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THE INTEGRAL AS AN ANTI-DIFFERENTIAL. AN ASPECT OF EULER'S ATTEMPT TO TRANSFORM THE CALCULUS INTO AN ALGEBRAIC CALCULUS

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1.- Introduction.

The integration of differential formulas was one of the main fields of Euler's activity. He wrote many papers on the subject. In his *Opera omnia*, about 54 articles concern the integration of functions and about 42 regard the integration of differential equations. There are also about 33 papers on elliptic integrals (the theory of elliptic integrals was part of the integral calculus) and three volumes of the massive *Institutionum calculi integralis*, published between 1668 and 1770. Moreover, Euler dealt with integration in many other papers which, even though they were devoted to different subjects, involved differential equations (especially papers that regard geometric or mechanical problems).

In this paper, I will dwell upon an important aspect of Euler's work on integration: the notion of integration as anti-differentiation. I will show that this notion requires examination within the context of Euler's strategy that aimed at transforming integral calculus into an exclusively algebraic theory[1] and that it produced several problems, the most important of which concerned the existence of the anti-differential and the nature of the functions involved in the operation of integration. I will also consider the role of general and particular integrals in Euler's theory and stress that the importance attributed to indefinite integration and general integrals was linked to the conception of analysis as the science that investigated mathematical objects in

[1] On the calculus in the 18th century, see FRASER, Craig (1989) "The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century", *Archive for History of Exact Sciences*, 39, 317-335; FERRARO, Giovanni (2007a) "The foundational aspects of Gauss's work on the hypergeometric, factorial and digamma functions", *Archive for History of Exact Sciences*, 61, 457-518, in particular pp. 459-479, and FERRARO, Giovanni (2008), *The rise and development of the theory of series up to the early 1820s*, New York, Spinger, in particular Chapter 18.

an abstract and general way. Finally, I will discuss the implication of Euler's concept on his treatment of elliptic integrals.

2.- Integration as anti-differentiation.

At the very beginning of his *Institutionum calculi integralis*[2], Euler defined the integral calculus as the method for solving the following problem:

(**P**) Given a relation between differentials, find the relation between the quantities (EULER, 1768-70, vol. 1: §. 1).

This problem was viewed as the inverse of another problem, which was the object of the differential calculus:

(**D**) Given a relation between quantities, find the relation between their differentials[3].

Problem (**P**) was solved by the operation of integration, which was merely defined as the inverse operation of differentiation; therefore the integral was conceived as an anti-differential[4].

This concept of integral[5] clearly differs from Leibniz's original one. Leibniz had defined the integral (*summatrix*) ∫f(x)dx of the function *f(x)* as the sum of

2 EULER, Leonhard (1768-70) *Institutionum calculi integralis,* Petropoli, Impensis Academiae Imperialis Scientiarum. Reprinted *Leonhardi Euleri opera omnia*, Basel, Birkhäuser (afterwards: Opera), series 1, vols. 11-13.

3 Similar definitions are found in other works by Euler. See EULER, Leonhard (1765) "De usu functionum discontinarum in Analysi", *Novi Commentarii academiae scientiarum Petropolitanae* (later: Novi Comm.), 11, 6-27, or *Opera*, ser. 1, vol. 23, 74-91; EULER, Leonhard (1769), "De formulis integralibus duplicatis", *Novi Comm.*, 14, 72-103, or *Opera*, ser. 1, vol. 17, 289-315; EULER, Leonhard (1755), *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, Petropoli, Impensis Academiae Imperialis Scientiarum, or *Opera*, ser. 1, vol. 10. For instance, in *Institutiones calculi differentialis*, after having defined the differential calculus as the calculus by which differentials are investigated and applied (1755, §. 115), Euler stated: "Just as in differential calculus the differential of any given quantity is investigated, so there is a kind of calculus that consists in finding a quantity whose differential is one that already given, and that is called integral calculus. If any quantity is given, the quantity whose differential is the proposed quantity is called its integral" (EULER, 1755: §.139).

4 I would emphasise that the term 'anti-differential' is not found in Euler's papers. I use it for reasons of brevity.

5 On the concept of integral as anti-differential, see FERRARO, Giovanni (2007b) "Euler's treatises on infinitesimal analysis: *Introductio in analysin infinitorum, Institutiones calculi differentialis, Institutionum calculi integralis"* in *Euler Reconsidered. Tercentenary essays*, Heber City UT, Kendrick Press, 39-101, and FRASER, Craig (2003) "Mathematics", *The Cambridge History of Science Volume 4 The Eighteenth Century*, in particular pp. 315-317. In this paper Fraser showed that "Euler held this concept from a very early stage of his career" (FRASER, 2003: 316).

infinite infinitesimal rectangles *f(x)dx*. Starting from this definition, he had deduced the relationship between integration and differentiation. As early as the early 1690s, Johann Bernoulli had preferred to define integration as anti-differentiation and, during the eighteenth-century, this definition was the prevailing one[6].

Euler's definition was therefore not new; what was innovative was the way Euler justified the superiority of this concept of the integral with respect to Leibniz's concept. In *Institutionum calculi integralis*, he did this in two short comments (EULER, 1768-70, vol. 1: §§.11 and 302). The second of these comments is found in Chapter 7, Section 2, Volume 1, where Euler faced the problem of finding the approximate value of

$$y(x) = b + \int_a^x X(x)dx\,.$$

He considered a sequence of numbers $a_1<a_2<a_3<\ldots$ lying between a and x such that the differences a_i-a_{i-1} were extremely small and stated that the function X(x) could be considered as a constant when x varied between a_{i-1} and a_i. Set $y(a_i) = b_i$ for i = 1, 2, …, n-1 and $a_0 = a$, $a_n = x$, $b_0 = b$, one obtained

$$y(a_i) = b_i = b_{i-1} + \int_{a_{i-1}}^{a_1} X(x)dx = b_{i-1} + X(a_{i-1})(a_i - a_{i-1})\,, \text{ for } i=1, 2, \ldots, n.$$

Hence,

$$y(a_1) = b_1 = b + \int_a^{a_1} X(x)dx = b + X(a)(a_1 - a)\,,$$

$$y(a_2) = b_2 = b_1 + \int_{a_1}^{a_2} X(x)dx = b_1 + X(a_1)(a_2 - a_1) =$$

$$= b + X(a)(a_1 - a) + X(a_1)(a_2 - a_1)\,, \ldots,$$

$$y(x) = b + \int_a^x X(x)dx = b_n = b + \sum_{i=0}^{n} X(a_{i-1})(a_i - a_{i-1}).$$

Moreover, if the differences a_i-a_{i-1} were all equal to α, one obtained

6 See, for instance, HERMANN, Jacob (1726) "De calculo Integrali", *Commentarii academiae scientiarum Petropolitanae* (later: *Comm.*), vol. 1, 169-167; D'ALEMBERT, Jean Le Ronde (1765) «Integral» in *Encyclopédie, ou dictionnaire raisonné des sciences, des arts et des métiers*, vol. 3, Paris, Durand, pp. 805a-805b; and LAGRANGE, Joseph-Louis (1797) *Théorie des fonctions analytiques*, Paris, Impr. de la République, 1797.

(1) $y(x) = b + \int X(x)dx = b+\alpha\big(X(a) + X(a+\alpha) + X(a+2\alpha) +\ldots+ X(a+n\alpha)\big)$

(EULER, 1768-70, vol. 1: §§. 297-304).

Euler also observed that the smaller the differences a_i-a_{i-1}, the more accurate the value of y(x), provided the differences $X(a_i)$ -$X(a_{i-1})$ between the values of the integrand function X(x) are also extremely small[7].

In a *scholium* to this theorem, Euler observed that (1) could be assumed as the definition of an integral, taking α as an infinitesimal. However, he immediately rejected this possibility because it involved considering infinitesimals as really existing entities: in his opinion, the calculus as an exact science could not be based upon the notion of actually existing infinitesimals. He provided a lengthy explanation of his point of view in his treatise on the differential calculus, in which he defined differentials as evanescent quantities or zero or nothing. According to Euler, a differential *dx* was only a way of denoting that a variable x vanished (namely, it tended to zero) and the numerical value of *dx* could only be zero. Therefore, the subject-matter of the differential calculus was not differentials (which were always equal to 0 and were not regarded as worthy of investigation) but their differential ratios *dy/dx*: "Differential calculus … is not concerned with investigating the magnitude of differentials, which is nothing, but with defining their mutual ratio, which has a determinate quantity in any case" (EULER, 1765: 80).

This allowed Euler to consider differentiation and integration as operations between finite quantities (and not between differentials or infinitesimals). The differentiation of a function $y = f(x)$ was the operation that associated another function $p(x)$, usually denoted by dy/dx, with the given function. Only this meaning of differentiation was correct whereas the usual definition (differentiation is the operation by which the differential *dy* of a function $y = f(x)$ is found) was not. In Euler's opinion, it was possible to write $dy = p(x)dx$, but it was necessary to know that this expression meant 0 = 0 and that, in reality, it was only a way of denoting $dy/dx = p(x)$. In the same way, integration was the operation that associated a finite quantity y such that $dy/dx = X$ with the finite quantity X and the symbol $\int Xdx$ was merely a conventional way of denoting this operation.

