_n)\mathbf{x}_2)_j, \dots, (\rho(g_1)\mathbf{x}_k)_j, \dots, (\rho(g_n)\mathbf{x}_k)_j$$

whose entry $(\rho(g_h), \mathbf{x}_i)_j$ corresponds to the vertex (g_h, u_i) of the lift Γ^α .

4. For each $\rho \in \text{Irep}(G)$, there are exactly $\text{rank}(\rho(H)) \cdot d_\rho$ independent eigenvectors of U^α obtained at step **3**, and they also are independent of the other eigenvectors obtained from any other irreducible representation $\rho' \in \text{Irep}(G)$.
5. Reasoning as in [7], use Proposition 2.2 to show that the total number of obtained independent eigenvectors of U^α , that is $\sum_{\rho \in \text{Irep}(G)} \text{rank}(\rho(H)) \cdot d_\rho$, equals kn , the number of vertices of Γ^α .

Notice that the proof is constructive, in the sense that we determine bases for all the eigenspaces of the relative lift Γ^α . \square

In particular, when H is the trivial subgroup, we get an ordinary voltage assignment on G , and (4) becomes

$$\mathrm{Sp}(U^\alpha) = \bigcup_{\rho \in \mathrm{Irep}(G)} \dim(\rho) \cdot \mathrm{Sp}(\rho(B)) \quad (5)$$

since $\mathrm{rank}(\rho(H)) = \mathrm{rank}(e) = \dim(\rho)$.

When we are dealing with a permutation voltage assignment on the group $G = \mathrm{Sym}(n)$, so that the subgroup H is the stabilizer of a given point in the set $[n]$ and $H \simeq \mathrm{Sym}(n-1)$, we only need to consider the trivial and standard representations of G , as shown in the following result.

Theorem 3.3. *Let Γ be a base graph of order k and let α be a permutation voltage assignment on Γ in the group $G = \mathrm{Sym}(n)$. Let ρ_0 and ρ_1 be the trivial and standard representations of $\mathrm{Sym}(n)$, respectively, with dimensions $\dim(\rho_0) = 1$ and $\dim(\rho_1) = n-1$. Then, the spectrum multiset of the universal adjacency matrix U^α for the relative lift Γ^α is*

$$\mathrm{Sp}(\Gamma^\alpha) = \mathrm{Sp}(\rho_0(B)) \cup \mathrm{Sp}(\rho_1(B)). \quad (6)$$

Proof. First, notice that, since $\dim(\rho_0) = 1$ and $\dim(\rho_1) = n-1$, we have that $|\mathrm{Sp}(\Gamma^\alpha)| = |\mathrm{Sp}(\rho_0(B))| + |\mathrm{Sp}(\rho_1(B))| = k + k(n-1) = kn = |V^\alpha|$, as expected. So, from Theorem 3.3, we only need to prove that $\mathrm{rank}(\rho_0(H)) = 1$, $\mathrm{rank}(\rho_1(H)) = 1$, and $\mathrm{rank}(\rho_i(H)) = 0$ for $i \neq 0, 1$, where H is the stabilizer of, say, 1 in $\mathrm{Sym}(n)$. Since $\rho_0(h) = 1$ for any $h \in \mathrm{Sym}(n)$, the first equality is clear. Now, in the standard representation ρ_1 of $\mathrm{Sym}(n)$, we see that all the matrices fixing 1, and hence e_1 , have their $(1, 1)$ -element equal to 1 (see, for instance, the matrices corresponding to ι and (23) in (2)). Hence, $\mathrm{rank}(\rho_1(H)) \geq 1$ and, from the above equality giving $|\mathrm{Sp}(\Gamma^\alpha)|$, we conclude that $\mathrm{rank}(\rho_1(H)) = 1$, and $\mathrm{rank}(\rho_i(H)) = 0$ for $i \neq 0, 1$. \square

Notice that the above result is also a consequence of the fact that ρ_1 is a faithful representation of $\mathrm{Sym}(n)$ (or of any subgroup of it). Hence, all eigenvalues of the matrix $\rho_1(B)$ must also be eigenvalues of the lift.

