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Multiplier ideals in two-dimensional local rings with rational singularities

Ferran Dachs Cadefau

Supervisors:
Prof. dr. M. Alberich Carramiñana
(Universitat Politècnica de Catalunya)
Prof. dr. J. Àlvarez Montaner
(Universitat Politècnica de Catalunya)
Prof. dr. W. Veys
(KU Leuven)

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Abstract

The aim of this memoir is to study multiplier ideals in two-dimensional local rings having at worst rational singularities. We also want to extend this study to the case of mixed multiplier ideals. The main achievements in the memoir are the following.

We introduce a new method to compute the antinef closure of any given divisor, generalizing previous versions of Casas-Alvero [CA00] and Reguera [Reg97].

We reveal which information encoded in a multiplier ideal determines the next jumping number. This leads to an algorithm to compute sequentially the jumping numbers and the whole chain of multiplier ideals in any desired range.

As a consequence of our method, we develop the notion of *jumping divisor* that allows to describe the jump between two consecutive multiplier ideals. In particular, we find unique minimal and maximal jumping divisors that are studied extensively.

We study the multiplicities of jumping numbers of $\mathfrak{m}$-primary ideals. The formula we provide for the multiplicities leads to a very simple and efficient method to detect whether a given rational number is a jumping number. We also give an explicit description of the Poincaré series of multiplier ideals associated to any ideal, proving, that it is a rational function.

The results obtained above are generalized to the case of mixed multiplier ideals. More precisely, we present a method to compute the jumping walls and the different mixed multiplier ideals in any compact of $\mathbb{R}^r_{\geqslant 0}$. This method is implemented as an algorithm that computes the jumping walls for a given family of ideals. We also generalize the notion of jumping divisor and we endow the jumping walls with a notion of multiplicity.

Beknopte samenvatting

Het doel van dit proefschrift is het bestuderen van multiplieridealen in een tweedimensionale lokale ring met een rationale singulariteit. We willen ook de resultaten hierover uitbreiden naar de studie van de gemengde multiplieridealen. De belangrijkste resultaten van dit proefschrift zijn de volgende.

We introduceren een nieuwe methode om de antinefsluiting van een zekere divisor te berekenen. Hiermee veralgemenen we vorige versies van Casas-Alvero [CA00] en Reguera [Reg97].

We onthullen welke informatie, gecodeerd in een multiplierideaal, het volgende jumpinggetal bepaalt. Dit leidt tot een algoritme dat een voor een de jumpinggetallen in elk gewenst bereik berekent.

Als gevolg van onze werkwijze ontwikkelen we de notie van *jumpingdivisor*, hetgeen ons toestaat de sprong tussen twee opeenvolgende multiplieridealen te beschrijven. In het bijzonder vinden we unieke minimale en maximale jumpingdivisoren, die uitgebreid worden bestudeerd.

We bestuderen ook de multipliciteiten van jumpinggetallen van $\mathfrak{m}$-primaire idealen. De formule die we opstellen voor de multipliciteiten leidt tot een zeer eenvoudige en efficiënte methode om te detecteren of een bepaald rationaal getal een jumpinggetal is. We geven ook een expliciete beschrijving van de Poincaréreeks van de multiplieridealen geassocieerd aan een ideaal. We bewijzen in het bijzonder dat het een rationale functie is.

De resultaten hierboven, breiden we uit tot gemengde multiplieridealen. Meer precies, presenteren we een methode om de jumpingwanden en gemengde multiplieridealen voor een gegeven punt in $\mathbb{R}^r_{\geqslant 0}$ te berekenen. Dit staat ons toe een algoritme te beschrijven dat de jumpingwanden berekent voor een bepaalde familie van idealen. We veralgemenen ook het begrip van de jumpingdivisor om de multipliciteit te kunnen berekenen op een manier die vergelijkbaar is met de reeds besproken methode.

Resum

L'objectiu d'aquesta memòria és estudiar els ideals multiplicadors en un anell local dos-dimensional que pot tenir una singularitat racional. Estenent també aquests resultats al cas d'ideals multiplicadors mixts. En particular, els principals resultats són els descrits a continuació.

S'introdueix un nou mètode per calcular la clausura antinef d'un divisor donat, generalitzant els resultats de Casas-Alvero [CA00] i Reguera [Reg97].

S'estudia quina informació codificada en un ideal multiplicador determina el següent nombre de salt. Això permet introduir a un algoritme per calcular seqüencialment els números de salt i tota la cadena d'ideals multiplicadors en qualsevol rang desitjat.

Es desenvolupa, com a conseqüència del mètode presentat, el concepte de *divisor de salt*, que permet descriure el salt entre dos ideals multiplicadors consecutius. En particular, s'estudia el divisor de salt minimal i maximal juntament amb la seva unicitat.

S'estudien les multiplicitats dels nombres de salt dels ideals $\mathfrak{m}$-primaris. La fórmula que es presenta per les multiplicitats porta a un mètode molt simple i eficient per detectar si donat un nombre racional, és un número de salt. També es fa una descripció explícita de la Sèrie de Poincaré dels ideals multiplicadors associats a un ideal qualsevol. Demostrant, en particular, que és una funció racional.

Es generalitzen els resultats obtinguts al cas d'ideals multiplicadors mixts. En particular, es presenta un mètode per tal de calcular les parets de salt i ideals multiplicadors mixts d'un punt donat a $\mathbb{R}^r_{\geqslant 0}$. Això permet introduir un algoritme que calcula les parets de salt per a una determinada família d'ideals. També es generalitza la noció de divisor de salt amb la finalitat de permetre

calcular la multiplicitat utilitzant mètodes similars als ja descrits.

Introduction

In recent years the theory of *multiplier ideals* has emerged as a fundamental tool in Algebraic Geometry and Commutative Algebra. These ideals were introduced by Lipman, under the terminology of *adjoint ideals*, in order to study the so-called Briançon-Skoda theorem. In the analytic context, multiplier ideals were introduced by Nadel to study plurisubharmonic functions.

Let X be a two-dimensional complex algebraic variety with mild singularities and $\mathcal{O}_{X,O}$ the local ring of X at a point $O \in X$, and denote by $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$ the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. To any ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ one may associate a family of *multiplier ideals* $\mathcal{J}(\mathfrak{a}^{\lambda})$ parametrized by positive rational numbers $\lambda \in \mathbb{Q}_{>0}$. They form a nested sequence of ideals

$$\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \ldots$$

and the rational numbers $0 < \lambda_1 < \lambda_2 < \cdots$ where the multiplier ideals change are called *jumping numbers*. The first jumping number λ_1 is also known as the *log-canonical threshold*. Multiplier ideals and their associated jumping numbers have proven to be a powerful tool to understand the geometry of singularities. They are defined using a log-resolution of the pair $(X, \mathfrak{a})$. In fact, smaller or more dense jumping numbers can be thought to correspond to 'worse' singularities.

We present a new approach to the understanding of multiplier ideals and jumping numbers of any ideal $\mathfrak{a}$ in the local ring $\mathcal{O}_{X,O}$ of a complex surface X, having at worst a rational singularity at O. This is a case, especially when X is smooth, that has received a lot of attention in recent years because of the interesting properties these invariants satisfy (see the works of Favre-Jonsson [FJ04], [FJ05], Lipman-Watanabe[LW03] or Tucker [Tuc09]). This is also one of the few cases where explicit computations have been done.

For simple complete ideals or irreducible plane curves in a smooth surface, Järviletho [Jär11] and Naie [Nai09] provide a closed formula for the set of jumping numbers in terms of some invariants of the singularity, the *Zariski exponents*. To give a closed formula for any general ideal is beyond the scope of this memoir. A formula for the log-canonical threshold already becomes quite complicated as one may see in the papers of Kuwata [Kuw99] and Galindo-Hernando-Monserrat [GHM16].

For the case of any ideal in a surface with a rational singularity, we must refer to the work of Tucker [Tuc10] where he gives a simple algorithm (see §6 in loc. cit.) to compute the set of jumping numbers. To such purpose, he developed the notion of divisors that *(critically) contribute*, building upon previous work of Smith-Thompson [ST07]. We may interpret jumping numbers as being parametrized by contributing divisors. However, critical divisors are more economic to detect since the complete ideals they define are very close to their corresponding multiplier ideal. The algorithm Tucker proposes uses a characterization of critical divisors that allows them to be found explicitly, and consequently allows the corresponding jumping numbers to be computed.

A similar strategy is used by Hyry-Järvilehto in [HJ11] where they proved that jumping numbers are parametrized by more general complete ideals[2]. Moreover, they provide a combinatorial criterion to detect a suitable ideal and its corresponding jumping number.

If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, its associated multiplier ideals are $\mathfrak{m}$-primary as well, so they have finite codimension, as $\mathbb{C}$-vector spaces, in $\mathcal{O}_{X,O}$. This fact prompted Ein-Lazarsfeld-Smith-Varolin [ELSV04] to define the *multiplicity* of a jumping number as the codimension, as $\mathbb{C}$-vector spaces, of two consecutive multiplier ideals. In general, for any positive real number c, we can define its multiplicity as

$$m(c) := \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^{c})}$$

where ε is small enough. In particular, c is a jumping number whenever $m(c) > 0$. In order to gather all the information given by all jumping numbers and their corresponding multiplicities, Galindo-Monserrat [GM10] introduced the so-called *Poincaré series of multiplier ideals* associated to $\mathfrak{a}$ as the series with fractional exponents

$$P_{\mathfrak{a}}(t) = \sum_{c\in\mathbb{R}_{>0}} m(c)\, t^{c}.$$

[2]Contributing divisors describe complete ideals nested in between consecutive multiplier ideals. The ideals considered in [HJ11] are not necessarily nested.

The main result in [GM10] is the fact that the Poincaré series of a simple complete $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, for a smooth point O, is rational, in the sense that it belongs to the field of fractional functions $\mathbb{C}(z)$ where the indeterminate z corresponds to a fractional power $t^{1/e}$ for a suitable $e \in \mathbb{N}_{>0}$. They also provide a closed formula for $P_{\mathfrak{a}}(t)$ that relies on Järviletho's formula [Jär11] for the set of jumping numbers.

Instead of considering one ideal $\mathfrak{a}$ and a real number λ, we can consider a tuple of ideals $(\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq \mathcal{O}_{X,O}^r$ and a point $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$, and associate to them a family of *mixed multiplier ideals* $\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r})$. If one considers the mixed multiplier ideals over the points of any line passing through the origin of $\mathbb{R}^r_{\geqslant 0}$, then, one gets the mixed multiplier ideal of a product of different powers of your ideals. In this case, we consider the *jumping walls*, i.e., the points where the mixed multiplier ideal changes. These jumping walls divide $\mathbb{R}^r_{\geqslant 0}$ in *constancy regions*, where any two points in that region have the same mixed multiplier ideal. Notice that the multiplier ideals are totally ordered while the mixed multiplier ideals are not. However, in the mixed case, you can consider a partial order on them.

More specifically, for any tuple of ideals $(\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq \mathcal{O}_{X,O}^r$, we define the *region* of $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ as all the points $\boldsymbol{\lambda'} = (\lambda'_1, ..., \lambda'_r) \in \mathbb{R}^r_{\geqslant 0}$ such that $\mathcal{J}\left(\mathfrak{a}_1^{\lambda'_1} \cdots \mathfrak{a}_r^{\lambda'_r}\right) \supseteq \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right)$ and its *constancy region* as all the points $\boldsymbol{\lambda'} = (\lambda'_1, ..., \lambda'_r) \in \mathbb{R}^r_{\geqslant 0}$ such that $\mathcal{J}\left(\mathfrak{a}_1^{\lambda'_1} \cdots \mathfrak{a}_r^{\lambda'_r}\right) = \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right)$. The *jumping walls* are defined as the boundaries of these regions.

Mixed multiplier ideals have not received that much attention as the multiplier ideals. They have been specially studied for their connections to other invariants. For instance, Libgober and Mustaţă in [LM11] investigated the properties of the region associated to the origin $\boldsymbol{\lambda}_0 = (0, ..., 0)$, that they call LCT-polytope. In their paper, they present some properties of this region and proved that this LCT-polytope satisfies a strong form of the ascending chain condition (see [LM11, Theorem 3.3]). Naie in [Nai13] uses the mixed multiplier ideals to establish a formula for the irregularity of abelian coverings of smooth projective surfaces.

Cassou-Noguès and Libgober study in [CNL11, CNL14] an analogous notion to the mixed multiplier ideals, the ideals of quasiadjunction, associated to germs of plane curves. In [CNL11], they describe some methods for the computation of the regions (see Proposition 2.2 and (2.3) in loc. cit.) and also the relations with other invariants such as the mixed Hodge structure or the Brenstein-Sato ideals. In [CNL14] they improve their results, showing that the jumping walls must satisfy certain conditions (see Theorem 4.1 in loc. cit.). They also characterize the log-canonical wall (see Theorem 4.22 in loc. cit.), and show an example where a jumping wall different from the log-canonical that does not satisfy the ascending chain condition.

The main goals of this memoir are the following.

- To compute jumping numbers and multiplier ideals from a careful understanding of the relation between consecutive multiplier ideals.
- To study the multiplicities of jumping numbers and describe the Poincaré series of any $\mathfrak{m}$-primary ideal.
- To extend the problems considered above to the case of mixed multiplier ideals.

These goals are achieved as follows.

i) To understand the whole change from a multiplier ideal to the next one, we reveal what information encoded in a multiplier ideal determines the next jumping number (see Theorem 2.1.5).

ii) Theorem 2.1.5 gives rise to an algorithm (see Algorithm 2.2.1) to compute the ordered sequence of multiplier ideals in any desired range of the real line. The algorithm avoids considering candidates and computes sequentially at each step a jumping number and its associated multiplier ideal. This new algorithm improves in efficiency the computation of jumping numbers when compared with Tucker's algorithm.

iii) Perhaps the most important contribution of the method presented above lies in finding out a divisor, that we name the *minimal jumping divisor*, tightly related to the aforementioned algorithm, which enables one to obtain a multiplier ideal from the previous one, and vice versa. Jointly with the *maximal jumping divisor*, they are studied, in particular its geometric structure on the dual graph, and they are compared with the previously known critically contributing divisors.

iv) To provide a systematic study of the multiplicity of c as a jumping number, i.e., the codimensions between the multiplier ideal associated to c and the multiplier ideal associated to $c-\varepsilon$ for $\varepsilon > 0$ small enough, using the theory of *jumping divisors*. This is done in Theorem 4.1.1, where we present a formula to compute them in terms of the maximal jumping divisor.

v) To give a simple numerical criterion (see Theorem 4.2.2) which characterizes whether any given rational number is a jumping number, allowing to present an algorithm to compute the jumping numbers of any given $\mathfrak{m}$-primary ideal (see Section 4.2).

vi) To prove that the Poincaré Series of any $\mathfrak{m}$-primary ideal is rational (see Theorem 4.3.1), in the sense that it belongs to the field of fractional functions $\mathbb{C}(z)$ where the indeterminate z corresponds to a fractional power $t^{1/e}$ for a suitable $e \in \mathbb{N}_{>0}$.

vii) To extend the results of multiplier ideals to mixed multiplier ideals presenting a characterization of the region and jumping walls of a given point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ (see Theorem 5.2.3). We also provide an algorithm to compute the jumping walls and the mixed multiplier ideals associated to any tuple of ideals (see Algorithm 5.2.11).

viii) To reformulate for the case of mixed multiplier ideals the definitions of minimal and maximal jumping divisors and present some properties of them in this case. This allows us to present also a formula for the multiplicity (see Theorem5.4.3).

This memoir will be divided in two main parts, the first one devoted to present results for multiplier ideals and jumping numbers, and a second one devoted to mixed multiplier ideals. In Chapter 1, we introduce the basics of the theory of multiplier ideals and some of the tools in the theory of singularities that we will need in the rest of the memoir. We pay special attention to the equivalence between complete ideals and antinef divisors developed by Lipman in [Lip69] since this is the way we will present multiplier ideals. In particular, we provide a new method to compute the antinef closure of any given divisor, generalizing previous versions of Casas-Alvero [CA00] and Reguera [Reg97].

In Chapter 2 we present one of the main results of this memoir, Theorem 2.1.5. It gives a generalization of a well-known formula for the log-canonical threshold and allows us to compute a jumping number from the data given by the preceding multiplier ideal. This leads us to develop an algorithm that computes sequentially the chain of multiplier ideals (see Algorithm 2.2.1).

In Chapter 3 we develop the theory of *jumping divisors*, which allows us to describe the whole jump between two consecutive multiplier ideals. Quite surprisingly, the algorithm developed in Chapter 2 allows us to construct the unique *minimal jumping divisor* associated to every jumping number. It is minimal in the sense that no proper subdivisor gives the whole jump between consecutive multiplier ideals. Moreover, in Section 3.1 we prove (see Theorem 3.1.1) that minimal jumping divisors are *generically* invariant with respect to log-resolutions of the ideal. Section 3.2 serves to show that they satisfy some nice geometric properties when viewed in the dual graph. In Section 3.3, we give a geometrical description of the maximal jumping divisor. We also point out that, *en passant*, we provide several technical results that will be crucial in Chapter 4.

Also in chapter 3, namely in Section 3.4, we present the theory of jumping divisors in a more general framework. We study their relation to the results of Hyry-Järvilehto [HJ11] and to the theory of contributing divisors of Tucker [Tuc10]. The main result of this section is the fact that, among all the contributing divisors associated to a jumping number that give the same ideal, there is a minimal one. For example, critical divisors are of this type. It turns out that these minimal contributing divisors are all contained in the minimal jumping divisor and inherit the same invariance property with respect to log-resolutions of the ideal.

In Chapter 4, we assume $\mathfrak{a}$ is $\mathfrak{m}$-primary. We provide two different formulae to describe the multiplicity for any $c \in \mathbb{R}_{>0}$. The first one (see Theorem 4.1.1) is described in terms of the maximal jumping divisor associated to c. The periodicity of this divisor leads to Proposition 4.1.5, which provides a very clean description of the growth of multiplicities in terms of dicritical components of the maximal jumping divisor. This is the key result which is used in the description of the Poincaré series associated to $\mathfrak{a}$ in the final section of this chapter. The second formula for the multiplicity (see Proposition 4.1.10) is given using the notion of *virtual codimension* introduced in [CA00] and [Reg97].

In Section 4.2, we provide a very simple (and efficient) algorithm to compute the set of jumping numbers of $\mathfrak{a}$. It boils down to compute the multiplicities of the rational numbers in the set of *candidate jumping numbers*. This relies on a simple numerical criterion to characterize jumping numbers (see Theorem 4.2.2). Another consequence of the formulae for the multiplicities is that we can describe those jumping numbers contributed by dicritical divisors. In particular we give in Theorem 4.2.5 a full description of the jumping numbers in the interval $(1, 2]$.

The main result of Section 4.3 is a description of the Poincaré series of multiplier ideals for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$. As a consequence, we can easily recover the case of simple ideals obtained by Galindo-Monserrat [GM10] in the smooth case. Finally we relate the Poincaré series to the Hodge spectrum of a generic element $f \in \mathfrak{a}$. In particular we recover an old result of Lê Văn Thành-Steenbrink

[TS89], describing the Hodge spectrum of a plane curve.

Finally, in Chapter 5 we consider mixed multiplier ideals. We start the chapter by introducing some basic properties and definitions in order to present Theorem 5.2.3, which gives for any point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ its jumping walls. This theorem is a generalization in this context of Theorem 2.1.5: for any given $\boldsymbol{\lambda}$, we can compute its region and also the associated jumping wall. Theorem 5.2.3 is the key ingredient for an algorithm (see Algorithm 5.2.11) which computes the jumping walls in any bounded subset of $\mathbb{R}^r_{\geqslant 0}$

In Section 5.3 we present a generalization of the results of Chapter 3 to the case of mixed multiplier ideals. As before, we introduce a nice geometrical description of the maximal and minimal jumping divisors over the dual graph. The last section of this chapter is devoted to present a formula to compute the multiplicity of any point (see Theorem 5.4.3). For this, we use the jumping divisors already introduced in Section 5.3. We illustrate this chapter with several examples included in Appendix A.

To conclude the memoir, we include Appendix B, which contains an implementation in the Computer Algebra System *Macaulay2* of the Algorithms 1.11.9, 2.2.1 and the one consequence of Theorem 4.1.1.

Chapter 1

Preliminaries

Throughout this work we will consider (X, O) to be a germ of complex surface with at worst a rational singularity. We denote by $\mathcal{O}_{X,O}$ the local ring at O and $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$ the maximal ideal. Given an ideal (or a sheaf of ideals) $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ we want to associate a collection of *multiplier ideals* $\mathcal{J}(X, \mathfrak{a}^c)$ depending on a (real) parameter $c > 0$. If X is clear from the context we will simply denote $\mathcal{J}(\mathfrak{a}^c)$.

First we will review some notations that will be needed for the construction of these ideals. In this section we have omitted most proofs, which can be found in [Laz04] (see also [BL04], [ELSV04]).

1.1 Divisors

Let $Div(X)$ be the free abelian group generated by the reduced and irreducible (i.e., integral) subvarieties of codimension one. A Weil divisor is a formal sum

$$D = \sum d_i D_i$$

where

- D_i's are reduced and irreducible subvarieties of codimension one, and
- $d_i \in \mathbb{Z}$ and only finitely many d_i are different from zero.

Also important in this context is the notion of Cartier divisor.

Definition 1.1.1. A divisor D that is locally principal is said to be a Cartier divisor on X.

Over smooth varieties these two notions are equivalent, however this is no longer true when dealing with singular varieties. For practicity, we adopt the convention that "divisor" means a Weil divisor.

Definition 1.1.2. We say that a divisor D is effective if and only if all $d_i \geqslant 0$.

For simplicity, we will use the following notations.

- We write an effective divisor D as $D \geqslant 0$.
- We write $D_1 \geqslant D_2$ if $D_2 - D_1$ is effective.

We are also interested in divisors with coefficients in $\mathbb{Q}$.

Definition 1.1.3. A $\mathbb{Q}$-divisor on X is an element of the $\mathbb{Q}$-vector space

$$Div_{\mathbb{Q}}(X) := Div(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

One can represent an element $D \in Div(X)$ as

$$D = \sum d_i D_i\,,$$

where

- D_i's are reduced and irreducible subvarieties of codimension one, and
- $d_i \in \mathbb{Q}$ and only finitely many d_i are different from zero.

We say that D is integral if, for all i, one has $d_i \in \mathbb{Z}$. One says that an integer cleans all the denominators of D if mD is integral.

For any $\mathbb{Q}$-divisor $D = \sum_i d_i D_i \in \mathrm{Div}_{\mathbb{Q}}(X)$, we denote its *round-down* and *round-up* as

$$\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i \quad \text{and} \quad \lceil D \rceil = \sum_i \lceil d_i \rceil D_i\,.$$

The *fractional part* of D is then $\{D\} = D - \lfloor D \rfloor = \sum_i \{d_i\} D_i$.

1.2 Log-resolutions

Definition 1.2.1. A divisor $D = \sum D_i$ on a smooth X is a simple normal crossings (SNC) divisor if each D_i is smooth, and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdot \ldots \cdot z_k = 0$$

for some $k \leqslant n$. We say that a ($\mathbb{Q}$-)divisor $\sum a_i D_i$ has simple normal crossings support if $\sum D_i$ is a SNC divisor.

Definition 1.2.2. Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. Denote $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$ the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. A *log-resolution* of the pair $(X, \mathfrak{a})$ (or of $\mathfrak{a}$, for short) is a proper birational morphism $\pi : X' \to X$ such that

i) X' is smooth,

ii) the preimage of $\mathfrak{a}$ is locally principal, that is, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for some effective Cartier divisor F, and

iii) $F + E$ is a divisor with simple normal crossings support where $E = Exc\,(\pi)$ is the exceptional locus.

The existence of a log-resolution for any sheaf of ideals in any variety over a field of characteristic zero is a result of Hironaka [Hir64]. Moreover, one can find always a log resolution which is a composition of blow-ups along smooth centers. If we consider a subvariety $Z \subset X$ we denote $Bl_Z X$ the blow-up along Z. In fact, all resolutions are dominated by one of this type.

Proposition 1.2.3. [Laz04, Example 9.1.16] *Let $\pi' : X' \longrightarrow X$ and $\pi'' : X'' \longrightarrow X$ be two log-resolutions of an ideal $\mathfrak{a}$. Then, there exists a third log-resolution $\pi : \tilde{X} \longrightarrow X$ dominating these two, i.e., there exist proper birational maps θ', θ'' that makes the following diagram commutative.*

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\theta'"'] \arrow[dd, "\pi"] \arrow[dr, "\theta''"] & \\ X' \arrow[dr, "\pi'"'] & & X'' \arrow[dl, "\pi''"] \\ & X & \end{tikzcd}$$

Moreover, this log-resolution π can be expressed as a composition of blow-ups.

Let $\mathrm{Div}(X')$ be the group of integral divisors in X', i.e. divisors of the form $D = \sum_i d_i E_i$ where the E_i are pairwise different (non necessarily exceptional) prime divisors and $d_i \in \mathbb{Z}$. Among them, we will consider divisors in the lattice $\Lambda := \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$ of exceptional divisors and we will simply refer them as divisors with *exceptional support*. Any divisor $D \in \mathrm{Div}(X')$ has a decomposition $D = D_{\mathrm{exc}} + D_{\mathrm{aff}}$ into its *exceptional* and *affine* part[1] according to its support. Our main example is the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. In this case we will denote its exceptional and affine part as

$$F_{\mathrm{exc}} = \sum_{i=1}^{r} e_i E_i \quad \text{and} \quad F_{\mathrm{aff}} = \sum_{i=r+1}^{s} e_i E_i$$

where, by definition, the e_i are non-negative integers. Whenever $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the divisor F is just supported on the exceptional locus. i.e. $F = F_{\mathrm{exc}}$.

Remark 1.2.4. Let $C : f = 0$ be a curve defined by an element $f \in \mathcal{O}_{X,O}$. The *total transform* of C is the pull-back $\overline{C} := \pi^* C$ and its *strict transform* C' is the closure of $\pi^{-1}(C - \{O\})$. The total transform has a presentation $\overline{C} = C' + \overline{C}_{\mathrm{exc}} = C' + \sum d_i E_i$ where the weights $v_i(f) := d_i$ are the *values* of the curve C at E_i. Recall that $f \in \mathfrak{a}$ whenever $C' + \overline{C}_{\mathrm{exc}} \geqslant F$ and f is *generic* in $\mathfrak{a}$ if $\overline{C}_{\mathrm{exc}} = F_{\mathrm{exc}}$ and $C' - F_{\mathrm{aff}}$ has no singular points.

1.3 Rational singularities

The theory of rational singularities was introduced by Artin in [Art66] and further developed by Lipman in [Lip69]. We recall that the point O being (at worst) a rational singularity means that $R^1\pi_*\mathcal{O}_{X'} = 0$ for some (hence any) desingularization. A first consequence of Artin's results is that the exceptional divisor of any desingularization is a tree of rational curves. Indeed, according to [Art66, Proposition 1] a singularity is rational if and only if any effective divisor D with exceptional support has arithmetic genus (see [Art62, Page 486])

$$p_a(D) = 1 + \frac{1}{2}(K_{X'} + D) \cdot D \leqslant 0\,,$$

where K_X is the canonical divisor on X, i.e., let ω_X be the canonical line bundle on X, thus K_X is a divisor such that $\mathcal{O}_X(K_X) = \omega_X$. Since the components E_i of the exceptional divisor are smooth, we have $p_a(E_i) \geq 0$, hence $p_a(E_i) = 0$, which means that they are rational. Furthermore, there cannot be a *cycle* $E_1, \ldots, E_k$ of exceptional components (i.e., such that $E_1 \cdot E_2 = E_2 \cdot E_3 =$

[1] We follow the terminology of Lipman-Watanabe [LW03]

$\cdots = E_1 \cdot E_k = 1$ and $E_i \cdot E_j = 0$ for any other $i \neq j$), since the formula $p_a(A+B) = p_a(A) + p_a(B) + A \cdot B - 1$ would give $p_a(E_1 + \cdots + E_k) = 1$.

The above numerical characterization [Art66, Proposition 1] of rational singularities is not satisfying enough, since it involves testing every effective exceptional divisor. In the same work, Artin proved in [Art66, Theorem 3] that it is enough to check the *fundamental cycle*, the unique smallest non-zero effective divisor Z (with exceptional support) such that

$$Z \cdot E_i \leqslant 0 \qquad \text{for every } \ i = 1, \ldots, r.$$

1.4 Relative canonical divisor

We will also consider $\mathbb{Q}$-divisors in $\mathrm{Div}_{\mathbb{Q}}(X') = \mathrm{Div}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ or divisors in the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} := \mathbb{Q}E_1 \oplus \cdots \oplus \mathbb{Q}E_r$. The main example will be the *relative canonical divisor* K_π. The definition of K_π is quite subtle if O is singular, because at first sight one can only define a canonical divisor K_X of X as a Weil divisor. Since rational singularities are in particular $\mathbb{Q}$-factorial, there exists in that case a positive integer m such that mK_X is Cartier, which can be pulled back to X' and allows to define $K_\pi = K_{X'} - \frac{1}{m}\pi^*(mK_X)$. Alternatively,

$$K_\pi = \sum_{i=1}^{r} k_i E_i$$

is supported on the exceptional locus E, and must satisfy

$$(K_\pi + E_i) \cdot E_i = \sum_{j=1}^{r} k_j E_j \cdot E_i + E_i^2 = -2 \tag{1.4.1}$$

for every exceptional component E_i because of the adjunction formula. This property indeed characterizes K_π because the intersection form on E is negative-definite, and therefore the system defined by equations (1.4.1) has a unique solution $(k_1, \ldots, k_r)$. However, the k_i are not necessarily integral, and can even be negative. In the case that $k_i > -1$ (resp. $k_i \geqslant -1$) for all E_i, one says that X has a *log-terminal singularity* (resp. *log-canonical singularity*) at O. Moreover, due to this numerical characterization, $K_{X'}$ can be replaced by K_π to compute the arithmetic genus as $p_a(Z) = 1 + \frac{1}{2}(K_\pi + Z) \cdot Z$.

1.5 Dual graph

The combinatorics of the log-resolution of $\mathfrak{a}$ can be encoded using the so-called *dual graph*. This is a rooted tree where the vertices represent the irreducible components $E_i \leqslant F$ and two vertices are joined by an edge if the corresponding divisors intersect.

Given any component E_i, we will denote by $\operatorname{Adj}(E_i)$ the set of components E_j, $j \neq i$, sharing an edge with E_i, i.e. $E_i \cdot E_j = 1$, and by

$$a(E_i) = \#\operatorname{Adj}(E_i) = E_i \cdot \left(F^{\mathrm{red}} - E_i\right)$$

the number of such components which is the *valency* of the vertex representing E_i, where F^{red} denotes the reduced divisor with the same support as F. An *end* of the dual graph is nothing but a vertex with valence 1, i.e., a vertex E_i such that $a(E_i) = 1$. More generally, for any effective subdivisor $D = E_{i_1} + \cdots + E_{i_m} \leqslant F$ we define

$$\operatorname{Adj}_D(E_i) = \{E_j \leqslant D \mid E_i \cdot E_j = 1\}$$

and $a_D(E_i) = \#\operatorname{Adj}_D(E_i)$. We denote by $v_D = m$ (resp. a_D) the number of components of D (resp. the number of intersections between different components of D). Since the dual graph is a tree it is clear that

$$\sum_{E_i \leqslant D} a_D(E_i) = 2a_D$$

and that $v_D - a_D$ equals the number of connected components of D. An end of the subgraph associated to D is a vertex with valence 1 or 0, the latter meaning that E_i is an isolated component of D.

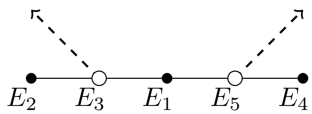
For any exceptional component E_i, we define the *excess* (of $\mathfrak{a}$) at E_i as $\rho_i = -F \cdot E_i$. It can be interpreted as the number of branches of the strict transform of a curve C defined by a generic element $f \in \mathfrak{a}$ that intersect the component E_i. Indeed, if its total transform is $\overline{C} = C' + F$, then $0 = \overline{C} \cdot E_i = C' \cdot E_i + F \cdot E_i = C' \cdot E_i - \rho_i$, which proves the claim.

There are two kinds of exceptional components that will play a special role.

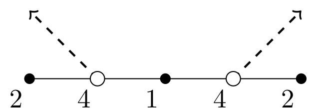
- A component E_i of E is a *rupture* component if $a(E_i) \geqslant 3$, that is, it intersects at least three components of F (different from E_i).
- We say that E_i is *dicritical* if $\rho_i > 0$. By [Lip69], dicritical components correspond to *Rees valuations*.

It is important to mention that non-exceptional components also correspond to *Rees valuations.*

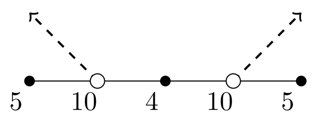
Example 1.5.1. Consider the ideal $\mathfrak{a} = (x^2y^2, x^5, y^5, xy^4, x^4y) \subseteq \mathbb{C}\{x, y\}$. After computing its log-resolution, one can encode the information gathered about the divisors and their intersections in the dual graph of the resolution.



The blank dots correspond to dicritical divisors and their excesses are represented by broken arrows. Another information that we can encode is the relative canonical divisor K_π. More precisely, $K_\pi = 1E_1 + 2E_2 + 4E_3 + 2E_4 + 4E_5$ can be encoded as follows.



Another option that we will use is collecting the values of any divisor in a vector, i.e., we encode K_π as $(1, 2, 4, 2, 4)$. Similarly, we can encode the multiplicities of the divisor F, i.e., the divisor defined as $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$.



The intersection matrix associated to this resolution, i.e., $M = (E_i \cdot E_j)_{1 \leqslant i,j \leqslant 5}$, is in our case

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

1.6 Complete ideals and antinef divisors

Given an effective $\mathbb{Q}$-divisor $D = \sum d_i E_i \in \mathrm{Div}_{\mathbb{Q}}(X')$, we may consider its associated (sheaf) ideal $\pi_* \mathcal{O}_{X'}(-D) := \pi_* \mathcal{O}_{X'}(-\lceil D \rceil)$. Its stalk at O is

$$I_D := \{ f \in \mathcal{O}_{X,O} \mid v_i(f) \geqslant \lceil d_i \rceil \text{ for all } E_i \leqslant D \},$$

where the weights $v_i(f) := d_i$ are the *values* of the curve associated to f at E_i. This is a complete ideal of $\mathcal{O}_{X,O}$ that is $\mathfrak{m}$-primary whenever D has exceptional support, i.e., $D \in \Lambda_{\mathbb{Q}}$. Any two divisors $D, D' \in \mathrm{Div}_{\mathbb{Q}}(X')$ defining the same complete ideal $\pi_* \mathcal{O}_{X'}(-D) = \pi_* \mathcal{O}_{X'}(-D')$ are called *equivalent divisors*.

In the equivalence class of a given divisor one may find a unique maximal representative. First, recall that an effective divisor with integral coefficients $D \in \mathrm{Div}(X')$ is called *antinef* if $-D \cdot E_i \geqslant 0$, for every exceptional prime divisor E_i. It is worth to point out that the affine part of $D = D_{\text{exc}} + D_{\text{aff}}$ satisfies $D_{\text{aff}} \cdot E_i \geqslant 0$. Therefore D is antinef whenever $-D_{\text{exc}} \cdot E_i \geqslant D_{\text{aff}} \cdot E_i$.

In the work of Lipman (see [Lip69, §18]) one may find the following correspondence that we will heavily use throughout this work.

Theorem 1.6.1. *There is a one to one correspondence between antinef divisors in* $\mathrm{Div}(X')$ *and complete ideals in* $\mathcal{O}_{X,O}$*. In particular, antinef divisors in* Λ *correspond to* $\mathfrak{m}$*-primary complete ideals.*

In order to find the representative in the equivalence class of a given divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X')$, we will consider its so-called *antinef closure* $\widetilde{D}$. The existence of such a divisor is a consequence of the following results that can be found in [Lip69, §18], but we also refer to [Tuc09] and [LW03] for more insight.

Lemma 1.6.2. *For any effective* $\mathbb{Q}$*-divisor* $D \in \mathrm{Div}_{\mathbb{Q}}(X')$*, there exists a unique minimal integral antinef divisor* $\widetilde{D} \in \mathrm{Div}(X')$ *satisfying* $\widetilde{D} \geqslant D$ *that is called the* antinef closure *of* D*. In particular, any antinef divisor* D' *such that* $D' \geqslant D$ *must satisfy* $D' \geqslant \widetilde{D} \geqslant D$

Proposition 1.6.3. *An effective* $\mathbb{Q}$*-divisor* $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ *and its antinef closure* $\widetilde{D} \in \mathrm{Div}(X')$ *are equivalent, i.e.*

$$\pi_* \mathcal{O}_{X'}(-D) = \pi_* \mathcal{O}_{X'}(-\widetilde{D}).$$

One of the advantages of working with antinef divisors is that they provide the following characterization for the inclusion (or strict inclusion) of two given complete ideals.

Proposition 1.6.4. *Let D_1, D_2 be two antinef divisors in* $\mathrm{Div}(X')$*. Then*

i) $\pi_*\mathcal{O}_{X'}(-D_1) \supseteq \pi_*\mathcal{O}_{X'}(-D_2)$ *if and only if* $D_1 \leqslant D_2$*,*

ii) $\pi_*\mathcal{O}_{X'}(-D_1) \supsetneq \pi_*\mathcal{O}_{X'}(-D_2)$ *if and only if* $D_1 < D_2$*.*

For non-antinef divisors we can only claim the following implication.

Proposition 1.6.5. *Let D_1, D_2 be two divisors in* $\mathrm{Div}_{\mathbb{Q}}(X')$ *such that $D_1 \leqslant D_2$. Then*

i) $\pi_*\mathcal{O}_{X'}(-D_1) \supseteq \pi_*\mathcal{O}_{X'}(-D_2)$*,*

ii) $\widetilde{D_1} \leqslant \widetilde{D_2}$*.*

The converses to these results are no longer true.

1.7 Unloading

In general, the divisors that will be considered in this work are not antinef. In order to compute their antinef closure we will use an inductive procedure called *unloading* that was already described in the work of Enriques [EC85, IV.II.17] (for the case of smooth varieties) and it is known as Laufer's procedure to compute the fundamental cycle [Lau72] (for varieties with rational singularities). Here we will present a new version that is a generalization of both the unloading procedures described by Casas-Alvero [CA00, §4.6] (for smooth varieties) and Reguera [Reg97] (for varieties with rational singularities).

Unloading procedure. Let $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ be any $\mathbb{Q}$-divisor. Its *excess* at the exceptional prime divisor E_i is the integer $\rho_i = -\lceil D \rceil \cdot E_i$. Denote by Θ the set of exceptional components $E_i \leqslant D$ with negative excesses, i.e.

$$\Theta := \{E_i \leqslant D_{\mathrm{exc}} \ \mid \ \rho_i = -\lceil D \rceil \cdot E_i < 0\}.$$

To *unload values* on this set is to consider the new divisor

$$D' = \lceil D \rceil + \sum_{E_i \in \Theta} n_i E_i,$$

where $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$. Notice that n_i is the least integer number such that

$$(\lceil D \rceil + n_i E_i) \cdot E_i = -\rho_i + n_i E_i^2 \leqslant 0.$$

Remark 1.7.1. Casas-Alvero considered at each step just one component with negative excess. Reguera also considered one component with negative excess but in her case she also imposed $n_i = 1$ at each step. In this sense, our approach is more economic from a computational point of view. Furthermore, our procedure allows unloading on divisors with affine part[2], which will enable us to treat in a unified way multiplier ideals of both curves and not necessarily $\mathfrak{m}$-primary complete ideals.

The correctness of the unloading procedure is a consequence of the following results.

Proposition 1.7.2. *Let D' be the divisor obtained from a divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ after one single unloading step. Then $I_{D'} = I_D$.*

Proof. It is clear from its construction that $I_{D'} \subseteq I_D$. Pick $f \in I_D$ and let $\overline{C} = C' + \overline{C}_{\mathrm{exc}}$ be the total transform of the curve C defined by $f = 0$. We have $v_i(f) \geqslant v_i(\lceil D \rceil) \geqslant v_i(D)$ for all E_i. Consider any exceptional divisor E_i where D has negative excess. From the inequality

$$(\overline{C}_{\mathrm{exc}} - v_i(f)E_i) \cdot E_i \geqslant (\lceil D \rceil - v_i(\lceil D \rceil)E_i) \cdot E_i \,,$$

we deduce

$$-v_i(f)E_i \cdot E_i \geqslant (\lceil D \rceil - v_i(\lceil D \rceil)E_i) \cdot E_i \,,$$

just because $\overline{C}_{\mathrm{exc}} \cdot E_i \leqslant 0$. Equivalently, $(\lceil D \rceil + (v_i(f) - v_i(\lceil D \rceil))E_i) \cdot E_i \leqslant 0$, so it follows that $n_i \leqslant v_i(f) - v_i(\lceil D \rceil)$. In particular, $n_i + v_i(\lceil D \rceil) \leqslant v_i(f)$ and $f \in I_{D'}$. $\square$

Proposition 1.7.3. *The antinef closure $\widetilde{D}$ of a divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ is achieved after finitely many unloading steps.*

Proof. We want to show that the divisors in the sequence

$$D \leqslant D_1 = \lceil D \rceil < \cdots < D_t < D_{t+1} < \cdots$$

obtained during the unloading procedure are all contained in the antinef closure $\widetilde{D}$, then the result will follow since both D_1 and $\widetilde{D}$ have integral coefficients and the inequalities in the unloading sequence are strict. Clearly $D_1 \leqslant \widetilde{D}$. Suppose that $D_t \leqslant \widetilde{D}$. Notice that for any component $E_i \leqslant D_t$ with negative excess we have $(\widetilde{D} - D_t) \cdot E_i \leqslant -D_t \cdot E_i$. Then, if we denote $\widetilde{D} - D_t = \sum_i m_i E_i$, the previous inequality becomes

$$\begin{array}{rcll} (\widetilde{D} - D_t) \cdot E_i & = & (m_i E_i + \sum_{j \neq i} m_j E_j) \cdot E_i & \\ & = & m_i E_i^2 + \sum_{j \neq i} m_j E_j \cdot E_i & \leqslant \quad -D_t \cdot E_i. \end{array}$$

[2]Our method also differs from the one considered by Lipman-Watanabe in [LW03].

