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Induced Hopf Galois structures

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A B S T R A C T

For a finite Galois extension K/k and an intermediate field F such that $\mathrm{Gal}(K/F)$ has a normal complement in $\mathrm{Gal}(K/k)$, we construct and characterize Hopf Galois structures on K/k which are induced by a pair of Hopf Galois structures on K/F and F/k.

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1. Introduction

A finite extension of fields K/k is a Hopf Galois extension if there exist a finite cocommutative k-Hopf algebra $\mathcal{H}$ and a Hopf action of $\mathcal{H}$ on K, i.e. a k-linear map $\mu : \mathcal{H} \to \mathrm{End}_k(K)$ giving K a left $\mathcal{H}$-module algebra structure and inducing a bijection

$K \otimes_k \mathcal{H} \to \mathrm{End}_k(K)$. We shall call such a pair $(\mathcal{H}, \mu)$ a Hopf Galois structure on K/k. Hopf Galois extensions were introduced by Chase and Sweedler in [7]. For separable field extensions, Greither and Pareigis [14] give the following group-theoretic characterization of the Hopf Galois property.

Theorem 1. *Let K/k be a separable field extension of degree n, $\widetilde{K}$ its Galois closure, $G = \mathrm{Gal}(\widetilde{K}/k)$, $G' = \mathrm{Gal}(\widetilde{K}/K)$. Then K/k is a Hopf Galois extension if and only if there exists a regular subgroup N of S_n normalized by $\lambda(G)$, where $\lambda : G \to S_n$ is the morphism given by the action of G on the left cosets G/G'.*

For a given Hopf Galois structure on K/k, we will refer to the isomorphism class of the corresponding group N as the type of the Hopf Galois structure. The Hopf algebra $\mathcal{H}$ corresponding to a regular subgroup N of S_n normalized by $\lambda(G)$ is the subalgebra of the group algebra $\widetilde{K}[N]$ fixed under the action of G, where G acts on $\widetilde{K}$ by k-automorphisms and on N by conjugation through λ. It is known that the Hopf subalgebras of $\mathcal{H}$ are in one-to-one correspondence with the subgroups of N stable under the action of G. In the sequel for an extension K/k endowed with a Hopf Galois structure with Hopf algebra $\mathcal{H}$ corresponding to a group N, and N' a G-stable subgroup of N, we shall denote by $K^{N'}$ the subfield of K fixed by the Hopf subalgebra of $\mathcal{H}$ corresponding to N' and refer to it as the subfield of K fixed by N' (see [13, Theorem 2.3]).

Childs [8] gives an equivalent more effective condition to the Hopf Galois property introducing the holomorph of the regular subgroup N of S_n. We state the more precise formulation of this result due to Byott [1] (see also [9, Theorem 7.3]).

Theorem 2. *Let G be a finite group, $G' \subset G$ a subgroup and $\lambda : G \to \mathrm{Sym}(G/G')$ the morphism given by the action of G on the left cosets G/G'. Let N be a group of order $[G : G']$ with identity element e_N. Then there is a bijection between*

$$\mathcal{N} = \{\alpha : N \hookrightarrow \mathrm{Sym}(G/G') \text{ such that } \alpha(N) \text{ is regular}\}$$

and

$$\mathcal{G} = \{\beta : G \hookrightarrow \mathrm{Sym}(N) \text{ such that } \beta(G') \text{ is the stabilizer of } e_N\}.$$

Under this bijection, if $\alpha \in \mathcal{N}$ corresponds to $\beta \in \mathcal{G}$, then $\alpha(N)$ is normalized by $\lambda(G)$ if and only if $\beta(G)$ is contained in the holomorph $\mathrm{Hol}(N)$ of N.

Let us recall that the inclusion of $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}\, N$ in $\mathrm{Sym}(N)$ is given by sending $n \in N$ to left translation by N and $\sigma \in \mathrm{Aut}\, N$ to itself considered as a permutation.