This attempt to transform the traditional calculus of differentials into a calculus of finite quantities was not only due to the obscurity of the notion of infinitesimals but also to the fact that infinitesimals represented a crucial problem in one of the main objectives of Euler's mathematics: the transformation of the calculus into an algebraic calculus. Here I take the term 'algebraic calculus' to refer to a calculus which is based upon rules that are appropriate extensions of the rules of algebra of finite quantities to infinite processes. Algebraic calculus could use infinite (power) series since they were viewed as infinite polynomials upon which one could apply the rule that was valid for finite polynomials. By contrast, differentials or infinitesimals had nothing in common with the algebra of finite quantities and were viewed as non-algebraic objects: they were not acceptable in a truly algebraic calculus. The same difficulty occurred for the theory of limits, since the notion of limits was also considered non-algebraic. In Euler's opinion, limits could only provide an intuitive justification for the rules of the calculus[8].

Euler was not able to provide a comprehensive answer to such a problem; however, he developed a strategy that aimed to reduce the use of differentials in mathematics to a minimum. This strategy had two main aspects. The first aspect was the creation of a corpus of knowledge which could be treated using only the infinite extension of the rules that were valid in the algebra of finities and avoided the notion of differentials. This corpus of knowledge was the subject-matter of the first part of *Introductio* (I will refer to it as the introduction of analysis of infinities or algebraic analysis, as later Lacroix named it[9]): it investigated functions, their transformations and their expansion into series, without using the operation of differentiation and integration. In *Introductio in analysin infinitorum*[10], Euler was not able to avoid infinitesimal considerations in various proofs although algebraic analysis, as a particular field of mathematics, lying midway between the calculus and the algebra of finite quantities, was clearly set out. This led to the subdivision of analysis into three parts:

– the analysis of finite quantities,
– the introduction of the analysis of infinities or algebraic analysis,

7 In modern terms, Euler supposed that the function X is continuous.

8 On the relation between Euler's concept of limits and infinitesimals, see FERRARO, Giovanni (2004) "Differentials and differential coefficients in the Eulerian foundations of the calculus", *Historia Mathematica*, vol. 31, 34-61.

9 LACROIX, Silvestre François (1797-1798) *Traité du calcul différentiel et du calcul intégral*, 2 vols., Paris, Duprat.

10 EULER, Leonhard (1748) *Introductio in analysin infinitorum*, Lausannae, M. M. Bousquet et Soc., or *Opera*, ser. 1, vols. 8-9.

– the calculus, where the operation of differentiation and integration were investigated.

This subdivision was widely accepted by all mathematicians during the eighteenth century[11].

The second aspect of Euler's strategy regarded the employment of differentials in the calculus. Although the notion of differentiation involved finite quantities, differentials remained essential in defining this operation and Euler was fully aware of this; however, he hoped to limit the use of differentials exclusively to the definition and determination of the differential coefficients. This is how he introduced the differential coefficients in his *Institutiones calculi differentialis*. Given a function $f(x)$, Euler considered its increment $\Delta f = f(x+\omega) - f(x)$, where ω is a finite quantity; he stated that $\Delta f = f(x+\omega) - f(x)$ could be expanded into series

$$\Delta f = p\omega+q\omega^2+r\omega^3+\ldots.$$

Then he set $\omega = dx$, where dx is a differential, and obtained

$$df = pdx+qdx^2+rdx^3+\ldots$$

The application of the principle of cancellation of higher-order infinitesimals enabled him to derive $df = pdx$ and determine the differential coefficient p of the function $f(x)$. Once the differential coefficient was determined, he (at least in principle) could focus on considering the finite quantity p rather than the infinitesimals dy and dx[12].

Within the context of this strategy, which was aimed at reducing the use of differentials, Euler's definition of the integral as anti-differential requires

11 At the end of the eighteenth century, Euler's plan to undertake an algebraic treatment of the broadest possible part of analysis of infinity had far-reaching consequences when Lagrange tried to reduce the whole of calculus to algebraic notions (see Lagrange, 1797).

12 One might note that, in the same way as Euler introduced the differential coefficient $p(x)$ by using differentials, he could introduce the integral by means of differentials. Indeed, the integral $\int_a^x f(x)dx$ can be thought as the sum $\sum_{i=0}^{n} X(a_{i-1})(a_i - a_{i-1})$ when the differences a_i-a_{i-1} vanish. This definition, which agreed with Euler's definition of differentials, would have provided a non-geometrical version of Leibniz's notion of integral. However, just because Euler wanted to reduce the use of differentials (and limits) as far as possible he was not interested in such a possibility, which several decades later exploited by Cauchy as basis for his definition of integral.

examination: it aimed to eliminate differentials from the integral calculus and, thus, constituted an important step in the process of transforming integral calculus into an exclusively algebraic theory.

Finally, I observe that, in order to substantiate his rejection of the integral as the sum of infinitesimals, Euler drew a comparison between the notion of integral as the sum of an infinite number of infinitesimals and the notion of lines as aggregates of infinite points. In his opinion, the concept of an integral as a sum of infinitesimals was no more strongly founded than the idea that lines were made up of points (EULER, 1768-70, vol. 1: §.11). Indeed, following a commonly held view, Euler thought that a line was not made up of points but generated by the motion of a point. In the same way, variable quantities, which were the basic entities upon which the calculus was constructed, were not conceived of as sets of points or numbers; they were abstract entities which could be increased or diminished continuously[13]. For this reason Euler thought that the Newtonian term 'fluent' was more appropriate than the expression 'variable quantity' used in the continental tradition of the calculus (EULER,1768-70, vol. 1: §. 6). According to Euler, both the notion of integral as the sum of infinitesimals and that of lines as aggregates of infinite points were useful in applications but they were only imprecise and approximate versions of the true notions of integrals and lines[14].

* * *

To modern eyes, the definition of integration as an anti-differential poses the crucial question of the existence of the integral; Euler did not perceive this as a problem and was not concerned to prove that the anti-differential of

13 This notion of continuous quantity was substantially a primitive notion in Euler's calculus.

14 It is interesting to note that Euler tried to apply the notion of an integral as an anti-differential to double integrals. He introduced the concept of double integrals in a paper that was presented to the St. Petersburg Academy on August 18, 1768 (see ENESTRÖM, Gustav (1913) "Die Schriften Eulers chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Leipzig, Teubner) and published in *Novi Commentarii* (EULER, Leonhard (1769) "De formulis integralibus duplicatis", *Novi Comm.*, vol. 14, 72-103 or *Opera*, Ser. 1, Vol. 17, 289–315). No mention of multiple integration is found in the *Institutionum calculi integralis*. According to Euler, the double integral ∬Zdxdy is a function of two variables which when twice differentiated, first with respect to x alone, second with respect to y alone, can be reduced to the formula $Zdxdy$ (1769, §.1). For example, if Z=a then $\iint adxdy=axy+X(x)+Y(y)$, where $X(x)$ is a function of y and Y is a function of y (1769, §. 3).

a given function existed. In the *Institutionum calculi integralis*, Euler limited himself to comparing integration with inverse arithmetical operations. He stated that analytical operations are always opposed in pairs. In the same way as addition was opposed to subtraction, multiplication to division or the raising to power to the extraction of the root, differentiation and integration were also opposed (EULER, 1768-70, vol. 1: §. 3). In certain cases, the inverse operation between numbers could not be performed and this led to the idea of new numbers. Thus, the inverse operation of subtraction could not always be performed and this led to the idea of negative numbers. Similarly, the operation of division and extraction of a root led to the ideas of rational and irrational numbers. In the same manner, when integration could not be performed by means of known functions this operation led to new transcendent 'quantities' (EULER, 1768-70, vol. 1: §. 29) (see below). It is clear that there is a profound difference between the way Euler perceived inverse operations and how they are perceived today.

Today, when an operation

$$T: O \in S \to A \in V,$$

transforming an object O into an object A = T(O), is given and we consider the inverse operation I_T such that $I_T(A) = O$, *first*, we must be sure that given an object A∈V there exists an element O∈S such that $I_T(A) = O$, *then*, we can operate on $O = I_T(A)$.

If the object $O = I_T(A)$ does not exist,

first, we must construct an appropriate set of objects S* including S and such that $I_T(A)$ exists, for every A,

then, we can operate on the new object $I_T(A)$.

Instead, Euler did not construct the appropriate class of the objects S* that made the inversion of the operation T possible. When the object $I_T(A)$ did not exist, he considered the undefined symbol $I_T(A)$ as a formal object that had to satisfy the condition

$$T(I_T(A)) = A.$$

This condition allowed him to manipulate $I_T(A)$. Euler therefore *operated on the unknown object* $I_T(A)$ *as if it were known*[15]. In most cases the operation T and its converse I_T had an immediate interpretation in geometric or arithmetical terms. But in some cases, such as the extraction of the root of a negative number, the inverse operation had no intuitive interpretation and its meaning derived from establishing a formal connection between mathematical objects. The intuitive interpretation, when it existed, helped the understanding of $I_T(A)$ although it was of interest mainly in applications and, as far as possible, they were not used in analysis.

For instance, according to Euler's definition of the sum of a series (FERRARO, 2008, Chapter 19), the operation of the sum $\sum(f_n)$ of a function series f_n is the inverse operation of the operation δ(f) of development of a function *f*. If *f(x)* does not exist (i.e, it is not one of known functions), we can handle the symbol $\sum f_n(x)$ subject to condition

$$\delta\left(\sum f_n(x)\right) = \sum f_n(x).$$

If $\sum f_n(x)$ is a convergent series[16], it has an immediate, numerical meaning (even if the sum is unknown).