Then, using that $\sum_{j\neq i} m_j E_j \cdot E_i \geqslant 0$, we get

$$m_i \geqslant \left\lceil \frac{-D_t \cdot E_i}{E_i^2} \right\rceil ,$$

where we used the fact that D_t and $\widetilde{D}$ have integer coefficients. It follows that D_{t+1} is also contained in $\widetilde{D}$. □

Example 1.7.4. Continuing Example 1.5.1, recall we associated to it the intersection matrix $M = (E_i \cdot E_j)_{1\leqslant i,j \leqslant 5}$

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now, we consider the divisors $D := (1,0,1,0,1)$ and $D' := (1,1,3,1,3)$, and we want to consider its antinef closure.

- We start by computing the unloading of $D := (1,0,1,0,1)$. It is not antinef because it has excess -1 at E_2 and E_4. The first unloading step is to consider the divisor $D + E_2 + E_4 = (1,1,1,1,1)$. This divisor has excess -1 at E_3 and E_5, so we need to perform a second unloading step to obtain the antinef closure $\widetilde{D} = (1,1,2,1,2)$.
- The second example is with $D' = (1,1,3,1,3)$. It has excess -1 at E_1, E_2 and E_4 and we obtain the divisor $(2,2,3,2,3)$ after the first unloading step. This divisor has excess -1 at E_3 and E_5 and, after a second unloading step, we obtain the antinef closure $\widetilde{D'} = (2,2,4,2,4)$.

1.8 Multiplier ideals

Now we can present the central object of this memoir, the multiplier ideals.

Definition 1.8.1. Let $\pi : X' \to X$ be a log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ and let F be the divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. The *multiplier ideal (sheaf)* associated to $\mathfrak{a}$ and some rational number $c \in \mathbb{Q}_{>0}$ is defined as[3]

$$\mathcal{J}(\mathfrak{a}^c) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) .$$

Remark 1.8.2. If some confusion arises over which variety the multiplier ideal is defined, we will denote it as $\mathcal{J}(X, \mathfrak{a})$ if the multiplier ideal is defined over X.

[3]By an abuse of notation, we will also denote $\mathcal{J}(\mathfrak{a}^c)$ its stalk at O, so we will omit the word "sheaf" if no confusion arises.

1.8.1 Mixed multiplier ideals

If instead of considering only an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, one considers a tuple of ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_X)^r$, one can generalize the notion of multiplier ideal and introduce the mixed multiplier ideals.

Definition 1.8.3. Given a tuple of ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_X)^r$, consider a log-resolution $\pi : X' \to X$ of this tuple and let F_i be the divisors such that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for $1 \leqslant i \leqslant r$. The *mixed multiplier ideal* associated to a point $\boldsymbol{\lambda} := (\lambda_1, .., \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ and $\boldsymbol{\mathfrak{a}}$ is defined as[4]

$$\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) := \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil\right) .$$

Mixed multiplier ideals satisfy similar properties as the ones of multiplier ideals.

1.8.2 First properties

We begin this section introducing some basic properties of the multiplier ideals.

Proposition 1.8.4.

- *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal and $c > 0$, then*

$$\mathcal{J}(\mathfrak{a}^c) \subseteq \mathcal{O}_{X,O} .$$

This property implies that $\mathcal{J}(\mathfrak{a}^c)$ is actually a (sheaf) ideal.

- *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal, then*

$$\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a}) .$$

- *Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathcal{O}_{X,O}$ be two ideals, then for any $c > 0$ we have*

$$\mathcal{J}(\mathfrak{a}_1^c) \subseteq \mathcal{J}(\mathfrak{a}_2^c) .$$

- *If $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_{X,O}$ are two ideals, then for any $c > 0$ and a sufficiently small $d > 0$ we have*

$$\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\mathfrak{a}^c \mathfrak{b}^d) .$$

A first, non-obvious, property is that the definition of multiplier ideal (and mixed multiplier ideal) is independent of the choice of log resolution.

[4]If no confusion arises, we abuse notation similarly as in footnote 3.

Theorem 1.8.5 (Esnault-Viehweg)**.** *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal, let $\pi' : X' \to X$ and $\pi'' : X'' \to X$ be two log-resolutions of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F')$ and $\mathfrak{a} \cdot \mathcal{O}_{X''} = \mathcal{O}_{X''}(-F'')$, then*

$$\pi'_* \mathcal{O}_{X'}(\lceil K_{\pi'} - c \cdot F' \rceil) = \pi''_* \mathcal{O}_{X''}(\lceil K_{\pi''} - c \cdot F'' \rceil) .$$

The main idea behind the proof of this result is that any two different log-resolutions are dominated by a third one (see Proposition 1.2.3). Then, for simplicity, we will always fix a given resolution.

Another property that we are interested in is that multiplier ideals are invariant up to integral closure.

Proposition 1.8.6. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal sheaf. Then*

$$\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\overline{\mathfrak{a}}^c) .$$

Therefore, without loss of generality, we may always assume that the ideal $\mathfrak{a}$ is complete. Another interesting property is that multiplier ideals are complete.

Proposition 1.8.7. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal sheaf and $\mathcal{J}(\mathfrak{a}^c)$ be the associated multiplier ideal. Then, $\mathcal{J}(\mathfrak{a}^c)$ is integrally closed for all c.*

Moreover, if $\mathfrak{a}$ is $\mathfrak{m}$-primary it follows that its associated multiplier ideals $\mathcal{J}(\mathfrak{a}^c)$ are $\mathfrak{m}$-primary as well for all c.

Lipman and Watanabe in [LW03] and independently Favre and Jonsson in [FJ05] proved that every integrally closed ideal in a two-dimensional regular local ring is a multiplier ideal. Tucker in [Tuc09] proved the same result but for integrally closed ideals on log terminal surfaces.

On the other hand, Lazarsfeld and Lee in [LL07] have shown that, in dimension three or higher, integrally closed ideals need to satisfy certain conditions in order to be realized as multiplier ideals. These conditions are not sufficient, so they give examples of integrally closed ideals that cannot be realized as multiplier ideals.

1.8.3 Vanishing theorems

Another interesting result that we will use through this work are vanishing theorems. The multiplier ideal is the zeroth derived image of $\mathcal{O}_{X'}(\lceil K_\pi - cE \rceil)$ under π_*. It turns out that the higher derived functors vanish. The following theorem is a special case of Kawamata-Viehweg vanishing (see [Laz04]).

Theorem 1.8.8 (Local vanishing). *Consider an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and a log-resolution $\pi : X' \longrightarrow X$ of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. Then*

$$R^i \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) = 0$$

for all $i > 0$ and $c > 0$.

Another vanishing theorem which is satisfied by multiplier ideals is the following one.

Theorem 1.8.9 (Nadel Vanishing Theorem). *Let $c > 0$ be a positive rational number, L and A two integral divisors on X such that $L - c \cdot A$ is big and nef, and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal such that $\mathfrak{a} \otimes \mathcal{O}_X(A)$ is globally generated. Then,*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(\mathfrak{a}^c)) = 0\,.$$

1.8.4 Skoda's Theorem

A theorem that will play a central role in our work is the so-called Skoda's Theorem.

Theorem 1.8.10 (Skoda). *Let X be a two-dimensional complex algebraic variety and $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal on X.*

i) Given $m \geqslant 2$ one has that:

$$\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1})\,.$$

Iterating this process, one get:

$$\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}^{m-1}\mathcal{J}(\mathfrak{a})$$

for $m \geqslant 2$.

ii) Let $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$, $\mathfrak{b} \neq (0)$ and $c \in \mathbb{Q}_{>0}$. Then:

$$\mathcal{J}(\mathfrak{a}^m \mathfrak{b}^c) = \mathfrak{a}^{m-1}\mathcal{J}(\mathfrak{a}\mathfrak{b}^c)$$

for $m \geqslant 2$.

Remark 1.8.11. We have as a consequence of this theorem the following.

i) $\mathcal{J}(\mathfrak{a}^m) \subseteq \mathfrak{a}^{m-1}$ for all $m \geqslant 2$.

ii) The second part can be generalized: given $\mathfrak{b}_1, ..., \mathfrak{b}_t \subseteq \mathcal{O}_{X,O}$ and $c_1, ..., c_t \in Q_{\geqslant 0}$, not all zero, then

$$\mathcal{J}(\mathfrak{a}^m \cdot \mathfrak{b}_1^{c_1} \cdots \mathfrak{b}_t^{c_t}) = \mathfrak{a}^{m-1}\mathcal{J}(\mathfrak{a} \cdot \mathfrak{b}_1^{c_1} \cdots \mathfrak{b}_t^{c_t}) .$$

This theorem can also be stated using, instead of the dimension, the number of generators of an ideal.

Theorem 1.8.12 (Skoda)**.** *Let X be a two-dimensional complex algebraic variety and $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal on X generated by r elements. Then, given $m \geqslant r$, one has that:*

$$\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{m-1})$$

for $m \geqslant r$.

In particular, for the case of principal ideals, one has that $\mathcal{J}\left(\mathfrak{a}^c\right) = \mathfrak{a} \cdot \mathcal{J}\left(\mathfrak{a}^{c-1}\right)$ for all $c \geqslant 1$.

1.9 Properties

We introduce some further properties of multiplier ideals.

1.9.1 Restriction Theorem

The first result explains the behavior of the multiplier ideals if we change the ambient space.

Theorem 1.9.1 (Restriction Theorem)**.** *Let $Y \subseteq X$ be a smooth subvariety of X and $\mathfrak{a}$ an ideal of $\mathcal{O}_{X,O}$ such that Y is not contained in the zero locus of $\mathfrak{a}$. Then*

$$\mathcal{J}\left(Y, (\mathfrak{a} \cdot \mathcal{O}_Y)^c\right) \subseteq \mathcal{J}(X, \mathfrak{a}^c) \cdot \mathcal{O}_Y ,$$

with equality if Y satisfies some transversality conditions (see [DMST06]).

1.9.2 Subadditivity formula

The Restriction Theorem is used to prove the following result due to Demailly, Ein and Lazarsfeld (see [DEL00]).

Theorem 1.9.2 (Subadditivity)**.** *Let* $\mathfrak{a}, \mathfrak{b}$ *be ideals in* $\mathcal{O}_{X,O}$ *and* c*,* $d > 0$ *two rational numbers. Then we have*

$$\mathcal{J}(\mathfrak{a}^c \cdot \mathfrak{b}^d) \subseteq \mathcal{J}(\mathfrak{a}^c) \cdot \mathcal{J}(\mathfrak{b}^d)\,.$$

In particular, for $m \in \mathbb{N}$ *we have*

$$\mathcal{J}(\mathfrak{a}^{cm}) = \mathcal{J}(\mathfrak{a}^c)^m\,.$$

The main idea of the proof is to pull back the data to $X \times X$ and restrict to the diagonal (see [DEL00] and also [BL04]). We can also consider the multiplier ideal associated to the intersection of two ideals.

Lemma 1.9.3 (Intersections of ideals)**.** *Let* $\mathfrak{a}, \mathfrak{b}$ *ideals in* $\mathcal{O}_{X,O}$*. Then we have*

$$\mathcal{J}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{J}(\mathfrak{a}) \cap \mathcal{J}(\mathfrak{b})\,.$$

1.9.3 Summation formula

The Subadditivity Formula gives a relation between the multiplier ideal of the product of two ideals and the product of the multiplier ideals associated to each ideal. Mustaţă (see [Mus02]) presented a similar statement for the case when we consider the sum of two ideals.

Theorem 1.9.4 (Mustaţă)**.** *Let* X *be a two-dimensional smooth complex algebraic variety and* $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_{X,O}$ *and* $c > 0$ *a rational number. Then*

$$\mathcal{J}\left((\mathfrak{a} + \mathfrak{b})^c\right) \subseteq \sum_{c_1+c_2=c} \mathcal{J}(\mathfrak{a}^{c_1})\mathcal{J}\left(\mathfrak{a}^{c_2}\right)\,.$$

Takagi in [Tak06] gave a refinement of this result.

Theorem 1.9.5 (Takagi)**.** *Let* X *be a two-dimensional smooth complex algebraic variety and* $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_{X,O}$ *be two ideals, and let* $c > 0$ *be a rational number. Then*

$$\mathcal{J}((\mathfrak{a} + \mathfrak{b})^c) = \sum_{c_1+c_2=c} \mathcal{J}(\mathfrak{a}^{c_1}\mathfrak{a}^{c_2})\,.$$

This result was improved by Jow and Miller in [JM08].

Theorem 1.9.6 (Jow and Miller). *Let X be a two-dimensional smooth complex algebraic variety and $\mathfrak{a}_1, ..., \mathfrak{a}_r, \mathfrak{b} \subseteq \mathcal{O}_{X,O}$ be ideals and $c, d > 0$ two rational numbers. Then,*

$$\mathcal{J}\left((\mathfrak{a}_1 + ... + \mathfrak{a}_r)^c \mathfrak{b}^d\right) = \sum_{c_1+...+c_r=c} \mathcal{J}\left(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r} \mathfrak{b}^d\right) .$$

1.10 Invariants

Multiplier ideals come with an attached set of invariants that were studied systematically by Ein-Lazarsfeld-Smith-Varolin in [ELSV04]. Clearly

$$\lceil K_\pi - cF \rceil \geqslant \lceil K_\pi - (c+\varepsilon) F \rceil$$

for any $\varepsilon > 0$, with equality if ε is small enough. Therefore the multiplier ideals form a discrete nested sequence of ideals

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq ... \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq ...$$

indexed by an increasing sequence of rational numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ such that for any $c \in [\lambda_i, \lambda_{i+1})$ one has

$$\mathcal{J}(\mathfrak{a}^{\lambda_i}) = \mathcal{J}(\mathfrak{a}^{c}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}}).$$

The λ_i are the so-called *jumping numbers* of the ideal $\mathfrak{a}$, and the first jumping number $\lambda_1 = \mathrm{lct}(\mathfrak{a})$ is the *log-canonical threshold* of $\mathfrak{a}$.

A first result concerning jumping numbers is a direct consequence of Skoda's Theorem (see Subsection 1.8.4).

Proposition 1.10.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal. Then $\lambda > n-1$ is a jumping number of $\mathfrak{a}$ if and only if $\lambda + 1$ is a jumping number of $\mathfrak{a}$.*

Even with this restriction, the distance between two jumping numbers cannot has an upper bound, as the following resultr eflects.

Proposition 1.10.2. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal. Then,*

$$\lambda_{i+1} \leqslant \lambda_1 + \lambda_i$$

for every $i \geqslant 1$, where λ_i is the i-th jumping number.

Finally, as a corollary of the Mustaţă Summation Formula (see Theorem 1.9.4), we get the following inequality when considering the log canonical threshold of a sum of ideals.

Corollary 1.10.3. *Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_{X,O}$ be two ideal sheaves. Then,*

$$\lambda_1(\mathfrak{a}+\mathfrak{b}) \leqslant \lambda_1(\mathfrak{a}) + \lambda_1(\mathfrak{b})\,.$$

1.10.1 Jumping length

If we restrict to the case of hypersurfaces, one can find as a consequence of the version of Theorem 1.8.12 that in this case 1 is always a jumping number.

Lemma 1.10.4. *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a non-zero polynomial. Then $\mathcal{J}((f)) = (f)$, but for $c < 1$ we have that $(f) \subsetneq \mathcal{J}((f)^c)$. Thus $c = 1$ is a jumping number.*

This allows us to define the jumping length.

Definition 1.10.5. The jumping length is the number of jumping numbers in the interval $(0, 1]$. We denote this invariant by $\ell(f)$.

We can give an upper bound for the jumping length due to the following inclusion of the Jacobian ideal:

$$\left(f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) \subseteq \mathcal{J}((f)^{1-\varepsilon})\,,$$

for a sufficiently small $\varepsilon > 0$, proved in [ELSV04].

Proposition 1.10.6. *Assume that f has at worst an isolated singularity at $O \in X$. Then*

$$\ell(f) \leqslant \tau(f, x) + 1\,,$$

where $\tau(f, x)$ is the Tjurina number of f at x, that is, the codimension of $\left(f, \frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right)$ in $\mathcal{O}_{X,x}$.

Budur pointed out (see [ELSV04, Remark 3.10]) that, given the relation of multiplier ideals with other invariants of the singularity, we have the following result.

Proposition 1.10.7. *Assume that f has at worst an isolated singularity at $x \in X$. Then*

$$\ell(f) \leqslant \frac{\mu}{2} + 1$$

where $\mu = \mu(f, x)$ is the Milnor number of f at x, that is the codimension

$$\mu := \dim_{\mathbb{C}} \frac{\mathcal{O}_X X}{\left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right)}\,.$$

1.10.2 Jumping multiplicity

We have that λ is a jumping number if and only if $\mathcal{J}(\mathfrak{a}^{\lambda}) \subsetneq \mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon})$. However, this does not imply that the codimension between these two ideals is one. The jumping multiplicity measures the codimension between these two ideals.

Definition 1.10.8. Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal. We define the multiplicity attached to a point $c \in \mathbb{R}_{\geqslant 0}$ as the codimension of $\mathcal{J}(\mathfrak{a}^{c})$ in $\mathcal{J}(\mathfrak{a}^{(c-\varepsilon)})$ for $\varepsilon > 0$ small enough. We denote it as

$$m(c) := \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^{c})}\,.$$

Remark 1.10.9. If $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ is an $\mathfrak{m}$-primary ideal, then $m(c) < \infty$.

In order to gather all the information given by all jumping numbers and their corresponding multiplicities, Galindo-Monserrat [GM10] introduced the so-called *Poincaré series of multiplier ideals* associated to $\mathfrak{a}$ as the series with fractional exponents

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^{c}\,.$$

1.10.3 Invariants of mixed multiplier ideals

Before introducing a generalization of the notion of jumping numbers to the case on mixed multiplier ideals, we need to introduce the following definition.

Definition 1.10.10. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. Then, for each $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, we define:

- The *region* of $\boldsymbol{\lambda}$:
$$\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\mathfrak{a}_1^{\lambda'_1} \cdots \mathfrak{a}_r^{\lambda'_r}\right) \supseteq \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right) \right\}$$

- The *constancy region* of $\boldsymbol{\lambda}$:
$$\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\mathfrak{a}_1^{\lambda'_1} \cdots \mathfrak{a}_r^{\lambda'_r}\right) = \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right) \right\}$$

Reading this definition in the case of multiplier ideals, means that for any $c \in [\lambda_i, \lambda_{i+1})$ it holds

$$\mathcal{J}(\mathfrak{a}^{\lambda_i}) = \mathcal{J}(\mathfrak{a}^{c}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}}).$$

Therefore, the region and the constancy region of c are $\mathcal{R}_{\mathfrak{a}}(c) = [\lambda_0, \lambda_{i+1})$ and $\mathcal{C}_{\mathfrak{a}}(c) = [\lambda_i, \lambda_{i+1})$ respectively.

Thus the analog of jumping numbers for mixed multiplier ideals is to consider the boundary of the regions. Then it is natural to give the following definition.

Definition 1.10.11. Let $\mathfrak{a} := (\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. The *jumping wall* associated to $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is the boundary of the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. One usually refers to the jumping wall of the origin as the *log-canonical wall.*

1.11 State of the art

Multiplier ideals on a surface X have been widely studied, especially in the case X smooth. For example, we have results from Smith-Thompson [ST07], Favre-Jonsson [FJ05], Järvilehto [Jär11], Hyry-Järvilehto [HJ11], Naie [Nai09], Tucker [Tuc09, Tuc10], Galindo-Monserrat [GM10], Galindo-Hernando-Monserrat [GHM16], Kuwata [Kuw99], to cite some of them. Mixed multiplier ideals, have not been that much studied, and most of the times for the relations with other invariants (see Libgober-Mustaţă in [LM11], Cassou-Noguès and Libgober[CNL11, CNL14] and Naie [Nai13]).

1.11.1 Case of simple ideals and irreducible plane curves

For the case of simple ideals and irreducible plane curves over smooth varieties, there are two main results. The first known formula was given by Järvilehto [Jär11], and there is also another result of Naie [Nai09] in terms of the Enriques diagram.

Before introducing the results, we need some notation. Let $\left\{\frac{m_1}{n}, ..., \frac{m_k}{n}\right\}$ be the characteristic exponents of an irreducible curve (see [CA00, Section 5.8]). We define the intersection multiplicities as:

$$\breve{m}_i = \sum_{j=1}^{i-1} \frac{(n^{j-1} - n^j)m_j}{n^{i-1}} + m_i \text{ for } i = 1, ..., k\,.$$

If $\Sigma(C)$ is the semigroup of the singularity (see [CA00, Section 5.8]), one can prove that the set $n, \breve{m}_1, ..., \breve{m}_k$ is a minimal set of generators of $\Sigma(C)$.

The formula given by Järvilehto can be expressed in terms of the semigroup.

Theorem 1.11.1. [Jär11, Theorem 6.2] *Let $\mathfrak{a}$ be a simple complete $\mathfrak{m}$-primary ideal and consider $\left\{\frac{m_1}{n},...,\frac{m_k}{n}\right\}$ be the characteristic exponents of a generic curve of the ideal. Then the jumping numbers of C are given by the union of the sets*

$$\mathcal{H}_i = \left\{ \frac{r+1}{n^{i-1}} + \frac{s+1}{\breve{m}_i} + \frac{m}{n^i} \middle| r,s,m, \in \mathbb{N} \text{ with } \frac{r+1}{n^{i-1}} + \frac{s+1}{\breve{m}_i} \leqslant \frac{1}{n^i} \right\}$$

where $n^i = \gcd(n, \breve{m}_1, ..., \breve{m}_i)$ for $i = 1, ..., k$, and all of the jumping numbers in $\mathcal{H}_i$ are (critically) contributed by E_i, which is a rupture divisor different from the origin.

If we consider an ideal with only one characteristic exponent, we have the following result.

Corollary 1.11.2. [Jär11, Corollary 6.3] *Let $\mathfrak{a}$ be a simple complete $\mathfrak{m}$-primary ideal such that any generic curve of $\mathfrak{a}$ has only one characteristic exponent. Then the set of jumping numbers of $\mathfrak{a}$ are*

$$\left\{ \frac{r+1}{n} + \frac{s+1}{\breve{m}_1} + m \,|r,s \in \mathbb{N} \right\},$$

and all of the jumping numbers are (critically) contributed by E_i, which is the last rupture divisor.

Naie in [Nai09] prsents two formulas to compute the jumping numbers. One is given in terms of $\Sigma(C)$, the semigroup of the singularity.

Theorem 1.11.3. [Nai09, Theorem 3.1] *Let C be an irreducible curve at O. Let $n, \breve{m}_1, ..., \breve{m}_k$ be a minimal set of generators of $\Sigma(C)$. Then the jumping numbers of the multiplier ideal of C in $(0,1]$ are given by*

$$\bigcup_{j=1}^{k} \frac{1}{[n^{j-1}, \breve{m}_j]} R^{n^j} \left(\frac{n^{j-1}}{n^j}, \frac{\breve{m}_j}{n^j} \right),$$

where

$$R^{\ell}(x,y) = \bigcup_{k=0}^{\ell-1} \left(kxy + \{ax + by | a, b \in \mathbb{N}, \quad ax + by < xy\} \right),$$

$n^0 = n$, $n^j = \gcd(n^{j-1}, \breve{m}_j)$ for any $j \geqslant 0$, and $[n^{j-1}, \breve{m}_j]$ denotes the least common multiple of two integers.

Naie also presents this result using the notion of *connected sum* of Enriques diagrams to reduce to the case of curves of the form $x^r - y^q = 0$ (see Theorem 2.3 in [Nai09]).

1.11.2 General case

Tucker [Tuc10] considers the case where X has at worst a rational singularity and refines Smith's and Thompson's notion of contribution (see [ST07]). In order to introduce this notion, we begin by observing that the jumps between multiplier ideals necessarily must occur at rational numbers $c \in \mathbb{Q}$ which cause the strict inclusion of divisors

$$\lceil K_\pi - cF \rceil < \lceil K_\pi - (c - \varepsilon) F \rceil$$

for any ε. If we take a close look at $F = F_{\text{exc}} + F_{\text{aff}}$ these rational numbers must belong to the set of *candidate jumping numbers*

$$\left\{ \frac{k_i + m}{e_i} \mid m \in \mathbb{Z}_{>0} \right\},$$

where $F = \sum e_i E_i$ and $K_\pi = \sum k_i E_i$ Notice that for non-exceptional components $E_i \leqslant F_{\text{aff}}$ we have $k_i = 0$, and their corresponding candidates $\left\{ \frac{m}{e_i} \mid m \in \mathbb{Z}_{>0} \right\}$ are indeed jumping numbers.

It is easy to check that not every candidate jumping number (coming from the exceptional part) is necessarily a jumping number. To separate the wheat from the chaff, Tucker [Tuc10] developed the notion of *divisor that contributes* to a jumping number, building upon previous work by Smith-Thompson [ST07].

Definition 1.11.4. A positive rational number λ is a *candidate jumping number* for a reduced divisor $G \leqslant F$ if it satisfies $\lambda e_i - k_i \in \mathbb{Z}_{>0}$ for any component $E_i \leqslant G$.

Definition 1.11.5. [Tuc10, Definition 3.1] (compare with [ST07, Definition 5]) A reduced divisor $G \leqslant F$ for which λ is a candidate jumping number is said to *contribute* to λ if

$$\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^\lambda)$$

Moreover, this contribution is *critical* if for any divisor $0 \leqslant G' < G$ we have

$$\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G') = \mathcal{J}(\mathfrak{a}^\lambda).$$

Most often we will simply say that G is just a *contributing* or a *critical divisor* associated to λ. Critical divisors define complete ideals very close to a multiplier ideal in a precise sense that will be explained in the forthcoming Corollary 3.4.5 in Section §3.4. One may identify critical divisors with exceptional support through the following numerical characterization.

Theorem 1.11.6. [Tuc10, Theorem 4.3] *Let λ be a candidate jumping number for a reduced divisor $G \in \Lambda$ with connected support.*

- *If $G = E_i$ is prime, then E_i is a critical divisor for λ if and only if*

$$(\lceil K_\pi - \lambda F \rceil + E_i) \cdot E_i \geqslant 0.$$

- *If G is reducible, then G is a critical divisor for λ if and only if*

$$(\lceil K_\pi - \lambda F \rceil + G) \cdot E_i = 0$$

  for all divisors E_i in the support of G.

This result is a consequence of a more general statement about the following cohomological characterization.

Proposition 1.11.7. [Tuc10, Proposition 4.1] *Let G be a reduced divisor, let λ be a candidate jumping number for G. Then λ is a jumping number of $\mathfrak{a}$ contributed by G if and only if*

$$H_0(G, (\lceil K_\pi - \lambda F \rceil + G)|_G) \neq 0.$$

Moreover, the contribution is critical if and only if we have

$$H_0(G, (\lceil K_\pi - \lambda F \rceil + G')|_{G'}) = 0$$

for all G' on X' as that are proper sub-divisors of G $(0 \preccurlyeq G' \prec G)$.

Moreover, critical divisors with exceptional support satisfy a nice geometric property when viewed in the dual graph.

Theorem 1.11.8. [Tuc10, Corollary 4.2 & Theorem 5.1] *Let G be a critical divisor for a jumping number λ. Then G is a connected chain in the dual graph of the log-resolution of $\mathfrak{a}$ whose ends must be either rupture or dicritical divisors.*

Using all these properties, Tucker provides a simple algorithm to compute the set of all jumping numbers (see [Tuc10, Section 5.5]). It boils down to the following steps.

Algorithm 1.11.9. [Tuc10, Section 6]

`Input:` *a log-resolution of an ideal* $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$.

`Output:` *the list of Jumping Numbers of* $\mathfrak{a}$.

- **Jumping number**
  - *Compute the candidate jumping numbers for* F_{exc}.
  - *Find all possible critical divisors using Theorem 1.11.8.*
  - *Find which candidate jumping numbers can be realized as jumping number associated to these critical divisors using Theorem 1.11.6.*
  - *Plug in those jumping numbers coming from* F_{aff}.

1.11.3 Mixed multiplier ideals

Mixed multiplier ideals have not received that much attention as the multiplier ideals. Mostly, they have been specially studied for their connections to other invariants. Libgober and Mustaţă in [LM11] investigated the properties of the constancy region of the origin, what they call the LCT-polytope. In their paper, they present some properties of this region and proved that this LCT-polytope satisfy a strong form of the ascending chain condition (see [LM11, Theorem 3.3]).

Naie in [Nai13] uses the mixed multiplier ideals to establish a formula for the irregularity of abelian coverings of smooth projective surfaces. He pointed out that the jumping walls of a tuple of ideals $\boldsymbol{\mathfrak{a}}$ are supported on hyperplanes of the form

$$\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} = \ell$$

corresponding to either rupture or dicritical divisors E_j (see Proposition 2.2 in loc. cit.). Naie defines a set of values ℓ, such that the hyperplane $\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} = \ell$ supports a jumping wall (see Proposition 2.7 in loc. cit.).

Cassou-Noguès and Libgober study in [CNL11, CNL14] an analogous notion to the mixed multiplier ideals, the ideals of quasiadjunction. In their work, they consider the mixed multiplier ideal associated to irreducible plane curves. They are able to characterize (see Theorem 4.1 in [CNL14]) the polytope of quasi-adjunction associated to a germ of curve, or equivalently, the region of a given point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, and in particular the corresponding jumping wall.

Theorem 1.11.10. [CNL14, Theorem 4.1] *Given a tuple of simple ideals* $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$*, a log-resolution* $\pi : X' \to X$ *of this tuple and* F_i *the divisors such that* $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ *for* $1 \leqslant i \leqslant r$.

- *Let* ϕ *be a germ of a plane curve, then the region associated to* ϕ *is defined by the set of inequalities*

$$e_{1,i}z_1 + \cdots + e_{r,i}z_r < k_i + 1 + e_i(\phi)$$

  for all i *such that* E_i *is a rupture or dicritical divisor, where* $e_i(\phi)$ *is the multiplicity of* ϕ *in* E_i.

- *For all* i *such that* E_i *is a rupture or dicritical divisor there exists* ϕ *and a face of the jumping wall associated to* ϕ *supported on a hyperplane of the form*

$$e_{1,i}z_1 + \cdots + e_{r,i}z_r = k_i + 1 + e_i(\phi)$$

In [CNL11], they describe some other methods for computing of the regions (see Proposition 2.2 and (2.3) in loc. cit.) and also the relations with other invariants such as the mixed Hodge structure or the Bernstein-Sato ideals.

In [CNL14], Cassou-Noguès and Libgober provide a characterization of the log-canonical wall (see Theorem 4.22 in loc. cit.). They are able to give a characterization of the hyperplanes where the log-canonical wall is supported (see Definition 4.14 and Theorem 4.22 in loc. cit.). They finish their paper presenting an example where the ascending chain condition is not satisfied for a jumping wall different from the log-canonical wall.

Chapter 2

An algorithm to compute jumping numbers and multiplier ideals

Let X be a complex surface with at worst a rational singularity and $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ an ideal. The aim of this chapter is to compute the jumping numbers and their corresponding multiplier ideals of any given ideal $\mathcal{O}_{X,O}$. To such purpose, we fix a log-resolution $\pi : X' \longrightarrow X$ of our ideal $\mathfrak{a}$. The main ingredients we will have to deal with are the relative canonical divisor $K_\pi = \sum_{i=1}^r k_i E_i \in \Lambda_{\mathbb{Q}}$, and the divisor $F \in \mathrm{Div}(X')$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. Recall that we have a decomposition

$$F = F_{\mathrm{exc}} + F_{\mathrm{aff}} = \sum_{i=1}^{r} e_i E_i + \sum_{i=r+1}^{s} e_i E_i$$

in terms of its exceptional and affine support.

We will provide a very simple algorithm that allows to construct sequentially the chain of multiplier ideals[1]

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \dots$$

[1] In fact, we can compute the chain inside any desired fixed range $[c, c'] \subseteq \mathbb{R}$:

$$\mathcal{J}(\mathfrak{a}^{c}) = \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_r}) = \mathcal{J}(\mathfrak{a}^{c'}) .$$

When X is a smooth surface, or even when X has a log-terminal singularity at O, the multiplier ideal associated to $\lambda_0 = 0$ is the whole ring, i.e. $\mathcal{O}_{X,O} = \mathcal{J}(\mathfrak{a}^{\lambda_0})$. In general, when X has a rational singularity we may have an strict inclusion $\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_0})$. The starting point of our method will be describing this ideal by means of the antinef closure $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ of $\lfloor -K_\pi \rfloor$ that we compute using the unloading procedure described in Section 1.7.

As a consequence of our main result (see Theorem 2.1.5), the log-canonical threshold satisfies the following formula[2]

$$\lambda_1 = \mathrm{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_0}}{e_i} \right\}. \tag{2.0.1}$$

Then we describe its associated multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_1})$ just computing the antinef closure D_{λ_1} of $\lfloor \lambda_1 F - K_\pi \rfloor$ using the unloading procedure. Once we have the divisor D_{λ_1}, we use an extension of Formula 2.0.1 given by Theorem 2.1.5, that computes the next jumping number λ_2. Then we only have to follow the same strategy: the antinef closure D_{λ_2} of $\lfloor \lambda_2 F - K_\pi \rfloor$, i.e. the multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_2})$, will allow us to compute λ_3 and so on.

The main idea behind our method is a simple comparison between complete ideals. Whenever we have two antinef divisors it is easy to check whether their corresponding complete ideals satisfy a strict inclusion (see Proposition 1.6.4). To compare the ideals associated to an antinef and a non-antinef divisor is more subtle and this is the situation that we will have to deal with in this section.

The results presented in this chapter correspond to [AADG14, Section 3].

2.1 Some technical results

To address this problem we will need some preliminary technical results.

Lemma 2.1.1. *Let D_1, D_2 be two divisors in $\mathrm{Div}(X')$ such that $D_1 \leqslant D_2$. Then, they have the same antinef closure $\widetilde{D_1} = \widetilde{D_2}$ if and only if $\widetilde{D_1} \geqslant D_2$.*

Proof. Recall that, by Proposition 1.6.5, we already have $\widetilde{D_1} \leqslant \widetilde{D_2}$ just because $D_1 \leqslant D_2$.

[2] When X is smooth, or even when it has log-terminal singularities, we have $D_{\lambda_0} = 0$ so one recovers the well-known formula for the log-canonical threshold.

Assume $\widetilde{D_1} \geqslant D_2$ then, by the definition of antinef closure (see Lemma 1.6.2), we also have $\widetilde{D_1} \geqslant \widetilde{D_2} \geqslant D_2$ and thus $\widetilde{D_1} = \widetilde{D_2}$. On the other hand, assume that $\widetilde{D_1} = \widetilde{D_2}$. Then, since the antinef closure of a divisor always contains it, we have $\widetilde{D_1} = \widetilde{D_2} \geqslant D_2$ as desired. □

Corollary 2.1.2. *Let D_1, D_2 be two divisors in* $\mathrm{Div}(X')$ *such that $D_1 \leqslant D_2$. Then, $\widetilde{D_1} < \widetilde{D_2}$ if and only if $v_i(\widetilde{D_1}) < v_i(D_2)$ for some E_i.*

Proof. As $D_1 \leqslant D_2$, the inclusion $\widetilde{D_1} \leqslant \widetilde{D_2}$ also holds. The result then follows from Lemma 2.1.1. □

Translated into the language of complete ideals, these results give a characterization of the jump between two nested ideals, which will be a key ingredient in the proof of our results.

Proposition 2.1.3. *Let D_1, D_2 be two divisors in* $\mathrm{Div}(X')$ *such that $D_1 \leqslant D_2$. Then:*

i) $\pi_* \mathcal{O}_{X'}(-D_1) = \pi_* \mathcal{O}_{X'}(-D_2)$ if and only if $\widetilde{D_1} \geqslant D_2$.

ii) $\pi_* \mathcal{O}_{X'}(-D_1) \supsetneq \pi_* \mathcal{O}_{X'}(-D_2)$ if and only if $v_i(\widetilde{D_1}) < v_i(D_2)$ for some E_i.

For convenience we also present this result in the form we will most commonly use it.

Corollary 2.1.4. *Let $\lambda' < \lambda$ be rational numbers. Let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$ be the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$. Then:*

i) $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \mathcal{J}(\mathfrak{a}^{\lambda})$ if and only if $\lfloor \lambda e_i - k_i \rfloor \leqslant e_i^{\lambda'}$ for all E_i.

ii) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ if and only if $\lfloor \lambda e_i - k_i \rfloor > e_i^{\lambda'}$ for some E_i.

With the technical tools stated above we are ready for the main result of this section.

Theorem 2.1.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal and let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$ be the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$ for a given $\lambda' \in \mathbb{Q}_{>0}$. Then,*

$$\lambda = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\}$$

is the jumping number consecutive to λ'.

Proof. Let us check first that $\lambda' < \lambda$. Indeed, by the definition of antinef closure, the integers $e_i^{\lambda'}$ satisfy $\lfloor \lambda' e_i - k_i \rfloor \leqslant e_i^{\lambda'}$ for any E_i, and hence:

$$\lambda' < \frac{k_i + 1 + e_i^{\lambda'}}{e_i}.$$

Thus, we have an inclusion of ideals $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supseteq \mathcal{J}(\mathfrak{a}^{\lambda})$. Notice that for those divisors E_i where the minimum is achieved we have

$$\lfloor \lambda e_i - k_i \rfloor = 1 + e_i^{\lambda'} > e_i^{\lambda'}$$

so the above inclusion of ideals is strict by Corollary 2.1.4. To conclude that λ is the jumping number immediately after λ', we have to show that for any $c \in \mathbb{R}$ with $\lambda' \leqslant c < \lambda$ we have $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \mathcal{J}(\mathfrak{a}^{c})$. Suppose the contrary, i.e., $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^{c})$. By Corollary 2.1.4, this c should satisfy $\lfloor \lambda e_i - k_i \rfloor > e_i^{\lambda'}$ or equivalently $c \geqslant \frac{k_i+1+e_i^{\lambda'}}{e_i}$ for some E_i, and this contradicts the fact that λ is the minimum of these rational numbers. □

The above result for the case $\lambda' = 0$ gives a mild generalization of the well-known formula for the log-canonical threshold in the smooth case. We point out that the antinef closure of $\lfloor -K_\pi \rfloor$ is 0 whenever X is smooth or, more generally, when it has log-terminal singularities.

Corollary 2.1.6. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal. Let $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ be the antinef closure of $\lfloor -K_\pi \rfloor$. Then,*

$$\operatorname{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_0}}{e_i} \right\}.$$

Another easy application of the results above is the following result that should be well-known to experts.

Corollary 2.1.7. *Let λ_1 be the log-canonical threshold of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and assume that X has at most a log-terminal singularity at O. Then $\mathcal{J}(\mathfrak{a}^{\lambda_1}) = \mathfrak{m}$.*

Proof. Since X has at most a log-terminal singularity, the log-canonical threshold is

$$\operatorname{lct}(\mathfrak{a}) = \lambda_1 = \min_i \left\{ \frac{k_i + 1}{e_i} \right\}$$

so it satisfies $\lambda_1 \leqslant \frac{k_i+1}{e_i}$ for any divisor E_i and equality is achieved at least for a given divisor. In particular, for all E_i we have

$$\lfloor \lambda_1 e_i - k_i \rfloor \leqslant 1.$$

It follows from Proposition 1.6.5 that $\mathfrak{m} \subseteq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \subsetneq \mathcal{O}_{X,O}$ and we get the desired result. $\square$

For non log-terminal singularities we may find examples where the codimension as $\mathbb{C}$-vector spaces of $\mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1})$ might be bigger than 1 (see Example 2.2.3).

2.2 An algorithm to compute jumping numbers and multiplier ideals

Combining Theorem 2.1.5 and the unloading procedure described in Section 1.7 we can describe a very simple algorithm that allows us to compute the chain of multiplier ideals:

Algorithm 2.2.1. *(Jumping Numbers and Multiplier Ideals)*

`Input:` *A log-resolution of an ideal* $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$.