As an example, consider the base graph Γ constituted by two vertices u, v , the edge $a_1 = uv$, and the loops $a_2 = uu$ and $a_3 = vv$. Let the permutation voltage assignment α be defined on $\mathrm{Sym}(5)$ as follows: $\alpha(a_1^\pm) = \iota$, $\alpha(a_2^\pm) = g^{\pm 1}$ and $\alpha(a_3^\pm) = h^{\pm 1}$, where ι is the identity element, $g = (12345)$ and $h = (13524)$. Then, it is easily checked that the resulting lift graph Γ^α is the Petersen graph P . The base matrix turns out to be

$$B = \begin{pmatrix} g + g^{-1} & \iota \\ \iota & h + h^{-1} \end{pmatrix}.$$

The trivial representation ρ_0 assigns 1 to every element, whereas the standard representation ρ_1 goes as follows:

$$\begin{aligned} \iota \mapsto I, \quad g \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad g^{-1} \mapsto \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ h \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad h^{-1} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then, the matrices

$$\rho_0(B) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \rho_1(B) = \begin{pmatrix} \rho_1(g) + \rho_1(g^{-1}) & I \\ I & \rho_1(h) + \rho_1(h^{-1}) \end{pmatrix}$$

have spectra

$$\text{Sp}(\rho_0(B)) = \{3, 1\} \quad \text{and} \quad \text{Sp}(\rho_1(B)) = \{1^{(4)}, -2^{(4)}\},$$

where the superscripts indicate the eigenvalue multiplicities. Hence, the spectrum of P is $\{3, 1^{(5)}, -2^{(4)}\}$.

4 Some applications

4.1 An example

Following the method of Theorem 3.3, we now work out some different spectra of the relative lift shown in Figure 1. Referring to this figure, we consider the base graph Γ with vertex set $V = \{u, v\}$ and arc set $X = \{a, a^-, b, b^-\}$, where the pairs $\{a, a^-\}$, $\{b, b^-\}$ determine an edge joining u to v , and a loop at v , respectively. The permutation voltage assignment α on Γ in the group $\text{Sym}(3)$ is given by $\alpha(a) = \alpha(a^-) = e$, $\alpha(b) = (123) = s$, and $\alpha(b^-) = (132) = s^2 = t$. An equivalent description is to regard α as a relative voltage assignment, with values of α on arcs acting on the right cosets of G/H for $H = \text{Stab}_G(1) = \{e, (23)\}$ by right multiplication. Then, the different base matrices with entries in the group algebra $\mathbb{C}[G]$ have the form

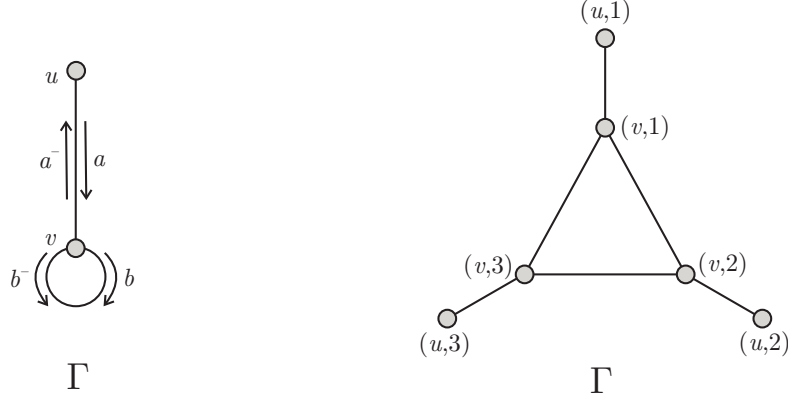


Figure 1: A graph Γ and its relative lift Γ^α in the group $\text{Sym}(3)$.

$$\begin{aligned}
B(A) &= \begin{pmatrix} 0 & \alpha(a) \\ \alpha(a^-) & \alpha(b) + \alpha(b^-) \end{pmatrix} = \begin{pmatrix} 0 & e \\ e & s+t \end{pmatrix}; \\
B(D) &= \begin{pmatrix} \deg(u)e & 0 \\ 0 & \deg(v)e \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 3e \end{pmatrix}; \\
B(I) &= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}; \\
B(J) &= \begin{pmatrix} e+s+t & e+s+t \\ e+s+t & e+s+t \end{pmatrix}.
\end{aligned}$$