In a similar way, even if the existence of the function

$$F(x) = \int f(x)dx$$

was not proven, one could handle the symbol $\int f(x)dx$ subject to the condition

$$d\left(\int f(x)dx\right) = f(x)dx$$

and so one could determine the properties of the unknown object $\int f(x)dx$.

* * *

Let us return to the general problem (**P**). In the *Institutionum calculi integralis* Euler divided it into two more specific problems, according to the number of variables contained in the solution. These more specific problems can be formulated as follows:

15 I note a certain similarity with the classical analytical method in Pappus' sense (one operates upon unknown objects subjects to appropriate conditions).

16 In the case of divergent series, similarly to imaginary numbers, there was no intuitive meaning.

($\mathbf{P_{ODE}}$) given a relation R between differentials of *y* and *x*, find the function of one variable *y*=f(*x*) that leads to this relation.

($\mathbf{P_{PDE}}$) given a relation R between differentials of Ψ, *x, y, z*, …, find the function Ψ = f(*x, y, z*,…) of more than one variable that leads to this relation (EULER, 1768-70, vol. 1: §. 13).

This distinction is analogous to the difference between ordinary and partial differential equations in modern mathematics; however, in Euler's mathematics it is of greater significance. Indeed, the solutions to the second problem involved a new type of functions which never featured in the solutions to the first problem[17].

Since the differentials could be of the first order or of higher order, each of the problems ($\mathbf{P_{ODE}}$) and ($\mathbf{P_{PDE}}$) was broken down into other two sub-problems. Even in this case, the distinction not only stems from obvious differences between first and higher order equations, but hides difficulties and troubles concerning the nature of higher-order differentials (FERRARO, 2004). Therefore, the integral calculus was conceived as the method for solving the following four problems:

($\mathbf{P_{ODE1}}$) given a relation between the first-order differentials of the variables *y* and *x*, find the function of one variable *y = f(x)* that leads to this relation.

($\mathbf{P_{ODE2}}$) given a relation which involves certain higher-order differentials of the variables y and x, find the function of one variable *y = f(x)* that leads to this relation.

($\mathbf{P_{PDE1}}$) given a relation between the first-order differentials of the variables Ψ, *x, y, z*, …, find the function Ψ = f(*x, y, z*,…) of more than one variable that leads to this relation.

($\mathbf{P_{PDE2}}$) given a relation which involves certain higher-order differentials of the variables Ψ, *x, y, z*, …, …, find the function Ψ = f(*x, y, z*,…) of one variable that originates this relation[18].

This double subdivision corresponds to the structure of Euler's *Institutionum calculi integralis*, which is subdivided into two books, each of which is divided into two parts[19].

17 See FERRARO, Giovanni (2000a) "Functions, Functional Relations and the Laws of Continuity in Euler", *Historia mathematica*, vol. 27, 107-132.

18 Here my use of terms "relation" and "function" precisely corresponds to Euler's use (EULER, 1768-70, vol. 1: §.13).

19 The first book consists of two volumes (vols. I and II), the second book consists of one volume (vol. III).

A special case of ($\mathbf{P_{ODE1}}$) was the problem of the integration of a function, which Euler formulated as follows:

($\mathbf{P_{IN}}$) given the differential formula *X(x)dx*, where *X(x)* is a function, find a function *y(x)* such that *dy = Xdx* (EULER, 1768-70, vol. 1: §. 7).

However, it is sufficient to glance at the volumes of Euler's *Opera omnia* to realize that when Euler addressed the problem of calculating the integral *∫X(x)dx* of a function *X(x)*, he was considering the problem of the integration of an elementary function (rational functions, irrational functions, exponential and logarithm functions, trigonometric functions or a composition of a finite number of the previous functions)[20].

For this reason, the problem ($\mathbf{P_{IN}}$) should be understood as follows.

($\mathbf{P'_{IN}}$) given an elementary function *X*, find a function *y* such that $\frac{dy}{dx} = X$.

The same thing occurs for all four problems ($\mathbf{P_{ODE1}}$), ($\mathbf{P_{ODE2}}$), ($\mathbf{P_{PDE1}}$), and ($\mathbf{P_{PDE2}}$): the term 'relation between differentials' always denoted a relation expressed by means of elementary functions[21].

Euler was aware of the fact that the solution to problem ($\mathbf{P_{IN}}$) is not always an elementary function, even when the integrand function *X* is an algebraic function. In the Introduction to the *Institutionum calculi integralis*, he asserted that when "integration is not successful", the sought function is to be considered transcendent: "Thus if the differential formula *Xdx* does not admit integration[22], its integral … is a transcendent function of *x*". According to Euler, the integral calculus generated an infinite number of transcendental functions: the simplest of them were logarithmic, exponential and trigonometric functions (elementary transcendent functions) but there were many

20 Euler also considered the integration of series and the integration of integrals; however, such integrations only occurred as an intermediate step of a calculation procedure or an (unsatisfactory) way of expressing the result of a problem.

21 In his (1768-70), Euler considered a single problem not regarding differential equations expressed by means of elementary functions. This problem regarded certain differential equations expressed by series of the type $\sum_{n=0}^{\infty} a_n \frac{d^n y}{dx^n} = 0$ (a_n constant) upon which one could apply an infinite extension of procedures valid for the differential equations of the type $\sum_{n=0}^{k} a_n \frac{d^n y}{dx^n} = 0$ with a_n constant (see also EULER, Leonhard (1743b) "De integratione aequationum differentialium altiorum graduum", *Miscellanea Berolinensia*, 7, 193-242, or *Opera*, ser. 1, vol. 22, 108-149).

22 I explicitly note that here the expression "not to admit integration" means that the integral is not an algebraic function.

others (I term these functions "non-elementary transcendent functions"). Consequently, the set of integral functions (algebraic, elementary transcendent and non-elementary transcendent functions) was larger than the set of integrand functions (algebraic and elementary transcendent functions). A crucial point of this concept is that the functions in the first set and not in the second set (non-elementary transcendent functions) had a different status from the others (algebraic and elementary transcendent functions). Indeed, in Euler's mathematics, one can distinguish different kinds of entities named "functions", but only some of them were considered as functions in the true sense of the term.

The first kind of functions consisted of algebraic functions. They were the starting point of the integral calculus, which mainly consisted in investigating the integration of algebraic functions and differential equations expressed by an algebraic function $f(x, y, dy/dx, \ldots)$ of x, y, dy/dx,.... Algebraic functions were the most perfect type of functions.

The second kind of functions consisted of elementary transcendent functions. Euler devoted considerable space to the integration of these functions in his integral calculus. He considered elementary transcendent functions to have the same status as algebraic functions since they were well-known. By the expression 'well-known' I mean that a) there existed a group of algorithmic rules related to the analytical expressions of these functions which allowed them to be manipulated; b) the values of these functions were considered as given since they could be calculated by performing algebraic operations and using tables of values.

The first and second kind of functions (namely, the elementary functions) constituted the only functions in the true sense of the term and were the genuine object of the calculus. However, throughout his mathematical career, Euler did not always maintain the same opinion about the functions belonging to the second kind of functions. He initially considered only exponential and logarithmic functions as true functions, whereas trigonometric quantities were thought of as geometric entities (lines in a circle[23]). Later, when investigating differential equations, Euler introduced the trigonometric functions as actual analytical objects[24] constituting the classic set of functions which lay at the basis of Euler's three great analytical treatises.

23 See, for instance, the pages 3-4 of EULER, Leonhard (1730-31) "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt", *Comm.*, vol. 5, 36-57, or *Opera*, ser. 1, vol. 14, 1-24.

The third[25] kind of functions consisted of certain functions that could be represented by an integral ∫Xdx or that expressed the solution to a differential equation. (I will refer to them as non-elementary transcendent functions.) *These functions were not considered functions in the proper sense* since they were not sufficiently well-known, namely they did not satisfy conditions a) and b). Euler did not think that a function belonged to the third group in a definitive and permanent sense; indeed, he believed that an integral ∫Xdx could be investigated so that it became known and could be treated as an elementary function; once an integral ∫Xdx was known, it became part of the second kind and could be considered a true function.

In the *Institutionum calculi integralis,* in order to investigate non-elementarily integrable functions, Euler devised an approach which can be divided into the following four parts:

1) broad classes of integrals $\int X_s dx$, where X_s was a non-integrable function depending on one or more parameters, were to be specified;
2) for each class $\int X_s dx$, a special integral $\int X dx$ belonging to the class $\int X_s dx$, was to be identified;
3) all the integrals of the class $\int X_s dx$ were to be reduced to the integral $\int X dx$;
4) the special integral $\int X dx$ was to be investigated in order to obtain a known function.

The achievement of this result would have been a remarkable contribution to the attempt to give a merely algebraic form to the calculus.