`Output:` *List of Jumping Numbers of* $\mathfrak{a}$ *and its corresponding Multiplier Ideals.*

Set $\lambda_0 = 0$ *and compute the antinef closure* $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ *of* $\lfloor -K_\pi \rfloor$ *using the unloading procedure. From* $j = 1$ *, incrementing by* 1

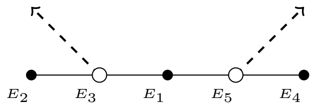
(Step j) · **Jumping number***: Compute*

$$\lambda_j = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_{j-1}}}{e_i} \right\}.$$

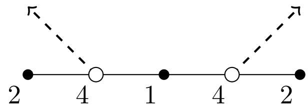
· **Multiplier ideal***: Compute the antinef closure* $D_{\lambda_j} = \sum e_i^{\lambda_j} E_i$ *of* $\lfloor \lambda_j F - K_\pi \rfloor$ *using the unloading procedure.*

Notice that we may also find all the multiplier ideals in any given interval $[c', c]$ of the real line. In this case, our starting point would be computing the antinef closure $D_{c'}$ of $\lfloor c'F - K_\pi \rfloor$. To illustrate this method we consider an easy example in a smooth variety.

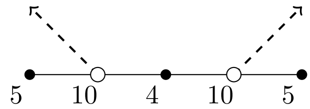
Example 2.2.2. Consider the ideal $\mathfrak{a} = (x^2y^2, x^5, y^5, xy^4, x^4y) \subseteq \mathbb{C}\{x, y\}$. We represent the relative canonical divisor K_π and the divisor F in the dual graph as follows:



Vertex ordering



K_π



F

The blank dots correspond to dicritical divisors and their excesses are represented by broken arrows[3]. For simplicity we will collect the values of any divisor in a vector. To start with we have $K_\pi = (1,2,4,2,4)$ and $F = (4,5,10,5,10)$. In the algorithm we will have to perform some unloading steps so we will have to consider the intersection matrix $M = (E_i \cdot E_j)_{1\leqslant i,j \leqslant 5}$

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The algorithm is performed as follows:

- We start computing the log-canonical threshold:

$$\lambda_1 = \mathrm{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i+1}{e_i} \right\} = \min_i \left\{ \frac{2}{4}, \frac{3}{5}, \frac{5}{10}, \frac{3}{5}, \frac{5}{10} \right\} = \frac{1}{2}.$$

The divisor $\left\lfloor \frac{1}{2}F - K_\pi \right\rfloor = (1,0,1,0,1)$ is not antinef since it has excess -1 at E_2 and E_4. The first unloading step is to consider the divisor

$$\left\lfloor \frac{1}{2}F - K_\pi \right\rfloor + E_2 + E_4 = (1,1,1,1,1)\,.$$

This divisor has excess -1 at E_3 and E_5 so we need to perform a second unloading step to obtain the antinef closure $D_{\lambda_1} = (1,1,2,1,2)$.

[3]The broken arrows also represent the branches of the strict transform of a curve defined by a generic $f \in \mathfrak{a}$.

- The second Jumping Number is:

$$\lambda_2 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_1}}{e_i} \right\} = \min_i \left\{ \frac{2+1}{4}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{3+1}{5}, \frac{5+2}{10} \right\} = \frac{7}{10}.$$

Then we get $\left\lfloor \frac{7}{10} F - K_\pi \right\rfloor = (1,1,3,1,3)$. It has excess -1 at E_1, E_2 and E_4 and we obtain the divisor $(2,2,3,2,3)$ after the first unloading step. This divisor has excess -1 at E_3 and E_5 and, after a second unloading step, we obtain the antinef closure $D_{\lambda_2} = (2,2,4,2,4)$.

- The third Jumping Number is:

$$\lambda_3 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_2}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{3+2}{5}, \frac{5+4}{10} \right\} = \frac{9}{10}.$$

Then we get $\left\lfloor \frac{9}{10} F - K_\pi \right\rfloor = (2,2,5,2,5)$ that has excess -1 at E_3 and E_5. After a single unloading step we get the antinef closure $D_{\lambda_3} = (2,3,5,3,5)$.

- The fourth Jumping Number is:

$$\lambda_4 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_3}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{3+3}{5}, \frac{5+5}{10} \right\} = 1.$$

Then we get $\lfloor F - K_\pi \rfloor = D_{\lambda_4} = (3,3,6,3,6)$ since this divisor is antinef.

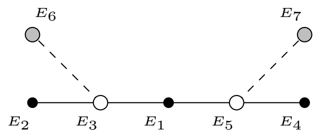
- The fifth Jumping Number is:

$$\lambda_5 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_4}}{e_i} \right\} = \min_i \left\{ \frac{2+3}{4}, \frac{3+3}{5}, \frac{5+6}{10}, \frac{3+3}{5}, \frac{5+6}{10} \right\} = \frac{11}{10}.$$

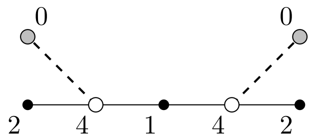
Then we get $\left\lfloor \frac{11}{10} F - K_\pi \right\rfloor = (3,3,7,3,7)$ and, after a single unloading step, we obtain the antinef closure $D_{\lambda_5} = (3,4,7,4,7)$.

Now we will compute the chain of multiplier ideals of the plane curve defined by $f = (x^2 - y^3)(y^2 - x^3) \in \mathbb{C}\{x,y\}$. The product of two cusps sharing the origin O is a generic element of the ideal $\mathfrak{a} = (x^2y^2, x^5, y^5, xy^4, x^4y)$ considered above, so $\mathcal{J}(f^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ for $\lambda < 1$. This example will illustrate how the non-exceptional components affect the unloading procedure and, consequently, the list of jumping numbers for $\lambda > 1$.

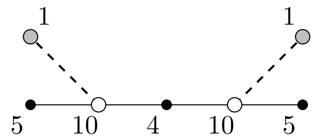
Denote the total transform of the curve defined by f simply as F. We represent the relative canonical divisor K_π and the divisor F in the dual graph as follows:



Vertex ordering



K_π



F

The gray dots will represent here the affine components belonging to the strict transform of the curve. The intersection matrix is now

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

The algorithm is performed as follows:

- The log-canonical threshold is:

$$\lambda_1 = \operatorname{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1}{e_i} \right\} = \min_i \left\{ \frac{2}{4}, \frac{3}{5}, \frac{5}{10}, \frac{3}{5}, \frac{5}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{1}{2}.$$

We get $\left\lfloor \frac{1}{2} F - K_\pi \right\rfloor = (1,0,1,0,1,0,0)$ and, as in the previous example, its antinef closure is $D_{\lambda_1} = (1,1,2,1,2,0,0)$.

- The second Jumping Number is:

$$\lambda_2 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_1}}{e_i} \right\}$$

$$= \min_i \left\{ \frac{2+1}{4}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{7}{10}.$$

Then we get $\lfloor \frac{7}{10} F - K_\pi \rfloor = (1,1,3,1,3,0,0)$ and its antinef closure is $D_{\lambda_2} = (2,2,4,2,4,0,0)$.

- The third Jumping Number is:

$$\lambda_3 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_2}}{e_i} \right\}$$

$$= \min_i \left\{ \frac{2+2}{4}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{9}{10}.$$

Then we get $\lfloor \frac{9}{10} F - K_\pi \rfloor = (2,2,5,2,5,0,0)$ and its antinef closure is $D_{\lambda_3} = (2,3,5,3,5,0,0)$.

- The fourth Jumping Number is:

$$\lambda_4 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_3}}{e_i} \right\}$$

$$= \min_i \left\{ \frac{2+2}{4}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{1}{1}, \frac{1}{1} \right\} = 1.$$

Then we get $\lfloor F - K_\pi \rfloor = (3,3,6,3,6,1,1)$ but this divisor is not antinef because of the non-exceptional components. Namely, we have excess -1 at E_3 and E_5. To obtain the antinef closure $D_{\lambda_4} = (4,5,10,5,10,1,1)$ we need to perform seven unloading steps with the intermediate divisors:

- $(3,3,7,3,7,1,1)$ with excess -1 at E_2 and E_4.
- $(3,4,7,4,7,1,1)$ with excess -1 at E_3 and E_5.
- $(3,4,8,4,8,1,1)$ with excess -1 at E_1.
- $(4,4,8,4,8,1,1)$ with excess -1 at E_3 and E_5.

- $(4,4,9,4,9,1,1)$ with excess -1 at E_2 and E_4.
- $(4,5,9,5,9,1,1)$ with excess -1 at E_3 and E_5.

If we compare with the $\mathfrak{m}$-primary ideal $\mathfrak{a}$ we should notice that the affine components of $\lfloor F - K_\pi \rfloor$ force us to add more exceptional components when computing its antinef closure and consequently, this will give a different jumping number in the next step.

- The fifth Jumping Number is:

$$\lambda_5 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_4}}{e_i} \right\}$$
$$= \min_i \left\{ \frac{2+4}{4}, \frac{3+5}{5}, \frac{5+10}{10}, \frac{3+5}{5}, \frac{5+10}{10}, \frac{2}{1}, \frac{2}{1} \right\} = \frac{3}{2}.$$

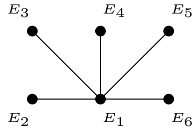
Then we get $\lfloor \frac{3}{2}F - K_\pi \rfloor = (5,5,11,5,11,1,1)$ and its antinef closure is $D_{\lambda_5} = (5,6,12,6,12,1,1)$.

Consider a normal surface X with a singularity at O. Given a minimal resolution $\pi : X' \longrightarrow X$ of X, Artin [Art66] introduced the *fundamental cycle* as the unique smallest non-zero effective divisor with exceptional support that is antinef. Moreover he proved that the singularity is rational if and only if the arithmetical genus of the fundamental cycle is zero.

We have that π is also a minimal log-resolution of the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{X,O}$ and the fundamental cycle is the divisor F such that $\mathfrak{m} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. To compute its arithmetical genus we can use the formula $p_a(F) = 1 + \frac{(K_\pi + F)\cdot F}{2}$ (see [Art66]).

This characterization gives us a good source of examples of surfaces with rational singularities.

Example 2.2.3. Consider a surface X with a rational singularity at O whose minimal resolution $\pi : X' \longrightarrow X$ has six exceptional components $E_1, \ldots, E_6$ with the following dual graph and intersection matrix:



$$\begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & 0 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 & 0 \\ 1 & 0 & 0 & -5 & 0 & 0 \\ 1 & 0 & 0 & 0 & -5 & 0 \\ 1 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

The fundamental cycle is the divisor $F = (2,1,1,1,1,1)$ and the relative canonical divisor is $K_\pi = (-\frac{5}{3}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15})$ so the singularity is not even log-canonical.

The multiplier ideals corresponding to $\lambda_0 = 0$ and $\lambda_1 = \mathrm{lct}(\mathfrak{m}) = \frac{4}{9}$ are given by the antinef divisors $D_{\lambda_0} = (2,1,1,1,1,1)$ and $D_{\lambda_1} = (3,1,1,1,1,1)$. Notice that $\mathcal{J}(\mathfrak{m}^{\lambda_0}) = \mathfrak{m}$ and, using the techniques of Section 4.2, we get that the codimension between these multiplier ideals is 4.

2.3 Implementation

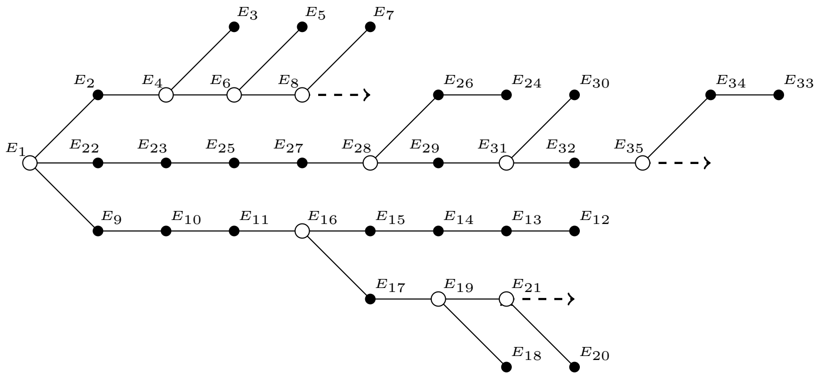
We have implemented Algorithm 2.2.1 in the Computer Algebra system `Macaulay 2` [GS]. The scripts of the source code as well as the output in full detail of some examples are available at the web page

`www.pagines.ma1.upc.edu/~jalvz/multiplier.html`

and also in Appendix B. We implemented Tucker's Algorithm 1.11.9 as well in order to compare both approaches. Of course, once we have the list of jumping numbers we may use the unloading procedure of Section 1.7 to describe the corresponding multiplier ideals. We have also implemented this extended version of Tucker's algorithm and it turns out that our method is much faster.

For example, we have tested the case of an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ whose corresponding dual graph has 35 vertices distributed in three branches only sharing the origin and each branch has three rupture divisors (see its dual graph in the next page).

This example has 56986 jumping numbers in the interval $(0,2]$. Using the extended version of Tucker's algorithm it takes 897.298 seconds to compute the whole list of jumping numbers and their corresponding multiplier ideals. Using our method it only takes 372.165 seconds, i.e. it is roughly 9 minutes faster.



The main difference between the two algorithms stems in the fact that Tucker needs to find first all the possible critical divisors. We will see in the next chapter that our algorithm can be understood as a method to find a unique and very precise contributing divisor.

The input that we use in both algorithms, i.e. the log-resolution $\pi : X' \to X$ of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, is encoded using the intersection matrix and the vector of values for the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. An algorithm to compute this data from a set of generators of the ideal $\mathfrak{a}$ has been described in [AAB15]. An implementation in `Macaulay 2` will be available soon. For principal ideals this can be done using the `Singular` [DGPS15] package `alexpoly.lib`.

Chapter 3

Jumping Divisors

The theory of critical divisors was introduced by Smith-Thompson in [ST07] and further developed by Tucker [Tuc10] focuses on complete ideals very close to a given multiplier ideal. The aim of this chapter is to understand the whole jump between two consecutive multiplier ideals. The results presented in this chapter correspond to [AAD14, Sections 4 and 5] and [AADG14, Section 3]. Consider then X to be a complex surface with at worst a rational singularity and $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ an ideal.

To understand the jump, we introduce the following natural definition.

Definition 3.0.1. Let λ be a jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. A reduced divisor $G \leqslant F$ for which λ is a candidate jumping number is called a *jumping divisor* for λ if

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G),$$

for ε small enough. We say that a jumping divisor is minimal if no proper subdivisor is a jumping divisor for λ, i.e.

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G')$$

for any $0 \leqslant G' < G$.

Remark 3.0.2. Any reduced divisor $G \leqslant F$ for which λ is a candidate jumping number defines an ideal nested between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supseteq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

Hence, a jumping divisor for λ is a contributing divisor to λ. In particular, a minimal jumping divisor can be understood as the minimal contribution which defines the preceding multiplier ideal.

It is a striking fact that the methods used in the previous chapter, in particular our main result Theorem 2.1.5, will allow us to construct the unique minimal jumping divisor associated to a jumping number. In fact, we will see in Corollary 3.0.9 that the only jumping divisors are those reduced divisors $D \leqslant F$ satisfying $G_\lambda \leqslant D \leqslant H_\lambda$, where G_λ and H_λ are defined as follows:

Definition 3.0.3. Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Let $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ be the antinef closure of $\lfloor (\lambda-\varepsilon)F - K_\pi \rfloor$ for ε small enough. Then we define:

- *Maximal jumping divisor:* Is the reduced divisor $H_\lambda \leqslant F$ supported on those components E_i for which $\lambda e_i - k_i \in \mathbb{Z}$. Equivalently
$$H_\lambda = \lceil K_\pi - (\lambda-\varepsilon)F \rceil - \lceil K_\pi - \lambda F \rceil.$$
- *Minimal jumping divisor:* Is the reduced divisor $G_\lambda \leqslant F$ supported on those components E_i for which
$$\lambda = \frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i},$$
i.e. supported on those divisors where the minimum considered in Theorem 2.1.5 is achieved.

Notice that one can generalize the definition of maximal jumping divisor to any real number $c > 0$.

Definition 3.0.4. Given any real number $c > 0$, we define its associated *maximal jumping divisor* as

$$H_c = \lceil K_\pi - (c-\varepsilon)\, F \rceil - \lceil K_\pi - cF \rceil \tag{3.0.1}$$

for a sufficiently small $\varepsilon > 0$. Alternatively, it can be defined as the reduced divisor whose components are the exceptional curves E_i such that $k_i - ce_i \in \mathbb{Z}$.

Remark 3.0.5. The definition of minimal jumping divisors given in Definition 3.0.3 is more involved and is closely related to the algorithm given in section 2.2 for the computation of the chain of multiplier ideals. Is for this reason that minimal jumping divisors are only defined for jumping numbers. However one may extend the definition to any positive real number c if we consider $G_c = 0$ for any non-jumping number $c > 0$. Notice that the equality of Definition 3.0.1 is still trivially satisfied for any divisor G such that $G_c \leqslant G \leqslant H_c$.

It is clear that H_λ is a jumping divisor and $G_\lambda \leqslant H_\lambda$. In fact, any reduced divisor $G \leqslant F$ that contributes to λ satisfies $G \leqslant H_\lambda$. We will prove next that G_λ deserves the given name.

Proposition 3.0.6. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. The reduced divisor G_λ is a jumping divisor.*

Proof. Since $G_\lambda \leqslant H_\lambda$, we have $\lfloor (\lambda - \varepsilon)F - K_\pi \rfloor \leqslant \lfloor \lambda F - K_\pi \rfloor - G_\lambda$ and therefore

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda).$$

For the reverse inclusion, let $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ be the antinef closure of $\lfloor (\lambda - \varepsilon)F - K_\pi \rfloor$. We want to check that $\lfloor \lambda F - K_\pi \rfloor - G_\lambda \leqslant D_{\lambda-\varepsilon}$. To this purpose we only need to consider the following cases:

· If $E_i \leqslant G_\lambda$ then we have $\lambda = \frac{k_i+1+e_i^{\lambda-\varepsilon}}{e_i}$. In particular,

$$\lfloor \lambda e_i - k_i \rfloor - 1 = e_i^{\lambda-\varepsilon}.$$

· If $E_i \nleqslant G_\lambda$ then we have $\lambda < \frac{k_i+1+e_i^{\lambda-\varepsilon}}{e_i}$. Thus

$$\lfloor \lambda e_i - k_i \rfloor < 1 + e_i^{\lambda-\varepsilon}$$

and the result follows.

□

The unicity of the jumping divisor G_λ is a consequence of the following more general statement

Theorem 3.0.7. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Any contributing divisor $G \leqslant F$ associated to λ satisfies either:*

· $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ *if and only if* $G_\lambda \leqslant G$, *or*

· $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ *otherwise.*

Proof. Since $G \leqslant H_\lambda$, we have $\lfloor (\lambda - \varepsilon)F - K_\pi \rfloor \leqslant \lfloor \lambda F - K_\pi \rfloor - G$ and therefore

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G).$$

Now assume $G_\lambda \leqslant G$. Then $\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_\lambda$, and using the fact that G_λ is a jumping divisor we obtain the equality

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G)\,.$$

If $G_\lambda \nleqslant G$ we may consider a component $E_i \leqslant G_\lambda$ such that $E_i \nleqslant G$. Notice that we have

$$v_i(D_{\lambda-\varepsilon}) = e_i^{\lambda-\varepsilon} = \lambda e_i - k_i - 1 < \lambda e_i - k_i = v_i(\lfloor \lambda F - K_\pi \rfloor - G)$$

where $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ is the antinef closure of $\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor$. Therefore, by Proposition 2.1.3, we get the strict inclusion

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G).$$

□

Corollary 3.0.8. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then G_λ is the unique minimal jumping divisor associated to λ.*

Notice that Theorem 3.0.7 also describes all the jumping divisors associated to a given jumping number. Namely, we have

Corollary 3.0.9. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then, any reduced divisor in the interval $G_\lambda \leqslant D \leqslant H_\lambda$ is a jumping divisor.*

It is clear from its definition that maximal jumping divisors are periodic for any $c > 0$.

Lemma 3.0.10. *For any real number $c > 0$, we have $H_c = H_{c+1}$.*

On the other hand, critical divisors do not satisfy any periodicity condition. One may find examples where a divisor G is a critical divisor for the jumping number λ but not for $\lambda + 1$ and vice versa. For minimal jumping divisors we have:

Proposition 3.0.11. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and G_λ its associated minimal jumping divisor. Then we have:*

i) If $\lambda \leqslant 1$ then $G_\lambda \leqslant G_{\lambda+1}$.

ii) If $\lambda > 1$ then $G_\lambda = G_{\lambda+1}$.

Proof. Assume that there exists a prime divisor $E_i \leqslant G_\lambda$ such that $E_i \nleqslant G_{\lambda+1}$. Then, for a sufficiently small $\varepsilon > 0$ we have

$$\lambda = \frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i} \quad \text{and} \quad \lambda + 1 < \frac{k_i + 1 + e_i^{(\lambda-\varepsilon)+1}}{e_i}$$

where $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ denotes the antinef closure of $\lfloor (\lambda - \varepsilon)F - K_\pi \rfloor$ and equivalently, $D_{(\lambda-\varepsilon)+1} = \sum e_i^{(\lambda-\varepsilon)+1} E_i$ is the antinef closure of

$$\lfloor ((\lambda - \varepsilon) + 1)F - K_\pi \rfloor .$$

Therefore

$$\frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i} + 1 < \frac{k_i + 1 + e_i^{(\lambda-\varepsilon)+1}}{e_i}$$

or equivalently $e_i^{\lambda-\varepsilon} + e_i < e_i^{(\lambda-\varepsilon)+1}$. Then we have $\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \nsubseteq \mathcal{J}(\mathfrak{a}^{(\lambda-\varepsilon)+1})$ so we get a contradiction.

For $\lambda > 1$ we have an equality $e_i^{\lambda-\varepsilon} + e_i = e_i^{(\lambda-\varepsilon)+1}$ because of Skoda's theorem so the result follows. □

Let $\lambda' < \lambda$ be two consecutive jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. It is quite surprising that the minimal jumping divisor G_λ gives such nice approach to the understanding of the jump from $\mathcal{J}(\mathfrak{a}^\lambda)$ to its preceding multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda'})$. Taking into account that its construction is based on Theorem 2.1.5, where λ is obtained from the antinef divisor associated to $\mathcal{J}(\mathfrak{a}^{\lambda'})$, it would seem more natural to consider the jump in the other direction. It turns out that the jump from $\mathcal{J}(\mathfrak{a}^{\lambda'})$ to $\mathcal{J}(\mathfrak{a}^\lambda)$ does not behave that nicely.

Proposition 3.0.12. *Let $\lambda' < \lambda$ be two consecutive jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and $D_{\lambda'}$ be the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$. Then we have:*

i) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \pi_* \mathcal{O}_{X'}(-D_{\lambda'} - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$.

ii) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - (\lambda - \varepsilon)F \rceil - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$

Proof. Let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$, $D_\lambda = \sum e_i^\lambda E_i$ be the antinef closures of $\lfloor \lambda' F - K_\pi \rfloor$ and $\lfloor \lambda F - K_\pi \rfloor$ respectively.

i) Since G_λ is a jumping divisor we have $\lfloor \lambda F - K_\pi \rfloor - G_\lambda \leqslant D_{\lambda'}$, and hence $\lfloor \lambda F - K_\pi \rfloor \leqslant D_{\lambda'} + G_\lambda$. This gives the inclusion $\pi_* \mathcal{O}_{X'}(-D_{\lambda'} - G_\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)$.

In order to check the reverse inclusion $\pi_*\mathcal{O}_{X'}(-D_{\lambda'}-G_\lambda) \supseteq \mathcal{J}(\mathfrak{a}^\lambda)$, it is enough, using Proposition 2.1.3, to prove $v_i(D_{\lambda'}+G_\lambda) \leqslant v_i(D_\lambda) = e_i^\lambda$ for any component E_i. We have $e_i^{\lambda'} \leqslant e_i^\lambda$ just because $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^\lambda)$ and the inequality is strict when $E_i \leqslant G_\lambda$, so the result follows.

ii) Let D' be the antinef closure of $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda$. Since $G_\lambda \leqslant H_\lambda$ we have

$$\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda \leqslant \lfloor\lambda F - K_\pi\rfloor \leqslant D_\lambda$$

so the inclusion $\pi_*\mathcal{O}_{X'}(\lceil K_\pi - (\lambda-\varepsilon)F\rceil - G_\lambda) \supseteq \mathcal{J}(\mathfrak{a}^\lambda)$ holds. In order to prove the reverse inclusion we will introduce an auxiliary divisor $D = \sum d_i E_i \in \Lambda$ defined as follows:

- $d_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + 1$ if $E_i \leqslant G_\lambda$,
- $d_i = e_i^{\lambda'}$ if $E_i \leqslant H_\lambda$ but $E_i \nleqslant G_\lambda$,
- $d_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor$ otherwise.

Clearly we have $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda \leqslant D$, but we also have $\lfloor\lambda F - K_\pi\rfloor \leqslant D$. Indeed,

- For $E_i \leqslant G_\lambda$ we have $\lfloor\lambda e_i - k_i\rfloor = \lambda e_i - k_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + 1 = d_i$.
- If λ is a candidate for E_i but $E_i \nleqslant G_\lambda$, $\lfloor\lambda e_i - k_i\rfloor = \lambda e_i - k_i < 1 + e_i^{\lambda'}$, hence $\lfloor\lambda e_i - k_i\rfloor \leqslant e_i^{\lambda'} = d_i$.
- Otherwise $\lfloor\lambda e_i - k_i\rfloor = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor = d_i$.

Therefore, taking antinef closures, we have $D' \leqslant D_\lambda \leqslant \widetilde{D}$. On the other hand $D \leqslant D'$. Namely, $v_i(D') \geqslant e_i^{\lambda'}$ at any E_i because

$$\lfloor\lambda' F - K_\pi\rfloor \leqslant \lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda\,.$$

Moreover, $v_i(D') \geqslant \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + \delta_i^{G_\lambda}$ by definition of antinef closure. Here, $\delta_i^{G_\lambda} = 1$ if $E_i \leqslant G_\lambda$ and zero otherwise. Thus $v_i(D') \geqslant v_i(D)$ as desired. As a consequence $\widetilde{D} \leqslant D'$, which together with the previous $D' \leqslant D_\lambda \leqslant \widetilde{D}$, gives $\widetilde{D} = D' = D_\lambda$ and the result follows. □

Remark 3.0.13. Contrary to the case of Theorem 3.0.7, G_λ may not be minimal in this case. In fact, we will see in Example 3.4.8 a divisor $G < G_\lambda$ satisfying:

$$\mathcal{J}(\mathfrak{a}^{\lambda'}) = \pi_*\mathcal{O}_{X'}(-D_{\lambda'}) \supsetneq \pi_*\mathcal{O}_{X'}(-D_{\lambda'} - G) = \mathcal{J}(\mathfrak{a}^\lambda)\,.$$

Despite the fact that the antinef closure of both $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor$ and $\lfloor\lambda' F - K_\pi\rfloor$ is $D_{\lambda'}$, it is quite remarkable that the above jumping property does not hold taking $\lfloor\lambda' F - K_\pi\rfloor$, i.e. the equality $\pi_*\mathcal{O}_{X'}(\lfloor\lambda' F - K_\pi\rfloor - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ is not always true.

3.1 Invariance of the minimal jumping divisor with respect to the log-resolution

Multiplier ideals and jumping numbers are known to be independent of the chosen log-resolution of the initial ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. The aim of this section is to prove that the minimal jumping divisor is *generically* independent of the log-resolution in a sense that we will make precise below. As a consequence of Proposition 3.4.6 and Corollary 3.4.5 in Section 3.4, critical divisors will also be generically independent of the log-resolution. This is a remarkable fact since, as it was pointed out by Tucker in [Tuc10, Remark 3.4], there is no reason to believe that critical divisors (and by extension minimal jumping divisors) are independent of the resolution since they depend on all the divisorial valuations appearing in F.

We start fixing some notation that we will use in this section. Let $\pi' : X' \longrightarrow X$ be the minimal log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Any other log-resolution $\pi : Y \longrightarrow X$ factors through π', i.e. there is a birational morphism $g : Y \longrightarrow X'$ such that $\pi = \pi' \circ g$ (see [Lip69, Theorem 4.1]).

For a given jumping number λ of $\mathfrak{a}$ we will denote G'_λ the minimal jumping divisor of π' and $E'_1, \ldots, E'_r$ the exceptional components of $E' = Exc(\pi')$. If G_λ and $E_1, \ldots, E_s$ are the minimal jumping divisor and the exceptional components of $E = Exc(\pi)$ for any other log-resolution π, we will enumerate them setting E_i equal to the strict transform by g of E'_i for $1 \leqslant i \leqslant r$. If no confusion arise, we will use the same symbol to denote a divisor $D = \sum_{i=1}^r d_i E'_i$ on X' or its strict transform $D = \sum_{i=1}^r d_i E_i$ on Y.

Theorem 3.1.1. *With the previous notations, G_λ is independent of the log-resolution π if and only if π does not include any blowing-up at points in the intersection of two components of the minimal jumping divisor G'_λ of the minimal log-resolution.*

Actually, from the proof of this result, we can express the minimal jumping divisor of any resolution. To such purpose we need to fix some notation:

A reduced divisor with exceptional support $D = E_{i_1} + \cdots + E_{i_m} \leqslant E$ is a *chain with ends* E_{i_1} and E_{i_m} if $a_D(E_{i_1}) = a_D(E_{i_m}) = 1$ and $a_D(E_{i_k}) = 2$ for any other $1 < k < m$. Given $E_{j_1}, E_{j_2} \leqslant E$, we say that the chain above *connects* E_{j_1} and E_{j_2} if $E_{j_1} \in \operatorname{Adj}(E_{i_1})$ and $E_{j_2} \in \operatorname{Adj}(E_{i_m})$. Observe that if E_{j_1} and E_{j_2} are adjacent in E, a chain connecting them will be $D = 0$.

Corollary 3.1.2. *Keeping the above notations we have*

$$G_\lambda = G'_\lambda + \sum_{\substack{E'_i+E'_j \leqslant G'_\lambda \\ E'_i \cdot E'_j = 1}} D_{ij} \tag{3.1.1}$$

where D_{ij} is a chain connecting E_i and E_j.

Consider *generic* log-resolutions as those obtained from a minimal one by further blowing-ups at simple (and hence generic) points on the exceptional components. Then, Theorem 3.1.1 states that generic log-resolutions have the same minimal jumping divisor. This *generictiy* may be formulated, when X is smooth, in terms of valuations in the valuative tree $\mathcal{V}$ of Favre-Jonsson [FJ04]. Consider the dual graphs Γ and Γ' of E and E' respectively, embedded in the valuative tree $\mathcal{V}$ as in [FJ04, Chapter 6] and let ν_i denote the divisorial valuation centered at E_i.

Corollary 3.1.3. *The minimal jumping divisor G_λ of π equals the minimal jumping divisor G'_λ if and only if Γ has no vertex inside any segment $]\nu_i, \nu_j[$ for which E'_i and E'_j are adjacent in E' and belong to G'_λ.*

Proof of Theorem 3.1.1. Let $\lambda' < \lambda$ be two consecutive jumping numbers of $\mathfrak{a}$. We will argue by induction on the number of blowing-ups needed to reach Y from a minimal resolution. In order to simplify the notation, we will assume throughout this proof that X' also dominates a minimal log-resolution and that Y is obtained from X' by one blowing-up $g : Y \longrightarrow X'$ at a closed point $p \in X'$ giving the exceptional component E_s. Assume that (3.1.1) holds on X' and let us prove it on Y. Notice that, keeping the notation used in this section, we are in the case $r + 1 = s$.

Let $F' = \sum_{i=1}^r e_i E'_i$ and $F = \sum_{i=1}^s e_i E_i$ be the divisors in X' and Y respectively such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F')$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. We also consider the antinef divisors $D'_{\lambda'} = \sum_{i=1}^r e_i^{\lambda'} E'_i$ and $D_{\lambda'} = \sum_{i=1}^s e_i^{\lambda'} E_i$ for which $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \pi'_* \mathcal{O}_{X'}(-D'_{\lambda'}) = \pi_* \mathcal{O}_Y(-D_{\lambda'})$ sharing the first r coefficients since multiplier ideals are independent of the log-resolution. Moreover, by Theorem 2.1.5

$$\lambda = \min_{1 \leqslant i \leqslant r} \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\} = \min_{1 \leqslant i \leqslant s} \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\},$$

clearly demonstrating that the strict transform of G'_λ is contained in G_λ. In particular, $\lambda e_i - k_i = 1 + e_i^{\lambda'}$ if and only if $E_i \leqslant G_\lambda$ and $\lambda e_i - k_i < 1 + e_i^{\lambda'}$ otherwise.

We distinguish two cases:

i) The closed point p lies only on one exceptional divisor E'_j. Then we have $e_s = e_j$, $k_s = k_j + 1$ and $e_s^{\lambda'} = e_j^{\lambda'}$ and thus

$$v_s(\lfloor \lambda F - K_\pi \rfloor) = \lfloor \lambda e_s - k_i \rfloor = \lfloor \lambda e_j - k_j \rfloor - 1 \leqslant e_j^{\lambda'} = e_s^{\lambda'}.$$

Hence E_s can not belong to G_λ.

ii) The closed point p lies on the intersection of two exceptional divisors E'_{j_1} and E'_{j_2}. Then we have $e_s = e_{j_1} + e_{j_2}$, $k_s = k_{j_1} + k_{j_2} + 1$ and $e_s^{\lambda'} = e_{j_1}^{\lambda'} + e_{j_2}^{\lambda'}$ so

$$\begin{aligned} v_s(\lfloor \lambda F - K_\pi \rfloor) &= \lfloor \lambda e_s - k_s \rfloor \\ &= \lfloor \lambda e_{j_1} - k_{j_1} + \lambda e_{j_2} - k_{j_2} \rfloor - 1 \\ &\leqslant {j_1}^{\lambda'} + e_{j_2}^{\lambda'} + 1 \\ &= e_s^{\lambda'} + 1, \end{aligned}$$

and equality holds if and only if $E'_{j_1} + E'_{j_2} \leqslant G_\lambda$. In particular, E_s does not belong to G_λ whenever none or just one of the components E'_{j_1}, E'_{j_2} belong to G'_λ.

□

3.2 Geometric properties of minimal jumping divisors in the dual graph

Assume that a critical divisor G associated to a jumping number λ has exceptional support. One of the key ingredients in Tucker's algorithm for the computation of jumping numbers is that G satisfies some nice geometric conditions when viewed in the dual graph: G is a connected chain and its ends must be either rupture or dicritical divisors (see Theorem 1.11.8). Then, it is natural to ask whether jumping divisors satisfy analogous properties.

Throughout this section we will also assume that the minimal jumping divisor G_λ has exceptional support. Then, it may have several connected components in the dual graph and these components are not necessarily chains. However, we can still control the ends of each component. To prove the main result of this section (see Theorem 3.2.4) we need some preliminary results first. Keep the notations of Chapter 1.

Lemma 3.2.1. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. For any component E_i of the minimal jumping divisor G_λ we have*

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \lambda \rho_i + \sum_{E_j \in \mathrm{Adj}(E_i)} \{\lambda e_j - k_j\} + a_{G_\lambda}(E_i).$$

Proof. For any $E_i \leqslant G_\lambda$ we have

$$\begin{aligned}(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i &= ((K_\pi - \lambda F) + \{-K_\pi + \lambda F\} + G_\lambda - E_i + E_i) \cdot E_i \\ &= (K_\pi + E_i) \cdot E_i - \lambda F \cdot E_i + \{\lambda F - K_\pi\} \cdot E_i + (G_\lambda - E_i) \cdot E_i.\end{aligned}$$

Let us now compute each summand separately. Firstly, the adjunction formula gives $(K_\pi + E_i) \cdot E_i = -2$ because $E_i \cong \mathbb{P}^1$. As for the second and fourth terms, the equality $-\lambda F \cdot E_i = \lambda \rho_i$ follows from the definition of the excesses, and clearly $a_{G_\lambda}(E_i) = (G_\lambda - E_i) \cdot E_i$ because $E_i \leqslant G_\lambda$.

Therefore it only remains to prove that

$$\{\lambda F - K_\pi\} \cdot E_i = \sum_{E_j \in \mathrm{Adj}(E_i)} \{\lambda e_j - k_j\}, \tag{3.2.1}$$

which is also quite immediate. Indeed, writing

$$\{\lambda F - K_\pi\} = \sum_{j=1}^{r} \{\lambda e_j - k_j\} E_j,$$

equality (3.2.1) follows by observing that (for $j \neq i$), $E_j \cdot E_i = 1$ if and only if $E_j \in \mathrm{Adj}(E_i)$, and the term corresponding to $j = i$ vanishes because we have $\lambda e_i - k_i \in \mathbb{Z}$. □

Remark 3.2.2. It is important to notice that $(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i \in \mathbb{Z}$, that is $-2 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{\lambda e_j - k_j\} + \lambda \rho_i + a_{G_\lambda}(E_i) \in \mathbb{Z}$.

The following result is an analogue of the numerical conditions that critical divisors satisfy (see Proposition 3.2.6). Unfortunately it does not provide a characterization of minimal jumping divisors.

Proposition 3.2.3. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. For any component $E_i \leqslant G_\lambda$ of the minimal jumping divisor G_λ we have*

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i \geqslant 0.$$

Proof. Let G_λ be the minimal jumping divisor. Given a prime divisor $E_i \leqslant G_\lambda$ we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda - E_i\right) \longrightarrow \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \longrightarrow$$
$$\longrightarrow \mathcal{O}_{E_i}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \longrightarrow 0$$

Pushing it forward to X we get

$$0 \longrightarrow \pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda - E_i\right) \longrightarrow \pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \longrightarrow$$
$$\longrightarrow H^0\left(E_i, \mathcal{O}_{E_i}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right)\right) \otimes \mathbb{C}_O,$$

where $\mathbb{C}_O$ denotes the skyscraper sheaf supported at O with fiber $\mathbb{C}$. The minimality of G_λ (see Theorem 3.0.7) implies that

$$\pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda - E_i\right) \neq \pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right).$$

Thus $H^0\left(E_i, \mathcal{O}_{E_i}\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right)\right) \neq 0$, or equivalently $\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \cdot E_i \geqslant 0$. □

With the above ingredients we can provide the following geometric property of minimal jumping divisors when viewed in the dual graph.

Theorem 3.2.4. *Let G_λ be the minimal jumping divisor associated to a jumping number λ of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then the ends of a connected component of G_λ must be either rupture or dicritical divisors.*

Proof. Assume that an end E_i of a connected component of G_λ is neither a rupture nor a dicritical divisor. It means that E_i has no excess, i.e. $\rho_i = 0$, and that it has one or two adjacent divisors, say E_j and E_l, in the dual graph but at most one of them belongs to G_λ.

For the case that E_i has two adjacent divisors E_j and E_l the formula given in lemma 3.2.1 reduces to

$$\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} + \lambda\rho_i + a_{G_\lambda}(E_i)\,.$$

Then:

· If E_i has valence one in G_λ, e.g. $E_l \not\leqslant G_\lambda$ then

$$\left(\lceil K_\pi - \lambda F\rceil + G_\lambda\right) \cdot E_i = -2 + \{\lambda e_l - k_l\} + 1 < 0.$$

- If E_i is an isolated component of G_λ, i.e., $E_j, E_l \nleqslant G_\lambda$ then

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} < 0.$$

If E_i has just one adjacent divisor E_j, i.e. E_i is an end of the dual graph, the formula reduces to

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \lambda \rho_i + a_{G_\lambda}(E_i)\,.$$

Then:

- If E_i has valence one in G_λ then $(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + 1 < 0$
- If E_i is an isolated component of G_λ then

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \{\lambda e_j - k_j\} < 0.$$

In any case we get a contradiction with Proposition 3.2.3. □

Remark 3.2.5. It follows from [Vey95, Theorem 3.3] that the minimal jumping divisor associated to the log-canonical threshold is connected in the case that X is smooth.

As a consequence we may also give the following refinement of Proposition 3.2.3.

Proposition 3.2.6. *Let λ be a jumping number of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. If $E_i \leqslant G_\lambda$ is neither a rupture nor a dicritical component of the minimal jumping divisor G_λ we have*

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = 0.$$

Proof. Assume that $E_i \leqslant G_\lambda$ is neither a rupture or a dicritical component. In particular, it is not the end of a connected component of G_λ. Thus, E_i has exactly two adjacent components E_j and E_l in G_λ, and its excess is $\rho_i = 0$. The formula given in Lemma 3.2.1 reduces to

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \lambda \rho_i + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} + a_{G_\lambda}(E_i)\,.$$

Notice that $a_{G_\lambda}(E_i) = 2$, and also $\{\lambda e_j - k_j\} = \{\lambda e_l - k_l\} = 0$ because E_j and E_l are components of G_λ, so finally $(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = 0$. □

3.3 Geometric properties of maximal jumping divisors in the dual graph

We focus now on the structure of H_c. We first prove some formulas to compute its intersection with its irreducible and connected components.