The voltage group $G = \text{Sym}(3) = \{e, g, h, r, s, t\}$ with $g = (23)$, $h = (12)$, $r = (13)$, $s = (123)$, and $t = (132)$ has a complete set of irreducible unitary representations $\text{Irep}(G) = \{\iota, \pi, \sigma\}$ corresponding to the symmetries of an equilateral triangle with vertices positioned at the complex points $e^{i\frac{2\pi}{3}}$, 1 , and $e^{i\frac{4\pi}{3}}$, where

$$\begin{aligned}
\iota : G &\rightarrow \{1\}, & \dim(\iota) &= 1 \quad (\text{the trivial representation}), \\
\pi : G &\rightarrow \{\pm 1\}, & \dim(\pi) &= 1 \quad (\text{the parity permutation representation}), \text{ and,} \\
\sigma : G &\rightarrow GL(2, \mathbb{C}), & \dim(\sigma) &= 2, \text{ generated by the unitary matrices}
\end{aligned}$$

$$\sigma(g) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad \text{and} \quad \sigma(h) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

whence we obtain

$$\sigma(r) = \sigma(ghg) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma(s) = \sigma(gh) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \text{and}$$

$$\sigma(t) = \sigma(hg) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

To determine the ‘multiplication factors’ appearing in front of the spectra in the statement of Theorem 3.3, we evaluate $\iota(H) = \iota(e) + \iota(g) = 1 + 1 = 2$, of rank 1, $\pi(H) = \pi(e) + \pi(g) = 1 - 1 = 0$, of rank 0, and

$$\sigma(H) = \sigma(e) + \sigma(g) = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} \quad (7)$$

of rank 1. Then, again by Theorem 3.3, the spectra of the universal adjacency matrices U^α of the relative lift Γ^α is obtained by the union of the sets $\text{Sp}(\iota(U))$ and $\text{Sp}(\sigma(U))$. Let us show some different cases.

4.2 Adjacency spectrum

In this case, we only need to consider the images of $B = B(A)$ under the representations ι and σ , which are

$$\iota(B) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \sigma(B) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad (8)$$

with spectra $\text{Sp}(\iota(B)) = \{1 \pm \sqrt{2}\}$ and $\text{Sp}(\sigma(B)) = \{[\frac{1}{2}(-1 \pm \sqrt{5})]^{(2)}\}$. Then,

$$\text{Sp}(A^\alpha) = \{1 \pm \sqrt{2}, [\frac{1}{2}(-1 \pm \sqrt{5})]^{(2)}\}.$$

4.3 Laplacian spectrum

Since the Laplacian matrix can be written as

$$L = D - A = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix} \quad (9)$$

(with rows and columns indexed with $(u, 1), (u, 2), \dots, (v, 3)$), we consider the base matrix of $B(L) = B(D) - B(A) = \begin{pmatrix} e & -e \\ -e & 3e - s - t \end{pmatrix}$ under the representations ι and σ , which are

$$\iota(B(L)) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \sigma(B(L)) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}, \quad (10)$$

with spectra $\text{Sp}(\iota(B(L))) = \{0, 2\}$ and $\text{Sp}(\sigma(B(L))) = \{[\frac{1}{2}(5 \pm \sqrt{13})]^{(2)}\}$. Thus,

$$\text{Sp}(L^\alpha) = \{0, 2, [\frac{1}{2}(5 \pm \sqrt{13})]^{(2)}\}.$$

Here, let us show also how the $(\frac{1}{2}(5 \pm \sqrt{13}))$ -eigenvectors of L^α are obtained from the eigenvectors of $\sigma(B(L))$. We follow the procedure explained in step **3** of the proof of Theorem 3.3, but since we are dealing with four eigenvectors at the same time, we adopt a matricial approach. Thus, the eigenvectors of $\sigma(B(L))$ are the columns of the matrix

$$\begin{aligned} U_\sigma &= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2}(3 + \sqrt{13}) & 0 & \frac{1}{2}(-3 + \sqrt{13}) \\ -\frac{1}{2}(3 + \sqrt{13}) & 0 & \frac{1}{2}(-3 + \sqrt{13}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{x}_{u1} & \mathbf{x}_{u2} & \mathbf{x}_{u3} & \mathbf{x}_{u4} \\ \mathbf{x}_{v1} & \mathbf{x}_{v2} & \mathbf{x}_{v3} & \mathbf{x}_{v4} \end{pmatrix} = \begin{pmatrix} X_u \\ X_v \end{pmatrix}. \end{aligned}$$

Then, since all vertices of Γ^α have second coordinate e , s , or t , we only have to consider the matrices