An example of Euler's approach can be found in Chapter 4 of the *Institutionum calculi integralis*, where he faced the problem of the integration of

$$\int \frac{x^{m-1}}{\log^{n} x}\, dx.$$

Here he observed that the integration of this formula depended on the integration of $\int \frac{x^{m-1}}{lg\, x} dx$. Moreover, if one set $x^m = z$, then

24 See KATZ, Victor (1987) "The Calculus of the Trigonometric Functions", *Historia Mathematica*, vol. 14, 311-324.

25 Euler considered two further classes of functions (discontinuous functions and inexplicable functions) which also were not conceived as functions in the proper sense of the term. For a discussion of inexplicable functions, I refer to FERRARO, Giovanni (1998) "Some Aspects of Euler's Theory of Series. Inexplicable functions and the Euler-Maclaurin summation formula", *Historia mathematica*, vol. 25, 290-317; for discontinuous functions, see FERRARO (2000a).

$$x^{m-1}dx = \frac{1}{m}dz \quad \text{and} \quad logx = \frac{1}{m}logz,$$

consequently $\int \frac{x^{m-1}}{lg\ x}dx$ was reduced to the very simple form

$$\int \frac{dz}{log\ z} \text{ (EULER, 1768-70, vol. 1: §. 219).}$$

Euler also noted that the integrals $\int \frac{a^x dx}{x^n}$ depended on $\int \frac{a^x dx}{x}$ and that the substitution $z = e^x$ transformed $\int \frac{dz}{log\ z}$ into $\int \frac{e^x dx}{x}$ into. Therefore, all the integral of the types

$$\int \frac{x^{m-1}}{lg\ x}dx \text{ and } \int \frac{a^x dx}{x^n}$$

can be reduced to the study of

$$\int \frac{dz}{log\ z} \text{ or } \int \frac{e^x dx}{x}.$$

Of course, the crucial point is to investigate the function $\int \frac{dz}{log\ z}$ (or $\int \frac{e^x dx}{x}$) in order to transform it into a known function. Euler was able to obtain the following expansion of $\int \frac{dz}{log\ z}$:

(2) $$\int \frac{dz}{log\ z} = C + log\ log\ z + \frac{log\ z}{1} + \frac{1}{2}\frac{log^2 z}{2!} + \frac{1}{3}\frac{log^3 z}{3!} + \ldots$$ [26].

According to Euler, if we assume that the integral $\int \frac{dz}{log\ z}$ is real for $(0<)z<1$, since $log(logz)$ is imaginary for $0<z<1$, then the constant C is imaginary, therefore $\int \frac{dz}{log\ z}$ is imaginary for $z>1$. Vice versa, if we assumed that $\int \frac{dz}{log\ z}$ is real for $z>1$, since $log(logz)$ is real for $z>1$, then C is real and $\int \frac{dz}{log\ z}$ is imaginary for $z<1$. That led Euler to state that the nature of this function was not known enough (EULER, 1768-70, vol. 1: §. 228). For this reason the integral formula $\int \frac{dz}{log\ z}$ was never considered as a function in the strict sense of the term.

This occurred for all other transcendental functions, even for those that Euler studied most: gamma and beta functions, and elliptic integrals. Only in his "De plurimis quantitatibus transcendentibus quas nullo modo per formulas integrales exprimere licet"[27], a short note which was presented to the St. Petersburg Academy on October 16, 1775, did Euler suggest the consideration of elliptic integrals as new functions in the strict sense of the term since they had been analyzed to such a degree that they could be considered as known. However, this reference to elliptic integrals as functions in the strict sense of term was an isolated one and it remained a mere suggestion without practical consequences in Euler's work.

3.- Particular integrals and definite integration.

In the first theorem of his *Institutionum calculi integralis*, Euler stated: "All functions which are found by means of the integral calculus are indeterminate. They must be calculated according to the nature of the question, to which they provide the solution" (EULER, 1768-70, vol. 1: §. 31).

Euler's demonstration of this proposition merely consisted of the following observations:

a) the integral of the differential dP is $P+C$, where C is an arbitrary constant;
b) the function found by means of a differential equation always contains a constant, of which no trace remains in the relation between differentials.

In the following corollaries (EULER, 1768-70, vol. 1: §. 32-33), Euler stated that the solution $y(x)$ to a (first-order) differential equation, which was indeterminate in itself, could be determined by setting $y(a) = b$; if the equation was of the second order the determination required two conditions: $y(a)=b$ and $\left.\frac{dy}{dx}\right|_{x=a} = c$; and so on. Therefore the integral of a differential equation had to

26 Cf. EULER (1768-70, vol. 1, §. 228). Formula (2) contains an inaccuracy. Mascheroni proved that if 0<z<1, then $\int \frac{dz}{lg\ z} = C + lg(-lg\ z) + \frac{lg\ z}{1} + \frac{1}{2}\frac{lg^2 z}{2!} + \frac{1}{3}\frac{lg^3 z}{3!} + \ldots$
and the constant C=0,577… is the Euler constant (see MASCHERONI, L. (1790-92) *Adnotationes ad calculum integralem Euleri in quibus nonnulla problemata ab Eulero proposita resolvuntur*, in EULER (*Opera*, (1), 12: 415-542).

27 EULER, Leonhard (1780) "De plurimis quantitatibus transcendentibus quas nullo modo per formulas integrales exprimere licet", *Acta academiae scientiarum petropolitanae* (later: *Acta*) 2, 31-37, or *Opera*, ser. 1, vol. 15: 522-527 (see, in particular, p. 522).

contain an adequate number of arbitrary constants[28]. Euler termed the integral of a differential equation containing an adequate number of arbitrary constants as *complete integral* (in modern terms, general integral). When the complete integral was determined and the constants disappeared, the integral was referred to as "particular"[29].

It is clear that in the above theorem and its corollaries Euler referred to the two following problems:

Problem 1. The search for the general integral.

Problem 2. The search for integrals that satisfied initial conditions (namely, what today is named a Cauchy problem).

In the modern theory of differential equations, one can study the second problem without reference to the first (for example, the classical theorems of existence of solutions to Cauchy problems under certain conditions are proved without reference to general integrals). Instead Euler viewed the solution to the second problem simply as a consequence of the solution to the first one. One specific case of this view is the notion of the definite integral. According to Euler, the definite integral $\int_a^b f(x)dx$ is the value that was obtained when

a) one determines the arbitrary constant of $\int f(x)dx$ under the condition that the anti-differential of $f(x)$ is equal to zero for $x = a$ (so one determines a specific anti-differential $F(x)$);

b) one calculates the value of the specific anti-differential $F(x)$ for $x = b$.

Conceptually, a definite integration was a trivial exercise following an indefinite integration, and for this reason Euler devoted no space to a general discussion of definite integration in his *Institutionum calculi integralis*[30].

The importance attributed to problem 1) with respect to problem 2) and, in particular, to indefinite integration with respect to definite integration was obviously connected to the notion of integration as anti-differentiation, but it

28 Moreover, he observed that when the solution was a function $y(x, t)$ of two variables, the determination was such that when a certain value a is given to t, the solution $y(x, a)$ expressed the nature of a given curve (1768-70, vol. 1, §. 34).

29 EULER (1768-70, vol. 1, §.36). A different definition of particular integral is found in (1768-70, vol. 1, §. 540). Here he gave the name *particular integral* to a relation between the variables such that satisfies the equation and that does not contain any new constant quantity.

30 I observe that in modern real analysis a crucial role is played by the "fundamental theorem of calculus", where by "fundamental theorem of calculus", I mean a theorem asserting that, under appropriate conditions, the definite integral of $f(x)$ on the interval (a, b) exists and defines a primitive function $F(x)$ of $f(x)$. In Euler's analysis there is no room for a similar theorem.

was also linked to Euler's concept of analysis as the science that investigated mathematical objects in an abstract and general way. Euler's analysis was inspired by a desire for generality: an analytical problem had to be tackled and solved in all its generality. In the case of the solution to differential equations, this meant that one had to search for the general integral: this was the only problem of interest in pure analysis. Problem 2) mainly concerned the applications of the calculus: indeed, it was when analysis was applied to physics or geometry that it was believed that particular solutions were necessary and so the general solution was made specific and suited to the initial conditions.

In Euler's integral calculus, the tendency towards generality ran up against two main difficulties:

- the increasing importance that definite integration assumed in the second part of 18th century;
- the discovery of singular integrals.

In this paper I limit myself to discussing the definite integration. While definite integration had no conceptual independence from indefinite integration, Euler soon realized that it could be a powerful instrument in analysis. Indeed,

- it is possible to compute the definite integral $\int_a^b f(x)dx$, even if the anti-differential of a function $f(x)$ cannot be determined;
- it is possible to represent variable quantities by means of the integral of type $\int_a^b f(x,k)dx$ dependent on a parameter k.

Euler had already used the second possibility in one of his first papers. In his "De progressionibus transcendentibus" (EULER, 1730-31: §§. 8-14). Euler had tackled the problem of Wallis's interpolation, namely, the problem of extending a number sequence a_n defined for integral values of n to non-integral values of n. For instance, the interpolation formula for the factorial sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, $a_4 = 24$, amounted to finding the values of terms, like the terms $a_{1/2}$, $a_{3/2}$, …, corresponding to non-integral indices. He was able to pose Wallis's problem into integral form, namely he sought to express the general term of a sequence a_n by an integral formula of the kind

$$\int_0^b p(x,n)dx\ ,$$

where $p(x,n)$ is an appropriate functions of the variable x and n. In particular, he showed that

$$(3) \qquad n! = \int_0^1 (-\log x)^n dx.$$

Formula (3) was the first integral expression of the factorial function[31] $\Pi(z) = \int_0^\infty t^z e^{-t} dt$. (In the present paper I will refer to the integral

$$\int_0^\infty t^z e^{-t} dt = \int_0^1 (-\log x)^n dx$$

as G-integral rather as factorial function since Euler did not consider it as a true function. For the same reason I will term the integral $\int_0^1 x^{\frac{f}{g}} (1-x)^n dx$ as B-integral rather refer to it as beta function[32].)