In this paper we consider a finite Galois extension K/k with group G such that G is the semi-direct product of a subgroup G' and a normal subgroup H. For F the subfield of K fixed by G' we prove in Theorem 3 that a Hopf Galois structure on F/k of type N_1 together with a Hopf Galois structure on K/F of type N_2 induces a Hopf Galois structure on K/k of type the direct product $N_1 \times N_2$. We shall call such Hopf Galois structures induced. We shall refer to a Hopf Galois structure of type a direct product as a split Hopf Galois structure. In Theorem 9, we characterize the split Hopf Galois structures of K/k which are induced. In section 3 we give examples of induced and split non-induced Hopf Galois structures. In section 4 we determine the number of induced Hopf Galois extensions for some Galois extensions and obtain some cases in which all split structures are induced.

2. Induced Hopf Galois structures

Theorem 3. *Let K/k be a finite Galois field extension, $G = \mathrm{Gal}(K/k)$ and let F be a field with $k \subset F \subset K$ such that $G' = \mathrm{Gal}(K/F)$ has a normal complement in G. Let $r = [K : F]$, $t = [F : k]$, $n = [K : k]$ and assume that $N_1 \subset S_t$ gives F/k a Hopf Galois structure and $N_2 \subset S_r$ gives K/F a Hopf Galois structure. Then $N_1 \times N_2 \subset S_t \times S_r \subset S_n$ gives K/k a Hopf Galois structure.*

Proof. The action of G' on itself by left translation gives rise to a morphism $\lambda_r : G' \to S_r$. Let $\widetilde{F}$ be the normal closure of F/k in K, $G'' = \mathrm{Gal}(K/\widetilde{F})$. Then $\mathrm{Gal}(\widetilde{F}/k) \simeq G/G''$, $\mathrm{Gal}(\widetilde{F}/F) \simeq G'/G''$ and the action of G/G'' by left translation on the left cosets $(G/G'')/(G'/G'')$ induces an action of G on the left cosets G/G', since the sets $(G/G'')/(G'/G'')$ and G/G' are in bijection with each other. This gives rise to a morphism $\lambda_t : G \to S_t$.

Let $H = \{x_1, \dots, x_t\}$ be a normal complement of G' in G. Then $x_1, \dots, x_t$ is a left transversal for G/G'. The action of $g \in G$ on G/G' is given by $g \cdot x_i G' = x_{\lambda_t(g)(i)} G'$. Let $G' = \{y_1, \dots, y_r\}$. The action of $g' \in G'$ is given by $g' \cdot y_j = y_{\lambda_r(g')(j)}$.

Now we have $G = \{x_i y_j,\ 1 \le i \le t,\ 1 \le j \le r\}$. Let $g \in G$, and write $g = xy$, $x \in H$, $y \in G'$. Then, for each i, we have $y x_i y^{-1} \in H$, since $H \lhd G$, hence $y x_i = x_{\lambda_t(y)(i)} y$. The action of G on itself by left translation is then given by

$$g x_i y_j = x (y x_i) y_j = x x_{\lambda_t(y)(i)} y y_j = x x_{\lambda_t(y)(i)} y_{\lambda_r(y)(j)}.$$

But also $g x_i y_j \in g x_i G' = x_{\lambda_t(g)(i)} G'$, so that

$$g x_i y_j = x_{\lambda_t(g)(i)} y_{\lambda_r(y)(j)}. \tag{1}$$

The action of G on itself by left translation induces then a monomorphism

$$\lambda : G \hookrightarrow S_n = \mathrm{Sym}(\{1, \dots, t\} \times \{1, \dots, r\}).$$

We recall that we have an injective morphism $\iota : S_t \times S_r \hookrightarrow S_{tr}$. The element (σ, τ) in $S_t \times S_r$ is sent to the permutation of S_{tr} given by

$$\begin{array}{rccc} (\sigma,\tau): & \{1,\dots,t\} \times \{1,\dots,r\} & \to & \{1,\dots,t\} \times \{1,\dots,r\} \\ & (i_1, i_2) & \mapsto & (\sigma(i_1), \tau(i_2)) \end{array}$$