Lemma 3.3.1. *Fix $c \in \mathbb{R}_{>0}$ and consider a component E_i of the jumping divisor H_c. Then*

$$(\lceil K_\pi - cF \rceil + H_c) \cdot E_i = -2 + c\rho_i + a_{H_c}(E_i) + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}.$$

Proof. For any $E_i \leqslant H_c$ we have

$$\begin{aligned}(\lceil K_\pi - cF \rceil + H_c) \cdot E_i &= ((K_\pi - cF) + \{-K_\pi + cF\} + H_c - E_i + E_i) \cdot E_i \\ &= (K_\pi + E_i) \cdot E_i - cF \cdot E_i + (H_c - E_i) \cdot E_i + \{cF - K_\pi\} \cdot E_i.\end{aligned}$$

Let us now compute each summand separately. The first three terms are easy: $(K_\pi + E_i) \cdot E_i = -2$ follows from the adjunction formula, $-cF \cdot E_i = c\rho_i$ holds by definition, and clearly $a_{H_c}(E_i) = (H_c - E_i) \cdot E_i$ because $E_i \leqslant H_c$. It only remains to prove that

$$\{cF - K_\pi\} \cdot E_i = \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}, \tag{3.3.1}$$

which is also quite immediate. Indeed, writing $\{cF - K_\pi\} = \sum_{j=1}^{r} \{ce_i - k_i\} E_j$, (3.3.1) follows by observing that, for $j \neq i$, $E_j \cdot E_i = 1$ if and only if $E_j \in \mathrm{Adj}(E_i)$, and the term corresponding to $j = i$ vanishes because we assumed $E_i \leqslant H_c$, hence $ce_i - k_i \in \mathbb{Z}$. □

Corollary 3.3.2. *For any $c \in \mathbb{R}_{>0}$ and any $E_i \leqslant H_c$, the sum*

$$c\rho_i + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}$$

is an integer.

Proposition 3.3.3. *Fix any $c \in \mathbb{R}_{>0}$, and let H_c be its associated maximal jumping divisor. Then the following inequalities hold:*

- $(\lceil K_\pi - cF \rceil + H_c) \cdot E_i \geqslant -1$ *for all $E_i \leqslant H_c$, and*
- $(\lceil K_\pi - cF \rceil + H_c) \cdot H \geqslant -1$ *for any connected component $H \leqslant H_c$.*

Proof. From Lemma 3.3.1 we already know that $(\lceil K_\pi - cF\rceil + H_c)\cdot E_i \geqslant -2$ for all $E_i \leqslant H_c$. If equality holds, then it must also hold

- $a_{H_c}(E_i) = 0$, that is, E_i is an isolated component in H_c,
- $\{ce_j - k_j\} = 0$ for all $E_j \in \mathrm{Adj}(E_i)$, that is, every exceptional component E_j intersecting E_i is also contained in H_c, and
- $\rho_i = 0$.

The first two conditions imply that E_i is the only exceptional curve of the log-resolution. But in this case $\rho_i = \rho > 0$ and the third condition is not satisfied.

As for the second part, using Lemma 3.3.1 for all $E_i \leqslant H$ and summing up we obtain

$$
\begin{aligned}
(\lceil K_\pi - cF\rceil + H_c)\cdot H &= -2v_H + \sum_{E_i\leqslant H}\left(\sum_{E_j\in\mathrm{Adj}(E_i)}\{ce_j - k_j\} + c\rho_i\right) + 2a_H \\
&= -2 + \sum_{E_i\leqslant H}\left(\sum_{E_j\in\mathrm{Adj}(E_i)}\{ce_j - k_j\}\right) + c\sum_{E_i\leqslant H}\rho_i \\
&\geqslant -2,
\end{aligned}
$$

where $a_H - v_H = 1$ due to the tree structure of the exceptional divisor and the connectedness of H. Equality holds if and only if

$$\sum_{E_i\leqslant H}\sum_{E_j\in\mathrm{Adj}(E_i)}\{ce_j - k_j\} = \sum_{E_i\leqslant H} c\rho_i = 0.$$

The first condition implies that H is the whole exceptional divisor, and then the second condition implies that $\rho = 0$, which is impossible. Hence the inequality must be strict, and since $(\lceil K_\pi - cF\rceil + H_c)\cdot H \in \mathbb{Z}$, the claim follows. $\square$

We will now get some insight on the topology of the H_c.

Theorem 3.3.4. *Fix any $c \in \mathbb{R}_{>0}$, and let H_c be the corresponding maximal jumping divisor. Then:*

- *The isolated components of H_c must be either a rupture divisor, a dicritical divisor or a divisor E_i with $a(E_i) = 2$ such that*

$$\sum_{E_j\in\mathrm{Adj}(E_i)}\{ce_j - k_j\} = 1.$$

- *An end of a reducible connected component of H_c must be either a rupture divisor, a dicritical divisor or an end of the whole exceptional divisor.*

Proof. Let E_i be an isolated component of H_c. Assume that it is neither a rupture nor a dicritical component. Then it only has one or two adjacent components in the exceptional divisor. In the first case, if E_j is the only exceptional component in $\operatorname{Adj}(E_i)$, then the formula given in Lemma 3.3.1 reduces to $(\lceil K_\pi - cF\rceil + H_c) \cdot E_i = -2 + \{ce_j - k_j\}$. Since $\{ce_j - k_j\} < 1$, we would get $(\lceil K_\pi - cF\rceil + H_c) \cdot E_i < -1$, contradicting Proposition 3.3.3. The only possible remaining case is $a(E_i) = 2$. If $\operatorname{Adj}(E_i) = \{E_j, E_l\}$, then we have $(\lceil K_\pi - cF\rceil + H_c) \cdot E_i = -2 + \{ce_j - k_j\} + \{ce_l - k_l\}$. Since

$$0 \leqslant \{ce_j - k_j\} + \{ce_l - k_l\} < 2$$

must be an integer by Corollary 3.3.2 (we assumed E_i to be non-dicritical, i.e. $\rho_i = 0$), it must equal 0 or 1. But the former contradicts Proposition 3.3.3, hence the only possibility is that $\{ce_j - k_j\} + \{ce_l - k_l\} = 1$, which is the last possibility given in the statement.

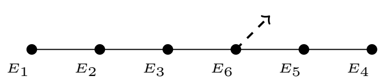
As for the second assertion, let E_i be an end of a reducible connected component of H_c that is neither a rupture divisor, nor a dicritical divisor nor an end of the whole exceptional divisor. Then it has two adjacent components in the whole exceptional divisor, say E_j and E_l, but only one of them, say E_j, is in H_c. Then we have

$$(\lceil K_\pi - cF\rceil + H_c) \cdot E_i = -2 + \{ce_l - k_l\} + 1 \notin \mathbb{Z},$$

which is impossible. ☐

There are examples where any of these cases is achieved, in particular we may find isolated components of H_c that are neither a rupture nor a dicritical divisor.

Example 3.3.5. Consider the ideal $\mathfrak{a} = (x^3, y^{10}) \subseteq \mathbb{C}\{x, y\}$. The minimal log-resolution has six exceptional components $E_1, \ldots, E_6$ indexed according to the order in which they are obtained by successive blow-ups. They are arranged as the following dual graph shows



where the dashed arrow indicates that E_6 is the only dicritical component, with excess $\rho_6 = 1$. The relative canonical divisor is

$$K_\pi = E_1 + 2E_2 + 3E_3 + 4E_4 + 8E_5 + 12E_6$$

and the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ is

$$F = 3E_1 + 6E_2 + 9E_3 + 10E_4 + 20E_5 + 30E_6 \,.$$

The maximal jumping divisor associated to $c = \frac{3}{2}$ is $H_{\frac{3}{2}} = E_2 + E_4 + E_5 + E_6$. It has two connected components, one of which (E_2) is as predicted at the first statement of Theorem 3.3.4.

3.4 Minimal contributing divisors

The theory of minimal jumping divisors introduced in Definition 3.0.3 can be included in a more general framework that we will describe in this section. To such purpose we will give our own perspective of the work of Hyry-Järviletho [HJ11] and its relation with the theory of contributing divisors of Tucker [Tuc10].

Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Recall that a reduced divisor $G \leqslant F$ that contributes to λ defines an ideal nested between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

We may interpret that λ is *parametrized* by the set of nested ideals defined by contributions but this is far from being a one-to-one correspondence. An easy way to detect such a nested ideal is finding a suitable critical divisor using Tucker's algorithm. The approach given in the previous sections is more economical in the sense that each jumping number is parametrized by its unique minimal jumping divisor G_λ or equivalently, its preceding multiplier ideal.

Hyry-Järviletho [HJ11] give a similar approach where jumping numbers are parametrized by general antinef divisors[1], or equivalently complete ideals not necessarily nested in the chain of multiplier ideals. We should point out that their results also hold for the case that X has rational singularities since their arguments are based on divisorial considerations. Given any antinef divisor $D = \sum d_i E_i \in \mathrm{Div}(X')$, they considered the following notions:

- *Jumping number corresponding to D:*

$$\lambda_D := \min_i \left\{ \frac{k_i + 1 + d_i}{e_i} \right\}.$$

[1] Hyry-Järviletho only consider the case of $\mathfrak{m}$-primary ideals on smooth surfaces and consequently antinef divisors with exceptional support but their ideas also hold in general

· *Support of a jumping number corresponding to D:*

$$S_D := \left\{ i \mid \lambda_D = \frac{k_i + 1 + d_i}{e_i} \right\}.$$

· *Contributing divisor associated to D:*

$$G_D := \sum_{i \in S_D} E_i.$$

Hyry-Järviletho proved in [HJ11, Proposition 1] that all jumping numbers of $\mathfrak{a}$ can be obtained in this way: as λ_D for a suitable antinef divisor $D \in \mathrm{Div}(X')$ (or equivalently a complete ideal I_D). Moreover, they give in [HJ11, Theorem 1] a combinatorial criterion that detects the existence of such antinef divisors. The simplest parametrizations they used to describe the set of jumping numbers are given by antinef divisors corresponding to critical divisors (see [HJ11, Theorem 2]).

In general, the complete ideal I_D associated to an antinef divisor $D \in \mathrm{Div}(X')$ satisfies $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) \supseteq I_D$ but does not necessarily contain $\mathcal{J}(\mathfrak{a}^{\lambda_D})$. However, if I_D is nested in between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda - \varepsilon}) \supseteq I_D \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$$

then it must satisfy $\lambda = \lambda_D$.

Remark 3.4.1. One can also interpret this framework through the generalized version of log-canonical thresholds already introduced by Järviletho in [Jär11]. Namely, the *log-canonical threshold* with respect to any other ideal $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$ is defined as follows:

$$\mathrm{lct}_{\mathfrak{b}}(\mathfrak{a}) := \inf\{c \in \mathbb{Q}_{>0} \mid \mathcal{J}(\mathfrak{a}^c) \not\supset \mathfrak{b}\}$$

Notice that whenever I_D is the complete ideal associated to an antinef divisor $D \in \mathrm{Div}(X')$, then $\lambda_D = \mathrm{lct}_{I_D}(\mathfrak{a})$.

Hyry-Järviletho [HJ11, Lemma 11] proved that if $D \in \mathrm{Div}(X')$ is an antinef divisor then G_D is a contributing divisor for λ_D. In fact, the contributing divisors obtained in this way satisfy some nice properties as we will see next.

Proposition 3.4.2. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Then $G_D \leqslant G$.*

Proof. Let $D = \sum d_i E_i$ be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Since I_D is a nested ideal in the chain of multiplier ideals, then we have

$$\lambda = \lambda_D = \min_i \left\{ \frac{k_i + 1 + d_i}{e_i} \right\}.$$

Hence $\lambda e_i - k_i \leqslant 1 + d_i$ and equality holds if and only if $i \in S_D$. In order to prove $G_D \leqslant G$ we will show that $E_i \nleqslant G$ implies $E_i \nleqslant G_D$. Indeed, if $E_i \nleqslant G$ and $E_i \leqslant G_D$ then $\lfloor \lambda e_i - k_i \rfloor \leqslant d_i$ (just because $\lfloor \lambda F - K_\pi \rfloor - G \leqslant D$ by Lemma 1.6.2) and $\lambda e_i - k_i - 1 = d_i$ so we get a contradiction. $\square$

Proposition 3.4.3. *Let $\lambda = \lambda_{D'}$ be a jumping number associated to an antinef divisor $D' \in \mathrm{Div}(X')$. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_{D'}$. Then we have $D \leqslant D'$, $\lambda_D = \lambda_{D'}$, $S_D = S_{D'}$ and $G_D = G_{D'}$.*

Proof. Using the definition of antinef closure (see Lemma 1.6.2), in order to get $D \leqslant D'$ we only need to prove that $\lfloor \lambda F - K_\pi \rfloor - G_{D'} \leqslant D'$. Set $D' = \sum d'_i E_i$. By hypothesis

$$\lambda = \lambda_{D'} = \min_i \left\{ \frac{k_i + 1 + d'_i}{e_i} \right\}$$

therefore we have $\lfloor \lambda e_i - k_i \rfloor \leqslant d'_i$ if $i \notin S_{D'}$, whereas $\lfloor \lambda e_i - k_i \rfloor - 1 = d'_i$ if $i \in S_{D'}$ as desired.

Notice then that we have $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_D \supseteq I_{D'}$ so, given the fact that $I_{D'} \nsubseteq \mathcal{J}(\mathfrak{a}^{\lambda})$, we get $\lambda_D = \lambda$. Now, the inclusion of divisors $D \leqslant D'$ having the same minimum $\lambda_D = \lambda_{D'}$, gives the inclusion of supports $S_D \supseteq S_{D'}$ and equivalently $G_D \geqslant G_{D'}$. On the other hand, taking $G = G_{D'}$ in Proposition 3.4.2, we get the reverse inequality of divisors $G_D \leqslant G_{D'}$ so we are done. $\square$

The main result of this section is that we can find a minimal contributing divisor among all contributing divisors defining the same nested ideal.

Theorem 3.4.4. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$, which gives a nested ideal*

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_D = \pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G \right) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

Then we also have $I_D = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G_D)$. *Furthermore,* G_D *is the minimal contributing divisor associated to* λ *that defines the same ideal* I_D*, that is:*

- *Any contribution* G' *to* λ *defining* $I_D = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G')$ *must satisfy* $G_D \leqslant G'$.
- *Any proper subdivisor* $G' < G_D$ *defines an strictly included ideal*

$$I_D \supsetneq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G').$$

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_D$. We will see first that $D = D'$ thus giving the desired equality of ideals

$$I_D = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G) = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G_D) = I_{D'}.$$

In virtue of Proposition 3.4.2, we have $G_D \leqslant G$ so

$$\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_D$$

and $D \leqslant D'$. The reverse inequality $D \geqslant D'$ is a consequence of Proposition 3.4.3.

To show that G_D is the minimal contributor to the jumping number λ that defines the same ideal I_D we will prove the following equivalent result:

Claim: Any contributor G' to λ for which $I_D \supseteq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G')$ also satisfies the reverse inclusion $I_D \subseteq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G')$ if and only if $G_D \leqslant G'$.

Proof of Claim: Suppose first that $G_D \leqslant G'$. Then

$$\lfloor \lambda F - K_\pi \rfloor - G' \leqslant \lfloor \lambda F - K_\pi \rfloor - G_D$$

and hence $D'' \leqslant D' = D$, where D'' is the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G'$. Therefore $I_D \subseteq I_{D''}$ as wanted.

Assume now that $G_D \nleqslant G'$ and pick a component $E_i \leqslant G_D$ such that $E_i \nleqslant G'$. By hypothesis $I_D \supseteq I_{D''}$ and equivalently $D \leqslant D''$ but in fact $D < D''$ since

$$v_i(D) = \lambda e_i - k_i - 1 < \lambda e_i - k_i = v_i(\lfloor \lambda F - K_\pi \rfloor - G') \leqslant v_i(D'').$$

The result follows then from Proposition 2.1.3. ∎

It turns out that critical divisors are also minimal in the above sense as we can see in the following generalization of [HJ11, Proposition 3].

Corollary 3.4.5. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Then G is a critical divisor if and only if $G_D = G$ and I_D and $\mathcal{J}(\mathfrak{a}^\lambda)$ do not admit strictly nested ideals between them defined by contributors to λ.*

Proof. Assume first that $G_D = G$. Then, by Theorem 3.4.4, any proper subdivisor $0 \leqslant G' < G$ defines an ideal strictly included in

$$I_D \supsetneq \pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G' \right) \supseteq \mathcal{J}(\mathfrak{a}^\lambda) \,.$$

Since I_D and $\mathcal{J}(\mathfrak{a}^\lambda)$ do not admit strictly nested ideals between them coming from contributors, we get $\pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G' \right) = \mathcal{J}(\mathfrak{a}^\lambda)$ so G is a critical divisor.

Assume now that G is a critical divisor. By Proposition 3.4.2 we have $G_D \leqslant G$. Both divisors define the same ideal by Theorem 3.4.4 so they must be equal otherwise we would have a contradiction with the fact that G is a critical divisor.

Finally we will see that there is no contributing divisor G' associated to λ defining a strictly nested ideal

$$I_D \supsetneq \pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G' \right) \supsetneq \mathcal{J}(\mathfrak{a}^\lambda).$$

Assume that such G' exists and let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G'$. Then the inclusion of divisors $D < D'$ having the same minimum $\lambda_D = \lambda_{D'} = \lambda$ implies $S_{D'} \subseteq S_D$ and $G_{D'} \leqslant G_D$. Since $G = G_D$ is minimal, applying Theorem 3.4.4, we must have $G = G_D = G_{D'} \leqslant G'$ contradicting the starting hypothesis of inclusion of ideals. $\square$

The minimal jumping divisor introduced in Definition 3.0.3 fits nicely in this theory. Given a jumping number λ of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, let $D_{\lambda - \varepsilon}$ be the antinef closure of $\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor$ for $\varepsilon > 0$ small enough. Then we have $\lambda = \lambda_{D_{\lambda-\varepsilon}}$ and the unique minimal jumping divisor is $G_\lambda = G_{D_{\lambda-\varepsilon}}$.

In general, a divisor $G \in \Lambda$ that contributes to the jumping number λ might not be contained in G_λ. For minimal contributing divisors we have the following:

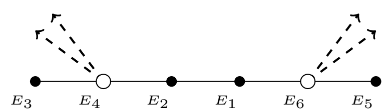
Proposition 3.4.6. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and G_λ be its associated minimal jumping divisor. Then $G_D \leqslant G_\lambda$ for any antinef divisor $D \in \operatorname{Div}(X')$ such that $\lambda = \lambda_D$.*

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_D$. By Proposition 3.4.3 we have $G_D = G_{D'}$ and $\lambda = \lambda_D = \lambda_{D'}$. Since the ideals $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_{D'}$ are nested, their corresponding antinef divisors satisfy $D_{\lambda-\varepsilon} \leqslant D'$ and they reach the same minimum $\lambda_{D_{\lambda-\varepsilon}} = \lambda_{D'} = \lambda$. Hence, $S_{D'} \subseteq S_{D_{\lambda-\varepsilon}}$ which implies $G_D = G_{D'} \leqslant G_\lambda$ as we wanted. $\square$

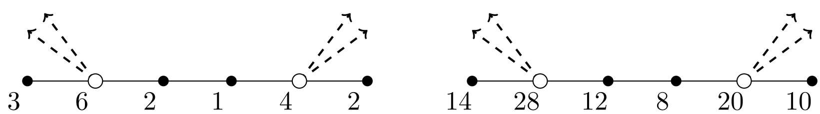
Corollary 3.4.7. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then we have $G \leqslant G_\lambda$ for any critical divisor G associated to λ.*

The reduced sum of all critical divisors equals the jumping divisor G_λ for simple complete ideals (see [GM10, Thm. 2.3] for the smooth case). However this is no longer true in general.

Example 3.4.8. Let X be a smooth surface and consider the $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ whose dual graph is

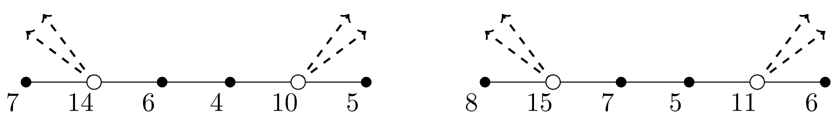


Vertex ordering



K_π F

The multiplier ideals corresponding to the consecutive jumping numbers $\frac{5}{7} < \frac{3}{4}$ are:



$\mathcal{J}(\mathfrak{a}^{\frac{5}{7}}) \supseteq \mathcal{J}(\mathfrak{a}^{\frac{3}{4}})$

The minimal jumping divisor corresponding to $\lambda = \frac{3}{4}$ is $G_{\frac{3}{4}} = E_1 + E_2 + E_4 + E_6$ but the only critical divisors are E_4 and E_6. In particular

$$\mathcal{J}(\mathfrak{a}^{\frac{5}{7}}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \frac{3}{4}F \rceil + E_4 + E_6).$$

It is worth pointing out that

$$\pi_* \mathcal{O}_{X'}(-D_{\frac{5}{7}} - E_4 - E_6) = \pi_* \mathcal{O}_{X'}(-D_{\frac{5}{7}} - G_{\frac{3}{4}}) = \mathcal{J}(\mathfrak{a}^{\frac{3}{4}})$$

where $D_{\frac{5}{7}}$ is the antinef closure of $\lfloor \frac{5}{7}F - K_\pi \rfloor$. So minimality is not always achieved for the divisor G_λ in Proposition 3.0.12.

In general, not every nested ideal between two consecutive multiplier ideals is given by a contributing divisor. The following result identifies them precisely.

Proposition 3.4.9. *Any nested ideal $\mathcal{J}(\mathfrak{a}^{\lambda - \varepsilon}) \supseteq I_{D'} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ comes from a contributing divisor G associated to λ, i.e. $I_{D'} = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G)$, if and only if $D' = D$ where D is the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$ and in this case $G = G_{D'}$.*

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. By Proposition 3.4.3 we have $D \leqslant D'$. On the other hand, Proposition 3.4.2 implies $G_{D'} \leqslant G$ which gives

$$\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_{D'}$$

and hence $D' \leqslant D$ so we get the desired result. The reverse implication is straightforward. □

Proposition 3.4.10. *Let I_D be the ideal associated to an antinef divisor $D \in \Lambda$. Then, I_D is a multiplier ideal for the ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ if and only if D is contained in the antinef closure of $\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor$. If this is the case, D is also the antinef closure of $\lfloor \lambda_D F - K_\pi \rfloor - G_D$.*

Proof. By definition, we have $\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor \leqslant D$ because $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) \supseteq I_D$. We also have $I_D \nsubseteq \mathcal{J}(\mathfrak{a}^{\lambda_D})$ so the only possibility for I_D of being a multiplier ideal is when $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) = I_D$ so, applying Lemma 2.1.1, D must be contained in the antinef closure of$\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor$. The rest of the statement follows from Theorem 3.4.4. □

Chapter 4

Multiplicities and Poincaré series

Let X be a complex surface with at worst a rational singularity and $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ an $\mathfrak{m}$-primary ideal. The aim of this section is to describe the multiplicity

$$m(c) = \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^{c})}$$

for any real exponent $c > 0$, where ε is small enough. In Theorem 4.1.1 we will give a formula described in terms of the maximal jumping divisor associated to c. This formula and Proposition 4.1.5 will be the key ingredients the description of the Poincaré series associated to $\mathfrak{a}$ that we will give in Theorem 4.3.1.

We will also provide a second formula for the multiplicity in Proposition 4.1.10 that is based on the concept of *virtual codimension* considered by Casas-Alvero [CA00] and Reguera [Reg97] for the smooth and the rational singularities case respectively. The results presented in this chapter correspond to [AADG14, Sections 4,5 and 6].

4.1 Multiplicities of Jumping Numbers

We start with the first formula.

Theorem 4.1.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$m(c) = (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \# \{\textit{connected components of } H_c\}.$$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow$$
$$\longrightarrow \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow 0$$

Pushing it forward to X and applying local vanishing for multiplier ideals we get the short exact sequence

$$0 \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow$$
$$\longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \otimes \mathbb{C}_O \longrightarrow 0$$

or equivalently, since $H_c = \lceil K_\pi - (c - \varepsilon) F \rceil - \lceil K_\pi - cF \rceil$ for ε small enough,

$$0 \longrightarrow \mathcal{J}(\mathfrak{a}^c) \longrightarrow \mathcal{J}(\mathfrak{a}^{(c-\varepsilon)}) \longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \otimes \mathbb{C}_O \longrightarrow 0$$

Therefore the multiplicity of c is just

$$\begin{aligned} m(c) &= h^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \\ &= \sum_{E_i \leqslant H_c} h^0(E_i, \mathcal{O}_{E_i}(\lceil K_\pi - cF \rceil + H_c)) - a_{H_c}, \end{aligned}$$

where in the second equality we have used that H_c has simple normal crossings, and hence the sections of the line bundle $\mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)$ correspond to sections over each component that agree on the a_{H_c} intersections.

Recall now that each exceptional component E_i is isomorphic to $\mathbb{P}^1$, and that the sections of a line bundle on $\mathbb{P}^1$ are determined by its degree (namely, $h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = d + 1$ if $d \geqslant -1$ and zero otherwise). Then, using that

$$\deg \mathcal{O}_{E_i}(\lceil K_\pi - cF \rceil + H_c) = (\lceil K_\pi - cF \rceil + H_c) \cdot E_i \geqslant -1$$

by Proposition 3.3.3, we get

$$\begin{aligned} m\left(c\right) &= \sum_{E_i \leqslant H_c} \left(\left(\lceil K_\pi - cF \rceil + H_c\right) \cdot E_i + 1\right) - a_{H_c} \\ &= \left(\lceil K_\pi - cF \rceil + H_c\right) \cdot H_c + v_{H_c} - a_{H_c} \\ &= \left(\lceil K_\pi - cF \rceil + H_c\right) \cdot H_c + \#\left\{\text{connected components of } H_c\right\}. \end{aligned}$$

□

Remark 4.1.2. When $c = \lambda$ is a jumping number, the same formula for the multiplicity can be described using the associated minimal jumping divisor G_λ. Namely,

$$m(\lambda) = (\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot G_\lambda + \#\{\text{connected components of } G_\lambda\}$$

The proof of this result holds verbatim to the one given for Theorem 4.1.1 but we have to refer to Proposition 3.2.3 instead of Proposition 3.3.3.

For reduced divisors in the interval $G_\lambda < G < H_\lambda$ we may have $E_i \leqslant G$ such that

$$(\lceil K_\pi - \lambda F \rceil + G) \cdot E_i = -2 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{\lambda e_j - k_j\} + \lambda \rho_i + a_G(E_i) = -2.$$

Namely, this happens when E_i is a non-dicritical isolated component of G with all adjacent divisors in H_λ. However, these divisors can also provide a formula for the multiplicity of a jumping number as follows. Refining the arguments used in the proof of Theorem 4.1.1 we obtain:

$$\begin{aligned} m(\lambda) = (\lceil K_\pi - \lambda F \rceil + G) \cdot G + \#\{\text{c.c. of } G\} & \\ + \#\left\{E_i \;\mid\; (\lceil K_\pi - \lambda F \rceil + G) \cdot E_i = -2\right\}. & \end{aligned}$$

In some cases it will be more convenient to use the following reinterpretation of the formula given in Theorem 4.1.1.

Corollary 4.1.3. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$\begin{aligned} m\left(c\right) = \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{c e_j - k_j\} + c \rho_i \right) & \\ - \#\left\{\textit{connected components of } H_c\right\}. & \end{aligned}$$

Proof. Using Lemma 3.3.1 we have:

$$m(c) = (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \# \{\text{connected components of } H_c\}$$

$$= \sum_{E_i \leqslant H_c} \left(-2 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i + a_{H_c}(E_i) \right) + \# \{\text{c.c. of } H_c\}$$

$$= -2v_H + \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i \right) + 2a_{H_c} + \# \{\text{c.c. of } H_c\}$$

$$= \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i \right) - \# \{\text{c.c. of } H_c\}$$

□

As an immediate consequence of this we obtain the following slight generalization of a result of Tucker [Tuc10, Proposition 7.3]. We point out that Järviletho already proved in [Jär11] that 1 is not a jumping number for simple $\mathfrak{m}$-primary ideals.

Corollary 4.1.4. *Suppose that O is a smooth point, and let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The multiplicity of $c = 1$ is*

$$m(1) = \rho - 1.$$

In particular, $c = 1$ is a jumping number if and only if $\mathfrak{a}$ is not simple.

Proof. The maximal jumping divisor for $c = 1$ has the same support as F, so the result follows from Corollary 4.1.3. □

From the formula given above and the periodicity of the maximal jumping divisor H_c, it is easy to control the growth of the multiplicities in terms of the excesses at dicritical components. This result is a key point in the proof of Theorem 4.3.1.

Proposition 4.1.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$m(c+1) - m(c) = \sum_{E_i \leqslant H_c} \rho_i.$$

In particular, $0 \leqslant m(c+1) - m(c) \leqslant \rho$.

Proof. Recall that c and $c+1$ have the same jumping divisor H_c (see Lemma 3.0.10). Therefore, by Theorem 4.1.1, we have

$$m(c+1) - m(c) = -F \cdot H_c = \sum_{E_i \leqslant H_c} \rho_i.$$

□

4.1.1 Virtual codimensions

Given an effective $\mathbb{R}$-divisor $D = \sum d_i E_i$ with exceptional support we may consider its associated (sheaf) ideal $\pi_* \mathcal{O}_{X'}(-D) := \pi_* \mathcal{O}_{X'}(-\lceil D \rceil)$. Its stalk at O is an $\mathfrak{m}$-primary complete ideal of $\mathcal{O}_{X,O}$ that we will simply denote as I_D. We say that two divisors are *equivalent* if they define the same ideal. In the equivalence class of a given divisor D one may find a unique maximal representative, its so-called *antinef closure* $\widetilde{D}$ (see [Lip69, §18]). First, recall that an effective divisor with integer coefficients D' is called *antinef* if $-D' \cdot E_i \geqslant 0$, for every exceptional prime divisor E_i.

The antinef closure of D can be computed using the *unloading* presented in Section 1.7. Recall that unloading values to any D is to consider the new divisor

$$D' = \lceil D \rceil + \sum_{E_i \in \Theta} n_i E_i,$$

where Θ is the set of components $E_i \leqslant D$ with negative excesses, i.e.

$$\Theta := \{ E_i \leqslant D \mid \rho_i = -\lceil D \rceil \cdot E_i < 0 \}$$

and $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$. We say that the unloading is *tame* if $\rho_i = -1$ for all $E_i \in \Theta$ and there are no adjacent divisors in Θ. This is a mild generalization of the notion of tameness introduced in [CA00]. The antinef closure $\widetilde{D}$ of D is achieved after finitely many unloading steps.

Given a divisor D with exceptional support, we will define its *virtual codimension* or *virtual number of conditions* as

$$\mathcal{C}(D) := -\frac{\lceil D \rceil \cdot (\lceil D \rceil + K_\pi)}{2}.$$

The main feature of this invariant is that it coincides with the codimension of the associated ideal when D is antinef. For a proof of this result one may consult [CA00, Proposition 4.7.1] for the smooth case and [Reg97, Proposition 3.7] for the rational singularities case.

Proposition 4.1.6. *Let D be an antinef divisor and I_D its associated ideal. Then:*

$$\mathcal{C}(D) = \dim_{\mathbb{C}} \mathcal{O}_{X,O}/I_D$$

This result is no longer true for arbitrary divisors. However, there are some non-antinef divisors for which this equality holds.

Proposition 4.1.7. *Assume that a divisor D' is obtained from a divisor D by performing a single unloading step. Then $\mathcal{C}(D) \geqslant \mathcal{C}(D')$ and the equality holds if and only if the unloading step is tame.*

Proof. Notice that, in order to compute the virtual codimension, we may always assume $D = \lceil D \rceil$. Hence, $D' = D + \sum_{E_i \in \Theta} n_i E_i$, where Θ and $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$ are defined as above. Therefore:

$$\begin{aligned}
\mathcal{C}(D) - \mathcal{C}(D') &= -\frac{1}{2}\left(D^2 - {D'}^2 + K_\pi \cdot (D - D')\right) \\
&= -\frac{1}{2}\left(-2\left(\sum_i n_i E_i\right) D - \left(\sum_i n_i E_i\right)^2 - K_\pi \cdot \left(\sum_i n_i E_i\right)\right) \\
&= \frac{1}{2}\left(2\left(\sum_i n_i E_i\right) D + \left(\sum_i n_i E_i\right)^2 - 2\sum_i n_i - \sum_i n_i E_i^2\right) \\
&= \sum_i \frac{n_i}{2}\left(-2\rho_i + (n_i - 1)E_i^2 - 2\right) + \sum_i \sum_{j>i} n_i n_j E_i \cdot E_j
\end{aligned}$$

We are assuming $n_i \geqslant 1$ for all $E_i \in \Theta$ so the summands

$$\frac{n_i}{2}\left(-2\rho_i + (n_i - 1)E_i^2 - 2\right)$$

are always $\geqslant 0$. Notice that they are zero if and only if $\rho_i = -1$ for all $E_i \in \Theta$. On the other hand, $\sum_i \sum_{j>i} n_i n_j E_i \cdot E_j \geqslant 0$ and equality holds if and only if $E_i \cdot E_j = 0$ for all $E_i \neq E_j \in \Theta$, i.e., there are no adjacent divisors in the set Θ. $\square$

Corollary 4.1.8. *Let $\widetilde{D}$ be the antinef closure of a divisor D and I_D their associated ideal, then:*

$$\mathcal{C}(D) \geqslant \mathcal{C}(\widetilde{D}) = \dim_{\mathbb{C}} \mathcal{O}_{X,O}/I_D$$

and the equality holds if and only if all the unloading steps performed to obtain $\widetilde{D}$ are tame.

When we deal with multiplier ideals we can extract a very simple formula for the multiplicity of any real number.

Proposition 4.1.9. *Let D_c and $D_{c-\varepsilon}$ be the antinef closures of $\lfloor cF - K_\pi \rfloor$ and $\lfloor (c-\varepsilon)F - K_\pi \rfloor$ respectively, for any $c \in \mathbb{R}_{\geqslant 0}$ and ε small enough. Then, the multiplicity of c is*

$$m(c) = \mathcal{C}(D_c) - \mathcal{C}(D_{c-\varepsilon}) = \frac{D_{c-\varepsilon} \cdot (D_{c-\varepsilon} + K_\pi)}{2} - \frac{D_c \cdot (D_c + K_\pi)}{2}.$$

Actually there is no need to compute the antinef closure of the aforementioned divisors to obtain the same result.

Proposition 4.1.10. *For any $c \in \mathbb{R}_{\geqslant 0}$ and ε small enough we have*

$$\begin{aligned} m(c) &= \mathcal{C}(\lfloor cF - K_\pi \rfloor) - \mathcal{C}(\lfloor (c-\varepsilon)F - K_\pi \rfloor) \\ &= \frac{\lfloor (c-\varepsilon)F - K_\pi \rfloor \cdot (\lfloor (c-\varepsilon)F - K_\pi \rfloor + K_\pi)}{2} \\ &\quad - \frac{\lfloor cF - K_\pi \rfloor \cdot (\lfloor cF - K_\pi \rfloor + K_\pi)}{2}. \end{aligned}$$

Proof. Recall that $\lceil K_\pi - (c-\varepsilon)F \rceil = \lceil K_\pi - cF \rceil + H_c$. Then:

$$\begin{aligned} &\mathcal{C}(\lfloor cF - K_\pi \rfloor) - \mathcal{C}(\lfloor cF - K_\pi \rfloor - H_c) \\ &\quad = \frac{1}{2}(\lfloor cF - K_\pi \rfloor - H_c) \cdot (\lfloor cF - K_\pi \rfloor - H_c + K_\pi) \\ &\qquad - \frac{1}{2}(\lfloor cF - K_\pi \rfloor) \cdot (\lfloor cF - K_\pi \rfloor + K_\pi) \\ &\quad = - \lfloor cF - K_\pi \rfloor \cdot H_c + \frac{H_c \cdot H_c}{2} - \frac{K_\pi \cdot H_c}{2} \end{aligned}$$

$$
\begin{aligned}
&= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c - \frac{(H_c + K_\pi) \cdot H_c}{2} \\
&= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \#\{\text{connected components of } H_c\} \\
&= m(c).
\end{aligned}
$$

Here we used the fact that

$$\frac{1}{2}(K_\pi + H_c) \cdot H_c = -v_{H_c} + a_{H_c} = -\#\{\text{connected components of } H_c\}$$

and Theorem 4.1.1.

□

Let $\lambda' < \lambda$ be two consecutive jumping numbers of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Despite the fact that $\lfloor \lambda' F - K_\pi \rfloor$ and $\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor$ have the same antinef closure their virtual codimensions may differ. However, we still have the following description of the multiplicity

Proposition 4.1.11. *Let $\lambda' < \lambda$ be two consecutive jumping numbers of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then, the multiplicity of λ is*

$$
\begin{gathered}
m(\lambda) = \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor \lambda' F - K_\pi \rfloor) = \\
= \frac{\lfloor \lambda' F - K_\pi \rfloor \cdot (\lfloor \lambda' F - K_\pi \rfloor + K_\pi)}{2} - \frac{\lfloor \lambda F - K_\pi \rfloor \cdot (\lfloor \lambda F - K_\pi \rfloor + K_\pi)}{2}.
\end{gathered}
$$

Proof. Consider all the rational numbers $\gamma \in (\lambda', \lambda)$ for which there exists at least one component E_i such that $\gamma e_i - k_i \in \mathbb{Z}$. We order them to form a finite sequence of rational numbers $\lambda' < \gamma_1 < \cdots < \gamma_r < \lambda$. Notice that these are the only rational numbers in this interval where the virtual codimension of $\lfloor \gamma F - K_\pi \rfloor$ may increase.

We have

$$
\begin{aligned}
m(\lambda) &= \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor) \\
&= \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor \gamma_r F - K_\pi \rfloor)
\end{aligned}
$$

and, at every step of the sequence, $m(\gamma_i) = \mathcal{C}(\lfloor \gamma_i F - K_\pi \rfloor) - \mathcal{C}(\lfloor \gamma_{i-1} F - K_\pi \rfloor)$. Therefore

$$m(\lambda) = m(\lambda) + \sum_{i>0} m(\gamma_i) = \mathcal{C}\left(\lfloor \lambda F - K_\pi \rfloor\right) - \mathcal{C}\left(\lfloor \lambda' F - K_\pi \rfloor\right)$$

due to the fact that $m(\gamma_i) = 0$ as these rational numbers are not jumping numbers. $\square$

Remark 4.1.12. In the case that X is smooth we can check that the unloading steps needed to compute the antinef closure of $\lfloor cF - K_\pi \rfloor$ for any $c \in \mathbb{R}_{\geqslant 0}$ are tame. Indeed, repeating the same arguments considered in the proof of Proposition 4.1.11 we may end up with the case $c = 0$. It is then easy to check that $\mathcal{C}\left(\lfloor -K_\pi \rfloor\right) = \mathcal{C}(D_0) = 0$ so we get

$$\mathcal{C}\left(\lfloor cF - K_\pi \rfloor\right) = \mathcal{C}(D_c)\,.$$

This concludes the remark thanks to Corollary 4.1.8.

4.2 Jumping Numbers via multiplicities

Fix a log-resolution $\pi : X' \longrightarrow X$ of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Consider the relative canonical divisor $K_\pi = \sum_{i=1}^r k_i E_i$, and the divisor $F = \sum_{i=1}^r e_i E_i$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. The jumps between multiplier ideals must occur at rational numbers that belong to the set of *candidate jumping numbers* (recall Definition 1.11.2)

$$\left\{ \frac{k_i + m}{e_i} \;\mid\; m \in \mathbb{Z}_{>0} \right\}.$$

Not every candidate jumping number is necessarily a jumping number. Using the formulas for the multiplicity given in the previous section we can easily extract the set of jumping numbers since we have:

Proposition 4.2.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and $c \in \mathbb{R}_{>0}$. Then, c is a jumping number if and only if $m(c) > 0$.*

In addition, we have the following simple criterion

Theorem 4.2.2. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and $c \in \mathbb{R}_{>0}$. Then, there exists a connected component $H \leqslant H_c$ such that*

$$\left(\lceil K_\pi - cF \rceil + H_c\right) \cdot H > -1$$

if and only if $m(c) > 0$.

Proof. The result follows from Theorem 4.1.1 and Proposition 3.3.3. $\square$

Therefore we have a simple algorithm to compute the set of jumping numbers of $\mathfrak{a}$ that boils down to compute the multiplicity of the rational numbers in the set of candidate jumping numbers by means of the formula given in Theorem 4.1.1 or the one given in Proposition 4.1.10. We have implemented this algorithm in the Computer Algebra system `Macaulay 2` [GS]. The scripts of the source codes as well as the output in full detail of some examples is available at the web page

`www.pagines.ma1.upc.edu/~jalvz/multiplier.html`

and also in Appendix B. It turns out that this algorithm is more efficient than the algorithms considered by Tucker in [Tuc10] and the one in Chapter 2.

4.2.1 Jumping numbers contributed by dicritical divisors

Another interesting consequence of the methods developed in the previous sections is the fact that we can describe a big chunk of the set of jumping numbers by means of an inspection of dicritical divisors. In the sequel we will consider a dicritical divisor E_i with excess $\rho_i = -F \cdot E_i > 0$ and value $v_i(F) = e_i$.

Theorem 4.2.3. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. Let $k \in \mathbb{N}$ be a non-negative integer number such that $\frac{k}{e_i} > \frac{1}{\rho_i}$. Then, $\lambda = \frac{k}{e_i}$ is a jumping number.*

Proof. Let $H \leqslant H_\lambda$ be the connected component that contains the dicritical divisor E_i. For $\lambda = \frac{k}{e_i} > \frac{1}{\rho_i}$ we have

$$\begin{aligned}(\lceil K_\pi - \lambda F \rceil + H_\lambda) \cdot H &= \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{E_j \leqslant H_\lambda} \lambda \rho_j - 2 \\ &> \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j + 1 - 2 \\ &\geqslant -1\end{aligned}$$

and the result follows from Theorem 4.2.2. $\square$

For the boundary case $\lambda = \frac{1}{\rho_i}$ we have the following criteria.