Euler investigated (3) at length. In "De progressionibus transcendentibus" (EULER, 1730-31: §. 16), he also attempted to reduce the calculation of the G-integral $\int_0^1 (-\log x)^n dx$ to the quadrature of certain algebraic curves. He indeed showed that

$$\int_0^1 (-\log x)^n dx = \frac{(f+g)(f+2g)\dots(f+(n+1)g)}{g^{n+1}} \int_0^1 x^{\frac{f}{g}} (1-x)^n dx^{33}.$$

In this way, Euler also started with the investigation of the B-integral[34]

$$\int_0^1 x^{\frac{f}{g}} (1-x)^n dx^{\ 35},$$

31 Euler later went on to provide the more usual integral expression $\int_0^\infty y^n e^{-y} dy$. See EULER, Leonhard (1785b) "Methodus inveniendi formulas integrales, quae certis casibus datam inter se teneant rationem, ubi sumul methodus traditur fractiones continuas summandi", *Opuscula Analytica*, Petropoli, typis academiae imperialis scientiarum (later: *Opuscula*), vol. 2, 178-216, or *Opera*, ser. 1, vol. 18, 209-243 (the formula is on p. 217); and EULER, Leonhard (1794) "De valoribus integralium a termino variabilis x = 0 usque ad x = ∞ extensorum", *Institutiones calculi integralis volumen quartum, continens supplementa partim inedita partim jam in operibus academiae imperialis scientiarum Petropolitanae impressa*, Petropoli, Impensis Academiae Imperialis Scientiarum, 337-345, or *Opera*, ser. 1, vol. 19, 217-227 (cf., in particular, p. 220). The factorial function $\Pi(z) = \int_0^\infty t^z e^{-t} dt$ is related to the gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ by the equation $z\Gamma(z) = \Pi(z)$.

32 The beta function is defined by the formula $B(\xi+1, \zeta+1) = \int_0^1 x^\xi (1-x)^\zeta dx$.

33 In a more modern notation, $\Gamma(n+1) = \frac{(f+g)(f+2g)\dots(f+(n+1)g)}{g^{n+1}} B(\frac{f}{g}+1, n+1)$.

34 On Euler's treatment of beta functions, see DELSHAMS, A. and MASSA ESTEVE, M. R. "Consideracions al voltant de la funció beta a l'obra de Leonhard Euler (1707-1783)", *Quaderns d'Història de l'Enginyeria*, Vol. IX, Barcelona.

35 I point out that, around 1730, Euler obtained several results that linked these functions with numerical sequences; for instance,

to which he devoted many pages. In "Observationes circa integralia formularum $\int x^{p-1} dx (1-x^n)^{q/n-1} dx$"[36], Euler stated that, since the definite integral

$$\int_0^1 x^{p-1} (1-x^n)^{\frac{q-n}{n}} dx$$

can be expressed in a very simple way, one generally does not seek the indefinite integral

$$\int x^{p-1} (1-x^n)^{\frac{q-n}{n}} dx$$

but the definite one (EULER, 1766: 268). For this reason, in this paper and in several other papers, he attempted to determine the value of the integrals $\int_0^1 x^{p-1} (1-x^n)^{\frac{q-n}{n}} dx$, where p, n, q are positive integers, and determine relations between these integrals so to try to make them known objects.

In his "Observationes circa integralia formularum $\int x^{p-1} dx (1-x^n)^{q/n-1} dx$" and *Institutionum calculi integralis*, Euler also introduced the symbol[37]

$$\left(\frac{p}{q}\right)$$

$$\frac{\prod_{i=1}^n (f+ig)}{\prod_{i=1}^n (h+ik)} = \frac{g^{n+1}(h+(n+1)k)\int_0^1 x^{h/k}(1-x)^n dx}{k^{n+1}(f+(n+1)g)\int_0^1 x^{f/g}(1-x)^n dx}$$

(EULER, 1730-31: §. 18) and

$$\frac{p(p+2r)\dots(p+2(n-1)r)}{(p+2q)(p+2q+2r)\dots(p+2q+2(n-1)r)} = \frac{\int_0^1 x^{\frac{p+2q-2r}{2r}}(1-x)^{n-1} dx}{\int_0^1 x^{\frac{p-2r}{2r}}(1-x)^{n-1} dx}$$

(see EULER, Leonhard (1739a) "De fractionibus continuis observationes", *Comm.*, vol. 11, 32-81, or *Opera*, ser. 1, vol. 14, 291-349, in particular p. 306).

36 EULER, Leonhard (1766) "Observationes circa integralia formularum $\int x^{p-1} dx (1-x^n)^{q/n-1}$ posito post integrationem $x = 1$", *Melanges de philosophie et de la mathématique de la société royale de Turin* 3, 156-177, or *Opera*, ser. 1, vol. 17, 268-288.

37 He later changed this symbol slightly and wrote (p, q) in place of $\left(\frac{p}{q}\right)$ (see EULER, Leonhard (1789) "Comparatio valorum formulae integralis $\int (x^{p-1} dx)/(\sqrt[n]{((1-x^n)^{n-q})})$ a termino $x = 0$ usque ad $x = 1$ extensae", *Nova Acta Academiae Scientarum Imperialis Petropolitinae*, vol. 5, 86-117, or *Opera*, ser. 1, vol. 18, 392-423).

to denote the integral $\int_0^1 x^{p-1}(1-x^n)^{\frac{q-n}{n}}dx$[38]. The various relationships that he found included the following[39]:

$$\left(\frac{p}{q}\right)=\int_0^1 x^{p-1}(1-x^n)^{\frac{q-n}{n}}dx=\frac{p+q}{pq}\,\frac{n(p+q+n)}{(p+n)(q+n)}\,\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\ldots;$$

$$\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right);$$

$$\left(\frac{p}{n-p}\right)=\int_0^1\frac{x^{p-1}}{\sqrt[n]{(1-x^n)^p}}dx=\frac{\pi}{n\sin\frac{p\pi}{n}};$$

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right);$$

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)$$

Euler also reduced some integrals to the computation of integrals of the kind $\int_0^1 x^{p-1}(1-x^n)^{\frac{q-n}{n}}dx$. For instance[40],

$$\int_0^1\frac{dx}{x\log x}\,\frac{x^\alpha-x^\beta}{1+x^n}=\log\frac{\int_0^1 x^{n-1}(1-x^{2n})^{\frac{\beta-2n}{2n}}dx}{\int_0^1 x^{n-1}(1-x^{2n})^{\frac{\alpha-2n}{2n}}dx}$$

and[41]

$$\int_0^1\frac{x^{a-1}dx}{\log x}\,\frac{(1-x^b)(1-x^c)}{1-x^n}=\log\frac{\int_0^1 x^{a+c-1}(1-x^n)^{\frac{b-n}{n}}dx}{\int_0^1 x^{a-1}(1-x^n)^{\frac{b-n}{n}}}=\log\frac{\int_0^1 x^{a+b-1}(1-x^n)^{\frac{c-n}{n}}dx}{\int_0^1 x^{a-1}(1-x^n)^{\frac{c-n}{n}}}.$$

38 By replacing x^n by y in the equation $\left(\frac{p}{q}\right)=\int_0^1 x^{p-1}(1-x^n)^{\frac{q-n}{n}}dx$, one obtains $\left(\frac{p}{q}\right)=\frac{1}{n}\int_0^1 y^{\frac{p}{n}-1}(1-y)^{\frac{q}{n}-1}dy$, therefore Euler's symbol $\left(\frac{p}{q}\right)$ is connected with the beta function $B(x,y)$ by the equation $\left(\frac{p}{q}\right)=\frac{1}{n}B\left(\frac{p}{n},\frac{q}{n}\right)$.

39 Cf. EULER (1766), (1768-1770, vol. 1, sect. 1, Chapter 8), and (1789).

40 EULER, Leonhard (1775a) "Speculationes analyticae", *Novi Comm*, vol. 20, 59-79, or *Opera*, ser. 1, vol. 18, 1-22. The formula is on p. 16.

Moreover, Euler tried to improve the knowledge of the G-integral. In "Evolutio formulae integralis $\int x^{f-1}dx(lx)^{m/n}$"[42], Euler used the sign [λ] to denote

$$\int_0^1\log^\lambda\frac{1}{x}dx$$

and employed this symbol to express some results in a short and elegant form, such as[43]

$$[\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin\lambda\pi};$$

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}(m-1)!\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n-1}{m}\right)};$$

$$\left[\frac{i+n}{n}\right]=\frac{i+n}{n}\left[\frac{i}{n}\right];$$

$$\left[\frac{i+2n}{n}\right]=\frac{i+n}{n}\,\frac{i+2n}{n}\left[\frac{i}{n}\right];\ \ldots..$$

* * *

Euler's technique was rather bold. For example, in "Nova methodus quantitates integrales determinandi"[44] to determine the value of the definite integral

$$\int_0^1\frac{z-1}{\log z}dz,$$

41 EULER, Leonhard (1777a) "De valore formulae integralis $\int\frac{x^{a-1}dx}{\log x}\frac{(1-x^b)(1-x^c)}{1-x^n}$ a termino $x=0$ usque ad $x=1$ extensae", *Acta*, vol. 2, 29-47, or *Opera*, ser. 1, vol. 18, 51-68. See, in particular, pp. 55-58.