If N_1 is a regular subgroup of S_t and N_2 is a regular subgroup of S_r, then under the above monomorphism, $N_1 \times N_2$ is a regular subgroup of S_n: it is transitive and its order is $tr = n$. Now, since $N_1 \subset S_t$ gives F/k a Hopf Galois structure, N_1 is normalized by $\lambda_t(G)$ and, since $N_2 \subset S_r$ gives K/F a Hopf Galois structure, N_2 is normalized by $\lambda_r(G')$. That is, for every $a \in N_1$, $g \in G$, there exists $a' \in N_1$ such that $\lambda_t(g)a = a'\lambda_t(g)$, and for every $b \in N_2$, $g' \in G'$, there exists $b' \in N_2$ such that $\lambda_r(g')b = b'\lambda_r(g')$. Let us check that $N_1 \times N_2$ is normalized by $\lambda(G)$. For $g = xy \in G$, $x \in H$, $y \in G'$, $a \in N_1$, $b \in N_2$, we have

$$\begin{aligned} \lambda(g)(a,b) &= (\lambda_t(g)a, \lambda_r(y)b) = (a'\lambda_t(g), b'\lambda_r(y)) \\ &= (a',b')(\lambda_t(g), \lambda_r(y)) = (a',b')\lambda(g). \qquad \square \end{aligned} \tag{2}$$

Remark 4. In the case $K = \widetilde{F}$, the result in Theorem 3 follows from Theorem 6.1 in [12]. Let us note that in Theorem 6.1 in [12] and with the notations there it has to be assumed that K/k is almost classically Galois.

Definition 5. A Hopf Galois structure on a Galois extension K/k with Galois group G will be called *induced* if it is obtained as in Theorem 3 for some field F with $k \subsetneq F \subsetneq K$ and given Hopf Galois structures on F/k and K/F. It will be called *split* if the corresponding regular subgroup of $Sym(G)$ is the direct product of two nontrivial subgroups.

Corollary 6. *A Galois extension K/k with Galois group $G = H \rtimes G'$ has at least one split Hopf Galois structure of type $H \times G'$.*

Proof. Let $F = K^{G'}$ and let $\widetilde{F}$ be the normal closure of F in K. Then K/F is Galois with group G' and F/k is almost classically Galois of type H since H is a normal complement of $\mathrm{Gal}(\widetilde{F}/F)$ in $\mathrm{Gal}(\widetilde{F}/k)$. By Theorem 3, these two Hopf Galois structures induce a Hopf Galois structure on K/k of type $H \times G'$. $\square$

The result in the corollary complements Lemma 7.1 in [4]. There a group G is considered having complementary subgroups H, J. Then a Galois extension with group $H \times J$ is proven to have a Hopf Galois structure of type G. Other results on the possible types of Hopf Galois structures on a classical Galois extension of fields are given in [5].

Remark 7. Let us note that under the action of G given by (2), both N_1 and N_2 are G-stable subgroups of $N_1 \times N_2$.

Taking into account Theorem 2, we can reformulate the construction of induced Hopf Galois structures in terms of holomorphs. The regular subgroup N_1 of S_t gives F/k a Hopf Galois structure if and only if there is a monomorphism $\varphi_1 : G \to \mathrm{Hol}(N_1)$ such that $\varphi_1(G')$ is the stabilizer of 1_{N_1} and the regular subgroup N_2 of S_r gives K/F a Hopf Galois structure if and only if there is a monomorphism $\varphi_2 : G' \to \mathrm{Hol}(N_2)$ such that