Proposition 4.2.4. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. Let $k \in \mathbb{N}$ be a non-negative integer number such that $\frac{k}{e_i} = \frac{1}{\rho_i}$. Then, the following are equivalent:*

i) $\lambda = \frac{1}{\rho_i}$ *is not a jumping number.*

ii) $H_\lambda = E$ *is the whole exceptional component, and E_i is the only dicritical divisor.*

Proof. Let $H \leqslant H_\lambda$ be the connected component that contains the dicritical divisor E_i. For $\lambda = \frac{k}{e_i} = \frac{1}{\rho_i}$ we have

$$(\lceil K_\pi - \lambda F \rceil + H_\lambda) \cdot H = \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j + 1 - 2$$

By Theorem 4.2.2, $\lambda = \frac{k}{e_i} = \frac{1}{\rho_i}$ is not a jumping number when this intersection multiplicity is -1. Notice that a divisor E_j satisfies $\{\lambda e_j - k_j\} = 0$ if and only if $E_j \leqslant H_\lambda$. Thus

$$\sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} = 0$$

if and only if $\mathrm{Adj}(H) = \emptyset$, or equivalently when $H_\lambda = E$. On the other hand

$$\sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j = 0$$

if and only if $\rho_j = 0$ for all $j \neq i$, i.e. when there are no dicritical divisors besides E_i.

□

Notice that the result above also generalizes the fact that 1 is not a jumping number for simple $\mathfrak{m}$-primary ideals. We can also extend to our setting Järviletho's result on the behavior of the jumping numbers in the interval $(1, 2]$ given in [Jär11, Theorem 9.9] for simple complete ideals in a smooth surface.

Theorem 4.2.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The only jumping numbers in the interval $(1, 2]$ are the following:*

- $\lambda + 1$, *where $\lambda \in (0, 1]$ is a jumping number.*
- $\lambda = \frac{k}{e_i}$, *for $e_i < k \leqslant 2e_i$ with E_i dicritical divisor.*

Proof. Assume that a jumping number $\lambda \in (1,2]$ is not of the announced types and consider its associated maximal jumping divisor H_λ. If λ is not of the first type then $m(\lambda) - m(\lambda - 1) > 0$. If it is not of the second type, then $\rho_i = 0$ for any $E_i \leqslant H_\lambda$. Both conditions cannot be satisfied simultaneously by Proposition 4.1.5 so we get a contradiction. $\square$

Remark 4.2.6. Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. A generic element $f \in \mathfrak{a}$ satisfies $\mathcal{J}(f^c) = \mathcal{J}(\mathfrak{a}^c)$ for any $c \in (0,1)$ so Theorem 4.2.5 says, roughly speaking, that the jumping numbers of $\mathfrak{a}$ are governed by the jumping numbers of a generic element $f \in \mathfrak{a}$ and the dicritical divisors of $\mathfrak{a}$.

4.3 Poincaré series of multiplier ideals

Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. In this section we will give a very simple description of the Poincaré series of multiplier ideals.

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^c = \sum_{c \in (0,1]} \sum_{k \in \mathbb{N}} m(c+k)\, t^{c+k}$$

To such purpose we only need to control the following two issues: First we have to describe the multiplicities of the jumping numbers in the interval $(0,1]$. This can be done using the formulas given in Theorem 4.1.1 or Proposition 4.1.10. Secondly, and equally important, we have to control the recurrence that these multiplicities satisfy. As shown in Proposition 4.1.5, dicritical components in the maximal jumping divisor allow us to describe the recurrence.

The main result of this section is the fact that the Poincaré series of multiplier ideals is rational in the sense that it belongs to the field of fractional functions $\mathbb{C}(z)$, where the indeterminate z corresponds to a fractional power $t^{1/e}$ for $e \in \mathbb{N}_{>0}$ being the least common multiple of the denominators of all jumping numbers. The formula for the Poincaré series that we obtain is the following:

Theorem 4.3.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The Poincaré series of $\mathfrak{a}$ can be expressed as*

$$P_{\mathfrak{a}}(t) = \sum_{c \in (0,1]} \left(\frac{m(c)}{1-t} + \rho_c \frac{t}{(1-t)^2} \right) t^c$$

where $\rho_c = -F \cdot H_c$ and H_c is the maximal jumping divisor associated to c.

Proof. Let $c \in (0,1]$ be a real number. For any $k \in \mathbb{N}$ we have, applying Proposition 4.1.5

$$m(c+k) = m(c) + k\rho_c,$$

where $\rho_c = m(c+1) - m(c) = -F \cdot H_c$. It follows that

$$\sum_{k \geqslant 0} m(c+k)\, t^{c+k} = m(c)\, t^c + (m(c) + \rho_c)\, t^{c+1} + (m(c) + 2\rho_c)\, t^{c+2} + \cdots$$

$$= \left(\frac{m(c)}{1-t} + \rho_c \frac{t}{(1-t)^2} \right) t^c$$

Thus we get the desired result. □

For the case of simple $\mathfrak{m}$-primary ideals we can easily recover the extension to the case where X has rational singularities of the main result of Galindo-Monserrat [GM10]. Our formulation slightly differs from theirs because we collect jumping numbers by the growth of the multiplicities instead of its critical divisors.

Corollary 4.3.2. [GM10, Theorem 2.1] *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be a simple $\mathfrak{m}$-primary ideal. The Poincaré series of $\mathfrak{a}$ can be expressed as*

$$P_{\mathfrak{a}}(t) = \sum_{\substack{c \in (0,1] \\ \rho_c = 0}} \frac{m(c)}{1-t} t^c + \sum_{\substack{c \in (0,1] \\ \rho_c = 1}} \left(\frac{m(c)}{1-t} + \frac{t}{(1-t)^2} \right) t^c$$

Proof. Simple $\mathfrak{m}$-primary ideals only have one dicritical divisor with excess 1 so the result follows. □

4.3.1 Hodge Spectrum

Let X be a smooth complex variety of dimension d and consider an hypersurface with an isolated singularity at O defined by $f \in \mathcal{O}_{X,O}$. The Hodge spectrum $Sp(f)$ associated to f was introduced by Steenbrink [Ste77] using the canonical mixed Hodge structure of the cohomology groups of the Milnor fiber of f. It is a fractional polynomial

$$Sp(f) = \sum_{c \in [0,d]} n(c)\, t^c,$$

where the rational number $c \in \mathbb{Q}$ is an *exponent* or *spectral number* if its associated multiplicity $n(c)$ is strictly positive. It is also known that the sum of all spectral numbers, counted with multiplicity, is equal to the Milnor number of f and that they are symmetric with respect to $\frac{d}{2}$, i.e. $n(c) = n(d-c)$

Budur [Bud03] established a nice relation between the Hodge spectrum and the set of multiplier ideals. More precisely, the multiplicity of spectral numbers and the multiplicity of the so-called *inner* jumping numbers coincide in the interval $(0,1]$. We point out that the usual jumping numbers are inner jumping numbers whenever they are not integer numbers in the case of hypersurfaces with isolated singularities.

In the case where X has dimension two we can make a closer relationship between the Hodge spectrum of a plane curve $f \in \mathcal{O}_{X,O}$, that we assume as a generic element of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, and the Poincaré series of multiplier ideals of $\mathfrak{a}$. Roughly speaking, the information given by the Hodge spectrum is equivalent, taking into account the symmetry with respect to 1, to the information given by the terms of the Poincaré series in the interval $(0,1)$. The aim of this section is to strengthen this relationship recovering some old results on the Hodge spectrum of a plane curve by using our methods.

The spectrum of a plane curve has been described by Lê Văn Thành and Steenbrink in [TS89] (see also [Thà88], [Sch90]). For the convenience of the reader we will reformulate their result using the terminology we are considering in this Thesis. To this aim, we consider a partial order on the exceptional components of the log-resolution. Since we are assuming that O is a smooth point, the exceptional divisor is naturally a *rooted* tree of rational curves, where the root E_1 is the (strict transform of) the exceptional divisor of the blow-up of O. The partial order is then defined by the paths from E_1, i.e. E_i precedes E_j if E_i belongs to the chain of components connecting E_1 and E_j. For any $i \neq 1$, we denote by $p\,(i)$ the index of the exceptional component immediately preceding E_i, so that $E_{p(i)}$ belongs to the chain connecting E_1 and E_i, and $E_i \cdot E_{p(i)} = 1$. The set of rupture or dicritical divisors different from the root E_1 will be denoted $\mathcal{R}$, i.e.

$$\mathcal{R} = \{i \,|\, E_i \neq E_1 \text{ is a rupture or dicritical divisor}\}.$$

Theorem 4.3.3. *[TS89, Theorem 1.5] Let $f \in \mathcal{O}_{X,O}$ be the equation of a plane curve with an isolated singularity at the origin O. Let $c \in \mathbb{Q}$ be a rational number. Then, its associated multiplicity $n(c)$ in the Hodge spectrum of f is $n(c) = n'(c) + n''(c)$, where:*

- $n'(c) = \#\left\{E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c\right\}$
- $$n''(c) = \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(-1 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right)$$

If we assume f as a generic element of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,0}$ we can recover this result using the formula given in Theorem 4.1.1.

Proposition 4.3.4. *Let $f \in \mathcal{O}_{X,O}$ be the equation of a plane curve with an isolated singularity at the origin O. For any $c \in (0,1)$ we have $n(c) = m(c)$.*

Proof. Lê Văn Thành and Steenbrink's formula states that:

$$\begin{aligned} n(c) = {} & \#\left\{E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c\right\} \\ & + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(-1 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \\ = {} & \#\left\{E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c\right\} \\ & - \#\left\{E_i \,|\, i \in \mathcal{R}\cup\{1\} \text{ and } E_i \leqslant H_c\right\} \\ & + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \\ = {} & -\#\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\} - \delta \\ & + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \end{aligned}$$

where $\delta = 1$ if $E_1 \leqslant H_c$ and $\delta = 0$ otherwise. Due to the rooted tree structure of the exceptional divisor, every connected component of H_c has exactly one minimal component E_i (the closest to E_1), and clearly $E_{p(i)} \nleqslant H_c$ if $i \neq 1$. There is therefore a bijection between the set $\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\}$ and the connected components of H_c that contain some rupture or dicritical component but do not contain E_1. Hence we have proved

$$\begin{aligned} & \#\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\} + \delta \\ & \quad = \#\left\{\substack{\text{connected components of } H_c \\ \text{containing a divisor } E_i,\ \ i \in \mathcal{R}\cup\{1\}}\right\}, \end{aligned}$$

which gives the following expression for $n\,(c)$:

$$n(c) = \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R} \cup \{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i \right) - \# \left\{ \substack{\text{connected components of } H_c \\ \text{containing a divisor } E_i, \ \ i \in \mathcal{R} \cup \{1\}} \right\} \tag{4.3.1}$$

On the other hand, Corollary 4.1.3 gives (recall that $k_i \in \mathbb{Z}$ because O is a smooth point)

$$m\,(c) = \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i \right) - \# \{\text{connected components of } H_c\} \,. \tag{4.3.2}$$

To prove that both formulas coincide, we have to consider the terms

$$\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i$$

for the $E_i \leqslant H_c$ with $i \notin \mathcal{R} \cup \{1\}$, as well as the connected components of H_c containing only components of this kind.

Consider first an E_i which is not an isolated component of H_c. On the one hand, by Theorem 3.3.4, all its adjacent components are contained in H_c, and hence $\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} = 0$. Since it is not dicritical, $\rho_i = 0$, and therefore E_i does not contribute to the first summand of $m\,(c)$. On the other hand, the connected component H of H_c containing E_i contains also either a rupture or dicritical component (again by Theorem 3.3.4), and hence its contribution to the second summand of (4.3.2) is already taken into account in (4.3.1).

To finish the proof, it remains to consider the E_i which are isolated components of H_c. In this case, Theorem 3.3.4 says that the contribution of E_i to the first term of (4.3.2) is $\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} = 1$, which cancels with the contribution to the number of connected components. □

Chapter 5

Mixed multiplier ideals

The aim of this chapter is to present an algorithm to compute the jumping walls associated to a family of ideals. We begin this chapter by recalling the definition of mixed multiplier ideal and announcing some of its properties. The second part is devoted to present an algorithm to compute the jumping walls and the associated ideals. In the third part we develop the concept of jumping divisor, that will allow us to endow the jumping walls with a notion of multiplicity. In the last part we introduce some results regarding the multiplicity.

5.1 Definitions and first properties

As stated, we begin by recalling the definition of mixed multiplier ideal (see definition 1.8.3).

Definition 5.1.1. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals and $\pi : X' \to X$ a log-resolution of this tuple with F_i the divisors such that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for $1 \leqslant i \leqslant r$. The *mixed multiplier ideal* associated to a point $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ and $\boldsymbol{\mathfrak{a}}$ is defined as[1]

$$\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) := \mathcal{J}\left(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil\right),$$

where $K_\pi = \sum_{j=1}^s k_j E_j$ is the relative canonical divisor and $F_i = \sum_{j=1}^s e_{i,j} E_j$.

[1] By abuse of notation, we will also denote by $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right)$ its stalk at O, so we will omit the word "sheaf" if no confusion arises.

Whenever we only consider a single ideal $\mathfrak{a} = \mathfrak{a}_1 \subseteq \mathcal{O}_{X,O}$ we recover the usual notion of *multiplier ideal*, and is not difficult to check out that mixed multiplier ideals satisfy analogous properties. For example, the definition of mixed multiplier ideals is independent of the choice of log resolution, they are complete ideals and are invariants up to integral closure, so we can always assume that the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are complete. For a detailed overview we refer to Chapter 1.

Remark 5.1.2. The mixed multiplier ideals of a tuple $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ associated to the ray passing through the origin $\boldsymbol{\lambda}_0 = (0, \ldots, 0)$ in the direction of a vector $(u_1, \ldots, u_r) \in \mathbb{Q}^r_{\geqslant 0}$ are the usual multiplier ideals of the ideal $\mathfrak{a}_1^{\alpha u_1} \cdots \mathfrak{a}_r^{\alpha u_r}$, with a convenient $\alpha \in \mathbb{Z}$ such that $\alpha \cdot u_i \in \mathbb{Z}$ for all i.

From the definition of mixed multiplier ideal one can easily deduce properties on the contention of the ideal corresponding to a fixed point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, with respect to those ideals of points in its neighborhood. These properties are rather obvious, but we include them here for completeness.

In the sequel, $B_\varepsilon(\boldsymbol{\lambda})$ will denote the Euclidean open ball centered in $\boldsymbol{\lambda}$ with radius $\varepsilon > 0$. We start by giving some properties of points in the positive orthant with respect to a given point.

Proposition 5.1.3 (Positive orthant properties)**.** *Fix a point* $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$.

i) We have $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}'})$ *for any* $\boldsymbol{\lambda}' \in \boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}$.

ii) We have $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}'})$ *for any* $\boldsymbol{\lambda}' \in \left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap B_\varepsilon(\boldsymbol{\lambda})$ *with* $\varepsilon > 0$ *small enough.*

iii) Let $\boldsymbol{\lambda}' \in \boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}$ *be a point such that* $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}'})$. *Then,* $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}''})$ *for any* $\boldsymbol{\lambda}'' \in \left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap \left(\boldsymbol{\lambda}' - \mathbb{R}^r_{\geqslant 0}\right)$.

Proof. i) We have $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor \leqslant \lfloor \lambda'_1 F_1 + \cdots + \lambda'_r F_r - K_\pi \rfloor$, so by Proposition 1.6.4 the result follows.

ii) For every exceptional divisor E_i, consider the minimal $\varepsilon_i > 0$ such that

$$\lfloor (\lambda_1 + \varepsilon_i) e_{1,i} + \cdots + (\lambda_r + \varepsilon_i) e_{r,i} - k_i \rfloor = \lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor + 1.$$

Then, if we take $\varepsilon > 0$ such that $\varepsilon < \varepsilon_i$ for all i, we get

$$\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor = \lfloor (\lambda_1 + \varepsilon) F_1 + \cdots + (\lambda_r + \varepsilon) F_r - K_\pi \rfloor \ .$$

iii) Assume that there exists $\boldsymbol{\lambda''} \in \left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap \left(\boldsymbol{\lambda'} - \mathbb{R}^r_{\geqslant 0}\right)$ such that $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda''}})$. We have $\boldsymbol{\lambda'} \in \boldsymbol{\lambda''} + \mathbb{R}^r_{\geqslant 0}$, thus $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda''}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda'}})$ by using part i). Therefore we get $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda''}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda'}})$, contradicting our assumption.

□

We can give some properties for points in the negative orthant as well.

Proposition 5.1.4 (Negative orthant properties)**.** *Fix a point* $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$.

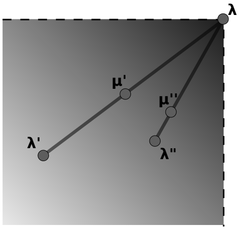
i) We have $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda'}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$ *for any* $\boldsymbol{\lambda'} \in \boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}$.

ii) Let $\boldsymbol{\lambda'} \in \boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}$ *be a point such that* $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\mu'}}\right) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$, *for any* $\boldsymbol{\mu'} \neq \boldsymbol{\lambda}$ *in the segment* $\overline{\boldsymbol{\lambda}\boldsymbol{\lambda'}}$. *Then, any* $\boldsymbol{\lambda''} \in \boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}$ *also satisfies* $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\mu''}}\right) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$, *for any* $\boldsymbol{\mu''} \neq \boldsymbol{\lambda}$ *in the segment* $\overline{\boldsymbol{\lambda}\boldsymbol{\lambda''}}$.

iii) For any $\boldsymbol{\lambda'}, \boldsymbol{\lambda''} \in \left(\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}\right) \cap B_\varepsilon(\boldsymbol{\lambda})$ *with* $\varepsilon > 0$ *small enough, we have* $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda'}}\right) = \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda''}}\right)$.

Proof. i) This is equivalent to i) in Proposition 5.1.3.

ii) Assume that there exists $\boldsymbol{\mu''}$ in the segment $\overline{\boldsymbol{\lambda}\boldsymbol{\lambda''}}$ such that $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\mu''}}\right) = \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$. Then, using Proposition 5.1.3 iii), we have $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\mu}}\right) = \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$ for any $\boldsymbol{\mu} \in (\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}) \cap (\boldsymbol{\mu''} + \mathbb{R}^r_{\geqslant 0})$. So we get a contradiction because this region contains points in the segment $\overline{\boldsymbol{\lambda}\boldsymbol{\lambda'}}$.



iii) For every exceptional divisor E_i, consider the maximal $\varepsilon_i > 0$ such that

$$\lfloor (\lambda_1 - \varepsilon_i)e_{1,i} + \cdots + (\lambda_r - \varepsilon_i)e_{r,i} - k_i \rfloor = \lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor - \delta_i,$$

where $\delta_i = 1$ if $\lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \in \mathbb{Z}_{\geqslant 0}$ and $\delta_i = 0$ otherwise. Then, if we take $\varepsilon > 0$ such that $\varepsilon < \varepsilon_i$ for all i, we obtain the desired result.

□

The above results give us some understanding of the behavior of the mixed multiplier ideals in the positive and negative orthants of a given point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$.

Indeed, we can give the following result for the rest of points in a small neighborhood of $\boldsymbol{\lambda}$.

Proposition 5.1.5 (Points in a small neighborhood). *The mixed multiplier ideal associated to some $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is the smallest among the mixed multiplier ideals in a small neighborhood. That is, we have $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}'}\right) \supseteq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$, for any $\boldsymbol{\lambda}' \in B_\varepsilon(\boldsymbol{\lambda})$ and $\varepsilon > 0$ small enough.*

Proof. We have $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\mu}'}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$ for any $\boldsymbol{\mu}' \in \left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap B_{\varepsilon'}(\boldsymbol{\lambda})$ and $\varepsilon' > 0$ small enough. Consider a ball $B_\varepsilon(\boldsymbol{\lambda})$ such that

$$\left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap B_\varepsilon(\boldsymbol{\lambda}) \subseteq \left(\boldsymbol{\lambda} + \mathbb{R}^r_{\geqslant 0}\right) \cap \left(\boldsymbol{\mu}' - \mathbb{R}^r_{\geqslant 0}\right) .$$

Then, using Proposition 5.1.4 *i*), we have that $\mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}'}\right) \supseteq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$, for any $\boldsymbol{\lambda}' \in B_\varepsilon(\boldsymbol{\lambda})$.

$\square$

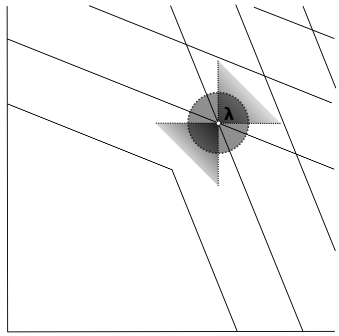


Figure 5.1: Small neighborhood of a given point $\boldsymbol{\lambda}$.

5.1.1 Jumping Walls

The most significative difference that we face when dealing with mixed multiplier ideals is that, whereas the usual multiplier ideals come with an attached set of numerical invariants, the *jumping numbers* (see Section 1.10), the corresponding notion for mixed multiplier ideals is more involved and is described in terms of the so-called *jumping walls* that we will introduce next. As these notions are based on the contention of multiplier ideals, it is then natural to consider the following.

Definition 5.1.6. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. Then, for each $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, we define

- the *region* of $\boldsymbol{\lambda}$: $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda'}}\right) \supseteq \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) \right\}$,
- the *constancy region* of $\boldsymbol{\lambda}$: $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda'}}\right) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) \right\}$.

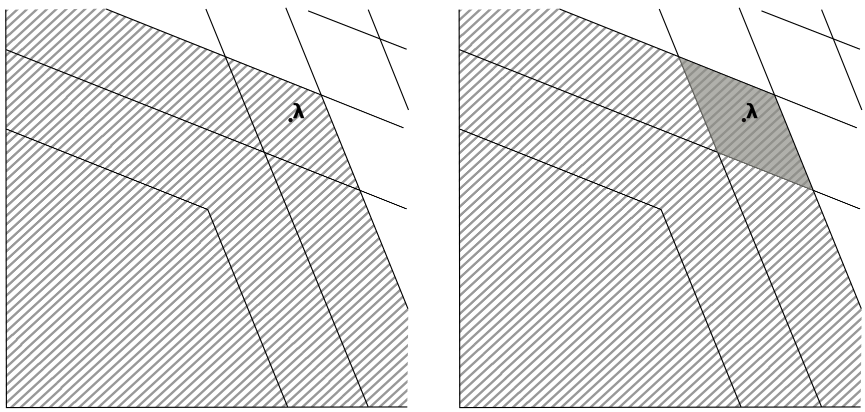


Figure 5.2: On the left, an example of $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ in striped gray lines. On the right, the corresponding $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ in gray.

Remark 5.1.7. For a single ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, the usual multiplier ideals form a discrete nested sequence of ideals

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \ldots$$

indexed by an increasing sequence of rational numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, the aforementioned *jumping numbers*, such that for any $c \in [\lambda_i, \lambda_{i+1})$ we have

$$\mathcal{J}(\mathfrak{a}^{\lambda_i}) = \mathcal{J}(\mathfrak{a}^c) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}}) .$$

Therefore, the region and the constancy region of c are respectively $\mathcal{R}_{\mathfrak{a}}(c) = [\lambda_0, \lambda_{i+1})$ and $\mathcal{C}_{\mathfrak{a}}(c) = [\lambda_i, \lambda_{i+1})$.

From now on we will consider $\mathbb{R}^r_{\geqslant 0}$ and its subsets endowed with the subspace topology from the Euclidean topology of $\mathbb{R}^r$. Thus, any region $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ is an open neighborhood of $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, by Propositions 5.1.3 and 5.1.5. Clearly, we have $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) \supseteq \mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) \ni \boldsymbol{\lambda}$, and the constancy regions are topological manifolds of dimension r with boundary.

The property that relates two points $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathbb{R}^r_{\geqslant 0}$ whenever $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right)$ defines an equivalence relation in $\mathbb{R}^r_{\geqslant 0}$, whose classes are the constancy regions. Hence the constancy regions provide a partition of the positive orthant, and any bounded set intersects only a finite number of them, due to the definition of mixed multiplier ideals in terms of a log-resolution.

There is a partial ordering on the constancy regions: $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}') \leqslant \mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ if and only if $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right) \supseteq \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right)$, or equivalently, if and only if $\boldsymbol{\lambda}' \in \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ (which is also equivalent to $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}') \subseteq \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ or to $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}') \subseteq \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$). Notice that the minimal element is the constancy region $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_0)$ of the origin $\boldsymbol{\lambda}_0 = (0, \ldots, 0)$. One of the aims of this work is to provide a set of points which includes at least one representative for each constancy region[2]. These points will be taken over the boundary of regions $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ associated to some $\boldsymbol{\lambda}$, i.e., the points where we have a change in the corresponding mixed multiplier ideals. Taking into account the behavior of these ideals in the neighborhood of a given point, we introduce the notion of jumping point

Definition 5.1.8. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. We say that $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is a *jumping point* of $\boldsymbol{\mathfrak{a}}$ if $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right) \supsetneq \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right)$ for all $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ and $\varepsilon > 0$ small enough.

It follows from the definition of mixed multiplier ideals that the jumping points $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ must lie on hyperplanes of the form

$$H_i : \; e_{1,i} z_1 + \cdots + e_{r,i} z_r = \ell_i + k_i \tag{5.1.1}$$

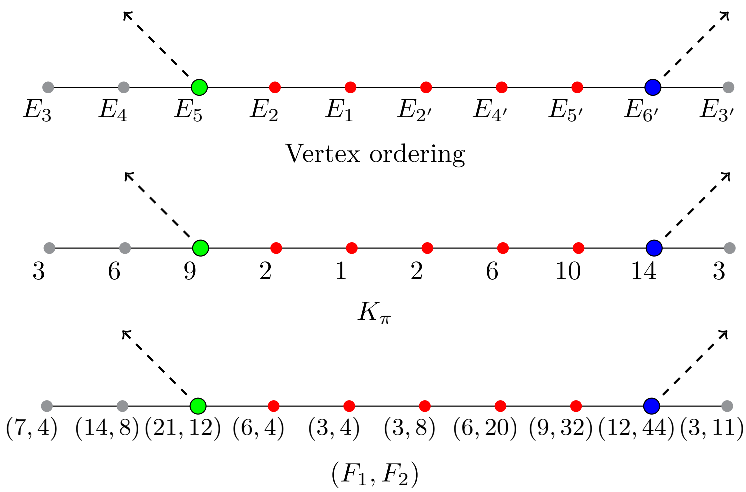
[2]For multiplier ideals we have a total order on the constancy regions, and the representative that we take is simply the corresponding jumping number.

for $i = 1, \dots, s$ where $\ell_i \in \mathbb{Z}_{>0}$. In particular, each hyperplane H_i is associated to an exceptional divisor E_i. Therefore, the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$ associated to a point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is a *rational convex polytope* defined by

$$e_{1,i} z_1 + \cdots + e_{r,i} z_r < \ell_i + k_i,$$

i.e., the minimal region in $\mathbb{R}^r_{\geqslant 0}$ described by these inequalities, for suitable ℓ_i.

Example 5.1.9. Let $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2) \subseteq (\mathcal{O}_{X,O})^2$ be a pair of ideals where X is smooth , $\mathfrak{a}_1 = (x^3, y^7)$ and $\mathfrak{a}_2 = (y^4, x^{11})$. We represent the relative canonical divisor K_π and F_1 and F_2 in the dual graph as follows.



Recall that the blank dots correspond to dicritical divisors and their excesses are represented by broken arrows. In order to describe the region associated to $\boldsymbol{\lambda}_0 = (0,0)$, we have to consider the hyperplanes $H_i : \; e_{1,i} z_1 + e_{2,i} z_2 = k_i + 1$, associated to the components of the exceptional divisor. The region is described by the hyperplanes associated to E_5 and $E_{6'}$. It is also worth noticing that the intersection point of these two hyperplanes is also contained in the hyperplanes corresponding to the divisors E_2, E_1, $E_{2'}$, $E_{4'}$ and $E_{5'}$.

Definition 5.1.10. Let $\mathfrak{a} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. The *jumping wall* associated to $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is the boundary of the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. One usually refers to the jumping wall of the origin as the *log-canonical wall.*

Notice that the facets of the jumping wall of $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ are also rational convex polytopes supported on the hyperplanes H_i considered in equation (5.1.1) that provide the minimal region. We will refer to them as the *supporting hyperplanes* of the jumping wall.

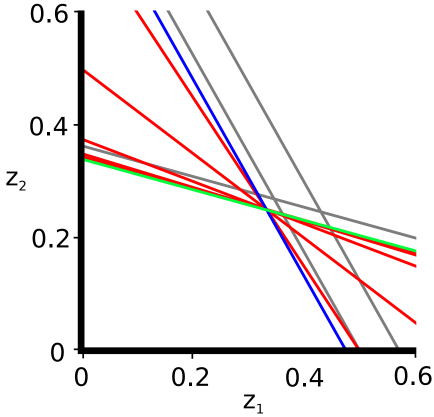


Figure 5.3: The region associated to $\boldsymbol{\lambda} = (0,0)$.

Remark 5.1.11. Whenever we intersect the jumping walls of a tuple $\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ with a ray from the origin in the direction of a vector $(u_1, \ldots, u_r) \in \mathbb{Q}^r_{\geqslant 0}$, we obtain (conveniently scaled) the jumping numbers of the ideal $\mathfrak{a}_1^{\alpha u_1} \cdots \mathfrak{a}_r^{\alpha u_r}$ with $\alpha \cdot u_i \in \mathbb{Z}$ for all i. In particular, the intersections of the coordinate axes with the jumping walls provide the jumping numbers of the ideals $\mathfrak{a}_i$, $i = 1, \ldots, r$.

Now we turn our attention to the constancy region of a given point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$. In general the constancy region $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ is not necessarily a convex polytope. Its boundary is entirely formed by jumping points and it has two components. Roughly speaking, the *inner* part of the boundary is $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) \backslash \mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})^{\circ}$, i.e., the non-interior points of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$, which are the points in $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ closest to the origin $\boldsymbol{\lambda}_0$. The *outer* part is $\overline{\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})} \backslash \mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$, formed by the points in the adherence of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ which are not in the constancy region, which are the points in $\overline{\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})}$ further away from the origin $\boldsymbol{\lambda}_0$. Notice that this latter component is contained in the boundary of the region $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$. In particular the facets of the outer boundary of the constancy region $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ are contained in the facets of the corresponding region, so they have the same supporting hyperplanes. However, it will be important to distinguish the outer facets of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ from the facets of $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$, and it is for this reason that we will refer to them as $\mathcal{C}$*-facets*. Namely, a $\mathcal{C}$-facet of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ is the intersection of the boundary of a connected component of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ with a supporting hyperplane of $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$. Indeed, every facet of a jumping wall decomposes into several $\mathcal{C}$-facets associated to different mixed multiplier ideals.

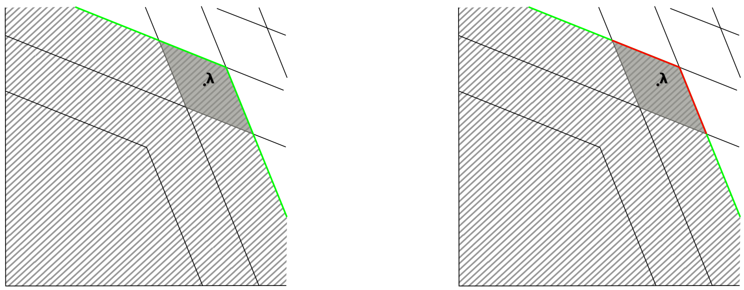


Figure 5.4: On the left, the jumping wall associated to $\boldsymbol{\lambda}$ in green. Notice that it has two facets. On the right, the corresponding $\mathcal{C}$-facets are in red.

Remark 5.1.12. It follows from its definition that the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$ associated to any given point is connected. We do not know whether the same property is satisfied by the constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$.

5.2 An algorithm to compute jumping numbers and multiplier ideals

In Chapter 2 we developed a very simple algorithm to construct sequentially the chain of multiplier ideals

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \dots$$

associated to a single ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. The key point is the fact proved in Theorem 2.1.5 that, given any $\lambda' \in \mathbb{R}_{\geqslant 0}$, the consecutive jumping number is

$$\lambda = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\},$$

where $D_{\lambda'} = \sum e_i^{\lambda'} E_i$ is the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$. In particular, the algorithm relies heavily on the unloading procedure described in Section 1.7.

The goal in this work is to adapt and extend the aforementioned methods to compute the constancy regions of a tuple of ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ and describe the corresponding mixed multiplier ideals. We start by fixing a common log-resolution $\pi : X' \longrightarrow X$ of $\boldsymbol{\mathfrak{a}}$. Then we have to consider the relative canonical divisor $K_\pi = \sum_{i=1}^{s} k_j E_j$ and the divisors F_i in X' such that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ decomposed as

$$F_i = F_i^{\text{exc}} + F_i^{\text{aff}} = \sum_{j=1}^{s} e_{i,j} E_j + \sum_{j=s+1}^{t} e_{i,j} E_j$$

in terms of its exceptional and affine support.

As in the case treated in Chapter 2, the key point of our method is how to compare the complete ideals defined by an antinef and a non-antinef divisor. First we recall the following result.

Proposition 5.2.1 (See Corollary 2.1.4)**.** *Let D_1, D_2 be two divisors in X' such that $D_1 \leqslant D_2$. Then*

i) $\pi_* \mathcal{O}_{X'}(-D_1) = \pi_* \mathcal{O}_{X'}(-D_2)$ *if and only if* $\widetilde{D_1} \geqslant D_2$,

ii) $\pi_* \mathcal{O}_{X'}(-D_1) \supsetneq \pi_* \mathcal{O}_{X'}(-D_2)$ *if and only if* $v_i(\widetilde{D_1}) < v_i(D_2)$ *for some* E_i.

Then we get the following generalization of Corollary 2.1.4 to the setting of mixed multiplier ideals.

Corollary 5.2.2. *Let* $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of ideals and* $\boldsymbol{\lambda}$, $\boldsymbol{\lambda'}$ *two points in* $\mathbb{R}^r_{\geqslant 0}$. *Let* $D_{\boldsymbol{\lambda}} = \sum e_j^{\boldsymbol{\lambda}} E_j$ *be the antinef closure of* $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor$. *Then*

$$\boldsymbol{\lambda'} \in \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}) \text{ if and only if } \lfloor \lambda'_1 e_{1,j} + \cdots + \lambda'_r e_{r,j} - k_j \rfloor \leqslant e_j^{\boldsymbol{\lambda}} \text{ for all } E_j.$$

Also, we need to recall some notions that we will use later on. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals, $\pi : X' \longrightarrow X$ a common log-resolution of $\boldsymbol{\mathfrak{a}}$ and the divisors F_i in X' such that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$. Then, for any exceptional component E_j, we define the *excess* (of $\mathfrak{a}_i$) at E_j as $\rho_{i,j} = -F_i \cdot E_j$. We also need the following concepts that will play a special role.

- A component E_j of $E = Exc(\pi)$ is a *rupture* component if it intersects at least three more components of E (different from E_j).
- We say that E_j is *dicritical* if $\rho_{i,j} > 0$ for some i.

With the technical tools stated above we are ready for the main result of this section. Namely, we provide a formula to compute the region associated to any given point, that is a generalization of Theorem 2.1.5 in the context of mixed multiplier ideals.

Theorem 5.2.3. *Let $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals and let $D_{\boldsymbol{\lambda}} = \sum e_j^{\boldsymbol{\lambda}} E_j$ be the antinef closure of $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor$ for a given $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$. Then the region of $\boldsymbol{\lambda}$ is the rational convex polytope determined by the inequalities*

$$e_{1,j} z_1 + \cdots + e_{r,j} z_r < k_j + 1 + e_j^{\boldsymbol{\lambda}} ,$$

corresponding to either rupture or dicritical divisors E_j.

In order to prove that we only have to consider the hyperplanes corresponding to either rupture or dicritical divisors, we need to invoke some results on *jumping divisors* that will be developed in Section 5.3.

Proof. It follows from Corollary 5.2.2 that $\boldsymbol{\lambda}'$ is not in the region if and only if there exists E_j such that

$$\lfloor \lambda'_1 e_{1,j} + \cdots + \lambda'_r e_{r,j} - k_j \rfloor > e_j^{\boldsymbol{\lambda}}.$$

This inequality is equivalent to, $-k_j + \lambda'_1 e_{1,j} + \cdots + \lambda'_r e_{r,j} \geqslant e_j^{\boldsymbol{\lambda}} + 1$ and therefore to $\lambda'_1 e_{1,j} + \cdots + \lambda'_r e_{r,j} \geqslant k_j + 1 + e_j^{\boldsymbol{\lambda}}$.

To finish the proof, we have to prove that we only need to consider the rupture or dicritical divisors. Let $H : e_{1,j} z_1 + \cdots + e_{r,j} z_r = k_j + 1 + e_j^{\boldsymbol{\lambda}}$ be the hyperplane associated to the divisor E_j considered above. Then, among all the exceptional divisors E_i such that $e_{1,i} z_1 + \cdots + e_{r,i} z_r = k_i + 1 + e_j^{\boldsymbol{\lambda}}$ gives the same hyperplane H, we may find a rupture or dicritical divisor by Theorem 5.3.13.

□

Remark 5.2.4. When X has a rational singularity at O, we may have a strict inclusion $\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_0})$ for $\boldsymbol{\lambda}_0 = (0, \ldots, 0)$. The above result for this case gives a mild generalization of the well-known formula for the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$ in the smooth case (see [LM11] where this region is denoted LCT-polytope). Namely, it is the rational convex polytope determined by the inequalities

$$e_{1,j} z_1 + \cdots + e_{r,j} z_r < k_j + 1 + e_j^{\boldsymbol{\lambda}_0} ,$$

corresponding to either rupture or dicritical divisors E_j.

Corollary 5.2.5. *Let $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. Then the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$ is bounded for any point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$.*

This property enables us to give a recursive way to compute the constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ from the finitely many constancy regions satisfying $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}') \leqslant \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$.

Corollary 5.2.6. *Let* $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of ideals. Given* $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$*, there exists finitely many points* $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k \in \mathbb{R}^r_{\geqslant 0}$ *such that*

$$\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) \setminus (\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) \cup \cdots \cup \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_k)) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) \setminus (\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) \cup \cdots \cup \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_k)) \,. \tag{5.2.1}$$

In particular, $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_1), \ldots, \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_k)$ *are all the constancy regions that are strictly smaller than* $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ *using the partial order* $\leqslant$*.*

Remark 5.2.7. To obtain a simpler expression in the first equation of (5.2.1) we may choose $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_s \in \mathbb{R}^r_{\geqslant 0}$ such that $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_1), \ldots, \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_s)$ are the maximal elements among those constancy regions which are strictly smaller than $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ using the partial order $\leqslant$. Then

$$\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) \setminus (\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) \cup \cdots \cup \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_s)) \,. \tag{5.2.2}$$

Theorem 5.2.3 is one of the key ingredients for the algorithm that we will present in Section 5.2.1. The other key ingredient comes from a careful study of the $\mathcal{C}$-facets of the components of a constancy regions that will show their subtelty.

For simplicity, due to the fact that, for a fixed jumping point $\boldsymbol{\lambda}$, any $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ for $\varepsilon > 0$ sufficiently small defines the same mixed multiplier ideal, we will denote this mixed multiplier ideal as the one associated to $(1-\varepsilon)\boldsymbol{\lambda}$ for $\varepsilon > 0$ sufficiently small.

We start with the following well-known fact.

Lemma 5.2.8. *Let* $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of ideals and* $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ *be a point.*

i) The interior of a $\mathcal{C}$*-facet, as a subspace of its supporting hyperplane, is non-empty.*

ii) Any constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ *different from the constancy region associated to the origin, has non-empty intersection with the interior of some* $\mathcal{C}$*-facets.*

iii) Any interior point $\boldsymbol{\lambda}'$ *of a* $\mathcal{C}$*-facet of* $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ *satisfies* $\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}'}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$

Proof. The key point in the proof of these three statements is that, for all $\varepsilon > 0$, we have that $B_\varepsilon(\boldsymbol{\lambda}) \cap \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$ contains an open ball $B_\varepsilon(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. To finish the proof of ii) we notice that the inner boundary $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}) \setminus \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})^\circ$ provides the points of $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ which are interior points of a $\mathcal{C}$-facet of some other constancy region, which is necessarily smaller than $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ using the partial order $\leqslant$. □

The key result states that a $\mathcal{C}$-facet cannot *be crossed* by any jumping wall.

Proposition 5.2.9. *Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ be interior points of the same $\mathcal{C}$-facet of a constancy region. Then we have $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right)$.*

Once again we need to use some results from Section 5.3 to prove this fact.

Proof. Let H be the supporting hyperplane of the $\mathcal{C}$-facet containing $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. Notice that both are jumping points coming from the same mixed multiplier ideal, namely $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{\lambda}}\right) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{\lambda}'}\right)$. For simplicity we take a point $\boldsymbol{\mu}$ as a representative of the constancy region of this ideal. Now, let $D_{\boldsymbol{\mu}} = \sum e_j^{\boldsymbol{\mu}} E_j$ be the antinef closure of $\lfloor \mu_1 F_1 + \cdots + \mu_r F_r - K_\pi \rfloor$. Consider the reduced divisor G supported on those exceptional components E_j such that the hyperplane H has equation

$$\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} = k_j + 1 + e_j^{\boldsymbol{\mu}} .$$

Then, using Lemma 5.3.5 and Proposition 5.3.9 we have

$$\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) = \pi_* \mathcal{O}_{X'}(-D_{(1-\varepsilon)\boldsymbol{\lambda}} - G) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right) .$$

□

Remark 5.2.10. It follows from the proof of Proposition 5.2.9 that, whenever $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are interior points of different $\mathcal{C}$-facets, but with the same supporting hyperplane, of a constancy region $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\mu})$, then $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}\right) = \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}\right)$.