42 EULER, Leonhard (1771) "Evolutio formulae integralis $\int x^{f-1}d(lx)^{m/n}$ integratione a valore $x=0$ ad $x=1$ extensa", *Novi Comm.*, vol. 16, 91-139, or *Opera*, ser. 1, vol. 17, 316-357.

43 Cfr. EULER (1771, 342-348). Moreover, Euler found many formulas that link G-integrals and B-integrals. For example, $\frac{\int_0^1\left(\log\frac{1}{x}\right)^{n-1}dx\cdot\int_0^1\left(\log\frac{1}{x}\right)^{m-1}dx}{\int_0^1\left(\log\frac{1}{x}\right)^{m+n-1}dx}=k\int_0^1 x^{mk-1}(1-x^k)^{n-1}dx$, namely, $\int_0^1 x^{\lambda-1}(1-x^\mu)^\nu dx=\frac{\Pi(\lambda/\mu)\cdot\Pi(\nu)}{\lambda\Pi(\nu+\lambda/\mu)}$ (EULER (1771, 331).

44 EULER, Leonhard (1775b) "Nova methodus quantitates integrales determinandi", *Novi Comm.*, 19 (1775), 66-102, or *Opera*, ser. 1, vol. 17, 421-457.

Euler set

$$log\ z = i(z^{1/i} - 1),$$

where i is a infinite number, and attempted to calculate

$$\int_0^1 \frac{z-1}{i(z^{1/i}-1)}dz.$$

Setting $z = x^i$, he obtained:

$$\int_0^1 \frac{z-1}{log\ z}dz = \int_0^1 \frac{x^i - 1}{(x-1)} x^{i-1} dx = \int_0^1 (x^{i-1} + x^i + \ldots + x^{2i-2})dx$$

$$= \left[\frac{x^i}{i} + \frac{x^{i+1}}{i+1} + \ldots + \frac{x^{2i-1}}{2i-1}\right]_0^1 = \frac{1}{i} + \frac{1}{i+1} + \ldots + \frac{1}{2i-1}$$

At this point, Euler set

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{2i-1}$$

and

$$B = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{i-1}.$$

Of course,

$$A - B = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{2i-1}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{i-1}\right)$$

$$= \frac{1}{i} + \frac{1}{i+1} + \ldots + \frac{1}{2i-1}$$

However, A-B can be calculated by subtracting B from A according to the following scheme:

$$\begin{array}{cccccccccccc}
A & = & 1 & +\frac{1}{2} & +\frac{1}{3} & +\frac{1}{4} & +\frac{1}{5} & +\frac{1}{6} & +\frac{1}{7} & +\ \ldots\ + & \frac{1}{2i-2} & +\ \frac{1}{2i-1} \\
B & = & & 1 & & +\frac{1}{2} & & +\frac{1}{3} & & +\ \ + & \frac{1}{i-1} & \\
A-B & = & 1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{2} & +\frac{1}{5} & -\frac{1}{3} & +\frac{1}{7} & +\ \ldots\ - & \frac{1}{i-1} & +\ \frac{1}{2i-1}
\end{array}$$

Hence,

$$A - B = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = log\ 2$$

and

$$\int_0^1 \frac{z-1}{log\ z}dz = log\ 2 \text{ (EULER, 1775b: 425-426).}$$

Euler then stated that in a similar way one can prove that

$$\int_0^1 \frac{z^m - 1}{log\ z}dz = log(1+m)$$

and

(4) $$\int_0^1 \frac{z^m - z^n}{log\ z}dz = log\frac{1+m}{1+n} \text{ (EULER, 1775b: 426-427).}$$

Euler calculated the definite integral without previously ensuring the existence of the integral. We have seen that the same approach was used for indefinite integration and, in effect, this approach was usual in Euler's analysis. However, while in the case of indefinite integration, this approach did not seem to yield any problem, in the case of definite integration, it gave rise to some paradoxical situations. This was mainly due to the fact that, with regard to indefinite integration, Euler formally manipulated analytical expressions and the result of manipulation was an analytical expression too: he ranged from one formal entity to another and the problem of the quantitative meaning of these analytical expressions did not arise. Instead, in the other case, the result of the formal manipulation of the integral was not an analytical expression but a determinate quantity and the quantitative meaning of analytical expressions was of importance[45]. A paradox concerning definite integration is found in "Observationes in aliquot theoremata illustrissimi de la Grange"[46]. Here, Euler observed that

45 This is a manifestation of the tension between the quantitative and the formal which characterize all of Euler's analysis (see FERRARO, 2000a; FERRARO, Giovanni (2000b) "The value of an infinite sum. Some Observations on the Eulerian Theory of Series", *Sciences et Techniques en Perspective*, ser. (2), vol. 4, 73-113; FERRARO, Giovanni (2007c) *L'evoluzione della matematica. Alcuni momenti critici*. Napoli. Ernesto Ummarino Editore.

46 See EULER, Leonhard (1785a) "Observationes in aliquot theoremata illustrissimi de la Grange", *Opuscula* vol. 2, 16-41, or *Opera*, ser. 1, vol. 18, 156-177, in particular pp. 159-161.

$$\int_0^1 \frac{z^{\alpha-1}-z^{\beta-1}}{\log z}dz=\int_0^1 \frac{z^{\alpha-1}}{\log z}dz-\int_0^1 \frac{z^{\beta-1}}{\log z}dz=\int_0^1 \frac{1}{\log x}\,dx-\int_0^1 \frac{1}{\log y}\,dy=0.^{47}$$

But Equation (4) implies

$$\int_0^1 \frac{z^{\alpha-1}-z^{\beta-1}}{\log z}dz=\log\frac{\alpha}{\beta}\,.$$

Therefore,

$$\log\frac{\alpha}{\beta}=0\,.$$

Euler stated that this paradox could be explained by observing that the difference

$$\int_0^1 \frac{1}{\log x}\,dx-\int_0^1 \frac{1}{\log y}\,dy=\infty-\infty$$

was not equal to 0 but indeterminate[48]: formula (4) precisely showed that it equal to log α/β.

* * *

The development of the technique of calculation of the definite integration led Euler to introduce a specific symbolism for definite integrals. Initially, Euler expressed the definite integral by means of a circumlocution: “if, after integration, a determinate value is given to a variable quantity”[49] or “integration extended from the value x=a to the value x=b” (EULER, 1771). Later, Euler introduced the symbol

$$\int f(x)dx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}$$

to denote the definite integral

47 Replacing z^{α} by x into $\int_0^1 \frac{z^{\alpha-1}}{\log z}dz$ and z^{β} by y into $\int_0^1 \frac{z^{\beta-1}}{\log z}dz$.

48 Euler seems to distinguish infinity as a number and infinity as an increasing quantity.

49 See, for instance, EULER,1766 and EULER, Leonhard (1743a) “De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur”, *Miscellanea Berolinensia*, vol. 7, 129-171, or *Opera*, ser. 1, vol. 17, 35-69.

$$\int_a^b f(x)dx^{50}.$$

In the final part of his mathematical career, Euler dealt with the properties of definite integration in an explicit way without reference to indefinite integration. This occurred in “Observationes in aliquot theoremata illustrissimi de la Grange”, a paper mainly devoted to the investigation of the above-mentioned paradox and of certain integrals. In particular, he showed that

$$\int_a^b (x^n-x^m)\frac{dx}{x\log x}=\int_m^n (a^y-b^y)\frac{dy}{y}\,.$$

To derive this equality, Euler observed that

$$\int_0^{\lambda} dy\int_a^b x^y\frac{dx}{x}=\int_0^{\lambda}\frac{b^y-a^y}{y}dy \text{ and } \int_a^b\frac{dx}{x}\int_0^{\lambda}x^y dy=\int_a^b\frac{x^{\lambda}-1}{x\log x}dx\,.$$

Since $\int_0^{\lambda} dy\int_a^b x^y\frac{dx}{x}=\int_a^b\frac{dx}{x}\int_0^{\lambda}x^y dy$[51], he obtained

$$\int_a^b (x^{\lambda}-1)\frac{dx}{x\log x}=\int_0^{\lambda}(a^y-b^y)\frac{dy}{y}\,.$$

Then, Euler operated as follows

$$\int_a^b (x^n-x^m)\frac{dx}{x\log x}=\int_a^b (x^n-1)\frac{dx}{x\log x}-\int_0^{\lambda}(x^m-1)\frac{dx}{x\log x}$$

$$=\int_0^n (a^y-b^y)\frac{dy}{y}-\int_0^m (a^y-b^y)\frac{dy}{y}=\int_m^n (a^y-b^y)\frac{dy}{y} \text{ (EULER, 1785a: 163-164).}$$

To justify the property used in the final step of the above derivation and certain other properties of definite integration, Euler employed a geometrical interpretation of the definite integral. He stated that if the ‘nature’ of the function P is represented by means of the line *ixabco* (see Fig. 1), then the integral $\int Pdx$ represents the area under the line and above the axis IO. Instead the integral $\int_a^b Pdx$ is the area of the figure *AaBb*.