$$\begin{aligned} \Phi_{(u,e)} &= \sigma(e)X_u = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \\ \Phi_{(u,s)} &= \sigma(s)X_u = \frac{1}{2} \begin{pmatrix} -\sqrt{3} & -1 & -\sqrt{3} & -1 \\ -1 & \sqrt{3} & -1 & \sqrt{3} \end{pmatrix}, \\ \Phi_{(u,t)} &= \sigma(t)X_u = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & -1 & -\sqrt{3} \end{pmatrix}, \\ \Phi_{(v,e)} &= \sigma(e)X_v = \frac{1}{2} \begin{pmatrix} 0 & -3 - \sqrt{13} & 0 & -3 + \sqrt{13} \\ -3 - \sqrt{13} & 0 & -3 + \sqrt{13} & 0 \end{pmatrix}, \\ \Phi_{(v,s)} &= \sigma(s)X_v = \frac{1}{4} \begin{pmatrix} \sqrt{3}(3 + \sqrt{13}) & 3 + \sqrt{13} & \sqrt{3}(3 - \sqrt{13}) & 3 - \sqrt{13} \\ 3 + \sqrt{13} & -\sqrt{3}(3 + \sqrt{13}) & 3 - \sqrt{13} & \sqrt{3}(-3 + \sqrt{13}) \end{pmatrix}, \\ \Phi_{(v,t)} &= \sigma(t)X_v = \frac{1}{2} \begin{pmatrix} -\sqrt{3}(3 + \sqrt{13}) & 3 + \sqrt{13} & \sqrt{3}(-3 + \sqrt{13}) & 3 - \sqrt{13} \\ 3 + \sqrt{13} & \sqrt{3}(3 + \sqrt{13}) & 3 - \sqrt{13} & \sqrt{3}(3 - \sqrt{13}) \end{pmatrix}. \end{aligned}$$

Hence, the matrix formed by the first rows of $\Phi_{(u,e)}$, $\Phi_{(u,s)}$, \dots , $\Phi_{(v,t)}$,

$$Z = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ 0 & -\frac{1}{2}(3 + \sqrt{13}) & 0 & \frac{1}{2}(-3 + \sqrt{13}) \\ \frac{1}{4}\sqrt{3}(3 + \sqrt{13}) & \frac{1}{4}(3 + \sqrt{13}) & \frac{1}{4}\sqrt{3}(3 - \sqrt{13}) & \frac{1}{4}(3 - \sqrt{13}) \\ -\frac{1}{4}\sqrt{3}(3 + \sqrt{13}) & \frac{1}{4}(3 + \sqrt{13}) & \frac{1}{4}\sqrt{3}(-3 + \sqrt{13}) & \frac{1}{4}(3 - \sqrt{13}). \end{pmatrix},$$

has first and second columns being eigenvectors of L in (9) with eigenvalue $\frac{1}{2}(5 + \sqrt{13})$, whereas its third and four columns are eigenvectors of L with eigenvalue $\frac{1}{2}(5 - \sqrt{13})$, as

claimed. Besides, these vectors are linearly independent since $\text{rank}(Z) = 4$. (The same comment applies when we take the second rows of $\Phi_{(u,e)}, \Phi_{(u,s)}, \dots, \Phi_{(v,t)}$, but the obtained eigenvectors are linearly dependent of the previous ones.)

4.4 Signless Laplacian spectrum

The signless Laplacian is $Q = A + D$. So, we consider the base matrix of $B(Q) = B(A) + B(D) = \begin{pmatrix} e & e \\ e & 3e + s + t \end{pmatrix}$ with the representations ι and σ as follows:

$$\iota(B(Q)) = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}, \quad \sigma(B(Q)) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad (11)$$

with spectra $\text{Sp}(\iota(B(Q))) = \{3 \pm \sqrt{5}\}$ and $\text{Sp}(\sigma(B(Q))) = \{[\frac{1}{2}(3 \pm \sqrt{5})]^{(2)}\}$. Thus,

$$\text{Sp}(Q^\alpha) = \{3 \pm \sqrt{5}, [\frac{1}{2}(3 \pm \sqrt{5})]^{(2)}\}.$$

4.5 Seidel spectrum

The Seidel matrix can be defined as $S = \bar{A} - A = J - 2A - I$. Then, we consider the base matrix of $B(S) = B(J) - 2B(A) - B(I) = \begin{pmatrix} s+t & -e+s+t \\ -e+s+t & -s-t \end{pmatrix}$ under the representations ι and σ , which are

$$\iota(B(S)) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \sigma(B(S)) = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}, \quad (12)$$

with spectra $\text{Sp}(\iota(B(S))) = \{\pm\sqrt{5}\}$ and $\text{Sp}(\sigma(B(S))) = \{[\pm\sqrt{5}]^{(2)}\}$. Thus,