In general, if we take two different constancy regions and points in their corresponding $\mathcal{C}$-facets, these points would give us different associated mixed multiplier ideals. Example A.1.1 in the Appendix shows a case where indeed two such points provide the same ideal, although we point out that both $\mathcal{C}$-facets have the same supporting hyperplane.

5.2.1 An algorithm to compute the constancy regions

The algorithm that we are going to present is a generalization of the one given in Algorithm 2.2.1 that we briefly recall. Given an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, we construct sequentially the chain of multiplier ideals

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \ldots$$

The starting point is to compute the multiplier ideal associated to $\lambda_0 = 0$ by means of the antinef closure $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ of $\lfloor -K_\pi \rfloor$ using the unloading procedure described in Section 1.7. The log-canonical threshold is

$$\lambda_1 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_0}}{e_i} \right\},$$

so we may describe its associated multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_1})$ just by computing the antinef closure $D_{\lambda_1} = \sum e_i^{\lambda_1} E_i$ of $\lfloor \lambda_1 F - K_\pi \rfloor$ using the unloading procedure. By Theorem 2.1.5, the next jumping number is

$$\lambda_2 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_1}}{e_i} \right\}.$$

Then we only have to follow the same strategy: the antinef closure D_{λ_2} of $\lfloor \lambda_2 F - K_\pi \rfloor$, i.e., the multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_2})$, allows us to compute λ_3 and so on.

We may interpret this as follows: at each step of the algorithm, the jumping number λ_i allows us to compute its region, and equivalently its constancy region $[\lambda_i, \lambda_{i+1})$. The boundary of this constancy region gives us the next jumping number λ_{i+1}. In particular we have a one-to-one correspondence between the constancy regions of the ideal $\mathfrak{a}$ and the jumping numbers.

The algorithm for mixed multiplier ideals is more involved. It starts with the computation of the mixed multiplier ideal associated to $\boldsymbol{\lambda}_0 = (0, \ldots, 0)$, using the unloading procedure. The region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$ is described by means of the formula given in Theorem 5.2.3. In this case the region coincides with the constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$, so we have a nice description of its boundary. For each $\mathcal{C}$-facet, using Proposition 5.2.9, we may take a single point as a representative. The next step of the algorithm is to compute the mixed multiplier ideals of these points in order to describe their corresponding regions, using Theorem 5.2.3 once again. Then we compute the corresponding constancy regions and their $\mathcal{C}$-facets and we follow the same strategy.

Roughly speaking, our strategy is to consider a discrete set of points comprising one interior point of each $\mathcal{C}$-facet. This gives a surjective correspondence with the partially ordered set of constancy regions. This correspondence is far from being one-to-one as in the case of a single ideal. To keep track of these points we will consider two sets N and D. N will contain the points for which we still have to compute the corresponding region and, once this region has been computed, we move it to D. In particular, we will start with $N = \{\boldsymbol{\lambda}_0\}$ and $D = \emptyset$ the empty set.

Algorithm 5.2.11. *(Constancy regions and mixed multiplier ideals)*

> `Input:` *a common log-resolution of the tuple of ideals* $\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$.
> `Output:` *list of constancy regions of* $\boldsymbol{\mathfrak{a}}$ *and their corresponding mixed multiplier ideals.*

Set $N = \{\boldsymbol{\lambda}_0 = (0, \ldots, 0)\}$ *and* $D = \emptyset$. *From* $j = 1$, *incrementing by* 1, *perform the following.*

(Step j)

(j.1) **Choosing a convenient point in the set** N.

- *Pick* $\boldsymbol{\lambda}_j$ *the first point in the set* N *and compute its region* $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j)$.
- *If there is some* $\boldsymbol{\lambda} \in N$ *such that* $\boldsymbol{\lambda} \in \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j)$ *and* $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}) \neq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}_j})$, *then put* $\boldsymbol{\lambda}$ *first in the list* N *and repeat this step* (j.1). *Otherwise continue with step* (j.2).

(j.2) **Checking out whether the region has been already computed**.

- *If some* $\boldsymbol{\lambda} \in D$ *satisfies* $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}}) = \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}_j})$, *then go to step* ($j$.4). *Otherwise continue with step* (j.3).

(j.3) **Picking new points for which we have to compute its region**.

- *Compute*

$$\mathcal{C}(j) = \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j) \setminus (\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_1) \cup \cdots \cup \mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_{j-1})) .$$

- *For each connected component of* $\mathcal{C}(j)$ *compute its outer facets*[3].
- *Pick one interior point in each* $\mathcal{C}$*-facet and add them as the last points in* N.

(j.4) **Update the sets** N **and** D.

- *Delete* $\boldsymbol{\lambda}_j$ *from* N *and add* $\boldsymbol{\lambda}_j$ *as the last point in* D.

Remark 5.2.12. Several points of the algorithm require a comparision between mixed multiplier ideals (an inequality in step (j.1) and an equality in step (j.2)). This can be done computing antinef closures of divisors using the unloading procedure. For the computation of the region $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda})$ (steps (j.1) and (j.3)) we use Theorem 5.2.3.

[3]The outer facets of $\mathcal{C}(j)$ are the intersection of the boundary of any connected component of $\mathcal{C}(j)$ with a supporting hyperplane of $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{\lambda}_j)$.

Remark 5.2.13. Step $(j.1)$ is equivalent to choosing a point whose constancy region is a minimal element by the ordering $\leqslant$ among those associated to the points in the set N. Any finite subset endowed with a partial ordering has some minimal element, thus there exists a convenient point in the set N that allows to continue with step $(j.2)$.

Lemma 5.2.14. *At each step j, the algorithm overcomes step $(j.1)$ and provides updated sets N and D.*

Theorem 5.2.15. *The constancy region of the point $\boldsymbol{\lambda}_j$ chosen at step $(j.1)$ is computed at step $(j.3)$ of the algorithm, i.e., $\mathcal{C}(j) = \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$, and one interior point for each $\mathcal{C}$-facet of $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ is added to the set N.*

Proof. We argue by induction on j. For the case $j = 1$ the statement holds since we pick $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0$ at step (1.1) and step (1.3) is performed.

Now assume that the statement is true all the steps up to $j-1$. We want to prove it for step j. Without loss of generality we may assume that step $(j.3)$ must be performed, so $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_i}) \neq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_j})$ for all $1 \leqslant i \leqslant j-1$. Notice that, by equation (5.2.2), $\mathcal{C}(j) = \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ is equivalent to the fulfillment of the following two conditions:

a) Each $\boldsymbol{\lambda}_i$, $1 \leqslant i \leqslant j-1$, satisfies either $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_i) \leqslant \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ or both constancy regions are not related by the partial order.

b) Consider a set $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_s\} \subset \mathbb{R}^r_{\geqslant 0}$ of representatives of the constancy regions which are maximal elements among those constancy regions smaller than $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$. Then, for each $k \in \{1, \ldots, s\}$ there is some $i_k \in \{1, \ldots, j-1\}$ such that $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_{i_k}) \leqslant \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\mu}_k)$.

First we are going to prove that condition a) is satisfied. Assume the contrary, i.e., there exists $i < j$ with $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_i) > \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$, that is $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_i) \supsetneq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$. Assume that $\boldsymbol{\lambda}_j$ was added to N at step $m < j$. Hence, by induction hypothesis $\boldsymbol{\lambda}_j$ is an interior point of some $\mathcal{C}$-facet of $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_m)$, and in particular $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_m) \subseteq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$. Thus $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_m) \subsetneq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_i)$, i.e., $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_m) < \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_i)$. We distinguish two cases:

· If $i < m$ we get a contradiction with the induction hypothesis at step m since condition a) is not fulfilled.

· If $i > m$, we have that $\boldsymbol{\lambda}_j$ already belongs to N at step i. This contradicts the requirement of step $(i.1)$ which says that $\boldsymbol{\lambda}_j$ should be treated before $\boldsymbol{\lambda}_i$.

Finally we prove condition b). Assume the contrary, i.e there exists $\boldsymbol{\mu}_i$ whose constancy region is not dominated by any $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_k)$, $1 \leqslant k \leqslant j-1$. Without loss of generality we may assume that the segment $\overline{\boldsymbol{\lambda}_0\boldsymbol{\mu}_i}$ intersect the jumping walls at interior points of the $\mathcal{C}$-facets, namely in the jumping points $\boldsymbol{\lambda}_0 = \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_m = \boldsymbol{\mu}_i$ with $\boldsymbol{\nu}_k \in \overline{\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\nu}_{k-1})}$, and thus $\boldsymbol{\nu}_{k-1} \in \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\nu}_k)$.

By induction hypothesis, representatives of each constancy region $\{\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\nu}_1), \ldots, \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\nu}_{m'})\}$, for $m' < m$, are added to N at some steps before step j, being $\boldsymbol{\lambda}'$ the last representative. Hence, we still have $\boldsymbol{\lambda}' \in N$ at step j and

$$\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}') = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\nu}_{m'}) \subseteq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\mu}_i) \subsetneq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j).$$

This contradicts the requirement of step $(j.1)$ for $\boldsymbol{\lambda}_j$. □

As a consequence of Theorem 5.2.15 we obtain the following

Corollary 5.2.16. *At step j of the algorithm, we have that:*

i) The set D contains at least a representative of each constancy region inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$.

ii) The set D contains a representative of all $\mathcal{C}$-facets inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$.

iii) A complete description of the jumping walls inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ is obtained by intersecting the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ with the jumping walls associated to the points $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{j-1}$.

Proof. From the proof of Theorem 5.2.15 we infer that at step j, the maximal elements among all the constancy regions inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ have already representants $\boldsymbol{\lambda}_{i_1}, \ldots, \boldsymbol{\lambda}_{i_s}$ in D, $i_1 < \cdots < i_s < j$. Arguing by reverse induction with any of these points $\boldsymbol{\lambda}_{i_k}$, the first claim follows.

Now, the statement of Theorem 5.2.15 asserts that at each step i of the algorithm, a representative of each $\mathcal{C}$-facet of $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_i)$ is added to N. If we only take into account the points $\boldsymbol{\lambda}_i$ of constancy regions inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$, the subsequent representatives in $\mathcal{C}$-facets still lying inside $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_j)$ must be treated (and added to D) before $\boldsymbol{\lambda}_j$, in virtue of step $(j.1)$ of the algorithm.

Part iii) of the statement is a direct consequence of claim i). □

Remark 5.2.17. Each point $\boldsymbol{\lambda}$ included in N at some step of the algorithm is treated after a finite number of steps and added to D. Indeed, the order of incorporation of the points in N is preserved unless step $(j.1)$ priorizes some other point. This happens only a finite amount of times since there is only a finite number of constancy regions inside any given region.

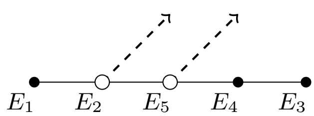
Proposition 5.2.18. *Once a point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$ is fixed, a set D which includes a representative of all constancy regions in the compact $(\boldsymbol{\lambda}_0 + \mathbb{R}^r_{\geqslant 0}) \cap (\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0})$ is achieved after a finite number of steps of the algorithm.*

Proof. Observe that $(\boldsymbol{\lambda}_0 + \mathbb{R}^r_{\geqslant 0}) \cap (\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}) \subseteq \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. In virtue of Corollary 5.2.16 and Remark 5.2.17, we only have to prove that some representative of $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ is added to N at some step. We may take $\boldsymbol{\lambda}' \in \mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda})$ such that the segment $\overline{\boldsymbol{\lambda}_0\boldsymbol{\lambda}'}$ intersects the jumping walls at interior points of $\mathcal{C}$-facets, namely in the jumping points $\boldsymbol{\lambda}_0 = \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_m = \boldsymbol{\lambda}'$. The algorithm starts with $\boldsymbol{\nu}_1$ and incorporates $\boldsymbol{\nu}_2$ to N. Since $\boldsymbol{\nu}_k \in \overline{\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\nu}_{k-1})}$, once $\boldsymbol{\nu}_k$ is selected at some finite step i_k, $\boldsymbol{\nu}_{k+1}$ is added to N at this same step. Hence, $\boldsymbol{\lambda}'$ is selected at some step $(j.1)$. Notice that this implies that no point in N lies in $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}') = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$, i.e. $N \cap \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \emptyset$.

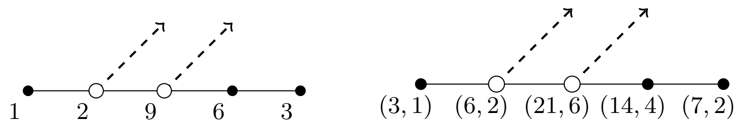
Conversely, if at some step j $N \cap \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \emptyset$, then the new N obtained at any forthcoming step still satisfies $N \cap \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \emptyset$. If some $\boldsymbol{\lambda}_i \notin \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$ with $i > j$ is chosen at step $(i.1)$, any new point $\boldsymbol{\mu}$ added to N at step $(i.3)$ satisfies $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\mu}}) \subsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_i}) \not\supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$ and hence $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\mu}}) \not\supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$, equivalently $\boldsymbol{\mu} \notin \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. Since the algorithm starts with $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 \in \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$, we may conclude that at a step where $N \cap \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}) = \emptyset$ necessarily the set D obtained at that step contains a representative of $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda})$. $\square$

We present the following simple example to highlight the nuances of the procedure. In the example, step $(j.1)$ is performed when computing the region associated to the point $\boldsymbol{\lambda}_5$ and step $(j.2)$ is performed for the points $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_4$, $\boldsymbol{\lambda}_7$ and $\boldsymbol{\lambda}_8$. In particular, step $(j.2)$ is included to avoid too many computations.

Example 5.2.19. Consider the set of ideals $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ with $\mathfrak{a}_1 = (x^3, y^7)$ and $\mathfrak{a}_2 = (x, y^2)$ on a smooth surface X. We represent the relative canonical divisor K_π and F_1 and F_2 in the dual graph as follows.



Vertex ordering



K_π $\qquad$ (F_1, F_2)

The blank dots correspond to dicritical divisors in one of the ideals and their excesses are represented by broken arrows. For simplicity we will collect the values of any divisor in a vector. Namely, we have $K_\pi = (1, 2, 3, 6, 9)$, $F_1 = (3, 6, 7, 14, 21)$ and $F_2 = (1, 2, 2, 4, 6)$. In the algorithm we will have to perform several times unloading steps, so we will have to consider the intersection matrix

$$M = (E_i \cdot E_j)_{1 \leqslant i,j \leqslant 5} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

Notice that E_2 and E_5 are the only dicritical divisors. Then, as a consequence of Theorem 5.2.3, the region of a given point $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is defined by

$$\begin{aligned} 6z_1 + 2z_2 &< 2 + 1 + e_2^{\boldsymbol{\lambda}}, \\ 21z_1 + 6z_2 &< 9 + 1 + e_5^{\boldsymbol{\lambda}}. \end{aligned}$$

We keep track of what we have to compute with the set N that for the moment will only contain $\boldsymbol{\lambda}_0 = (0, 0)$. The set D that keeps track of the points that we have already computed will be empty since we have not computed anything yet.

• **Step 0**. We start computing the multiplier ideal corresponding to $\boldsymbol{\lambda}_0 = (0, 0)$. Namely, the antinef closure of the divisor $\lfloor 0F_1 + 0F_2 - K_\pi \rfloor$ is $D_{\boldsymbol{\lambda}_0} = (0, 0, 0, 0, 0, 0)$. The corresponding region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$ is given by the inequalities

$$\begin{aligned} 6z_1 + 2z_2 &< 3, \\ 21z_1 + 6z_2 &< 10. \end{aligned}$$

Notice that the constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$ coincides with $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$. Its boundary, i.e., the corresponding jumping wall, has two $\mathcal{C}$-facets. So, according to Proposition 5.2.9, we only need to consider an interior point of each $\mathcal{C}$-facet in order to continue our procedure. For simplicity, we consider the barycenters $\left(\frac{1}{6}, 1\right)$ and $\left(\frac{17}{42}, \frac{1}{4}\right)$ corresponding to each segment:

- $N = \{\left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right)\}$,
- $D = \{(0,0)\}$.

• **Step 1**. We pick the first point $\boldsymbol{\lambda}_1 := \left(\frac{1}{6}, 1\right)$ in N and we compute its multiplier ideal. Namely, $\lfloor \frac{1}{6}F_1 + F_2 - K_\pi \rfloor = (0,1,0,0,0)$ and its antinef closure is $D_{\boldsymbol{\lambda}_1} = (1,1,1,2,3)$, so the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1)$ is given by the inequalities

$$\begin{array}{rcr} 6z_1 + 2z_2 & < & 4\,, \\ 21z_1 + 6z_2 & < & 13\,. \end{array}$$

The constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) \setminus \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0)$ has two $\mathcal{C}$-facets for which we pick the interior points $\left(\frac{1}{6}, \frac{3}{2}\right)$ and $\left(\frac{10}{21}, \frac{1}{2}\right)$ respectively. Then, the sets N and D are

- $N = \{\left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right)\}$,
- $D = \{(0,0), \left(\frac{1}{6}, 1\right)\}$.

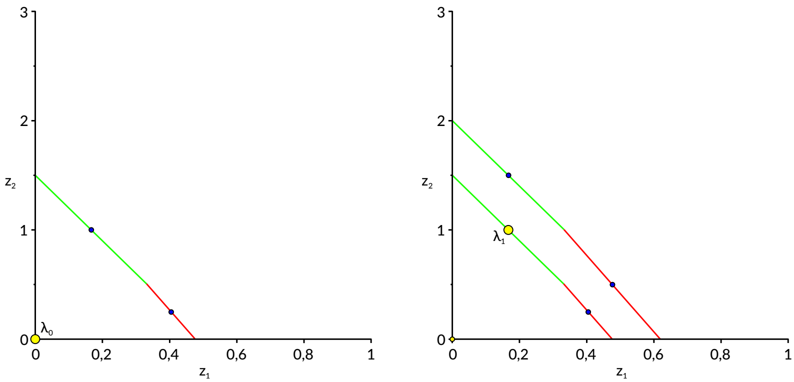


Figure 5.5: Constancy regions associated to $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_1$.

• **Step 2**. The point $\boldsymbol{\lambda}_2 := \left(\frac{17}{42}, \frac{1}{4}\right)$ satisfies $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_2}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_1})$, so they have the same region. In order to keep track of all the $\mathcal{C}$-facets we have to consider this point as well, so the sets N and D that we get after this step are:

· $N = \left\{\left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right)\right\}$.

· $D = \left\{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right)\right\}$.

• **Step 3**. We pick $\boldsymbol{\lambda}_3 := \left(\frac{1}{6}, \frac{3}{2}\right)$. We have $\lfloor \frac{1}{6}F_1 + \frac{3}{2}F_2 - K_\pi \rfloor = (1,2,1,2,3)$ and its antinef closure is $D_{\boldsymbol{\lambda}_3} = (1,2,2,4,6)$, so the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_3)$ is given by the inequalities

$$\begin{aligned} 6z_1 + 2z_2 &< 5\,, \\ 21z_1 + 6z_2 &< 16\,. \end{aligned}$$

The constancy region

$$\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_3) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_3) \backslash (\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_0) \cup \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1) \cup \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_2)) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_3) \backslash \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_1)$$

has two $\mathcal{C}$-facets, for which we pick the interior points $\left(\frac{1}{6}, 2\right)$ and $\left(\frac{23}{42}, \frac{3}{4}\right)$, respectively. Then, the sets N and D are

· $N = \left\{\left(\frac{10}{21}, \frac{1}{2}\right), \left(\frac{1}{6}, 2\right), \left(\frac{23}{42}, \frac{3}{4}\right)\right\}$,

· $D = \left\{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right)\right\}$.

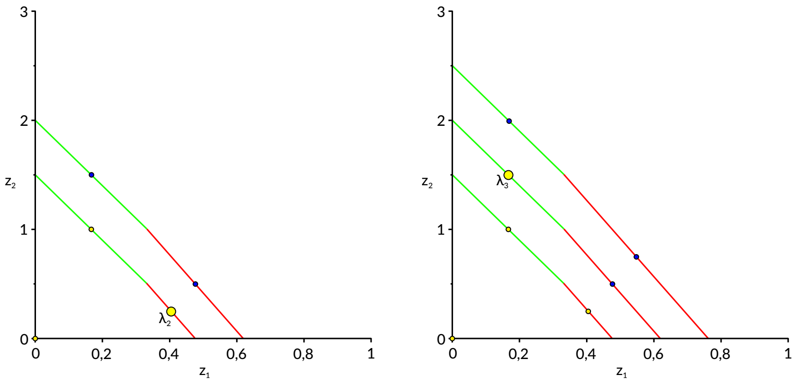


Figure 5.6: Constancy regions associated to $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$.

• **Step 4**. The point $\boldsymbol{\lambda}_4 := \left(\frac{10}{21}, \frac{1}{2}\right)$ satisfies $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_4}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_3})$, so they have the same region. We update the sets N and D to obtain

· $N = \left\{\left(\frac{1}{6}, 2\right), \left(\frac{23}{42}, \frac{3}{4}\right)\right\}$,

· $D = \left\{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right)\right\}$.

• **Step 5**. We have that the region associated to $\left(\frac{23}{42}, \frac{3}{4}\right)$ is contained in the region of $\left(\frac{1}{6}, 2\right)$. It is for this reason that we will consider first the point $\boldsymbol{\lambda}_5 := \left(\frac{23}{42}, \frac{3}{4}\right)$. We have $\lfloor \frac{23}{42}F_1 + \frac{3}{4}F_2 - K_\pi \rfloor = (1,2,2,4,7)$, and its antinef closure is $D_{\boldsymbol{\lambda}_5} = (1,2,3,5,7)$. So the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_3)$ is given by the inequalities

$$\begin{aligned} 6z_1 + 2z_2 &< 5\,, \\ 21z_1 + 6z_2 &< 17\,. \end{aligned}$$

The constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_5) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_5) \setminus \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_3)$ has two $\mathcal{C}$-facets for which we pick the interior points $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{31}{42}, \frac{1}{4}\right)$ respectively. Then, the sets N and D are

· $N = \left\{\left(\frac{1}{6}, 2\right), \left(\frac{1}{2}, 1\right), \left(\frac{31}{42}, \frac{1}{4}\right)\right\}$,

· $D = \left\{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right), \left(\frac{23}{42}, \frac{3}{4}\right)\right\}$.

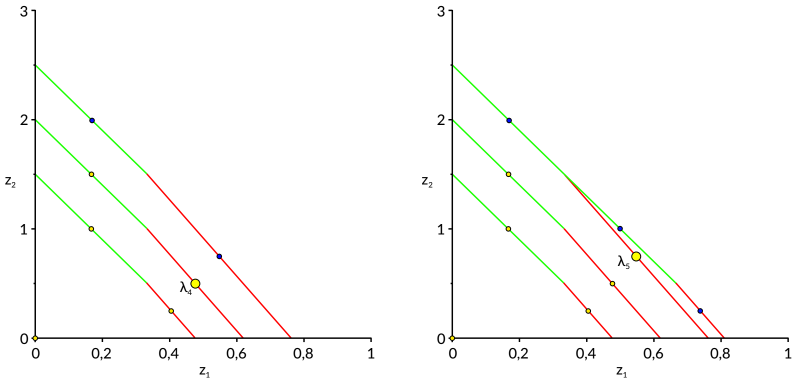


Figure 5.7: Constancy regions associated to $\boldsymbol{\lambda}_4$ and $\boldsymbol{\lambda}_5$.

• **Step 6**. We pick now $\boldsymbol{\lambda}_6 := \left(\frac{1}{6}, 2\right)$. We have $\lfloor \frac{1}{6}F_1 + 2F_2 - K_\pi \rfloor = (1,3,2,4,6)$ and its antinef closure is $D_{\boldsymbol{\lambda}_6} = (2,3,3,6,9)$, so the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_6)$ is given by the inequalities

$$\begin{aligned} 6z_1 + 2z_2 &< 6\,, \\ 21z_1 + 6z_2 &< 19\,. \end{aligned}$$

The constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\lambda}_6) = \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_6) \backslash \mathcal{R}_{\mathfrak{a}}(\boldsymbol{\lambda}_5)$ has two $\mathcal{C}$-facets for which we pick the interior points $\left(\frac{1}{6}, \frac{5}{2}\right)$ and $\left(\frac{13}{21}, 1\right)$. Then the sets N and D are

- $N = \{\left(\frac{1}{2}, 1\right), \left(\frac{31}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{5}{2}\right) \left(\frac{13}{21}, 1\right)\}$,
- $D = \{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right), \left(\frac{23}{42}, \frac{3}{4}\right), \left(\frac{1}{6}, 2\right)\}$.

• **Steps 7, 8 and 9**. The points $\boldsymbol{\lambda}_7 := \left(\frac{1}{2}, 1\right)$ and $\boldsymbol{\lambda}_8 := \left(\frac{31}{42}, \frac{1}{4}\right)$ satisfy $\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_8}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_7}) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}_6})$, so they have the same region. We update the sets N and D to obtain

- $N = \{\left(\frac{1}{6}, \frac{5}{2}\right) \left(\frac{13}{21}, 1\right)\}$,
- $D = \left\{(0,0), \left(\frac{1}{6}, 1\right), \left(\frac{17}{42}, \frac{1}{4}\right), \left(\frac{1}{6}, \frac{3}{2}\right), \left(\frac{10}{21}, \frac{1}{2}\right), \left(\frac{23}{42}, \frac{3}{4}\right), \left(\frac{1}{6}, 2\right), \left(\frac{1}{2}, 1\right), \left(\frac{31}{42}, \frac{1}{4}\right)\right\}$.

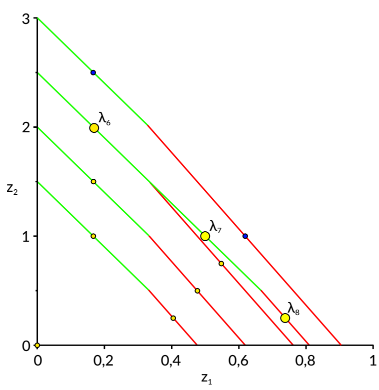


Figure 5.8: Constancy regions associated to $\boldsymbol{\lambda}_6$, $\boldsymbol{\lambda}_7$ and $\boldsymbol{\lambda}_8$.

5.3 Jumping divisors

The theory of jumping divisors was introduced in Chapter 3 in order to describe the jump between two consecutive multiplier ideals. The aim of this section is to extend these notions to the case of mixed multiplier ideals. Then, following the same arguments as in Chapter 3, we will be able to describe the multiplicity of a jumping point in Section 5.4. More importantly, the theory of jumping divisors is the right framework that provides the technical results needed in the proofs of the key results Theorem 5.2.3 and Proposition 5.2.9.

We begin with a generalization of the notion of contribution introduced by Smith and Thompson in [ST07] and further developed by Tucker in [Tuc10].

Definition 5.3.1. Let $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals, $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ a point and $G \leqslant \sum_{i=1}^r F_i$ a reduced divisor supported on those components E_j for which the point $\boldsymbol{\lambda}$ satisfies

$$\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j \in \mathbb{Z}\,.$$

Then it is said that G *contributes* to $\boldsymbol{\lambda}$ if

$$\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)\,.$$

Moreover, this contribution is *critical* if for any divisor $0 \leqslant G' < G$ we have

$$\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G') = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}).$$

The following is the natural extension of Definition 3.0.1 to the context of mixed multiplier ideals.

Definition 5.3.2. Let $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ be a jumping point of a tuple of ideals $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. A reduced divisor $G \leqslant \sum_{i=1}^r F_i$ for which any $E_j \leqslant G$ satisfies

$$\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j \in \mathbb{Z}_{>0}$$

is called a *jumping divisor* for $\boldsymbol{\lambda}$ if

$$\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}'}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G)\,,$$

for any $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ for ε small enough. We say that a jumping divisor is minimal if no proper subdivisor is a jumping divisor for $\boldsymbol{\lambda}$, i.e.,

$$\mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}'}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G')$$

for any $0 \leqslant G' < G$ and for any $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^r_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ for $\varepsilon > 0$ sufficiently small.

Next we introduce the extension of maximal and minimal jumping divisor given in Chapter 3. We remind that, in loc. cit., the maximal jumping divisor is defined over any real number, whereas the minimal jumping divisor only make sense for jumping numbers. This will also be the case in the framework of mixed multiplier ideals.

Definition 5.3.3. Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals.

- Given any real number $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$, the corresponding *maximal jumping divisor* is the reduced divisor $H_{\boldsymbol{c}} \leqslant \sum_{i=1}^r F_i$ supported on those components E_j such that

  $$c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j \in \mathbb{Z}\,.$$

  Equivalently, for a sufficiently small $\varepsilon > 0$,

  $$H_{\boldsymbol{\lambda}} = \lceil K_\pi - (c_1 - \varepsilon)F_1 - \cdots - (c_r - \varepsilon)F_r \rceil - \lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil\,.$$

- Given a jumping point $\boldsymbol{\lambda} \in \mathbb{R}^r_{\geqslant 0}$, the corresponding *minimal jumping divisor* is the reduced divisor $G_\lambda \leqslant \sum_{i=1}^r F_i$, supported on those components E_j for which the point $\boldsymbol{\lambda}$ satisfies

  $$\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} = k_j + 1 + e_j^{(1-\varepsilon)\boldsymbol{\lambda}}\,.$$

  where, for a sufficiently small $\varepsilon > 0$, $D_{(1-\varepsilon)\boldsymbol{\lambda}} = \sum e_j^{(1-\varepsilon)\boldsymbol{\lambda}} E_j$ is the antinef closure of

  $$\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor.$$

Remark 5.3.4. Let $\boldsymbol{\lambda}$ be a jumping point contained in some $\mathcal{C}$-facets of $\mathcal{C}_{\boldsymbol{\mathfrak{a}}}((1-\varepsilon)\boldsymbol{\lambda})$. The exceptional components E_j, such that

$$H_j : \qquad \lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} = k_j + 1 + e_j^{(1-\varepsilon)\boldsymbol{\lambda}}$$

are the supporting hyperplanes of these $\mathcal{C}$-facets, are the components of the minimal jumping divisor $G_{\boldsymbol{\lambda}}$.

We can say more about the minimal jumping divisor, it is not only related to a jumping point. Indeed, we can associate it to the interior of each $\mathcal{C}$-facet. The following property is no longer true for the maximal jumping divisor, as we can see in Example A.2.1 in the Appendix.

Lemma 5.3.5. *The interior points of a $\mathcal{C}$-facet have the same minimal jumping divisor.*

Proof. This is a direct consequence of Remark 5.3.4. □

Let $\boldsymbol{\lambda}$ be a jumping point. It follows from the definition that $H_{\boldsymbol{\lambda}}$ is a jumping divisor and $G_{\boldsymbol{\lambda}} \leqslant H_{\boldsymbol{\lambda}}$. We prove next that $G_{\boldsymbol{\lambda}}$ is also a jumping divisor and deserves its name.

Proposition 5.3.6. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ be a jumping point of a tuple of ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. Then the reduced divisor $G_{\boldsymbol{\lambda}}$ is a jumping divisor.*

Proof. Since $G_{\boldsymbol{\lambda}} \leqslant H_{\boldsymbol{\lambda}}$, we have

$$\lceil K_\pi - (1-\varepsilon)\lambda_1 F_1 - \cdots - (1-\varepsilon)\lambda_r F_r \rceil \geqslant \lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rfloor + G_{\boldsymbol{\lambda}}$$

for a sufficiently small $\varepsilon > 0$, and therefore

$$\mathcal{J}(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}).$$

For the reverse inclusion, let $D_{(1-\varepsilon)\boldsymbol{\lambda}} = \sum e_i^{(1-\varepsilon)\boldsymbol{\lambda}} E_i$ be the antinef closure of

$$\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor.$$

We want to check that $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G_{\boldsymbol{\lambda}} \leqslant D_{(1-\varepsilon)\boldsymbol{\lambda}}$. For this purpose we consider two cases.

- If $E_i \leqslant G_{\boldsymbol{\lambda}}$, then we have $-k_i + \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} = 1 + e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$. And, in particular, $\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor - 1 = e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$.
- If $E_i \not\leqslant G_{\boldsymbol{\lambda}}$, then we have $-k_i + \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} < 1 + e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$. Thus $\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor < 1 + e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$ and the result follows.

□

As in the case of multiplier ideals, the unicity of the jumping divisor $G_{\boldsymbol{\lambda}}$ is a consequence of a more general statement.

Theorem 5.3.7. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ be a jumping point of a tuple of ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. Any reduced contributing divisor $G \leqslant \sum_{i=1}^r F_i$ associated to $\boldsymbol{\lambda}$ satisfies either*

- $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{\lambda}}\right) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G) \supsetneq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}})$ *if and only if $G_{\boldsymbol{\lambda}} \leqslant G$, or*

- $\mathcal{J}\left(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}\right) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$ *otherwise.*

Proof. Since $G \leqslant H_{\boldsymbol{\lambda}}$, we have

$$\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor \leqslant \lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G$$

and therefore

$$\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G).$$

Now assume $G_{\boldsymbol{\lambda}} \leqslant G$. Then

$$\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G \leqslant \lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G_{\boldsymbol{\lambda}}\,,$$

and using the fact that $G_{\boldsymbol{\lambda}}$ is a jumping divisor we obtain the equality

$$\mathcal{J}\left(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}\right) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 + \cdots + \lambda_r F_r \rceil + G).$$

If $G_{\boldsymbol{\lambda}} \not\leqslant G$, we may consider a component $E_i \leqslant G_{\boldsymbol{\lambda}}$ such that $E_i \not\leqslant G$. Notice that we have

$$\begin{aligned} v_i(D_{(1-\varepsilon)\boldsymbol{\lambda}}) = e_i^{(1-\varepsilon)\boldsymbol{\lambda}} &= \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i - 1 < \\ &< \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i = v_i(\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G)\,, \end{aligned}$$

where $D_{(1-\varepsilon)\boldsymbol{\lambda}} = \sum e_i^{(1-\varepsilon)\boldsymbol{\lambda}} E_i$ is the antinef closure of the divisor

$$\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor\,.$$

Therefore, by Corollary 5.2.2, we get the strict inclusion

$$\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G).$$

□

Corollary 5.3.8. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ be a jumping point of a tuple of ideals $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. Then $G_{\boldsymbol{\lambda}}$ is the unique minimal jumping divisor associated to $\boldsymbol{\lambda}$.*

The minimal jumping divisor also allows to describe the jump of mixed multiplier ideals in the other direction, although in this case we do not have minimality for the jump.

Proposition 5.3.9. *Let $\boldsymbol{\lambda}$ be a jumping point of a tuple of ideals $\mathfrak{a} \subseteq (\mathcal{O}_{X,O})^r$ and $D_{(1-\varepsilon)\boldsymbol{\lambda}}$ the antinef closure of $\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor$. Then we have*

i) $\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supsetneq \pi_* \mathcal{O}_{X'}(-D_{(1-\varepsilon)\boldsymbol{\lambda}} - G_\lambda) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$,

ii) $\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - (1-\varepsilon)\lambda_1 F_1 - \cdots - (1-\varepsilon)\lambda_r F_r \rceil - G_\lambda) = \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$.

Proof. Let $D_{\boldsymbol{\lambda}} = \sum e_i^{\boldsymbol{\lambda}} E_i$ be the antinef closure of $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor$.

i) Since $G_{\boldsymbol{\lambda}}$ is a jumping divisor, we have $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor - G_\lambda \leqslant D_{(1-\varepsilon)\boldsymbol{\lambda}}$, and hence $\lfloor \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rfloor \leqslant D_{(1-\varepsilon)\boldsymbol{\lambda}} + G_{\boldsymbol{\lambda}}$. This gives the inclusion

$$\pi_* \mathcal{O}_{X'}(-D_{(1-\varepsilon)\boldsymbol{\lambda}} - G_{\boldsymbol{\lambda}}) \subseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}}).$$

In order to check the reverse inclusion $\pi_* \mathcal{O}_{X'}(-D_{(1-\varepsilon)\boldsymbol{\lambda}} - G_{\boldsymbol{\lambda}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$, it is enough, using Corollary 5.2.2, to prove $v_i(D_{(1-\varepsilon)\boldsymbol{\lambda}} + G_{\boldsymbol{\lambda}}) \leqslant v_i(D_{\boldsymbol{\lambda}}) = e_i^{\boldsymbol{\lambda}}$ for any component E_i. We have $e_i^{(1-\varepsilon)\boldsymbol{\lambda}} \leqslant e_i^{\boldsymbol{\lambda}}$, just because $\mathcal{J}(\mathfrak{a}^{(1-\varepsilon)\boldsymbol{\lambda}}) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$. This inequality is strict when $E_i \leqslant G_{\boldsymbol{\lambda}}$, so the result follows.

ii) Let D' be the antinef closure of $\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor + G_{\boldsymbol{\lambda}}$. Since $G_{\boldsymbol{\lambda}} \leqslant H_{\boldsymbol{\lambda}}$, we have

$$\begin{aligned} &\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor + G_{\boldsymbol{\lambda}} \\ &\qquad \leqslant \lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor \leqslant D_{\boldsymbol{\lambda}}\,, \end{aligned}$$

so the inclusion $\pi_* \mathcal{O}_{X'}(\lceil K_\pi - (1-\varepsilon)\lambda_1 F_1 - \cdots - (1-\varepsilon)\lambda_r F_r \rceil - G_{\boldsymbol{\lambda}}) \supseteq \mathcal{J}(\mathfrak{a}^{\boldsymbol{\lambda}})$ holds. In order to prove the reverse inclusion, we will introduce an auxiliary divisor $D = \sum d_i E_i \in \Lambda$ defined as follows:

- $d_i = \lfloor (1-\varepsilon)\lambda_1 e_{1,i} + \cdots + (1-\varepsilon)\lambda_r e_{r,i} - k_i \rfloor + 1$ if $E_i \leqslant G_{\boldsymbol{\lambda}}$,
- $d_i = e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$ if $E_i \leqslant H_{\boldsymbol{\lambda}}$ but $E_i \not\leqslant G_{\boldsymbol{\lambda}}$,
- $d_i = \lfloor (1-\varepsilon)\lambda_1 e_{1,i} + \cdots + (1-\varepsilon)\lambda_r e_{r,i} - k_i \rfloor$ otherwise.

Clearly we have $\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor + G_{\boldsymbol{\lambda}} \leqslant D$, but we also verify that $\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor \leqslant D$.

· For $E_i \leqslant G_{\boldsymbol{\lambda}}$ we have

$$\begin{aligned}
&\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor \\
&\qquad = \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \\
&\qquad = \lfloor (1-\varepsilon)\lambda_1 e_{1,i} + \cdots + (1-\varepsilon)\lambda_r e_{r,i} - k_i \rfloor + 1 = d_i \,.
\end{aligned}$$

· If λ is a candidate for E_i but $E_i \nleqslant G_{\boldsymbol{\lambda}}$, then

$$\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i = \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i < 1 + e_i^{(1-\varepsilon)\boldsymbol{\lambda}} \,,$$

and hence $\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i \rfloor \leqslant e_i^{(1-\varepsilon)\boldsymbol{\lambda}} = d_i$.

· Otherwise

$$\lfloor \lambda_1 e_{1,i} + \cdots + \lambda_r e_{r,i} - k_i = \lfloor (1-\varepsilon)\lambda_1 e_{1,i} + \cdots + (1-\varepsilon)\lambda_r e_{r,i} - k_i \rfloor = d_i \,.$$

Therefore, taking antinef closures, we have $D' \leqslant D_{\boldsymbol{\lambda}} \leqslant \widetilde{D}$. On the other hand $D \leqslant D'$. Namely, $v_i(D') \geqslant e_i^{(1-\varepsilon)\boldsymbol{\lambda}}$ for any E_i because

$$\lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor \leqslant \lfloor (1-\varepsilon)\lambda_1 F_1 + \cdots + (1-\varepsilon)\lambda_r F_r - K_\pi \rfloor + G_{\boldsymbol{\lambda}} \,.$$

Moreover, $v_i(D') \geqslant \lfloor (1-\varepsilon)\lambda_1 e_{1,i} + \cdots + (1-\varepsilon)\lambda_r e_{r,i} - k_i \rfloor + \delta_i^{G_{\boldsymbol{\lambda}}}$ by definition of antinef closure. Here $\delta_i^{G_{\boldsymbol{\lambda}}} = 1$ if $E_i \leqslant G_{\boldsymbol{\lambda}}$ and zero otherwise. Thus $v_i(D') \geqslant v_i(D)$ as desired. As a consequence $\widetilde{D} \leqslant D'$, which, together with the previous $D' \leqslant D_{\boldsymbol{\lambda}} \leqslant \widetilde{D}$, gives $\widetilde{D} = D' = D_{\boldsymbol{\lambda}}$ and the result follows. $\square$

5.3.1 Geometric properties of minimal and maximal jumping divisors in the dual graph

In the case of multiplier ideals, the minimal and maximal jumping divisors satisfy some geometric conditions in the dual graph (see Theorem 3.2.4 and Theorem 3.3.4). We will extend these results to the case of mixed multiplier ideals. In particular, the forthcoming Theorem 5.3.13 is the result we need in the proof of Theorem 5.2.3.