50 See, e.g., EULER,1789; EULER, 1785a, and EULER, Leonhard (1777b) “De integratione formulae ∫(*dx lx*)/√(1-*xx*) ab *x* = 0 ad *x* = 1 extensa”, *Acta*, vol. 2, 3-28, or *Opera*, ser. 1, vol. 18, 23-50.

51 Euler applied the formula for the change the order of integration which he had illustrated in his (1775b).

Based upon this diagram, he gave the following lemmas (EULER, 1785a: 157-159).

Lemma 1. $\int_a^b Pdx = -\int_b^a Pdx$.

Euler explained that if $a<b$, then $\int_b^a Pdx$ and $\int_a^b Pdx$ represent the same area AaBb, but in the case of the definite integral $\int_b^a Pdx$ the area is considered in the retrograde sense and, therefore, $\int_b^a Pdx$ must be considered as negative.

Lemma 2. $\int_a^b Pdx + \int_b^c Pdx = \int_a^c Pdx$.

According to Euler, the mere inspection of the figure makes the lemma manifest.

Lemma 3. $\int_a^c Pdx - \int_a^b Pdx = \int_b^c Pdx$.

Lemma 4. $\int_a^c Pdx - \int_b^c Pdx = \int_a^b Pdx$.

Lemma 5. $\int_a^b Pdx + \int_b^c Pdx + \int_a^c Pdx = 0$.

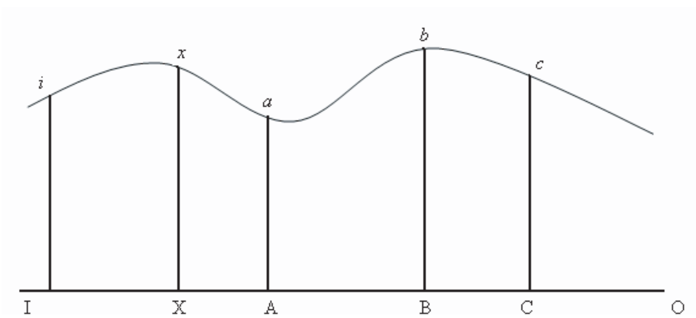

Fig.1

Euler had used the geometrical meaning of integration as an area or a length in several of his early papers, when he did not entirely exclude geometrical interpretations from analysis. By the 1740s, in particular from the publication of *Introductio* (1748) Euler developed an analytical program which not only aimed to separate analysis from geometry[52] but even sought to place analysis at the heart of all mathematics. Thus not only was analysis considered to be an independent discipline from geometry but, more importantly, analysis was thought to be the general part of mathematics - namely the part of mathematics where quantity was studied in the most general and pure form possible. Any other mathematical discipline, including geometry, was thought to be a part of mathematics that concerned a particular type of quantity and to which the findings of analysis could be applied. For this reason, analytical arguments could be used in geometry though, on the contrary, one could not use geometric arguments in analytical demonstrations. Indeed, Euler attempted to avoid a geometric interpretation of integrals in analysis and only employed it in geometric and physical applications. Thus, in the *Institutionum calculi integralis*, Euler did not provide an explicit explanation of the geometrical meaning of the integral, even though it could be easily derived from Equation (1).

52 On geometry and analysis in 18th century, see FERRARO, Giovanni (2001) "Analytical symbols and geometrical figures. Eighteenth Century Analysis as Nonfigural Geometry", *Studies in History and Philosophy of Science*, vol. 32, 535-555.

Euler was, however, unable to pursue this program of elimination of geometrical meaning to its natural conclusion; indeed, on some occasions, he resorted to the geometrical interpretation of the integral even in an analytical context. This is the case with discontinuous functions in Euler's sense (FERRARO, 2000a). This is also the case for definite integration as discussed in "Observationes in aliquot theoremata illustrissimi de la Grange" (1785a).

Euler's use of geometrical references in his analytical work is a sign of his difficulty in constructing a satisfactory analytical theory of integration. In effect, as long as the subject of this theory was the indefinite integration of elementary functions and ordinary differential equations, Euler succeeded in avoiding geometric interpretations but when definite integration and partial differential equations developed, Euler used geometric interpretations to strengthen his arguments; in this way he came into conflict with the fundamental objective of his analytical program.

4.- Elliptic integrals.

It is of interest to the purpose of this paper to examine briefly how Euler dealt with elliptic integrals and attempted to transform elliptic integrals in true functions. In an article published in 1761[53], Euler announced a new strat-

53 EULER, Leonhard (1756-57) "De integratione aequationis differentialis $(mdx)/\sqrt{(1-x^4)} = (ndy)/\sqrt{(1-y^4)}$", *Novi Comm.*, vol. 6, 37-57, or *Opera*, ser. 1, vol. 20, 58-79 (translation in LANGTON, Stacy (ET) "On the integration of the differential equation $(mdx)/\sqrt{(1-x^4)} = (ndy)/\sqrt{(1-y^4)}$)", home.sandiego.edu/~langton/e251.pdf).

egy for the investigation of certain integral formulas $\int Xdx$ which resisted all known techniques of integration. This new strategy consisted in considering a differential equation $Xdx = Ydy$, where the function $Y(y)$ were derived from $X(x)$ by changing x into y and multiplying it by a constant. In certain cases the general integral of the equation $Xdx = Ydy$ could be expressed algebraically and this allowed Euler to obtain results about the integrals of two formulas Xdx and Ydx which could not be individually integrated. For example, "the integral of the formula $\frac{dz}{\sqrt{1-z^4}}$ cannot be expressed either by means of angles or of logarithms, the only transcendental quantities thought suitable for such expressions", nevertheless if one considers the differential equation $\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$, the relation between x and y can be exhibited algebraically whenever the ratio $m{:}n$ is rational, "so that the curved line whose indefinite arc is expressed by the integral formula $\frac{dz}{\sqrt{1-z^4}}$ enjoys a property similar to that of the circle, namely that all its arcs can be compared with one another; or, if any of its arcs be given, it is possible to determine geometrically any other arc which has a given ratio with the first. Indeed, what amount to the same thing, the equation for the integral of the given differential equation, which gives the true relation between x and y, not only does not involve such an integral, but will in fact be algebraic" (EULER, 1756-57: 58). In his opinion, this strategy opened "an entirely new field … in Analysis" (EULER, 1756-57: 58).

Euler stated that he was not led to the integral of the equation

$$\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$$

by any definite method, but rather found it by guesswork, or by trial and error (EULER, 1756-57: §. 7). He was never satisfied at all of this fact. In effect his research was mainly inspired by the analogy between the equation $\frac{mdx}{\sqrt{1-x^2}} = \frac{ndy}{\sqrt{1-y^2}}$ (which has the solution $m\cdot\arcsin x = n\cdot\arcsin y + C$) and the equation $\frac{mdx}{\sqrt{1-x^4}} = \frac{ndy}{\sqrt{1-y^4}}$. Euler started by observing that two particular integrals of

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

were $x = y$ and $x^2+y^2+x^2y^2 = 1$. These two integrals suggested to Euler that the complete integral was (EULER, 1756-57: §. 9)

$$x^2+y^2+c^2x^2y^2 = c^2+2xy\sqrt{1-c^2}\,.$$

Euler then interpreted the integral $\int\frac{dx}{\sqrt{1-x^4}}$ as the length of a curve. If one set the abscissa $AP = u$, the corresponding arc was $AM = \int\frac{dx}{\sqrt{1-u^4}}$ (see Fig. 2).

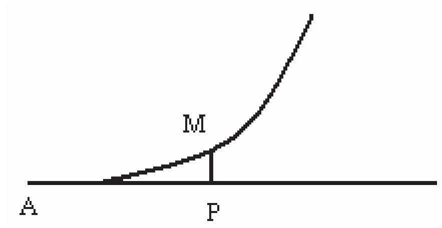

Fig. 2

Instead if one took the abscissa $ap = x$, the *arc am* was equal to $\int\frac{dx}{\sqrt{1-x^4}}$ (see Fig. 3) Therefore, if one let

$$x = \frac{u\sqrt{1-c^4} \pm c\sqrt{1-u^4}}{1+c^2u^2}\,,$$

it followed that *arc am* = *arc AM* + Const (EULER, 1756-57: §. 11).

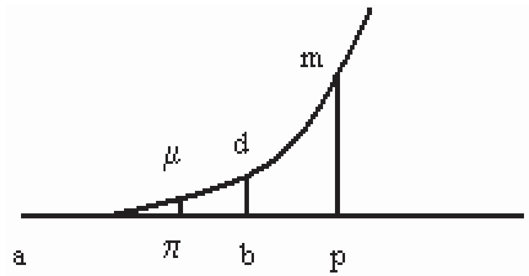

Fig. 3

To determine the value of the constant, Euler set $u = 0$, in which case the arc AM vanished. This produced $x = c$. Thus if the abscissa $ab = c$ was taken, to which the arc *ad* corresponded, it followed that arc dm = arc AM. Euler stated that in the given curve, from any given point d, it was possible to cut off arcs dm and $d\mu$ in both directions, which were equal to any given arc AM, corresponding to the abscissa $AP = u$ (EULER, 1756-57: §. 12). In analytical terms, Euler obtained the addition formula

(5) $$\int_0^x \frac{dx}{\sqrt{1-x^4}} = \int_0^c \frac{dx}{\sqrt{1-x^4}} + \int_0^u \frac{dx}{\sqrt{1-x^4}},$$

when $x = \dfrac{u\sqrt{1-c^4} \pm c\sqrt{1-u^4}}{1+c^2u^2}$.