$\varphi_2(1_{G'})$ is the stabilizer of 1_{N_2}. If we write an element $g \in G$ as $g = xy$, with $x \in H$, $y \in G'$, as above, the induced Hopf Galois structure on K/k is then given by

$$\begin{array}{rcccc} \varphi: & G & \rightarrow & \mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2) & \stackrel{\iota}{\hookrightarrow} & \mathrm{Hol}(N_1 \times N_2) \\ & g = xy & \mapsto & (\varphi_1(g), \varphi_2(y)) & & \end{array}$$

where the monomorphism ι of $\mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2)$ into $\mathrm{Hol}(N_1 \times N_2)$ is the restriction of $\iota : \mathrm{Sym}(N_1) \times \mathrm{Sym}(N_2) \to \mathrm{Sym}(N_1 \times N_2)$ to $\mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2)$ whose image is clearly contained in $\mathrm{Hol}(N_1 \times N_2)$. Let us check that $\varphi(1_G)$ is the stabilizer of $1_{N_1 \times N_2}$. Indeed, for $g = xy \in G$, $x \in H$, $y \in G'$, $\varphi(g)(1_{N_1 \times N_2}) = 1_{N_1 \times N_2}$ is equivalent to $\varphi_1(g)(1_{N_1}) = 1_{N_1}$ and $\varphi_2(y)(1_{N_2}) = 1_{N_2}$ then to $g \in G'$ and $y = 1_{G'}$, hence to $g = 1_G$.

Given a Galois extension K/k of degree n with Galois group G and a regular subgroup $N = N_1 \times N_2$ of S_n giving K/k a split Hopf Galois structure, we want to determine under which conditions this Hopf Galois structure is induced. A first step in this direction is the following.

Proposition 8. *Let K/k be a finite Galois field extension, $n = [K : k]$, $G = \mathrm{Gal}(K/k)$. Let K/k be given a Hopf Galois structure such that the corresponding regular subgroup N of S_n has a G-stable subgroup N_2. Let $F = K^{N_2}$, $G' = \mathrm{Gal}(K/F)$ and $r = [K : F]$. Then N_2 is a regular subgroup of S_r normalized by $\lambda_r(G')$.*

Proof. Let $H = K[N]^G$ be the Hopf algebra giving the Hopf Galois structure on K/k, $H_2 = K[N_2]^G$ the Hopf subalgebra of H corresponding to the G-stable subgroup N_2 of N. By [7, Theorem 7.6], K/F is a Hopf Galois extension with Hopf algebra $H_2 \otimes_k F$. By classical Galois theory, F is the subfield of K fixed by a subgroup G' of G and $G' = \mathrm{Gal}(K/F)$. Now $(H_2 \otimes_k F) \otimes_F K$ is isomorphic to $H_2 \otimes_k K$, hence to $K[N_2]$, since H_2 is a K-form of $k[N_2]$. Since K/F is Galois with Galois group G', by [14, Theorem 3.1], we have $H_2 \otimes_k F \simeq K[N_2]^{G'}$, hence N_2 is a regular subgroup of the symmetric group S_r normalized by $\lambda_r(G')$. □

Let K/k be a finite Galois extension of fields with Galois group G and let G' be a subgroup of G with normal complement H. Let $F = K^{G'}$ and assume that K/F is a Hopf Galois extension with Hopf algebra $K[N_2]^{G'}$. Suppose also that N_1 is some G-stable regular subgroup of $\mathrm{Sym}(G/G')$. Under these hypotheses, we have proved in Theorem 3 that if F/k is Hopf Galois with Hopf algebra $K[N_1]^G$, then K/k is Hopf Galois with Hopf algebra $K[N_1 \times N_2]^G$, that is, the Hopf Galois structure on K/k is induced by those on K/F and F/k. The next theorem establishes the opposite implication.

Theorem 9. *Let K/k be a finite Galois field extension, $n = [K : k]$, $G = \mathrm{Gal}(K/k)$. Let K/k be given a split Hopf Galois structure by a regular subgroup N of S_n such that $N = N_1 \times N_2$ with N_1 and N_2 G-stable subgroups of N. Let $F = K^{N_2}$ be the subfield of K fixed by N_2 and let us assume that $G' = \mathrm{Gal}(K/F)$ has a normal complement* *in G. Then K/F is Hopf Galois with group N_2 and F/k is Hopf Galois with group N_1. Moreover the Hopf Galois structure of K/k given by N is induced by the Hopf Galois structures given by N_1 and N_2.*