Lemma 5.3.10. *Let $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals and $G \leqslant \sum F_i$ a reduced divisor. For any $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ and any component $E_i \leqslant G$ we* *have*

$$(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + G)\cdot E_i$$
$$= -2 + c_1\rho_{1,i} + \cdots + c_r\rho_{r,i} + a_G\,(E_i) + \sum_{E_j\in \mathrm{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\}\ ,$$

where $\mathrm{Adj}(E_i)$ *denotes the adjacent components of* E_i *in the dual graph.*

Proof. For any $E_i \leqslant G_{\boldsymbol{\lambda}}$ we have

$$\begin{aligned}
(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + G)\cdot E_i&\\
&= (K_\pi - c_1F_1 - \cdots - c_rF_r)\cdot E_i\\
&\quad + (\{-K_\pi + c_1F_1 + \cdots + c_rF_r\} + G - E_i + E_i)\cdot E_i\\
&= (K_\pi + E_i)\cdot E_i - (c_1F_1 + \cdots + c_rF_r)\cdot E_i\\
&\quad + \{c_1F_1 + \cdots + c_rF_r - K_\pi\}\cdot E_i + (G - E_i)\cdot E_i\,.
\end{aligned}$$

Let us now compute each summand separately. Firstly, the adjunction formula gives $(K_\pi + E_i)\cdot E_i = -2$ because $E_i \cong \mathbb{P}^1$. As for the second and fourth terms, the equality $-(c_1F_1 + \cdots + c_rF_r)\cdot E_i = c_1\rho_{1,i} + \cdots + c_r\rho_{r,i}$ follows from the definition of the excesses, and clearly $a_G\,(E_i) = (G - E_i)\cdot E_i$ because $E_i \leqslant G$. Therefore it only remains to prove that

$$\{c_1F_1 + \cdots + c_rF_r - K_\pi\}\cdot E_i = \sum_{E_j\in \mathrm{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\}\,, \tag{5.3.1}$$

which is also quite immediate. Indeed, writing

$$\{c_1F_1 + \cdots + c_rF_r - K_\pi\} = \sum_{j=1}^{\ell} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\}\,E_j,$$

equality (5.3.1) follows by observing that (for $j \neq i$) $E_j\cdot E_i = 1$ if and only if $E_j \in \mathrm{Adj}\,(E_i)$, and the term corresponding to $j = i$ vanishes because we have $c_1e_{1,i} + \cdots + c_re_{r,i} - k_i \in \mathbb{Z}$. $\square$

It is important to notice that we have integers in both sides.

Corollary 5.3.11. *Let $\mathfrak{a} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of ideals and $G \leqslant \sum F_i$ a reduced divisor. For any $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ and any component $E_i \leqslant G$ we have*

$$c_1\rho_{1,i} + \cdots + c_r\rho_{r,i} + a_G(E_i) + \sum_{E_j \in \operatorname{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\} \in \mathbb{Z}$$

Minimal jumping divisor

As in the case of multiplier ideals treated in Section 3.2, minimal jumping divisors satisfy a nice numerical condition.

Proposition 5.3.12. *Let $\boldsymbol{\lambda}$ be a jumping point of a tuple of ideals $\mathfrak{a} \subseteq (\mathcal{O}_{X,O})^r$. For any component $E_i \leqslant G_{\boldsymbol{\lambda}}$ of the minimal jumping divisor $G_{\boldsymbol{\lambda}}$, we have*

$$(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \geqslant 0.$$

Proof. Given a prime divisor $E_i \leqslant G_{\boldsymbol{\lambda}}$, we consider the short exact sequence

$$\begin{aligned}0 \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}} - E_i) \longrightarrow \\ \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}}) \longrightarrow \\ \longrightarrow \mathcal{O}_{E_i}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}}) \longrightarrow 0\end{aligned}$$

Pushing it forward to X, we get

$$\begin{aligned}0 \longrightarrow \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_\lambda - E_i) \longrightarrow \\ \longrightarrow \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_\lambda) \longrightarrow \\ \longrightarrow H^0(E_i, \mathcal{O}_{E_i}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_\lambda)) \otimes \mathbb{C}_O,\end{aligned}$$

where $\mathbb{C}_O$ denotes the skyscraper sheaf supported at O with fiber $\mathbb{C}$. The minimality of $G_{\boldsymbol{\lambda}}$ (see Corollary 5.3.8) implies that

$$\begin{aligned}\pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}} - E_i) \\ \neq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}}).\end{aligned}$$

Thus $H^0(E_i, \mathcal{O}_{E_i}(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}})) \neq 0$, or equivalently

$$(\lceil K_\pi - \lambda_1F_1 - \cdots - \lambda_rF_r\rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \geqslant 0.$$

□

With the above ingredients we can provide, as in the case for multiplier ideals, a geometric property of the minimal jumping divisors when viewed in the dual graph.

Theorem 5.3.13. *Let $G_{\boldsymbol{\lambda}}$ be the minimal jumping divisor associated to a jumping point $\boldsymbol{\lambda}$ of a tuple of ideals $\mathfrak{a} := (\mathfrak{a}_1, \dots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. Then the ends of a connected component of $G_{\boldsymbol{\lambda}}$ must be either rupture or dicritical divisors.*

Proof. Assume that an end E_i of a connected component of $G_{\boldsymbol{\lambda}}$ is neither a rupture nor a dicritical divisor. It means that E_i has no excess, i.e., $\rho_{j,i} = 0$ for all E_j of the resolution, and that it has one or two adjacent divisorsin the dual graph, where at most one of them belongs to $G_{\boldsymbol{\lambda}}$.

For the case that E_i has two adjacent divisors E_j and E_l, the formula given in Lemma 5.3.10 reduces to

$$\begin{aligned}
&(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \\
&\quad = -2 + \{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} + \{\lambda_1 e_{1,l} + \cdots + \lambda_r e_{r,l} - k_l\} \\
&\qquad + \lambda_1 \rho_{1,i} + \cdots + \lambda_r \rho_{r,i} + a_{G_{\boldsymbol{\lambda}}}(E_i)\,.
\end{aligned}$$

· If E_i has valence one in $G_{\boldsymbol{\lambda}}$, e.g. $E_l \not\leq G_{\boldsymbol{\lambda}}$, then

$$\begin{aligned}
&(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \\
&\quad = -2 + \{\lambda_1 e_{1,l} + \cdots + \lambda_r e_{r,l} - k_l\} + 1 < 0\,.
\end{aligned}$$

· If E_i is an isolated component of $G_{\boldsymbol{\lambda}}$, i.e., $E_j, E_l \not\leq G_{\boldsymbol{\lambda}}$, then

$$\begin{aligned}
&(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \\
&\quad = -2 + \{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} \\
&\qquad + \{\lambda_1 e_{1,l} + \cdots + \lambda_r e_{r,l} - k_l\} < 0.
\end{aligned}$$

If E_i has just one adjacent divisor E_j, i.e., E_i is an end of the dual graph, the formula reduces to

$$\begin{aligned}
&(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}) \cdot E_i \\
&\quad = -2 + \{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} + \lambda_1 \rho_{1,i} + \cdots + \lambda_r \rho_{r,i} + a_{G_{\boldsymbol{\lambda}}}(E_i)\,.
\end{aligned}$$

· If E_i has valence one in $G_{\boldsymbol\lambda}$, then

$$(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol\lambda}) \cdot E_i = -2 + 1 < 0\,.$$

· If E_i is an isolated component of $G_{\boldsymbol\lambda}$, then

$$(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol\lambda}) \cdot E_i = -2 + \{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} < 0.$$

In any case we get a contradiction with Proposition 5.3.12. □

This result allows us to give a refinement of Proposition 5.3.12.

Proposition 5.3.14. *Let $\boldsymbol\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\geqslant 0}$ be a jumping point of a tuple of ideals $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$. If $E_i \leqslant G_{\boldsymbol\lambda}$ is neither a rupture nor a dicritical component of the minimal jumping divisor $G_{\boldsymbol\lambda}$, we have*

$$(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol\lambda}) \cdot E_i = 0.$$

Proof. Assume that $E_i \leqslant G_{\boldsymbol\lambda}$ is neither a rupture or a dicritical component. In particular, it is not the end of a connected component of $G_{\boldsymbol\lambda}$. Thus, E_i has exactly two adjacent components E_j and E_l in $G_{\boldsymbol\lambda}$, and its excesses are $\rho_{j,i} = 0$ for all $1 \leqslant j \leqslant r$. The formula given in Lemma 5.3.10 for $G_{\boldsymbol\lambda}$ reduces to

$$\begin{aligned}(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol\lambda}) \cdot E_i \\ = -2 + \lambda_1 \rho_{1,i} + \cdots + \lambda_r \rho_{r,i} + \{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} \\ + \{\lambda_1 e_{1,l} + \cdots + \lambda_r e_{r,l} - k_l\} + a_{G_\lambda}(E_i)\,.\end{aligned}$$

Notice that $a_{G_{\boldsymbol\lambda}}(E_i) = 2$, and also that

$$\{\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j\} = \{\lambda_1 e_{1,l} + \cdots + \lambda_r e_{r,l} - k_l\} = 0\,,$$

because E_j and E_l are components of G_λ, so finally

$$(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol\lambda}) \cdot E_i = 0\,.$$

□

Maximal jumping divisor

For multiplier ideals, the numerical properties that maximal jumping divisors satisfy were treated in Section 3.3. Their proofs are a little bit more involved than in the case of minimal jumping divisors.

Proposition 5.3.15. *Fix any* $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$*, and let* $H_{\boldsymbol{c}}$ *be its associated maximal jumping divisor. Then the following inequalities hold:*

- $(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil + H_{\boldsymbol{c}}) \cdot E_i \geqslant -1$ *for all* $E_i \leqslant H_{\boldsymbol{c}}$*, and*
- $(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil + H_{\boldsymbol{c}}) \cdot H \geqslant -1$ *for any connected component* $H \leqslant H_{\boldsymbol{c}}$.

Proof. From Lemma 5.3.10 we already know that

$$(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil + H_{\boldsymbol{c}}) \cdot E_i \geqslant -2$$

for all $E_i \leqslant H_{\boldsymbol{c}}$. If equality holds, then we must also have that

- $a_{H_{\boldsymbol{c}}}(E_i) = 0$, that is, E_i is an isolated component in $H_{\boldsymbol{c}}$,
- $\{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\} = 0$ for all $E_j \in \operatorname{Adj}(E_i)$, that is, every exceptional component E_j intersecting E_i is also contained in $H_{\boldsymbol{c}}$, and
- $\rho_{j,i} = 0$ for all $1 \leqslant j \leqslant r$.

The first two conditions imply that E_i is the only exceptional curve of the log-resolution. But in this case there must exist at least one j such that $\rho_{j,i} > 0$, and the third condition is not satisfied.

As for the second part, using Lemma 5.3.10 for all $E_i \leqslant H$ and summing up we obtain

$$\begin{aligned}
&(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil + H_{\boldsymbol{c}}) \cdot H \\
&\qquad = -2v_H + \sum_{E_i \leqslant H} \left(\sum_{E_j \in \operatorname{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\} + c\rho_i \right) + 2a_H \\
&\qquad = -2 + \sum_{E_i \leqslant H} \left(\sum_{E_j \in \operatorname{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\} \right) \\
&\qquad\qquad + c_1 \sum_{E_i \leqslant H} \rho_{1,i} + \cdots + c_r \sum_{E_i \leqslant H} \rho_{r,i} \geqslant -2,
\end{aligned}$$

where $a_H - v_H = 1$ due to the tree structure of the exceptional divisor and the connectedness of H. Equality holds if and only if

$$\sum_{E_i \leqslant H} \left(\sum_{E_j \in \operatorname{Adj}(E_i)} \{c_1e_{1,j} + \cdots + c_re_{r,j} - k_j\} \right) = 0$$

and

$$c_1 \sum_{E_i \leqslant H} \rho_{1,i} = ... = c_r \sum_{E_i \leqslant H} \rho_{r,i} = 0 \, .$$

The first condition implies that H is the whole exceptional divisor, and then the second condition implies that $\sum_i \rho_{j,i} = 0$ for all $1 \leqslant j \leqslant r$, which is impossible. Hence the inequality must be strict, and since

$$(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot H \in \mathbb{Z} \, ,$$

the claim follows. □

This allows us to give some geometrical properties of $H_{\boldsymbol{c}}$.

Theorem 5.3.16. *Fix any $\boldsymbol{c} \in \mathbb{R}^r_{>0}$, and let $H_{\boldsymbol{c}}$ be the corresponding maximal jumping divisor. Then we have the following*

- *The isolated components of $H_{\boldsymbol{c}}$ must be either a rupture divisor, a dicritical divisor or a divisor E_i with $a\,(E_i) = 2$ such that*

$$\sum_{E_j \in \operatorname{Adj}(E_i)} \{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} = 1.$$

- *An end of a reducible connected component of $H_{\boldsymbol{c}}$ must be either a rupture divisor, a dicritical divisor or an end of the whole exceptional divisor.*

Proof. Let E_i be an isolated component of $H_{\boldsymbol{c}}$. Assume that it is neither a rupture nor a dicritical component. Then it only has one or two adjacent components in the exceptional divisor. In the first case, if E_j is the only exceptional component in $\operatorname{Adj}(E_i)$, then the formula given in Lemma 5.3.10 reduces to

$$\begin{aligned}(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot E_i \\ = -2 + \{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} \, .\end{aligned}$$

Since $\{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} < 1$, we would get

$$(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot E_i < -1 \, ,$$

contradicting Proposition 5.3.15. The only possible remaining case is when $a\,(E_i) = 2$. If $\operatorname{Adj}(E_i) = \{E_j, E_l\}$, then we have

$$\begin{aligned}&(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot E_i \\ &\quad = -2 + \{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} + \{c_1 e_{1,l} + \cdots + c_r e_{r,l} - k_l\} \, .\end{aligned}$$

Since

$$0 \leqslant \{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} + \{c_1 e_{1,l} + \cdots + c_r e_{r,l} - k_l\} < 2$$

must be an integer by Corollary 5.3.11, therefore it should be equal to 0 or 1 (recall that we assumed that E_i satisfies $\rho_{\ell,i} = 0$ for all $1 \leqslant \ell \leqslant r$). But the former contradicts Proposition 5.3.15, hence the only possibility is that $\{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} + \{c_1 e_{1,l} + \cdots + c_r e_{r,l} - k_l\} = 1$, which is the last possibility given in the statement.

As for the second assertion, let E_i be an end of a reducible connected component of $H_{\boldsymbol{c}}$ that is neither a rupture divisor, nor a dicritical divisor nor an end of the whole exceptional divisor. Then it has two adjacent components in the whole exceptional divisor, say E_j and E_l, but only one of them, say E_j, is in $H_{\boldsymbol{c}}$. Then we have

$$(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot E_i = -2 + \{c_1 e_{1,l} + \cdots + c_r e_{r,l} - k_l\} + 1 \notin \mathbb{Z},$$

which is impossible. □

5.4 Multiplicities of jumping points

This last part is devoted to generalize the results about the multiplicity obtained in Chapter 4 in the case of mixed multiplier ideals. We define the multiplicity of a point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ as follows.

Definition 5.4.1. Let $\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be tuple of $\mathfrak{m}$-primary ideals. We define the multiplicity attached to a point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ as the codimension of $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}})$ in $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)$ for $\varepsilon > 0$ small enough, i.e.,

$$m(\boldsymbol{c}) := \dim_{\mathbb{C}} \frac{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)}{\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}})} .$$

Remark 5.4.2. Notice that $m(\boldsymbol{c}) > 0$ if and only if $\boldsymbol{c}$ is a jumping point.

The main goal of this section is to present a formula for the multiplicity in terms of the maximal jumping divisor, extending the one given in Theorem 4.1.1 for multiplier ideals.

Theorem 5.4.3. *Let $\boldsymbol{\mathfrak{a}} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of $\mathfrak{m}$-primary ideals and $H_{\boldsymbol{c}}$ the maximal jumping divisor associated to some $\boldsymbol{c} \in \mathbb{R}^r_{>0}$. Then*

$$m(\boldsymbol{c}) = (\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}) \cdot H_{\boldsymbol{c}} + \#\{\textit{connected components of } H_{\boldsymbol{c}}\} .$$

Proof. Consider the short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X'}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil\right) \longrightarrow \\ \longrightarrow \mathcal{O}_{X'}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right) \longrightarrow \\ \longrightarrow \mathcal{O}_{H_{\boldsymbol{c}}}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right) \longrightarrow 0\,. \end{aligned}$$

Pushing it forward to X and applying local vanishing for mixed multiplier ideals, we get the short exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil\right) \longrightarrow \\ \longrightarrow \pi_*\mathcal{O}_{X'}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right) \longrightarrow \\ \longrightarrow H^0\left(H_{\boldsymbol{c}}, \mathcal{O}_{H_{\boldsymbol{c}}}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right)\right) \otimes \mathbb{C}_O \longrightarrow 0 \end{aligned}$$

or, equivalently, since

$$H_{\boldsymbol{c}} = \lceil K_\pi - (c_1 - \varepsilon)F_1 - \cdots - (c_r - \varepsilon)F_r\rceil - \lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil$$

for $\varepsilon > 0$ small enough,

$$\begin{aligned} 0 \longrightarrow \mathcal{J}(\mathfrak{a}^{\boldsymbol{c}}) \longrightarrow \mathcal{J}(\mathfrak{a}^{((1-\varepsilon)\boldsymbol{c}}) \longrightarrow \\ \longrightarrow H^0\left(H_{\boldsymbol{c}}, \mathcal{O}_{H_{\boldsymbol{c}}}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right)\right) \otimes \mathbb{C}_O \longrightarrow 0\,. \end{aligned}$$

Therefore the multiplicity of $\boldsymbol{c}$ is just

$$\begin{aligned} m(\boldsymbol{c}) &= h^0\left(H_{\boldsymbol{c}}, \mathcal{O}_{H_{\boldsymbol{c}}}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right)\right) \\ &= \sum_{E_i \leqslant H_{\boldsymbol{c}}} h^0\left(E_i, \mathcal{O}_{E_i}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right)\right) - a_{H_{\boldsymbol{c}}}, \end{aligned}$$

where in the second equality we have used that $H_{\boldsymbol{c}}$ has simple normal crossings, and hence the sections of the line bundle $\mathcal{O}_{H_{\boldsymbol{c}}}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right)$ correspond to sections over each component that agree on the $a_{H_{\boldsymbol{c}}}$ intersections.

Recall now that each exceptional component E_i is isomorphic to $\mathbb{P}^1$, and that the sections of a line bundle on $\mathbb{P}^1$ are determined by its degree (namely, $h^0\left(\mathcal{O}_{\mathbb{P}^1}(d)\right) = d+1$ if $d \geqslant -1$ and zero otherwise). Then, using that

$$\begin{aligned} \deg \mathcal{O}_{E_i}\left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right) \\ = \left(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}\right) \cdot E_i \geqslant -1 \end{aligned}$$

by Proposition 5.3.15, we get

$$
\begin{aligned}
m(\boldsymbol{c}) &= \sum_{E_i \leqslant H_{\boldsymbol{c}}} \left(\left(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}\right) \cdot E_i + 1\right) - a_{H_{\boldsymbol{c}}} \\
&= \left(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}\right) \cdot H_{\boldsymbol{c}} + v_{H_{\boldsymbol{c}}} - a_{H_{\boldsymbol{c}}} \\
&= \left(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil + H_{\boldsymbol{c}}\right) \cdot H_{\boldsymbol{c}} \\
&\qquad + \# \left\{\text{connected components of } H_{\boldsymbol{c}}\right\}.
\end{aligned}
$$

□

The multiplicity of a jumping point $\boldsymbol{\lambda}$ can also be described using an analogous formula to the one given in the previous theorem, but using the minimal jumping divisor $G_{\boldsymbol{\lambda}}$ instead of the maximal jumping divisor $H_{\boldsymbol{\lambda}}$. Namely, using the same arguments as before, we have

$$
\begin{aligned}
m(\boldsymbol{\lambda}) = (\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil + G_{\boldsymbol{\lambda}}) \cdot G_{\boldsymbol{\lambda}} \\
+ \#\{\text{connected components of } G_{\boldsymbol{\lambda}}\}.
\end{aligned}
$$

Remark 5.4.4. This formula is handy to study the multiplicity over the points of a $\mathcal{C}$-facet. Recall that two interior points of a $\mathcal{C}$-facet have the same minimal jumping divisor (see Lemma 5.3.5). Therefore, the multiplicity is constant along the interior points. This is no longer true for points at the intersection of $\mathcal{C}$-facets.

We can rephrase Theorem 5.4.3 as follows.

Corollary 5.4.5. *Let $\boldsymbol{\mathfrak{a}} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of $\mathfrak{m}$-primary ideals and $H_{\boldsymbol{c}}$ the maximal jumping divisor associated to some $\boldsymbol{c} \in \mathbb{R}^r_{>0}$. Then*

$$
\begin{aligned}
m(\boldsymbol{c}) = \sum_{E_i \leqslant H_{\boldsymbol{c}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{c_1 e_{1,j} + \cdots + c_r e_{r,j} - k_j\} + c_1 \rho_{1,i} + \cdots + c_r \rho_{r,i} \right) \\
- \# \left\{\textit{connected components of } H_{\boldsymbol{c}}\right\}.
\end{aligned}
$$

The main consequence of these formulas is that we have a very simple numerical criterion to detect whether a given point $\boldsymbol{c} \in \mathbb{R}^r_{>0}$ is a jumping point, or equivalently, a point in a jumping wall.

Proposition 5.4.6. *Let* $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of* $\mathfrak{m}$*-primary ideals and* $\boldsymbol{c} \in \mathbb{R}^r_{>0}$*. Then* $\boldsymbol{c}$ *is a jumping point if and only if* $m(\boldsymbol{c}) > 0$*.*

Theorem 5.4.7. *Let* $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of* $\mathfrak{m}$*-primary ideals and* $\boldsymbol{c} \in \mathbb{R}^r_{>0}$*. Then there exists a connected component* $H \leqslant H_{\boldsymbol{c}}$ *such that*

$$(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}) \cdot H > -1$$

if and only if $m(\boldsymbol{c}) > 0$*.*

Proof. We have

$$\begin{aligned} m(\boldsymbol{c}) &= (\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}) \cdot H_{\boldsymbol{c}} \\ &\qquad + \#\{\text{connected components of } H_{\boldsymbol{c}}\} \\ &= \sum_{H \leqslant H_{\boldsymbol{c}}} \left((\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_{\boldsymbol{c}}) \cdot H + 1\right), \end{aligned}$$

where the sum is taken over all the connected components $H \leqslant H_{\boldsymbol{c}}$. Then the result follows since $(\lceil K_\pi - c_1F_1 - \cdots - c_rF_r\rceil + H_c) \cdot H \geqslant -1$ by Proposition 5.3.15. □

It follows from the definition of maximal jumping divisor that they satisfy the following periodicity property.

Lemma 5.4.8. *For any* $\boldsymbol{c} \in \mathbb{R}^r_{>0}$ *and* $\boldsymbol{\alpha} \subseteq \{0,1\}^r$ *we have* $H_{\boldsymbol{c}} = H_{\boldsymbol{c}+\boldsymbol{\alpha}}$*.*

Recall that Skoda's theorem (see Theorem 1.8.10) in this setting states that $\mathcal{J}(\mathfrak{a}^{\boldsymbol{c}+\boldsymbol{\alpha}}) = \mathfrak{a}^{\boldsymbol{\alpha}}\mathcal{J}(\mathfrak{a}^{\boldsymbol{c}})$, so we also have a periodicity property for mixed multiplier ideals. Using Theorem 5.4.3, we may control the growth of the multiplicities in terms of the excesses at dicritical components.

Proposition 5.4.9. *Let* $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ *be a tuple of* $\mathfrak{m}$*-primary ideals and* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \subseteq \{0,1\}^r$*. Then,*

$$m(\boldsymbol{c} + \boldsymbol{\alpha}) - m(\boldsymbol{c}) = \sum_{E_i \leqslant H_{\boldsymbol{c}}} \sum_{1 \leqslant j \leqslant r} \alpha_j \rho_{j,i}\,.$$

Proof. It is clear that $\boldsymbol{c}$ and $\boldsymbol{c}+\boldsymbol{\alpha}$ have the same jumping divisor $H_{\boldsymbol{c}}$. Therefore, by Theorem 5.4.3, we have

$$m(\boldsymbol{c} + \boldsymbol{\alpha}) - m(\boldsymbol{c}) = -(\alpha_1F_1 + \cdots + \alpha_rF_r) \cdot H_{\boldsymbol{c}} = \sum_{E_i \leqslant H_{\boldsymbol{c}}} \sum_{1 \leqslant j \leqslant r} \alpha_i \rho_{j,i}.$$

□

Appendix A

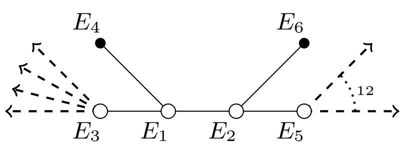
Examples of jumping walls

With this appendix we pretend to illustrate some results from Chapter 5. These examples include an example over an X that has a rational singularity, an example where the maximal jumping divisor is not constant over the $\mathcal{C}$-facet and examples of the log-canonical wall.

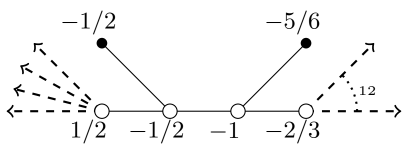
A.1 An example with rational singularities

The following example serves to illustrate that two points on different $\mathcal{C}$-facets supported on the same hyperplane can have the same mixed multiplier ideal associated.

Example A.1.1. Consider a surface X with a rational singularity at O whose minimal resolution $\pi : X' \longrightarrow X$ has six exceptional components $E_1, \ldots, E_6$ with the following dual graph and K_π:



Vertex ordering

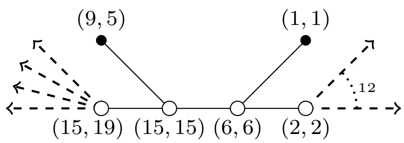


K_π

with fundamental cycle the divisor $Z = (3,2,3,1,1,1)$ and intersection matrix

$$M = (E_i \cdot E_j)_{1 \leqslant i,j \leqslant 6} = \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 \end{pmatrix}.$$

Over them consider the following pair of ideals $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$, non singular on X' given by the following divisors



(F_1, F_2)

For simplicity, we have $K_\pi = \left(-\frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}, -\frac{5}{6}\right)$, $F_1 = (15,6,15,9,2,1)$ and $F_2 = (15,6,19,5,2,1)$.

Notice that E_1, E_2, E_3 and E_6 are the dicritical and rupture divisors. Then, as a consequence of Theorem 5.2.3, the region of a given point $\boldsymbol{\lambda} = (z_1, z_2)$ is defined by

$$\begin{array}{lcl} 15z_1 + 15z_2 & < & -\frac{1}{2} + 1 + e_1^{\boldsymbol{\lambda}} \\ 6z_1 + 6z_2 & < & -1 + 1 + e_2^{\boldsymbol{\lambda}} \\ 15z_1 + 19z_2 & < & \frac{1}{2} + 1 + e_3^{\boldsymbol{\lambda}} \\ z_1 + z_2 & < & -\frac{5}{6} + 1 + e_6^{\boldsymbol{\lambda}} \end{array}.$$

In our case, we are interested on the line $15z_1 + 15z_2 = \frac{1}{2} + 1 + 6$. This line correspond to the case when $e_1^{\boldsymbol{\lambda}} = 6$. Notice that in this case, we have that the mixed multiplier ideals associated to $\boldsymbol{\lambda}_1 = \left(\frac{5}{96}, \frac{215}{608}\right)$ and $\boldsymbol{\lambda}_2 = \left(\frac{5}{24}, \frac{1}{8}\right)$ are different. Writing as $D_{\boldsymbol{\lambda}}$ the antinef closure of the divisor $\lceil \lambda_1 F_1 + \cdots + \lambda_r F_r - K_\pi \rceil$. We have that $D_{\boldsymbol{\lambda}_1} = (6,3,7,2,1,1)$ and $D_{\boldsymbol{\lambda}_2} = (6,3,6,3,1,1)$, but for any point on the $\mathcal{C}$-facets associated to $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ supported on H_1, the mixed multiplier ideal is the same. Consider for example $\boldsymbol{\lambda'}_1 = \left(\frac{1}{24}, \frac{47}{120}\right)$ and $\boldsymbol{\lambda'}_2 = \left(\frac{31}{120}, \frac{7}{40}\right)$ two points on the $\mathcal{C}$-facets supported on

H_1 associated to $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ respectively, the divisor associated to their mixed multiplier ideal is $D_{\boldsymbol{\lambda}'_1} = D_{\boldsymbol{\lambda}'_2} = (7, 3, 7, 3, 1, 1)$.

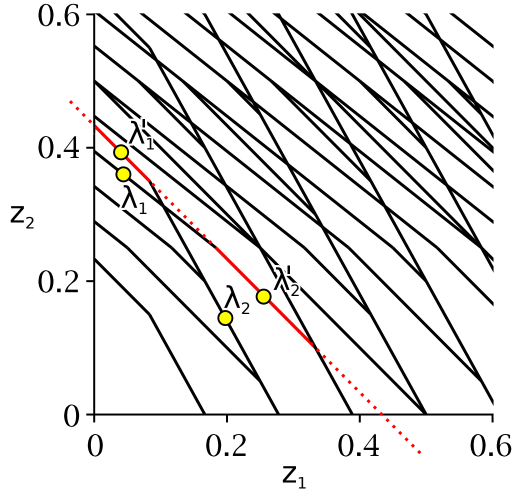
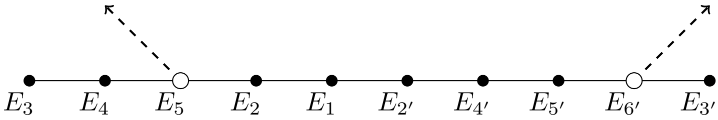


Figure A.1: The jumping walls of the example with the red dotted line $15z_1 + 15z_2 = -\frac{1}{2} + 1 + 6$ and the points $\boldsymbol{\lambda}_1 = \left(\frac{5}{96}, \frac{215}{608}\right)$, $\boldsymbol{\lambda}_2 = \left(\frac{5}{24}, \frac{1}{8}\right)$.

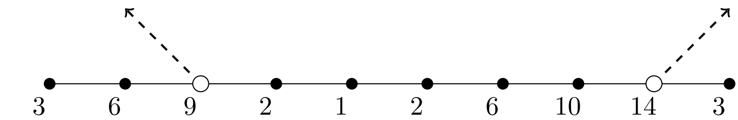
A.2 An example where maximal jumping divisor is not constant over a $\mathcal{C}$-facet

The following example serves to illustrate that contrarily to what happens with the minimal jumping divisor (see Lemma 5.3.5), the maximal jumping divisor can be non-constant over a $\mathcal{C}$-facet.

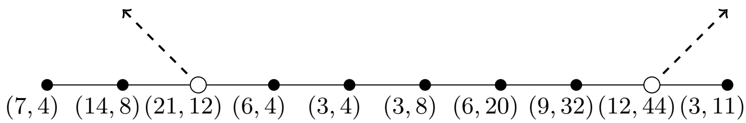
Example A.2.1. Consider the following set of ideals $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ with $\mathfrak{a}_1 = (x^3, y^7)$ and $\mathfrak{a}_2 = (y^4, x^{11})$. We represent the relative canonical divisor K_π and F_1 and F_2 in the dual graph as follows:



Vertex ordering



K_π



F_1, F_2

The hyperplanes where $\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j \in \mathbb{Z}_{>0}$ for $i \in \{1, ..., 5, 2', ..., 6'\}$ are

$$\begin{array}{ccccccc}
3z_1 & + & 4z_2 & = & 1 & + & \ell_1 \\
6z_1 & + & 4z_2 & = & 2 & + & \ell_2 \\
7z_1 & + & 4z_2 & = & 3 & + & \ell_3 \\
14z_1 & + & 8z_2 & = & 6 & + & \ell_4 \\
21z_1 & + & 12z_2 & = & 9 & + & \ell_5 \\
3z_1 & + & 8z_2 & = & 2 & + & \ell_{2'} \\
3z_1 & + & 11z_2 & = & 3 & + & \ell_{3'} \\
6z_1 & + & 20z_2 & = & 6 & + & \ell_{4'} \\
9z_1 & + & 32z_2 & = & 10 & + & \ell_{5'} \\
12z_1 & + & 44z_2 & = & 14 & + & \ell_{6'}
\end{array} \tag{A.2.1}$$

with $\ell_i \in \mathbb{Z}_{>0}$ for $i \in \{1, ..., 5, 2', ..., 6'\}$. If we draw them, we can see that the maximal jumping divisor could not be constant over any $\mathcal{C}$-facet (see Figure A.2). Consider for example the points $\boldsymbol{\lambda}_1 = \left(\frac{7}{12}, \frac{1}{16}\right)$, $\boldsymbol{\lambda}_2 = \left(\frac{11}{20}, \frac{29}{240}\right)$ and $\boldsymbol{\lambda}_3 = \left(\frac{98}{195}, \frac{53}{260}\right)$. The minimal jumping divisor for all of them is

$$G_{\boldsymbol{\lambda}_1} = G_{\boldsymbol{\lambda}_2} = G_{\boldsymbol{\lambda}_3} = E_5\,,$$

on the other hand, the maximal jumping divisors are

$$H_{\boldsymbol{\lambda}_1} = E_1 + E_5,\ H_{\boldsymbol{\lambda}_2} = E_5 \text{ and } H_{\boldsymbol{\lambda}_3} = E_5 + E_{6'}\,.$$

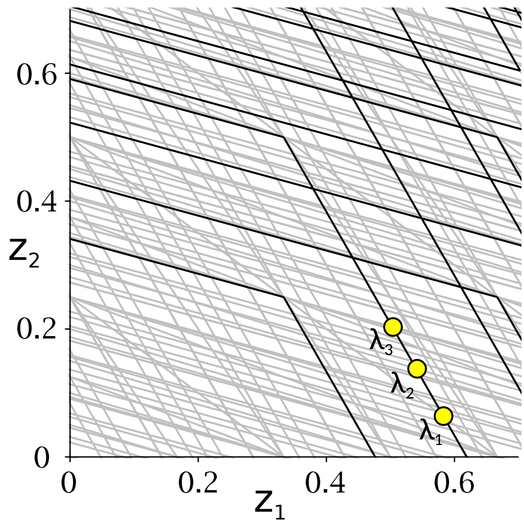


Figure A.2: The jumping walls (in black) and the hyperplanes (A.2.1) (in grey) where $\lambda_1 e_{1,j} + \cdots + \lambda_r e_{r,j} - k_j \in \mathbb{Z}_{>0}$ for $i \in \{1, ..., 5, 2', ..., 6'\}$.

A.3 The log-canonical wall

Let X be a smooth two dimensional variety, $\mathfrak{a} := (\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ a tuple of simple ideals. Consider the dual graph Γ associated to a common log-resolution of $\mathfrak{a}$. Notice that the dual graph Γ_i associated to each simple ideal $\mathfrak{a}_i$ can be identified with a subgraph of Γ. For every such subgraph $\Gamma_i \subseteq \Gamma$ we will only consider the divisors in the path connecting the origin E_1 and the first rupture or dicritical divisor. Collecting all these paths in the dual graph of $\mathfrak{a}$ we obtain a connected subgraph $\gamma \subseteq \Gamma$ whose ends are the rupture or dicritical divisors of the ideals $\mathfrak{a}_i$ and the origin[1]. Notice that γ may contain some extra rupture points depending on the *contact* of the subgraphs Γ_i.

Definition A.3.1. The *Newton nest* of the dual graph of $\mathfrak{a}$ is the set of rupture or dictritical divisors contained in the connected subgraph γ.

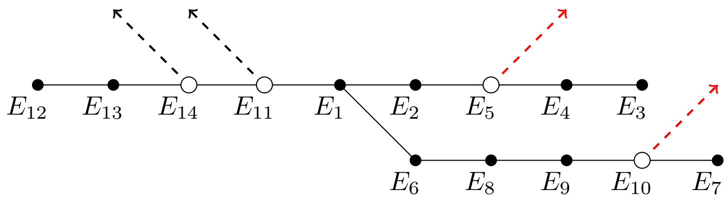
Cassou-Noguès and Libgober [CNL14, Theorem 4.22] gave a description of the log-canonical wall of mixed multiplier ideals in terms of the Newton nest that provides a generalization of the results cited above. Namely, their result is

[1] If the subgraphs corresponding to the ideals $\mathfrak{a}_i$ only share the origin then the origin will not be an end of γ.

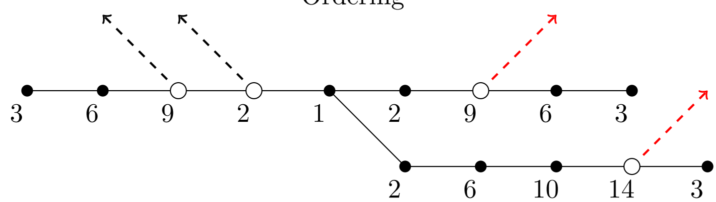
Theorem A.3.2. *[CNL14, Theorem 4.22] Let $\mathfrak{a} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of simple ideals. Then, there is a one-to-one correspondence between the exceptional divisors in the Newton nest of $\mathfrak{a}$ and the facets of the log-canonical wall.*

With the following example, we want to illustrate that this result is no longer true if the condition of being simple is dropped.

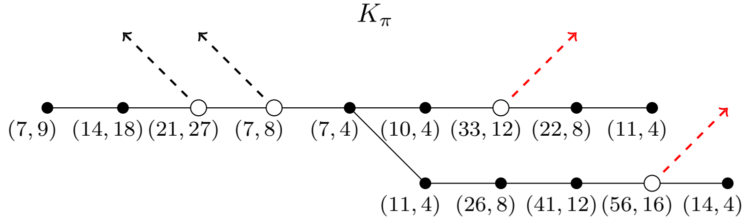
Example A.3.3. Consider the following tuple of non-simple ideals $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ over a smooth variety X with relative canonical divisor K_π and F_1 and F_2 in the dual graph as follows:



Ordering

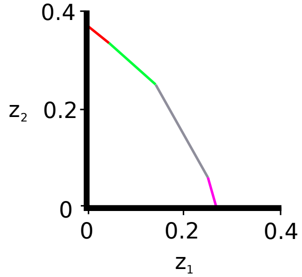


K_π

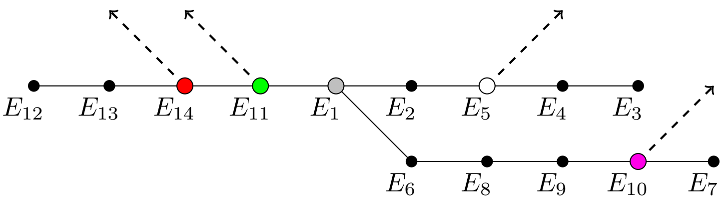


(F_1, F_2)

The associated log-canonical wall is the following

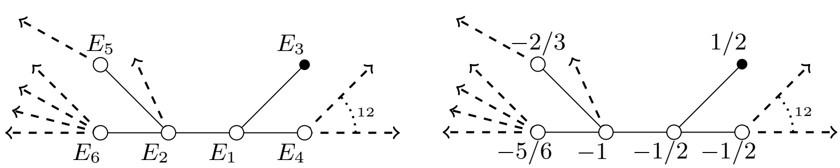


where the colors make reference to the divisors that have the same color in the next dual graph.



Furthermore, if we drop the condition of X smooth, we can have examples like the following.

Example A.3.4. Consider a surface X with a rational singularity at O whose minimal resolution $\pi : X' \longrightarrow X$ has six exceptional components $E_1, \dots, E_6$ with the following dual graph and K_π:

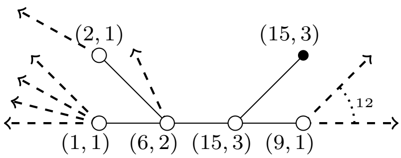


Vertex ordering $\qquad$ K_π

with fundamental cycle the divisor $Z = (3,2,3,1,1,1)$ and intersection matrix,

$$M = (E_i \cdot E_j)_{1 \leqslant i,j \leqslant 6} = \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 \end{pmatrix}.$$

Over them consider the following pair of ideals $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$, with $\mathfrak{a}_1$ non singular on X' and $\mathfrak{a}_2 = \mathfrak{m}$ given by the following divisors



(F_1, F_2)

For simplicity, we have $K_\pi = \left(-\frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}, -\frac{5}{6}\right)$, $F_1 = (15,6,15,9,2,1)$ and $F_2 = Z = (3,2,3,1,1,1)$.

With this example we illustrate that if we drop the condition of X being smooth, we loose the bijection of Theorem A.3.2 and also the fact that the mixed multiplier ideals associated to all the $\mathcal{C}$-facets of the log-canonical wall are equal.