$$\text{Sp}(S^\alpha) = \{[\pm\sqrt{5}]^{(3)}\}.$$

4.6 Using ordinary voltage assignments

Note that, in our example, the group G generated by the permutation $s = (123)$ is a regular permutation on $\{1, 2, 3\}$ that is isomorphic to the cyclic group \mathbb{Z}_3 . Then, the covering $\Gamma^\beta \rightarrow \Gamma$ is regular, and we can also construct the lift Γ^α from an ordinary voltage assignment on \mathbb{Z}_3 . As it is well-known, the irreducible representations ρ_i of the cyclic group \mathbb{Z}_n are $\rho_i : 1, \omega^i, \omega^{2i}, \dots, \omega^{(n-1)i}$, for $i = 0, \dots, n-1$, where ω is a primitive n -th root of unity. Thus, all these representations have dimension one. In our case, with $e = s^0$, s , and $s^2 = s^{-1}$ being the elements of \mathbb{Z}_3 , such representations are shown in Table 1. Then, by using the same base matrices $M \in \{B, B(L), B(Q), B(S)\}$ as before (with $t = s^2$), we get the following matrices $\rho_i(M)$, for $i = 0, 1, 2$, together with their spectra:

$g \in \mathbb{Z}_3$	e	s	s^2
$\rho_0 (d_1 = 1)$	1	1	1
$\rho_1 (d_2 = 1)$	1	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$\rho_2 (d_3 = 1)$	1	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{2\pi}{3}}$

Table 1: The irreducible representations of the cyclic group \mathbb{Z}_3 .

- **Adjacency spectrum**

$$\rho_0(B) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \text{Sp}(\rho_0(B)) = \{1 \pm \sqrt{2}\};$$

$$\rho_1(B) = \rho_2(B) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \text{Sp}(\rho_1(B)) = \text{Sp}(\rho_2(B)) = \{\frac{1}{2}(-1 \pm \sqrt{5})\}.$$

- **Laplacian spectrum**

$$\rho_0(L(B)) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \text{Sp}(\rho_0(L(B))) = \{0, 2\};$$

$$\rho_1(L(B)) = \rho_2(L(B)) = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}, \text{Sp}(\rho_1(L(B))) = \text{Sp}(\rho_2(L(B))) = \{\frac{1}{2}(5 \pm \sqrt{13})\}.$$

- **Signless Laplacian spectrum**

$$\rho_0(Q(B)) = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}, \text{Sp}(\rho_0(Q(B))) = \{3 \pm \sqrt{5}\};$$

$$\rho_1(Q(B)) = \rho_2(Q(B)) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \text{Sp}(\rho_1(Q(B))) = \text{Sp}(\rho_2(Q(B))) = \{\frac{1}{2}(3 \pm \sqrt{5})\}.$$

- **Seidel spectrum**

$$\rho_0(S(B)) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \text{Sp}(\rho_0(S(B))) = \{\pm\sqrt{5}\};$$

$$\rho_1(S(B)) = \rho_2(S(B)) = \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}, \text{Sp}(\rho_1(S(B))) = \text{Sp}(\rho_2(S(B))) = \{\pm\sqrt{5}\}.$$

From these spectra, and applying (5), we again obtain the different spectra of our example.

As a last comment, note that, in this context of ordinary voltage assignments on a group G of order n , of a base graph Γ , the base matrix of J has entries $(J)_{uv} = \sum_{g \in G} g$. Then, from representation theory, the matrices $\rho_0(J)$ and $\rho_i(J)$ for $i \neq 0$ turn out to be nJ and O (the 0-matrix with appropriate dimension), respectively. Thus, for $i \neq 0$, $\rho_i(S(B)) = -2\rho_i(A) - \rho_i(I) = -2\rho_i(A) - I$, and we can state the following remark (see the above results for an example).

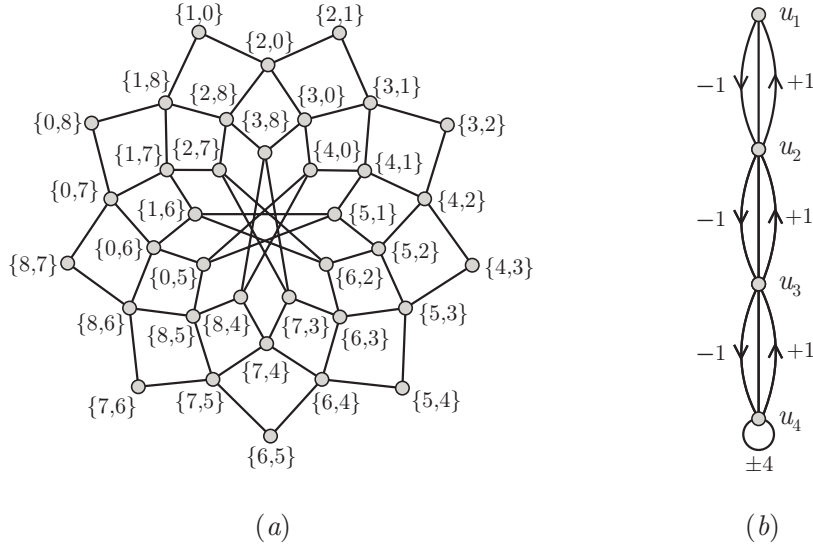


Figure 2: (a) The symmetric square graph $F_2(C_9)$ of the cycle graph C_9 with vertex set $V(C_9) = \mathbb{Z}_9$. (b) The base graph Γ with voltages in \mathbb{Z}_9 .