Proof. Since K/k is Hopf Galois with group N, we have a monomorphism

$$\begin{array}{rccl} \varphi: & G & \to & \mathrm{Hol}(N) = N \rtimes \mathrm{Aut}\, N \\ & g & \mapsto & \varphi(g) = (n(g), \sigma(g)) \end{array}$$

such that $\varphi(1_G)$ is the stabilizer of 1_N. If $n \in N$, $\sigma \in \mathrm{Aut}\, N$, we have $\sigma n \sigma^{-1} = \sigma(n)$ in $\mathrm{Sym}(N)$. Now, if N_1 and N_2 are G-stable, we have $\varphi(g)(N_i) \subset N_i$, $i = 1, 2$, for all $g \in G$, i.e. for $n_i \in N_i$, $i = 1, 2$, $\varphi(g) n_i \varphi(g)^{-1} = n(g)\sigma(g) n_i \sigma(g)^{-1} n(g)^{-1} = n(g)\sigma(g)(n_i) n(g)^{-1} \in N_i$ which implies $\sigma(g)(n_i) \in N_i$, since $N_i \lhd N$. Then, for $(n, \sigma) \in \varphi(G)$ with $n = (n_1, n_2)$, we have $(n, \sigma) = \iota((n_1, \sigma_{|N_1}), (n_2, \sigma_{|N_2}))$. Since $\varphi(G) \subset \iota(\mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2))$, we obtain morphisms

$$\varphi_1 : G \to \mathrm{Hol}(N_1), \quad \varphi_2 : G' \to \mathrm{Hol}(N_2).$$

Since F is the subfield of K fixed by N_2 and $G' = \mathrm{Gal}(K/F)$, we have for an element $g \in G$, $g \in G' \Leftrightarrow \varphi(g)(1_N) \in N_2$, taking into account the definition of the bijection b between G and N used in the proof of Theorem 7.3 in [9]. Hence $\varphi_1(G')$ is the stabilizer of 1_{N_1}. Now for $y \in G', \varphi_2(y)(1_{N_2}) = 1_{N_2} \Rightarrow \varphi_2(y)(1_N) \in N_1$. But we had $\varphi(y)(1_N) \in N_2$, hence $\varphi(y)(1_N) = 1_N$, which implies $y = 1_G$, so $\varphi_2(1_{G'})$ is the stabilizer of 1_{N_2}. $\square$

3. Examples

3.1. Examples of induced Hopf Galois structures

Applying Corollary 6, we obtain the following results.

- Galois extensions with Galois group $S_3 = C_3 \rtimes C_2$ have induced Hopf Galois structures of cyclic type $C_6 = C_3 \times C_2$.
- Galois extensions with Galois group $D_{2n} = C_n \rtimes C_2$ have induced Hopf Galois structures of type $C_n \times C_2$.
- Galois extensions with Galois group $S_n = A_n \rtimes C_2$ have induced Hopf Galois structures of type $A_n \times C_2$.
- Galois extensions with Galois group $A_4 = V_4 \rtimes C_3$ have induced Hopf Galois structures of type $V_4 \times C_3$.
- Galois extensions with Galois group a Frobenius group $G = H \rtimes G'$, where H is the Frobenius kernel and G' a Frobenius complement, have induced Hopf Galois structures of type $H \times G'$. Let us note that Sonn [17] has proved that all Frobenius groups occur as Galois groups over $\mathbb{Q}$.

- Galois extensions with Galois group $\mathrm{Hol}(M) = M \rtimes \mathrm{Aut}(M)$ have induced Hopf Galois structures of type $M \times \mathrm{Aut}(M)$.

A partial answer to the question of which groups are a semi-direct product is given by Theorem 10 below. A *Hall divisor* of an integer n is a divisor m of n such that $(m, n/m) = 1$. A *normal Hall subgroup* of a group G is a normal subgroup N such that its order $|N|$ is coprime with its index in G, i.e. such that $|N|$ is a Hall divisor of $|G|$.