From (5) Euler derived the duplication formula. Indeed if the arc *ad* is taken to be equal to the *arc AM*, or $c = u$, then the *arc am* is double the *arc AM*. Hence if we let

$$ap = x = \frac{2u\sqrt{1-u^4}}{1+u^4},$$

then *arc am* = 2*arc AM*, namely,

$$\int_0^x \frac{dx}{\sqrt{1-x^4}} = 2\int_0^u \frac{dx}{\sqrt{1-u^4}}.$$

Euler then generalized these results. He considered the *arc ad* (equal to $n \cdot$*arc AM*) and set the abscissa $ab = z$ so that he could write

$$\int_0^z \frac{dx}{\sqrt{1-z^4}} = n\int_0^u \frac{du}{\sqrt{1-u^4}}.$$

He stated that if one takes

$$x = \frac{z\sqrt{1-u^4} + u\sqrt{1-z^4}}{1+z^2u^2},$$

then

$$\int_0^x \frac{dx}{\sqrt{1-x^4}} = (n+1)\int_0^u \frac{du}{\sqrt{1-u^4}}.$$

In other terms, if we know the complete integral $z(u)$ of $\dfrac{dx}{\sqrt{1-z^4}} = n\dfrac{du}{\sqrt{1-u^4}}$, then

$$x(u) = \frac{z\sqrt{1-u^4} + u\sqrt{1-z^4}}{1+z^2u^2}$$

is the complete integral of $\dfrac{dx}{\sqrt{1-x^4}} = (n+1)\dfrac{du}{\sqrt{1-u^4}}$. Therefore, it is possible to find the complete integral of

$$\frac{dx}{\sqrt{1-x^4}} = n\frac{du}{\sqrt{1-u^4}}$$

for every integer number n. But, in a similar way, it is possible to determine the integral $y(u)$ of $\dfrac{dx}{\sqrt{1-y^4}} = m\dfrac{du}{\sqrt{1-u^4}}$; consequently if an equation between x and y is obtained by eliminating u, it is an integral of the equation $\dfrac{mdx}{\sqrt{1-x^4}} = \dfrac{ndy}{\sqrt{1-y^4}}$.

Euler observed that although the method which he had used in the proof of this theorem was not derived from the nature of the problem, but rather led indirectly to the desired result, it was nevertheless of much broader applicability; for in a similar way one could determine that the complete integral of other differential equations (EULER, 1756-57: §. 16), such as

$$\frac{mdx}{\sqrt{A+Cx^2+Ex^4}} = \frac{ndy}{\sqrt{A+Cy^2+Ey^4}}$$

and

$$\frac{mdx}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}} = \frac{ndy}{\sqrt{A+2By+Cy^2+2Dy^3+Ey^4}}.$$

In "De integratione aequationis differentialis" (1756-57) Euler used the geometrical interpretation as an essential step in his reasoning; in other papers[54] he attempted to give a more analytical form to his argument, indeed in these papers, he did not refer to a diagram and did not think the integral as a the arc of a curve, rather he thought it as a formula and reasoned on it directly. In the *Institutionum calculi integralis* (EULER, 1768-70, vol. 1: §. 579), for example, Euler denoted the integral

$$\int_0^x \frac{dx}{\sqrt{A+Cx^2+Ex^4}}$$

54 See EULER (1768-70) and EULER, Leonhard (1758-59) "Specimen novae methodi curvarum quadraturas et rectificationes aliasque quantitates transcendentes inter se comparandi", *Novi Comm.*, vol. 7, 83-127, or *Opera*, ser. 1, vol. 20, 108-152.

by the symbol $\Pi{:}x$ (I will write $\Pi(x)$). He considered the differential equation

$$\frac{dx}{\sqrt{A+Cx^2+Ex^4}}+\frac{dy}{\sqrt{A+Cy^2+Ey^4}}+=0 \tag{6}$$

and proved that the general integral is

$$y=\frac{b\sqrt{A(A+Cx^2+Ex^4)}-x\sqrt{A(A+Cb^2+Eb^4)}}{A-Eb^2x^2}.$$

By integrating Equation (6) Euler obtained

$$\Pi(x)+\Pi(y)+c=0. \tag{7}$$

He then set $x=0$ in Equation (7). Since $y(0)=b$ and $\Pi(0)=0$, it follows that $\Pi(0)+\Pi(b)+c=0$, $c=-\Pi(b)$, and

$$\Pi(x)+\Pi(y)=\Pi(b).$$

He then set $x=p$, $y=q$, $b=-r$, and obtained

$$\Pi(p)+\Pi(q)+\Pi(r)=0,$$

where r, p, and q satisfy the equation

$$(A-Eq^2r^2)\,p+r\sqrt{A(A+Cp^2+Ep^4)}+p\sqrt{A(A+Cr^2+Er^4)}=0\,.$$

He changed r into –r and obtained

$$\Pi(r)=\Pi(p)+\Pi(q),$$

where $r=\dfrac{q\sqrt{A(A+Cp^2+Ep^4)}+p\sqrt{A(A+Cq^2+Eq^4)}}{A-Ep^2q^2}$.

By setting $q=p$ he found the duplication formula

$$\Pi(r)=2\Pi(p)$$

for $r=\dfrac{2p\sqrt{A(A+Cp^2+Ep^4)}}{A-Ep^4}$ (EULER, 1768-70, vol. 1: §. 612-613).

He then generalized this result and stated $\Pi(z)$ is equal to $n\Pi(p)$ for appropriate values of z and p.

In a similar way, Euler dealt with the integral formulas

$$\int_0^z\frac{L+Mx^2+Nx^4}{\sqrt{A+Cx^2+Ex^4}}dx,\ \int_0^z\frac{dx}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}},$$

$\displaystyle\int_0^z\frac{a+bx+cx^2+dx^3+ex^4}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}}dx$ and (EULER, 1768-70, vol. 1 : Chapter 4).

5.- Conclusion.

The above observations make clear that the notion of the integral as an anti-differential was an important instrument in Euler's attempt to transform the calculus into an exclusively algebraic theory: it allowed him to avoid the use of infinitesimals and eliminate geometric references in the construction of the integral calculus. Euler obtained many interesting findings; however, he was unable to provide a satisfactory treatment of the new functions, such as gamma and beta functions, which he attempted to introduce in analysis. Moreover, definite integration did not play an adequate role in Euler's conception; this contrasted with the increasing importance of definite integrals in mathematical practice and, when Euler had to provide some properties of definite integration in an explicit way, he was forced to resort to the geometrical interpretation of the integral. The difficulties of Euler's methodology[55] in

55 On Euler's methodology (and, more generally, on 18th century methodology), see FERRARO, 2007a : 459-479 and FERRARO, 2008: Chapter 18).

going beyond the restrict domain of elementary functions and in pursuing his program of elimination of geometrical meaning to its natural conclusion helps to explain the reasons of the rejection of 18th century analysis.

CONSIDERACIONS AL VOLTANT DE LA FUNCIÓ BETA A L'OBRA DE LEONHARD EULER (1707-1783)

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1.- Introducció.


Figura 1. Leonhard Euler.

Leonhard Euler[1] (Fig. 1), considerat com un dels matemàtics més prolífics de la història, va contribuir al desenvolupament de la hidràulica, la hidrodinàmica, la teoria dels vaixells, l'elasticitat i la mecànica dels cossos rígids, així com a la teoria de nombres i de les sèries infinites, al concepte de funció, a les funcions de variable complexa, a les equacions diferencials, al càlcul de variacions, a l'astronomia, etc. Les seves obres completes contenen 25.000 pàgines, es divideixen en quatre grans parts o sèries i cadascuna té uns quants volums. Tanmateix, si el pes relatiu quantitatiu és considerable no ho és menys el seu pes qualitatiu.

1 Euler va néixer el 15 d'abril de 1707, a Basilea, en el si d'una família vinculada a la fe calvinista, tant el seu pare com el seu avi en varen ser ministres. Va entrar a la Universitat de Basilea als 13 anys i va estudiar matemàtiques amb Johann Bernoulli. Les seves primeres publicacions són de quan tenia 19 o 20 anys. Ja el 1727 va obtenir el segon premi en un concurs sobre temes d'investigació convocat per l' Académie des Sciences de París. En el seu treball estudiava la manera més eficient de col·locar els mastelers en un vaixell. El 5 d'abril de 1727 va abandonar Basilea per incorporar-se a la recentment inaugurada Acadèmia de Ciències a Sant Petersburg. D'aquesta primera etapa assenyalarem dos fets, el seu matrimoni a finals de 1733 amb Katharina Gsell, filla del pintor suís G. Gsell, amb qui va tenir 13 fills, dels quals cinc varen morir essent infants i la pèrdua de l'ull dret el 1738. No va tornar mai més a Basilea i va passar els anys successius entre dues estades a Sant Petersburg (1727-1741; 1766-1783) i una a Berlín (1741-1766). Sobre informació biogràfica d'Euler es pot consultar: GRAY, 1985: 171-192; CALINGER, 1996: 121-166; YOUSCHKEVITCH, 1970: 467-484; MASSA, 2007: 35-38.