Remark 4.1. Let Γ be a base graph with ordinary voltage assignment α on a group G of order n . Let B be the base matrix of Γ . Then, apart from the eigenvalues of $nJ - 2B - I$, the other Seidel eigenvalues of the lift Γ^α are of the form $-2\lambda - 1$, where λ is an adjacency eigenvalue of Γ^α (that is, $\lambda \in \text{Sp}(\rho_i(B))$ for $i \neq 0$).

4.7 Symmetric squares of odd cycles

In this subsection, we show how to use the above results to compute the Laplacian spectrum of an infinite family of the so-called symmetric powers. The *symmetric square* (or *2-token graph*) $F_2(\Gamma)$ of Γ is the graph with vertex set $V(F_2(\Gamma))$, the $\binom{n}{2}$ 2-subsets of vertices of Γ , and where two vertices A and B of $F_2(\Gamma)$ are adjacent whenever their symmetric difference $A \triangle B = \{a, b\}$, where $a \in A$, $b \in B$, and $\{a, b\} \in E(\Gamma)$. In particular, if Γ is the complete graph K_n , then $F_2(K_n) \cong J(n, 2)$, the (distance-regular) Johnson graph. In Figure 2(a), we show the symmetric square $F_2(C_9)$ of the cycle graph C_9 , with vertices $\{i, j\}$ for $i, j \in \mathbb{Z}_9$. For more details, see Audenaert, Godsil, Royle, and Rudolph [2], who proved that the (adjacency) spectra of the symmetric square of strongly regular graphs with the same parameters are equal. Also, for the more general concept of k -th power of a graph, see Fabila-Monroy, Flores-Peñalozza, Huemer, Hurtado, Urrutia, and Wood [9].

Given an integer k , let us consider the path graph $\Gamma = P_k$ with vertices u_1, u_2, \dots, u_k , arcs $a_i = u_i u_{i+1}$, $a_i^- = u_{i+1} u_i$, for $i = 1, 2, \dots, k-1$, and the loops b and b^- at u_k . Let α be the voltage assignment on Γ in the cyclic group \mathbb{Z}_{2k+1} , given by $\alpha(a_i) = -1$, $\alpha(a_i^-) = +1$, for $i = 1, \dots, k-1$, $\alpha(b) = k$, and $\alpha(b^-) = -k$. Figure 2(b) shows the base graph Γ for

$k = 4$. Thus, we have the following result.

Lemma 4.2. *Given $\Gamma = P_k$ with the above voltage assignment on \mathbb{Z}_{2k+1} , the symmetric square of the cycle graph C_{2k+1} is the lift graph Γ^α . That is,*

$$F_2(C_{2k+1}) \cong \Gamma^\alpha.$$

Proof. Let denote the $\binom{2k+1}{2}$ vertices of $F_2(C_{2k+1})$ as follows (all arithmetic must be understood modulo $2k + 1$):

$$\begin{array}{cccccc} u_1 = \{1, 0\}, & \{2, 1\}, & \{3, 2\}, & \dots, & \{0, 2k\}, \\ u_2 = \{2, 0\}, & \{3, 1\}, & \{4, 2\}, & \dots, & \{1, 2k\}, \\ u_3 = \{3, 0\}, & \{4, 1\}, & \{5, 2\}, & \dots, & \{2, 2k\}, \\ u_4 = \{4, 0\}, & \{5, 1\}, & \{6, 2\}, & \dots, & \{3, 2k\}, \\ \vdots & \vdots & \vdots & & \vdots \\ u_k = \{k, 0\}, & \{k + 1, 1\}, & \{k + 2, 2\}, & \dots, & \{k - 1, 2k\}, \end{array}$$

where the first column corresponds to the vertices of the 0-th ‘copy’ of Γ . Consider the following equivalence between these vertices and the vertices of the lift Γ^α :