Theorem 10 *(Schur–Zassenhaus). Let G be a finite group of order n and let m be a Hall divisor of n. If there exists a normal subgroup N of G with order m, then N is a characteristic subgroup of G and N has a complement H in G, i.e. $G = N \rtimes H$. If H and H' are two complements of N in G, then H and H' are conjugate.*

We have then that $G = N \rtimes H$ with $(|N|, |H|) = 1$ is equivalent to N being a normal Hall subgroup of G. Let us consider the class of groups G having (at least) one normal Hall subgroup. For p a prime integer, p-groups do not belong to this class; groups of order $2p^k$, with $p \geq 3$ prime, and also groups of order $4p^k$, with $p \geq 5$ prime, belong to this class since its unique p-Sylow subgroup is a Hall normal subgroup. Therefore a Galois extension with Galois group in one of these two last sets has induced Hopf Galois structures.

A group G of order $2m$ where m is odd, has a subgroup of order m. Indeed, by Cauchy's theorem, G has an element x of order 2. The image of x by the regular representation $\varphi : G \to S_{2m}$ is an odd permutation. Then $N := \varphi^{-1}(\varphi(G) \cap A_{2m})$ is a subgroup of order m of G, therefore a normal Hall subgroup. The group G is either a direct or semi-direct product of N and $\langle x \rangle$. Hence Galois extensions with Galois group G have at least one Hopf Galois structure of type $N \times C_2$, either Galois or induced.

3.2. Examples of split non-induced Hopf Galois structures

Not all split Hopf Galois extensions are induced. The quaternion group H_8 cannot be decomposed into a semi-direct product of two groups. However a Galois extension with Galois group H_8 has a Hopf Galois structure of type $C_2 \times C_2 \times C_2$. Let us write

$$H_8 = \langle i, j | i^4 = 1, i^2 = j^2, ij = ji^3 \rangle = \{1, i, i^2, i^3, j, ij, i^2j, i^3j\}.$$

The morphism $\lambda : H_8 \to \mathrm{Sym}(H_8)$ given by the action of H_8 on itself by left translation is determined by

$$\lambda(i) = (1, i, i^2, i^3)(j, ij, i^2j, i^3j)$$
$$\lambda(j) = (1, j, i^2, i^2j)(i, i^3j, i^3, ij).$$

Then, $\lambda(H_8)$ normalizes

$$N = \langle (1,i^2)(i,i^3)(j,i^2j)(ij,i^3j), (1,i^3)(i,i^2)(j,ij)(i^2j,i^3j), (1,i^3j)(i,j)(i^2,ij)(i^3,i^2j) \rangle$$

which is a regular subgroup of $\mathrm{Sym}(H_8)$ isomorphic to $C_2 \times C_2 \times C_2$.

Simple groups are another example of groups which are not a semi-direct product of two subgroups, hence a Galois extension with Galois group a simple group has no induced Hopf Galois structures. In fact, it is proven in [3] that a Galois extension with simple Galois group G only has Hopf Galois structures of type G.

4. Counting Hopf Galois structures

In this section we determine the number of split and induced Hopf Galois structures for some Galois extensions. We obtain some cases in which all split Hopf Galois structures are induced. The procedure to determine the number of split Hopf Galois structures is as follows. Given a Galois field extension with Galois group G, we fix a normal subgroup H of G and let G' be some subgroup of G such that $G = H \rtimes G'$. Let $F = K^{G'}$. We fix N_2 giving a Hopf Galois structure on K/F. By Theorems 3 and 9, for each N_1 that yields a Hopf Galois structure on F/k, there is a unique induced Hopf Galois structure on K/k by $N_1 \times N_2$, and every Hopf Galois structure on K/k by a split group $N_1 \times N_2$ corresponds to a unique Hopf Galois structure on F/k by N_1. So we can count the induced Hopf Galois structures on K/k arising from F by counting the Hopf Galois structures on K/F and on F/k, that is, by counting the N_2 and the N_1. Then we can vary G'.