$$\{h + i, h\} \leftrightarrow (\{0, i\}, h) = (u_i, h),$$

where $i = 1, \dots, k$ and $h \in \mathbb{Z}_{2k+1}$. Then, assuming that $i \neq 1, k$ (the other cases are similar), the vertex $\{h + i, h\}$ of the symmetric square of C_{2k+1} is adjacent to the vertices

$$\{h + i, h + 1\}, \quad \{h + i, h - 1\}, \quad \{h + i + 1, h\}, \quad \{h + i - 1, h\},$$

which, respectively, correspond to the vertices of the lift

$$(u_{i-1}, h + 1) = (u_{i-1}, h + \alpha(a_i^-)), \quad (u_{i+1}, h - 1) = (u_{i+1}, h + \alpha(a_i)), \quad (u_{i+1}, h), \quad (u_{i-1}, h),$$

as claimed. (Notice that the last two adjacencies correspond to the edges $\{u_i, u_{i\pm 1}\}$ of the same h -th ‘copy’ of Γ .) \square

As a consequence, for each $z = \omega^\ell$, where $\omega = e^{i\frac{2\pi}{2k+1}}$, with $\ell = 0, 1, \dots, 2k$, an irreducible representation of the Laplacian base matrix of $\Gamma^\alpha \cong F_2(C_{2k+1})$ is a tridiagonal matrix $B_L(z)$ with main diagonal $(2, 4, \dots, 4, 4 - z^4 - \frac{1}{z^4})$, main superdiagonal $(-1 - \frac{1}{z}, \dots, -1 - \frac{1}{z})$, and main subdiagonal $(-1 - z, \dots, -1 - z)$. For instance, in the case of $F_2(C_9)$ ($k = 4$), we have

$$B_L(z) = \rho_z(B(L)) = \begin{pmatrix} 2 & -1 - \frac{1}{z} & 0 & 0 \\ -1 - z & 4 & -1 - \frac{1}{z} & 0 \\ 0 & -1 - z & 4 & -1 - \frac{1}{z} \\ 0 & 0 & -1 - z & 4 - z^4 - \frac{1}{z^4} \end{pmatrix}.$$

In Table 2, we show the different eigenvalues of each $B_L(z)$ for $z = \omega^r$ with $r = 0, 1, 2, 3$.

$\omega = e^{i\frac{2\pi}{9}}, z = \omega^r$	$\lambda_{r,1}$	$\lambda_{r,2}$	$\lambda_{r,3}$	$\lambda_{r,4}$
$\text{Sp}(B_L(\omega^0))$	0	1.171572876	4	6.828427124
$\text{Sp}(B_L(\omega^1)) = \text{Sp}(B_L(\omega^8))$	0.4679111136	2.52079560	5.420264509	7.470414013
$\text{Sp}(B_L(\omega^2)) = \text{Sp}(B_L(\omega^7))$	0.783324839	1.65270363	3.895673125	6.136209510
$\text{Sp}(B_L(\omega^3)) = \text{Sp}(B_L(\omega^6))$	1.50913638	3	4.656620432	5.834243185
$\text{Sp}(B_L(\omega^4)) = \text{Sp}(B_L(\omega^5))$	1.939683655	3.382489411	3.87938479	4.451145779

Table 2: All the eigenvalues of matrices $B_L(\omega^r)$, which yield the eigenvalues the symmetric square $F_2(C_9)$. The values in bold face correspond to the eigenvalues of C_9 .

Proposition 4.3. *Given an integer k , the characteristic polynomial and spectrum of the symmetric power of C_{2k+1} satisfy the following.*

- (i) *The characteristic polynomial of the Laplacian matrix of $\Gamma^\alpha \cong F_2(C_{2k+1})$ is*

$$\phi(\Gamma^\alpha, x) = \phi_0(x) \prod_{r=1}^k \phi_r(x)^2,$$

where, for $r = 0, \dots, k$, $\phi_r(x)$ is the characteristic polynomial of $B_L(\omega^r)$, computed with the following recurrence. Starting from $f_0 = 1$ and $f_1 = x - 2$, let

$$f_i(x) = (x - 4)f_{i-1}(x) - 2 \cos\left(\frac{r2\pi}{2k+1}\right) f_{i-2}(x), \quad \text{for } i = 2, \dots, k-1,$$

and

$$\phi_r(x) = f_k(x) = \left(x - 4 \pm \cos\left(\frac{r\pi}{2k+1}\right)\right) f_{k-1}(x) - 2 \cos\left(\frac{r2\pi}{2k+1}\right) f_{k-2}(x),$$

where the plus and minus sign corresponds to the cases of even and odd r , respectively.

- (ii) *The spectrum of the symmetric square $F_2(C_{2k+1})$ is the union of the spectra of the tridiagonal matrices $B'_L(r)$, for $r = 0, \dots, 2k+1$, with main diagonal $(2, 4, \dots, 4 \pm 2 \cos(\frac{r\pi}{2k+1}))$, depending on whether r is odd or even, and constant main superdiagonal and main subdiagonal with entries $\sqrt{2(1 + \cos(\frac{r2\pi}{2k+1}))} = 2 \cos(\frac{r\pi}{2k+1})$.*

- (iii) *The Laplacian spectrum of the symmetric square $F_2(C_{2k+1})$ contains the k eigenvalues*

$$\lambda_j = 4 \left(1 - \cos\left(\frac{j\pi}{2k+1}\right)\right), \quad j = 0, 1, \dots, k-1, \quad (13)$$

and also all the $2k+1$ eigenvalues of the cycle C_{2k+1} ,

$$\theta_j = 2 \left(1 - \cos\left(\frac{j2\pi}{2k+1}\right)\right), \quad j = 0, 1, \dots, 2k. \quad (14)$$

Proof. (i) Let $\omega = e^{i\frac{2\pi}{2k+1}}$. Since, for every $r = 0, \dots, k$, the matrix $xI - B_L(\omega^r)$ is tridiagonal, its determinant $\phi_r(x)$ is the last term of the so-called *continuant*, which is the sequence $f_0(x), \dots, f_k(x)$ satisfying the following recurrence (with coefficients the entries of the matrix): Starting from $f_0 = 1$ and $f_1 = x - 2$, let

$$f_i(x) = (x - 4)f_{i-1}(x) - (1 + \omega^r) \left(1 + \frac{1}{\omega^r}\right) f_{i-2}(x), \quad \text{for } i = 2, \dots, k - 1,$$

$$f_k(x) = \left(x - 4 + \omega^{rk} + \frac{1}{\omega^{rk}}\right) f_{k-1}(x) - (1 + \omega^r) \left(1 + \frac{1}{\omega^r}\right) f_{k-2}(x).$$

Then, the result follows since $\frac{1}{\omega} = \bar{\omega}$ implies $(1 + \omega^r) \left(1 + \frac{1}{\omega^r}\right) = 2 \left(1 + \cos\left(\frac{r2\pi}{2k+1}\right)\right)$, $(1 + \omega^{rk}) \left(1 + \frac{1}{\omega^{rk}}\right) = \pm \cos\left(\frac{r\pi}{2k+1}\right)$, depending on whether r is even or odd, and moreover $B_L(\omega^r) = B_L(\omega^{2k+1-r})$ for $r = 0, \dots, k$.

(ii) From the previous reasoning, it is clear that the matrices $B_L(\omega^r)$ and $B'_L(\omega^r)$ have the same continuant and, hence, the same characteristic polynomial.

(iii) The eigenvalues in (13) are those of the matrix $B_L(1)$ ($r = 0$), which correspond to a kind of tridiagonal matrix whose spectrum was given by Yueh, see [20, Th. 4]. \square

Here, it is worth noting that, since the matrices $B'_L(r)$ are real, symmetric, and tridiagonal, we can apply the efficient algorithm proposed by Coakley and Rokhlin [5] for computing their eigenvalues, requiring only $O(k \ln k)$ operations.

We finish the paper with the following comment that can be seen as an open problem. In [6], it was shown that the Laplacian spectrum of a graph is contained in the Laplacian spectrum of its symmetric square. Thus, in our context, $\text{Sp}(C_{2k+1}) \subset \text{Sp}(F_2(C_{2k+1}))$, and computer results suggest that the following holds (see, for example, Table 2):

1. For every $r = 0, \dots, 2k$, the matrix $B_L(\omega^r)$ (or $B'_L(\omega^r)$) has exactly one eigenvalue of C_{2k+1} , which is $2 \left(1 - \cos\left(\frac{r2\pi}{2k+1}\right)\right)$.
2. In particular, the algebraic connectivity $a(F_2(C_{2k+1}))$ (that is, its smallest nonzero eigenvalue) equals the smallest eigenvalue of $B_L(\omega)$ ($r = 1$), which is also the algebraic connectivity of C_{2k+1} , $2 \left(1 - \cos\left(\frac{2\pi}{2k+1}\right)\right)$. (In fact, by applying the Gershgorin circle theorem to $B_L(\omega^r)$ when $r \neq 0, 1, 2k$, it can be shown that the smallest eigenvalue of $B_L(\omega^r)$ is at least $a(C_{2k+1})$).

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