4.1. The alternating group A_4

Let K/k be a Galois extension with Galois group A_4. We have checked that such an extension has only two types of Hopf Galois structures: A_4 and $V_4 \times C_3$.

In [6] it is shown that the number $e(A_4, A_4)$ of Hopf Galois structures of type A_4 is equal to 10. Let us determine the number of induced Hopf Galois structures of type $V_4 \times C_3$.

We have a unique choice for the nontrivial normal subgroup H, which is the Klein subgroup

$$V_4 = \{id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

It has four different complements in G

$$G'_1 = \langle (2,3,4) \rangle, \quad G'_2 = \langle (1,3,4) \rangle, \quad G'_3 = \langle (1,2,4) \rangle, \quad G'_4 = \langle (1,2,3) \rangle.$$

We have $G'_i = \mathrm{Stab}(i, A_4)$ and they are conjugate.

For a fixed complement G', if $F = K^{G'}$, then F/k is a quartic extension with Galois closure K. Since V_4 is a normal complement of G' in A_4, it has a Hopf Galois structure given by $\lambda(V_4) \subset \mathrm{Sym}(V_4)$. As a cyclic subgroup of order 4 of S_4 is not normalized by A_4, this is the unique Hopf Galois structure of F/k. The extension K/F is Galois of prime degree 3 with Galois group G'. This is also the unique Hopf Galois structure for K/F. We obtain then a unique induced Hopf Galois structure for each G'. Therefore K/k has four different induced Hopf Galois structures of type $V_4 \times C_3$. We obtain then that the number $e(A_4, V_4 \times C_3)$ of Hopf Galois structures of type $V_4 \times C_3$ of a Galois extension with Galois group A_4 satisfies

$$e(A_4, V_4 \times C_3) \geq 4.$$

4.2. Groups of order $4p$

Let us assume that p is an odd prime, $p \geq 5$, G a nonabelian group of order $4p$ and K/k a Galois extension with Galois group G. Such a group G has a unique p-Sylow subgroup H and p 2-Sylow subgroups which are isomorphic either to the cyclic group C_4 or to the elementary abelian group $C_2 \times C_2$.

Let G' be a 2-Sylow subgroup of G and F its fixed field $K^{G'}$. Since F/k has degree p and G is solvable, it is known (cf. [8]) that F/k is Hopf Galois. Furthermore, in this case F/k is almost classically Galois and has a unique Hopf Galois structure given by the normal complement H of G'. The number of induced Hopf Galois structures of type $N_1 \times N_2$ of K/k, with $N_1 \simeq H$, depends on the number of Hopf Galois structures of type N_2 of K/F.

The Hopf Galois structures for Galois extensions with group isomorphic to G' are known. For $G' \simeq C_2 \times C_2$, they are described in [11, Section 2.2]. For $G' \simeq C_4$, it is easy to check that the only regular subgroups of S_4 normalized by $\lambda(G')$ are $\lambda(G')$ itself (giving the Galois classical structure) and V_4. We give the number of these Hopf Galois structures in the following table.

	$N_2 \simeq C_4$	$N_2 \simeq C_2 \times C_2$
$G' \simeq C_4$	1	1
$G' \simeq C_2 \times C_2$	3	1

Putting all together we obtain the following numbers of induced Hopf Galois structures for K/k:

	Structures $C_4 \times C_p$	Structures $C_2 \times C_2 \times C_p$
2-Sylow subgroup $\simeq C_4$	p	p
2-Sylow subgroup $\simeq C_2 \times C_2$	$3p$	p

According to the results in [15] these are the numbers of split Hopf Galois structures for K/k of type $C_4 \times C_p$ or $C_2 \times C_2 \times C_p$.

Since the dihedral group of order $4p$ is $D_{4p} \simeq C_2 \times D_{2p}$, the dihedral Hopf Galois structures are also split structures and we wonder if they are also induced.

The groups D_{4p} and the generalized quaternion group Q_p have a center of order 2. Let $G' = Z(G)$ and $F = K^{G'}$. The extension F/k is Galois of order $2p$. If it is nonabelian it admits $2 + p$ Hopf Galois structures, 2 of them of type D_{2p} and p of them of type C_{2p} (cf. [2]). Therefore if $G = D_{4p}$, taking $G' = Z(G)$ we obtain 2 structures of type $C_2 \times D_{2p}$ and p structures of type $C_2 \times C_{2p} = C_2 \times C_2 \times C_p$. Hence all the split Hopf Galois structures of these types are induced.

4.3. Groups of order pq

Let us assume that G is a group of order pq, with p and q primes and $p > q$ and that K/k is a Galois extension with group G. If $q \nmid p - 1$, then pq is a Burnside number and K/k has a unique Hopf Galois structure, the classical Galois one (cf. [1]).

Assume that $q \mid p - 1$. Then, the group G is either cyclic or metacyclic $C_p \rtimes C_q$. In the cyclic case there are $2q - 1$ different Hopf Galois structures for K/k, the classical one with $N \simeq C_{pq}$ (split) and $2q - 2$ structures with $N \simeq C_p \rtimes C_q$ (nonsplit).

Let us consider the nonabelian case $G \simeq C_p \rtimes C_q$. Such a group G has a unique p-Sylow subgroup and p q-Sylow subgroups. Let G' be a q-Sylow subgroup of G and $F = K^{G'}$ the corresponding intermediate field. Since F/k has prime degree p and G is solvable, K/k is Hopf Galois (cf. [8]). Furthermore, in this case K/k is almost classically Galois and has a unique Hopf Galois structure. On the other hand, the same unicity property is true for the Galois extension K/F which has prime degree q.

Therefore, for each G', we obtain exactly one induced Hopf Galois structure for K/k and all together we obtain in this way p induced Hopf Galois structures for K/k. According to Theorem 6.2 in [2], this covers all split structures for K/k.

In particular, if p is an odd prime and K/k is a dihedral extension of degree $2p$, its Hopf Galois structures are the two given by G and G^{opp} (dihedral type) and the p split structures of type $C_2 \times C_p$ (cyclic type), induced by the structures of K/F and F/k, for $F = K^{G'}$ with G' ranging over the set of complements in G of the cyclic subgroup of order p (cf. [2, Corollary 6.5]).

4.4. Safe primes and Frobenius groups

Let p be a safe prime, that is, a prime such that $p = 2q + 1$ with q also a prime. A detailed discussion of Hopf Galois structures on Galois extensions with Galois groups of order $p(p-1)$, with p a safe prime, is given in Section 4 of [5]. Let G be a Frobenius group of degree p and order $p(p-1)$ and K/k a Galois extension with group G. This group G has a unique p-Sylow subgroup and p conjugate subgroups of order $p - 1 = 2q$. If G' is one of these subgroups and $F = K^{G'}$, then K/k is the Galois closure of F/k.

Again, F/k has a unique Hopf Galois structure, with $N \simeq C_p$. The Galois extension K/F has Galois group isomorphic to C_{2q}. If $q = 2$ it has 2 different Hopf Galois structures, one of them of cyclic type C_4 and the other one of dihedral type V_4. We obtain then 5 induced structures of type $C_4 \times C_5$ and 5 induced structures of type $V_4 \times C_5$ for a Frobenius extension of degree 20. This case was already treated when we considered groups of order $4p$ and we know that there are no further split structures.

If q is odd, we know from [2] (Corollary 6.4) that K/F has 3 Hopf Galois structures, namely the classical one and 2 of dihedral type. We obtain then p induced structures of type $C_{p-1} \times C_p$ and $2p$ induced structures of type $D_{p-1} \times C_p$ for a Frobenius group $F_{p(p-1)}$ with p a safe prime > 5. We know from [10, Theorem 4.1] or from [16, Theorem 5.1] that there are just p Hopf Galois structures of type $C_{p-1} \times C_p$ and just $2p$ of type $D_{p-1} \times C_p$. Hence all the (split) Hopf Galois structures of these types are induced.

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