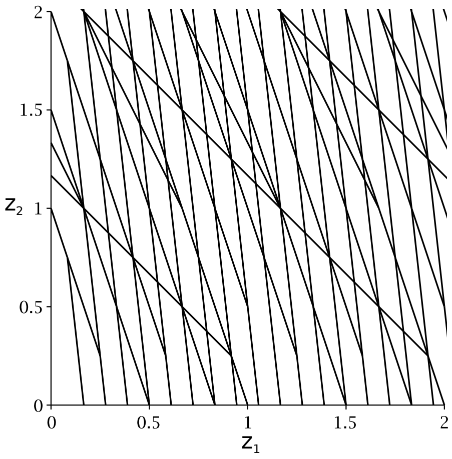


Figure A.3: The jumping walls of the example where we can appreciate that the mixed multiplier ideals associated to the two $\mathcal{C}$-facets of the log-canonical wall are different.

Appendix B

Package MultiplierIdealsDim2.m2

MultiplierIdealsDim2 is a package that contains several tools related with the computations of multiplier ideals. Given the self intersection matrix and the divisor associated to this ideal, using the function *MultiplierIdeals* one can compute the jumping numbers and their associated multiplier ideals in the interval $(SmallestJN, BiggestJN]$ using either the algorithms presented on [Tuc10], Chapter 2 or Chapter 4. However, if we want to know the multiplicity of a given number as a jumping number for a given ideal, one can use *MultiplicityJN*. Another option that this package offers is to compute the Multiplier Ideal associated to a given number, thanks to the function *MultIdeal*. The package also contains two extra functions: to compute the relative Canonical divisor of a resolution (*RelativeCanonicalDivisor*). And to compute the antinef closure of a given divisor (*Unloading*).

B.1 MultiplierIdeals

- Headline: Computes the jumping numbers and their ideals.
- Synopsis:
  - Usage:

    `MultiplierIdeals(F,E)`
  - Inputs:

    * F: Divisor associated to the ideal $\mathfrak{a}$ to whom we want to compute its associated jumping numbers and multiplier ideals.
    * E: Intersection matrix associated of the log-resolution of $\mathfrak{a}$.
  - Optional inputs:
    * algorithm: Method used to compute the jumping numbers and multiplier ideals.
    * BiggestJN: Upper bound of the interval where we want to compute the jumping numbers.
    * JumpingDivisor: Show or not the jumping divisors.
    * MaxIterations: Limits the number of iterations of the Unloading algorithm.
    * SmallestJN: Lower bound of the interval where we want to compute the jumping numbers.
  - Outputs: A table that contains at least the jumping number, their multiplicities and the ideals.

- Description: Starting form the divisor encoded as a matrix of dimensions $1 \times m$, and the intersection matrix as presented in the Thesis, the algorithm computes the jumping numbers for this ideal with their multiplicities and associated ideals in the interval $(SmallestJN, BiggestJN]$.

- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                 {  0, -2,  1,  0,  0},
                 {  1,  1, -1,  0,  0},
                 {  0,  0,  0, -2,  1},
                 {  1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : F = matrix({{4,5,10,5,10}})

o2 = | 4 5 10 5 10 |

              1        5
o2 : Matrix ZZ  <-- ZZ

i3 : MultiplierIdeals(F,E,BiggestJN => 1)
```

```
                        1
o3 =  Jumping number: -   Multiplicity: 1
                        2
                      Multiplier ideal: | 1 1 2 1 2 |
                      Maximal jumping divisor: {| 1 0 1 0 1 |}
                      Minimal jumping divisor: {| 1 0 1 0 1 |}

                         7
      Jumping number: -  Multiplicity: 2
                        10
                      Multiplier ideal: | 2 2 4 2 4 |
                      Maximal jumping divisor: {| 0 0 1 0 1 |}
                      Minimal jumping divisor: {| 0 0 1 0 1 |}

                         9
      Jumping number: -  Multiplicity: 2
                        10
                      Multiplier ideal: | 2 3 5 3 5 |
                      Maximal jumping divisor: {| 0 0 1 0 1 |}
                      Minimal jumping divisor: {| 0 0 1 0 1 |}

      Jumping number: 1   Multiplicity: 1
                      Multiplier ideal: | 3 3 6 3 6 |
                      Maximal jumping divisor: {| 1 1 1 1 1 |}
                      Minimal jumping divisor: {| 1 0 1 0 1 |}

o3 : Type{...1...}
```

- Code:

```
MultiplierIdeals = {SmallestJN => 0,MaxIterations => 10000,
      BiggestJN => 2,algorithm => "AlbAlvDac",JumpingDivisor => true}
      >> o -> (F,IntersectionMatrix) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  NumDiv := numgens source IntersectionMatrix;
  if o.algorithm == "AlbAlvDac" then(
     JNandMI = AlbAlvDac(F,IntersectionMatrix,SmallestJN =>
        o.SmallestJN,MaxIterations => o.MaxIterations,
        BiggestJN => o.BiggestJN);
  )else if o.algorithm == "Tucker" then(
     JNandMI = Tucker(F,IntersectionMatrix,SmallestJN =>
        o.SmallestJN,MaxIterations => o.MaxIterations,
        BiggestJN => o.BiggestJN);
  )else if o.algorithm == "Mult" and NumExceptionalDiv == NumDiv then(
     JNandMI = Multiplicities(F,IntersectionMatrix,SmallestJN =>
        o.SmallestJN,MaxIterations => o.MaxIterations,
        BiggestJN => o.BiggestJN);
  )else(
     print "Please choose another algorithm, this ideal is not m-primary";
     JNandMI = 0;
  );
JumpingNumbersTable := new Type of HashTable;
```

```
  if (o.JumpingDivisor == false) then(
           net JumpingNumbersTable  :=  CollectionJN -> stack apply
             (sort keys CollectionJN, k -> " Jumping number: "
             |net k_0 | " Multiplicity: " | net ((CollectionJN#k)_0)
             | " Multiplier ideal: " | net matrix (CollectionJN#k)_1);
  )else if (o.JumpingDivisor == true) and o.algorithm == "Mult" then(
           net JumpingNumbersTable  :=  CollectionJN -> stack apply
             (sort keys CollectionJN, k -> " Jumping number: "
             |net k_0 | "  Multiplicity: " | net ((CollectionJN#k)_0)
             | " Multiplier ideal: " | net (CollectionJN#k)_1|
             " Maximal jumping divisor: " | net matrix ((CollectionJN#k)_2));
  )else if (o.JumpingDivisor == true) and o.algorithm == "AlbAlvDac" then(
           net JumpingNumbersTable  :=  CollectionJN -> stack apply (sort
             keys CollectionJN, k -> " Jumping number: " |net k_0
             | "  Multiplicity: " | net (CollectionJN#k)_0
             | " Multiplier ideal: " | net matrix((CollectionJN#k)_1)
             | " Maximal jumping divisor:\n Minimal jumping divisor: "
             | net (VerticalList (new List from(matrix ((CollectionJN#k)_2),
             matrix ((CollectionJN#k)_3)))));
  )else if (o.JumpingDivisor == true) and o.algorithm == "Tucker" then(
           net JumpingNumbersTable  :=  CollectionJN -> stack apply (sort
             keys CollectionJN, k -> " Jumping number: " |net k_0
             | "  Multiplicity: " | net (CollectionJN#k)_0
             | " Multiplier ideal: " | net matrix((CollectionJN#k)_1)
             | " Maximal jumping divisor:\n Minimal jumping divisor:\n
             Critical divisor(s): " | net (VerticalList (new List
             from(matrix ((CollectionJN#k)_2), matrix (
             (CollectionJN#k)_3),matrix ((CollectionJN#k)_4)))));
  );
new JumpingNumbersTable from JNandMI
)
```

B.2 Unloading

- Headline: Computes the antinef closure of a divisor.
- Synopsis:
  - Usage:

    `Unloading(D,E)`
  - Inputs:
    - D: A divisor.
    - E: Intersection matrix associated of the log-resolution of $\mathfrak{a}$.
  - Optional inputs:
    - MaxIterations: Limits the number of iterations of the Unloading algorithm.

    * UnloadingValue: Show the maximum of the unloaded values.
  - Outputs: Antinef closure of D.

- Description: Starting form the divisor encoded as a matrix of dimensions $1 \times m$, and the intersection matrix as presented in Section 1.7, it returns the antinef closure of the divisor.

- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                 {  0, -2,  1,  0,  0},
                 {  1,  1, -1,  0,  0},
                 {  0,  0,  0, -2,  1},
                 {  1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : D = matrix({{1,9,8,8,5}})

o2 = | 1 9 8 8 5 |

              1        5
o2 : Matrix ZZ  <-- ZZ

i3 : Unloading(D,E)

o3 = | 6 9 15 8 14 |

o3 : MutableMatrix
```

- Code:

```
Unloading = {MaxIterations => 10000,UnloadingValue => false}»p ->
         (D,IntersectionMatrix) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  NumDiv := numgens source IntersectionMatrix;
  Tame := 0;
  isUnloaded := false;
  UnloadedDiv := mutableMatrix(ZZ,1,NumDiv);
  for j from 0 to NumDiv - 1 list(
     UnloadedDiv_(0,j) = ceiling(D_(0,j));
  );
  for cont from 0 to p.MaxIterations when (not isUnloaded) list(
    NumUnloadedDiv := 0;
```

```
    Condition := transpose(IntersectionMatrix * transpose(matrix(UnloadedDiv)));
    for j from 0 to NumExceptionalDiv - 1 list(
      if (Condition_(0,j) > 0) then(
UnloadedDiv_(0,j) = UnloadedDiv_(0,j)
     + ceiling( - Condition_(0,j) / IntersectionMatrix_(j,j));
        if (ceiling( - Condition_(0,j) / IntersectionMatrix_(j,j)) > Tame) then(
            Tame = ceiling( - Condition_(0,j) / IntersectionMatrix_(j,j));
        );
NumUnloadedDiv = 0;
      ) else(
NumUnloadedDiv = NumUnloadedDiv + 1;
      );
    ); -print peek(UnloadedDiv);
    if (NumUnloadedDiv == NumExceptionalDiv) then(
      isUnloaded = true;
    );
  );
  if (not isUnloaded) then(
    print "Not Unloaded, you need more iterations";
  );
  if (p.UnloadingValue == true) then(
    return(UnloadedDiv,Tame)
  )else if (p.UnloadingValue == false) then(
    UnloadedDiv
  )
)
```

B.3 RelativeCanonicalDivisor

- Headline: Computes the relative canonical divisor.
- Synopsis:
  - Usage:

    `RelativeCanonicalDivisor(E)`
  - Inputs:
    - E: Intersection matrix associated of the log-resolution of $\mathfrak{a}$.
  - Outputs: The relative canonical divisor of the resolution.
- Description: Starting form the intersection matrix as presented in Chapter 1, it returns the relative canonical divisor of the resolution as a matrix $1 \times m$.

- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                { 0, -2,  1,  0,  0},
                { 1,  1, -1,  0,  0},
                { 0,  0,  0, -2,  1},
                { 1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : RelativeCanonicalDivisor(E)

o2 = | 1 2 4 2 4 |

              1        5
o2 : Matrix QQ  <-- QQ
```

- Code:

```
RelativeCanonicalDivisor = (IntersectionMatrix) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  NumDiv := numgens source IntersectionMatrix;
  AuxRelCanDivisor := mutableMatrix(ZZ,1,NumExceptionalDiv);
  for i from 0 to NumExceptionalDiv - 1 do(
     AuxRelCanDivisor_(0,i) =  - 2 - IntersectionMatrix_(i,i);
  );
  matrix(AuxRelCanDivisor) * transpose(inverse(promote(submatrix
    (IntersectionMatrix, {0..NumExceptionalDiv - 1}),QQ)))
    |matrix(mutableMatrix(QQ,1,NumDiv - NumExceptionalDiv))
)
```

B.4 MultiplicityJN

- Headline: Computes the multiplicity as a jumping number.
- Synopsis:
  - Usage:

    ```
    MultiplicityJN(F,E,jn)
    ```

- Inputs:
  - F: Divisor associated to the ideal $\mathfrak{a}$ to whom we want to compute its associated jumping numbers and multiplier ideals.
  - E: Intersection matrix associated of the log-resolution of $\mathfrak{a}$.
  - jn: Real number.
- Outputs: The multiplicity of jn as a jumping number.

- Description: Starting form the divisor encoded as a matrix of dimensions $1 \times m$, the intersection matrix as presented in Chapter 1 and a real number jn, it returns the multiplicity of jn as a jumping number. It is important to notice that if jn is not a jumping number, then the multiplicity will be zero.

- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                 {  0, -2,  1,  0,  0},
                 {  1,  1, -1,  0,  0},
                 {  0,  0,  0, -2,  1},
                 {  1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : F = matrix({{4,5,10,5,10}})

o2 = | 4 5 10 5 10 |

              1        5
o2 : Matrix ZZ  <-- ZZ

i3 : MultiplicityJN(F,E,1 / 2)

o3 = 1

o3 : QQ
```

- Code:

```
MultiplicityJN = (F,IntersectionMatrix,JumpingNumber) -> (
  RelCanDivisor := RelativeCanonicalDivisor(IntersectionMatrix);
  NumDiv := numgens source IntersectionMatrix;
```

```
  Excess :=  - F * IntersectionMatrix;
    MultJN:= 0;
    FracPart := mutableMatrix(QQ,1,NumDiv);
    for j from 0 to NumDiv - 1 list(
       FracPart_(0,j) =  - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j)
              - floor( - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j));
    );
    NumDivIntersect := 0;
    for j from 0 to NumDiv - 1 list(
       if FracPart_(0,j) == 0 then(
          for k from 0 to NumDiv - 1 list(
              if IntersectionMatrix_(k,j) == 1 then(
                   if FracPart_(0,k) == 0 then(
                       NumDivIntersect = NumDivIntersect + 1;
                   )else(
                       MultJN = MultJN + FracPart_(0,k);
                   );
              );
          );
          MultJN = MultJN + JumpingNumber * Excess_(0,j) - 1;
       );
    );
    return(MultJN + NumDivIntersect / 2)
)
```

B.5 MultIdeal

- Headline: Computes the multiplier ideal of a given number.
- Synopsis:
  - Usage:

    `MultIdeal(F,E,jn)`
  - Inputs:
    - F: Divisor associated to the ideal $\mathfrak{a}$ to whom we want to compute its associated jumping numbers and multiplier ideals.
    - E: Intersection matrix associated of the log-resolution of $\mathfrak{a}$.
    - jn: Real number.
  - Outputs: The multiplier ideal associated to jn.
- Description: Starting form the divisor encoded as a matrix of dimensions $1 \times m$, the intersection matrix as presented in Chapter 1 and a real number, it returns the multiplier ideal associated to this number.

- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                 {  0, -2,  1,  0,  0},
                 {  1,  1, -1,  0,  0},
                 {  0,  0,  0, -2,  1},
                 {  1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : F = matrix({{4,5,10,5,10}})

o2 = | 4 5 10 5 10 |

              1        5
o2 : Matrix ZZ  <-- ZZ

i3 : MultIdeal(F,E,1 / 2)

o3 = | 1 1 2 1 2 |

o3 : MutableMatrix
```

- Code:

```
MultIdeal = {MaxIterations => 10000} >>
    o -> (F,IntersectionMatrix,JumpingNumber) -> (
  RelCanDivisor := RelativeCanonicalDivisor(IntersectionMatrix);
   NumDiv := numgens source IntersectionMatrix;
   Divisor := mutableMatrix(ZZ,1,NumDiv);
   for j from 0 to NumDiv - 1 list(
      Divisor_(0,j) = -(ceiling(RelCanDivisor_(0,j) - JumpingNumber * F_(0,j)));
   );
   Unloading(Divisor,IntersectionMatrix,MaxIterations => o.MaxIterations)
)
```

B.6 Multiplicity

- Headline: Technical routine to compute the multiplicity of a jumping number (similar to MultiplicityJN).
- Code:

```
Multiplicity = (F,IntersectionMatrix,RelCanDivisor,Excess,JumpingNumber) -> (
    MultJN := 0;
    NumDiv := numgens source IntersectionMatrix;
    FracPart := mutableMatrix(QQ,1,NumDiv);
    MaxJumpingDivisor := mutableMatrix(QQ,1,NumDiv);
    for j from 0 to NumDiv - 1 list(
       FracPart_(0,j) =  - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j)
           - floor( - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j));
    );
    NumDivIntersect := 0;
    SumExcess := 0;
    for j from 0 to NumDiv - 1 list(
       if FracPart_(0,j) == 0 then(
          MaxJumpingDivisor_(0,j) = 1;
          for k from 0 to NumDiv - 1 list(
              if IntersectionMatrix_(k,j) == 1 then(
                   if FracPart_(0,k) == 0 then(
                       NumDivIntersect = NumDivIntersect + 1;
                   )else(
                       MultJN = MultJN + FracPart_(0,k);
                   );
              );
          );
          MultJN = MultJN + JumpingNumber * Excess_(0,j) - 1;
          SumExcess = SumExcess + Excess_(0,j);
       );
    );
    return(MultJN + NumDivIntersect / 2,SumExcess,MaxJumpingDivisor)
)
```

B.7 CompIdeal

- Headline: Technical routine to compute the ideal associated to a jumping number (similar to MultIdeal).
- Code:

```
CompIdeal = {MaxIterations => 10000,UnloadingValue => false} >> o
     -> (JumpingNumber,IntersectionMatrix,F,RelCanDivisor) -> (
```

```
   NumDiv := numgens source IntersectionMatrix;
   Divisor := mutableMatrix(ZZ,1,NumDiv);
   for j from 0 to NumDiv - 1 list(
       Divisor_(0,j) =  - (ceiling(RelCanDivisor_(0,j)
         - JumpingNumber * F_(0,j)));
   );
   Unloading(Divisor,IntersectionMatrix,MaxIterations
      => o.MaxIterations,UnloadingValue => false)
)
```

B.8 Multiplicities

- Headline: Algorithm consequence of Theorem 4.1.1 (Technical routine).
- Code:

```
Multiplicities = {SmallestJN => 0,MaxIterations => 10000,BiggestJN => 2} » o
   -> (F,IntersectionMatrix) -> (
  NumDiv := numgens source IntersectionMatrix;
  RelCanDivisor := RelativeCanonicalDivisor(IntersectionMatrix);
  Excess :=  - F * IntersectionMatrix;
  JN := new MutableHashTable;
  Candidates := ();
  for j from 0 to NumDiv - 1 list(
     for k from ceiling(RelCanDivisor_(0,j) - F_(0,j)) to
        floor(RelCanDivisor_(0,j)) list(
        Candidates = append(Candidates,( - k + RelCanDivisor_(0,j)) / F_(0,j));
     );
  );
  for i from 0 to #Candidates - 1 list(
    JumpingNumber := Candidates#i;
    (MultJN,Increase,MaxJumpingDivisor) := Multiplicity(F,IntersectionMatrix,
        RelCanDivisor,Excess,JumpingNumber);
    for j from 0 to ceiling(o.BiggestJN) list(
        if MultJN + j * Increase > 0 and JumpingNumber + j <= o.BiggestJN and
           o.SmallestJN < JumpingNumber + j then(
           JN#{JumpingNumber + j} = {MultJN + j * Increase,
              CompIdeal(JumpingNumber + j,IntersectionMatrix,F,RelCanDivisor,
                 MaxIterations => o.MaxIterations,
                 UnloadingValue => false),MaxJumpingDivisor};
        );
    );
  );
new HashTable from JN
)
```

B.9 AlbAlvDac

- Headline: Algorithm 2.2.1 (technical routine).
- Code:

```
AlbAlvDac = {SmallestJN => 0,MaxIterations => 10000,BiggestJN => 2} »
      o -> (F,IntersectionMatrix) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  NumDiv := numgens source IntersectionMatrix;
  RelCanDivisor := RelativeCanonicalDivisor(IntersectionMatrix);
  JN := new MutableHashTable;
  StartingDiv := mutableMatrix(ZZ,1,NumDiv);
  for i from 0 to NumDiv - 1 do(
    StartingDiv_(0,i) = floor(o.SmallestJN * F_(0,i) - RelCanDivisor_(0,i));
  );
  StartingDiv = Unloading(StartingDiv,IntersectionMatrix);
  k := 1;
  if NumExceptionalDiv =!= NumDiv then(
      CodimPrevMI := ( - (submatrix(matrix(StartingDiv),
          {0..NumExceptionalDiv - 1}) * submatrix(IntersectionMatrix,
          {0..NumExceptionalDiv - 1}) * transpose(submatrix(matrix(StartingDiv),
          {0..NumExceptionalDiv - 1}) + submatrix(RelCanDivisor,
          {0..NumExceptionalDiv - 1})))_(0,0) / 2,submatrix(matrix(StartingDiv),
          {NumExceptionalDiv..NumDiv - 1}));
  ) else (
      CodimPrevMI = ( - (submatrix(matrix(StartingDiv),
          {0..NumExceptionalDiv - 1}) * submatrix(IntersectionMatrix,
          {0..NumExceptionalDiv - 1}) * transpose(submatrix(matrix(StartingDiv),
          {0..NumExceptionalDiv - 1}) + submatrix(RelCanDivisor,
          {0..NumExceptionalDiv - 1})))_(0,0) / 2,0);
        );
  isLastJN := false;
  while (not isLastJN) do(
    JumpingNumber := o.BiggestJN + 1;
    MinJumpingDivisor := mutableMatrix(ZZ,1,NumDiv);
    for i from 0 to NumDiv - 1 list(
        CandidateJN := (RelCanDivisor_(0,i) + StartingDiv_(0,i) + 1) / F_(0,i);
        if (JumpingNumber > CandidateJN) then (
            MinJumpingDivisor = mutableMatrix(ZZ,1,NumDiv);
            JumpingNumber = CandidateJN;
            MinJumpingDivisor_(0,i) = 1;
        )else if (JumpingNumber == CandidateJN) then (
            MinJumpingDivisor_(0,i) = 1;
        );
     );
     if (JumpingNumber > o.BiggestJN) then(
isLastJN = true;
     )else(
        FracPart := mutableMatrix(QQ,1,NumDiv);
        MaxJumpingDivisor := mutableMatrix(QQ,1,NumDiv);
        for j from 0 to NumDiv - 1 list(
```

```
            FracPart_(0,j) =  - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j)
                 - floor( - RelCanDivisor_(0,j) + JumpingNumber * F_(0,j));
            if FracPart_(0,j) == 0 then(
                MaxJumpingDivisor_(0,j) = 1;
            );
        );
      DivJN := CompIdeal(JumpingNumber,IntersectionMatrix,F,RelCanDivisor,
           MaxIterations => o.MaxIterations,UnloadingValue => false);
        if NumExceptionalDiv =!= NumDiv then(
            CodimActMI := ( - (submatrix(matrix(DivJN),
                {0..NumExceptionalDiv - 1}) * submatrix(IntersectionMatrix,
                {0..NumExceptionalDiv - 1}) * transpose(submatrix(matrix(DivJN),
                {0..NumExceptionalDiv - 1}) + submatrix(RelCanDivisor,
                {0..NumExceptionalDiv - 1})))_(0,0) / 2,submatrix(matrix(DivJN),
                {NumExceptionalDiv..NumDiv - 1}));
        )else(
            CodimActMI = ( - (submatrix(matrix(DivJN),
                {0..NumExceptionalDiv - 1}) * submatrix(IntersectionMatrix,
                {0..NumExceptionalDiv - 1}) * transpose(submatrix(matrix(DivJN),
                {0..NumExceptionalDiv - 1}) + submatrix(RelCanDivisor,
                {0..NumExceptionalDiv - 1})))_(0,0) / 2,0);
        );
        if CodimPrevMI_1 =!= CodimActMI_1 then(
            JN#{JumpingNumber} = {infinity,DivJN,MaxJumpingDivisor,
                MinJumpingDivisor};
        )else(
            JN#{JumpingNumber} = {CodimActMI_0 - CodimPrevMI_0,DivJN,
                 MaxJumpingDivisor,MinJumpingDivisor};
        );
        CodimPrevMI = CodimActMI;
        k = k + 1;
        StartingDiv = DivJN;
     );
  );
new HashTable from JN
)
```

B.10 Chain

- Headline: Recursive routine to generate the critical chains (technical routine).
- Code:

```
Chain = (actp, Combinations, Divisor, IntersectionMatrix,
    RuptureDivisors,NumDiv) -> (
  DivisorMutable := mutableMatrix Divisor;
  NumExceptionalDiv := numgens target IntersectionMatrix;
  if RuptureDivisors_(0,actp) == 1 then(
```

```
    isInCombination := false;
    for i from 0 to #Combinations - 1 when not isInCombination do(
        AreEqual := true;
        for j from 0 to NumExceptionalDiv - 1 when AreEqual do(
            AreEqual = (Combinations#i_(0,j) == DivisorMutable_(0,j));
        );
        isInCombination = AreEqual;
    );
    if not isInCombination then(
       Combinations#(#Combinations) = matrix DivisorMutable;
    );
  );
  for i from 0 to NumExceptionalDiv - 1 do(
      if IntersectionMatrix_(i,actp) == 1 and DivisorMutable_(0,i) == 0 then(
          DivisorMutable_(0,i) = 1;
          Combinations = Chain(i, Combinations, DivisorMutable,
             IntersectionMatrix, RuptureDivisors,NumDiv);
          -DivisorMutable_(0,i) = 0;
      );
  );
Combinations
)
```

B.11 GenChains

- Headline: Technical routine to generate the critical chains.
- Code:

```
- Tecnical routine to generate the critical chains
GenChains = (IntersectionMatrix,F,NumDiv) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  Combinations := new MutableList;
  Divisor := mutableMatrix(ZZ,1,NumDiv);
    RuptureDivisors := mutableMatrix(ZZ,1,NumDiv);
    Excess :=  - F * IntersectionMatrix;
    for i from 0 to NumExceptionalDiv - 1 list(
       if Excess_(0,i) > 0 then(
          RuptureDivisors_(0,i) = 1;
       );
       s  :=  0;
       for j from 0 to NumExceptionalDiv - 1 list(
          s = s + IntersectionMatrix_(j,i);
       );
       s = s - IntersectionMatrix_(i,i);
       if s > 2 then(
```

```
         RuptureDivisors_(0,i) = 1;
      );
      RuptureDivisors_(0,0) = 1;
   );
   for i from 0 to NumExceptionalDiv - 1 do(
      if RuptureDivisors_(0,i) == 1 then(
         Divisor_(0,i) = 1;
         Combinations = Chain(i,Combinations,Divisor,
            IntersectionMatrix,RuptureDivisors,NumDiv);
         Divisor_(0,i) = 0;
      );
   );
  Combinations
)
```

B.12 Tucker

- Headline: Tucker's Algorithm (see Algorithm 1.11.9, technical routine).
- Code:

```
Tucker = {SmallestJN => 0,MaxIterations => 10000,BiggestJN => 2} »
     o -> (F,IntersectionMatrix) -> (
  NumExceptionalDiv := numgens target IntersectionMatrix;
  NumDiv := numgens source IntersectionMatrix;
  RelCanDivisor := RelativeCanonicalDivisor(IntersectionMatrix);
  Excess :=  - F * transpose(IntersectionMatrix);
  CandidateJumpingNumbers := new MutableList;
  JNList := new MutableHashTable;
  for k from 0 to NumDiv - 1 do(
    Candidates := new MutableList;
    NextDivisor := false;
    j := 0;
    while not NextDivisor do(
      j = j + 1;
      CandidateJN := (RelCanDivisor_(0,k) + j) / F_(0,k);
      if (CandidateJN <= o.SmallestJN) then(
      ) else if (CandidateJN <= o.BiggestJN) and
           (CandidateJN > o.SmallestJN) then(
        Candidates#(#Candidates) = CandidateJN;
        if k >= NumExceptionalDiv then(
            Chain := mutableMatrix(ZZ,1,NumDiv);
            Chain_(0,k) = 1;
            if not JNList#?({CandidateJN}) then(
               JNList#({CandidateJN}) = new List from entries Chain;
            ) else(
               JNList#({CandidateJN}) = VerticalList((JNList#({CandidateJN}))
                      |new List from entries Chain);
            );
```

```
        );
      )else (
NextDivisor = true;
      );
    );
    if k <  NumExceptionalDiv then(
       CandidateJumpingNumbers#k = Candidates;
    );
  );
  NumIntersectingDiv := mutableMatrix(ZZ,1,NumDiv);
  for i from 0 to NumExceptionalDiv - 1 do(
    for j from 0 to NumDiv - 1 do(
      if i =!= j then(
        NumIntersectingDiv_(0,i) =
             NumIntersectingDiv_(0,i) + IntersectionMatrix_(i,j);
      );
    );
  );
  for i from 0 to NumExceptionalDiv - 1 do(
    if Excess_(0,i) >= 1 then(
       NumIntersectingDiv_(0,i) = NumIntersectingDiv_(0,i) + 1;
    );
  );
  Chains := new List from GenChains(
       submatrix(IntersectionMatrix,{0..NumExceptionalDiv - 1}),
       submatrix(F, {0..NumExceptionalDiv - 1}),NumDiv);
  AdmissibleChains := new MutableList;
  for i from 0 to #Chains - 1 do(
    Chain := mutableMatrix(ZZ,1,NumDiv);
    isAdmissibleChain := true;
    for j from 0 to NumExceptionalDiv - 1 when isAdmissibleChain do(
      if Chains#i_(0,j) == 1 then(
        for k from 0 to NumDiv - 1 do(
          if j =!= k then(
             Chain_(0,j) = Chain_(0,j)
                + Chains#i_(0,k) * IntersectionMatrix_(j,k););
        );
        if Chain_(0,j) == 1 and NumIntersectingDiv_(0,j) < 3 then(
           isAdmissibleChain = false;
        );
      );
    );
    if isAdmissibleChain then(
      NElems := 0;
      StartingElem := NumExceptionalDiv;
      for k from 0 to NumExceptionalDiv - 1 do(
        NElems = NElems + Chains#i_(0,k);
        if Chains#i_(0,k) > 0 and k < StartingElem then( StartingElem = k;);
      );
      AdmissibleChains#(#AdmissibleChains) = (Chains#i,NElems,StartingElem);
    );
  );
  JN := new MutableList;
```

```
- Part of JN
  for jj from 0 to #AdmissibleChains - 1 do(
    JNOptions := CandidateJumpingNumbers#(AdmissibleChains#jj#2);
    for i from (AdmissibleChains#jj#2) to NumExceptionalDiv - 1 do(
      if AdmissibleChains#jj#0_(0,i) == 1 then(
         Candidates := new MutableList;
         for j from 0 to #JNOptions - 1 do(
           SearchedJN := JNOptions#j;
           for k from 0 to #CandidateJumpingNumbers#i - 1 when SearchedJN
                >= CandidateJumpingNumbers#i#k do(
             if SearchedJN == CandidateJumpingNumbers#i#k then(
               Candidates#(#Candidates) = SearchedJN;
             );
           );
         );
         JNOptions = Candidates;
      );
    );
- Computation of  - G * E_i
    Candidates := new MutableList;
    for i from 0 to #JNOptions - 1 do(
      JNandValues := new MutableList;
      if AdmissibleChains#jj#1 > 1 then(
        Values := mutableMatrix(ZZ,1,AdmissibleChains#jj#1);
        CandidateJN := JNOptions#i;
        a  :=  0;
        for j from 0 to NumDiv - 1 do(
          if AdmissibleChains#jj#0_(0,j) == 1 then(
            R := 0;
            for k from 0 to NumDiv - 1 do(
              R = R + ceiling(RelCanDivisor_(0,k)
                 - CandidateJN * F_(0,k)) * IntersectionMatrix_(j,k);
            );
            Values_(0,a) = R;
            a = a + 1;
          );
        );
        JNandValues#(#JNandValues) = (CandidateJN,Values);
- Check if  - G * E_i satisfies the conditions of the reference
        IntersectionVal := new MutableList;
        for kk from 0 to NumDiv - 1 do(
          if AdmissibleChains#jj#0_(0,kk) == 1 then(
           R := 0;
           for k from 0 to NumDiv - 1 do(
               R = R - AdmissibleChains#jj#0_(0,k) * IntersectionMatrix_(kk,k);
           );
           IntersectionVal#(#IntersectionVal) = R;
          );
        );
        for kk from 0 to #JNandValues - 1 do(
          A := 0;
          for j from 0 to #IntersectionVal - 1 do(
            if IntersectionVal#j == JNandValues#kk#1_(0,j) then(
```

```
                A = A + 1;
             );
           );
           if A == #IntersectionVal then(
             Candidates#(#Candidates) = JNandValues#kk#0;
           );
         );
-  Part for G = E_i
       )else(
         R := 0;
         for l from 0 to NumDiv - 1 do(
           R = R + ceiling(RelCanDivisor_(0,l) - (JNOptions#i) * F_(0,l))
                 * IntersectionMatrix_(AdmissibleChains#jj#2,l);
         );
         if R  >=   - IntersectionMatrix_(AdmissibleChains#jj#2,
               AdmissibleChains#jj#2) then(
           Candidates#(#Candidates) = JNOptions#i;
         );
       );
     );
     JN#jj = (AdmissibleChains#jj#0,Candidates);
- Tecnical part for the output
   );
   for i from 0 to #JN - 1 do(
     for j from 0 to #(JN#i#1) - 1 do(
       if not JNList#?({JN#i#1#j}) then(
          JNList#({JN#i#1#j}) = new List from entries JN#i#0;
       ) else(
          JNList#({JN#i#1#j}) = VerticalList((JNList#({JN#i#1#j}))
               |new List from entries JN#i#0);
       );
     );
   );
   JN = new MutableHashTable;
   OrderedJN := sort keys JNList;
   StartingDiv := mutableMatrix(ZZ,1,NumDiv);
   for i from 0 to NumDiv - 1 do(
     StartingDiv_(0,i) = floor(o.SmallestJN * F_(0,i) - RelCanDivisor_(0,i));
   );
   StartingDiv = Unloading(StartingDiv,IntersectionMatrix);
   k := 1;
   if NumExceptionalDiv =!= NumDiv then(
      CodimPrevMI :=( -(submatrix(matrix(StartingDiv),{0..NumExceptionalDiv - 1})
           * submatrix(IntersectionMatrix, {0..NumExceptionalDiv - 1})
           * transpose(submatrix(matrix(StartingDiv), {0..NumExceptionalDiv - 1})
           + submatrix(RelCanDivisor, {0..NumExceptionalDiv - 1})))_(0,0) / 2,
           submatrix(matrix(StartingDiv), {NumExceptionalDiv..NumDiv - 1}));
   )else(
      CodimPrevMI =( -(submatrix(matrix(StartingDiv), {0..NumExceptionalDiv - 1})
           * submatrix(IntersectionMatrix, {0..NumExceptionalDiv - 1})
           * transpose(submatrix(matrix(StartingDiv), {0..NumExceptionalDiv - 1})
           + submatrix(RelCanDivisor, {0..NumExceptionalDiv - 1})))_(0,0) / 2,0);
   );
```

```
    StartingDiv = CompIdeal(o.SmallestJN,IntersectionMatrix,F,
        RelCanDivisor,MaxIterations => o.MaxIterations,UnloadingValue => false);
  for i from 0 to #OrderedJN - 1 do(
    MinJumpingDivisor := mutableMatrix(ZZ,1,NumDiv);
    for j from 0 to NumDiv - 1 list(
        CandidateJN := (RelCanDivisor_(0,j) + StartingDiv_(0,j) + 1) / F_(0,j);
        if (OrderedJN#i_0 == CandidateJN) then (
            MinJumpingDivisor_(0,j) = 1;
        );
    );
        FracPart := mutableMatrix(QQ,1,NumDiv);
        MaxJumpingDivisor := mutableMatrix(QQ,1,NumDiv);
        for j from 0 to NumDiv - 1 list(
            FracPart_(0,j) =  - RelCanDivisor_(0,j) + (OrderedJN#i_0) * F_(0,j)
                - floor( - RelCanDivisor_(0,j) + (OrderedJN#i_0) * F_(0,j));
            if FracPart_(0,j) == 0 then(
                MaxJumpingDivisor_(0,j) = 1;
            );
        );
    DivJN := CompIdeal(OrderedJN#i_0,IntersectionMatrix,F,RelCanDivisor,
         MaxIterations => o.MaxIterations,UnloadingValue => false);
        if NumExceptionalDiv =!= NumDiv then(
           CodimActMI := ( -(submatrix(matrix(DivJN), {0..NumExceptionalDiv -1})
                * submatrix(IntersectionMatrix, {0..NumExceptionalDiv - 1})
                * transpose(submatrix(matrix(DivJN), {0..NumExceptionalDiv -1})
                + submatrix(RelCanDivisor,{0..NumExceptionalDiv -1})))_(0,0)/2,
                submatrix(matrix(DivJN), {NumExceptionalDiv..NumDiv - 1}));
        )else(
            CodimActMI = ( -(submatrix(matrix(DivJN), {0..NumExceptionalDiv -1})
                * submatrix(IntersectionMatrix, {0..NumExceptionalDiv - 1})
                * transpose(submatrix(matrix(DivJN), {0..NumExceptionalDiv -1})
                +submatrix(RelCanDivisor,{0..NumExceptionalDiv-1})))_(0,0)/2,0);
        );
        if CodimPrevMI_1 =!= CodimActMI_1 then(
           JN#(OrderedJN#i)={infinity,DivJN,MaxJumpingDivisor,MinJumpingDivisor,
                 JNList#(OrderedJN#i)};
        )else(
           JN#(OrderedJN#i)={CodimActMI_0-CodimPrevMI_0,DivJN,MaxJumpingDivisor,
                 MinJumpingDivisor,JNList#(OrderedJN#i)};
        );
        CodimPrevMI = CodimActMI;
        k = k + 1;
  );
new HashTable from JN
)
```

B.13 Symbols

B.13.1 JNandMI

- Headline: HashTable containing all the information about the jumping numbers.
- Description: This HashTable contains all the information about the jumping numbers that computes *MultiplierIdeals*. As a key, it contains the Jumping Number and for each Jumping Number it contains the multiplicity, the divisor associated to the ideal, the maximal and minimal jumping divisors and (if it applies) the critical chains.
- Example:

```
i1 : E = matrix({{ -5,  0,  1,  0,  1},
                 {  0, -2,  1,  0,  0},
                 {  1,  1, -1,  0,  0},
                 {  0,  0,  0, -2,  1},
                 {  1,  0,  0,  1, -1}})

o1 = | -5 0  1  0  1  |
     | 0  -2 1  0  0  |
     | 1  1  -1 0  0  |
     | 0  0  0  -2 1  |
     | 1  0  0  1  -1 |

              5        5
o1 : Matrix ZZ  <-- ZZ

i2 : F = matrix({{4,5,10,5,10}})

o2 = | 4 5 10 5 10 |

              1        5
o2 : Matrix ZZ  <-- ZZ

i3 : MultiplierIdeals(F,E,BiggestJN => 1)

                      1
o3 =  Jumping number: -   Multiplicity: 1
                      2
                    Multiplier ideal: | 1 1 2 1 2 |
                    Maximal jumping divisor: {| 1 0 1 0 1 |}
                    Minimal jumping divisor: {| 1 0 1 0 1 |}

                       7
      Jumping number: -  Multiplicity: 2
                      10
                    Multiplier ideal: | 2 2 4 2 4 |
```

```
                  Maximal jumping divisor: {| 0 0 1 0 1 |}
                  Minimal jumping divisor: {| 0 0 1 0 1 |}

                     9
    Jumping number: -  Multiplicity: 2
                    10
                  Multiplier ideal: | 2 3 5 3 5 |
                  Maximal jumping divisor: {| 0 0 1 0 1 |}
                  Minimal jumping divisor: {| 0 0 1 0 1 |}

    Jumping number: 1   Multiplicity: 1
                  Multiplier ideal: | 3 3 6 3 6 |
                  Maximal jumping divisor: {| 1 1 1 1 1 |}
                  Minimal jumping divisor: {| 1 0 1 0 1 |}

o3 : Type{...1...}

i4 : JNandMI

o4 = HashTable{{1} => {1, | 3 3 6 3 6 |, | 1 1 1 1 1 |, | 1 . 1 . 1 |} }
                7
              {-} => {2, | 2 2 4 2 4 |, | . . 1 . 1 |, | . . 1 . 1 |}
               10
                9
              {-} => {2, | 2 3 5 3 5 |, | . . 1 . 1 |, | . . 1 . 1 |}
               10
               1
              {-} => {1, | 1 1 2 1 2 |, | 1 . 1 . 1 |, | 1 . 1 . 1 |}
               2

o4 : HashTable
```

B.13.2 algorithm

- Headline: Method used to compute the jumping numbers and multiplier ideals.
- Used in:
  - MultiplierIdeals: algorithm.
- Description: Default value “AlbAlvDac”. This variable is used to choose which method we want to use to compute the jumping numbers, the three options are “AlbAlvDac” for the algorithm in Chapter 2, “Mult” for the one in Chapter 4 and “Tucker” for Tucker’s Algorithm 1.11.9.

B.13.3 MaxIterations

- Headline: Limits the number of iterations of the Unloading algorithm.
- Used in:
  - MultiplierIdeals: MaxIterations,
  - Unloading: MaxIterations,
  - MultIdeal: MaxIterations.
- Description: Default value 10000. This variable is used to limitate the number of iterations of the Unloading algorithm. If the resulting divisor is not unloaded, a warning will appear.

B.13.4 BiggestJN

- Headline: Upper bound of the interval where we want to compute the JN.
- Used in:
  - MultiplierIdeals: BiggestJN.
- Description: Default value 2. Upper bound of the interval where be computed the jumping numbers. The lower bound is $SmallestJN$.

B.13.5 SmallestJN

- Headline: Lower bound of the interval where we want to compute the JN.
- Used in:
  - MultiplierIdeals: SmallestJN.
- Description: Default value 0. Lower bound of the interval where be computed the jumping numbers. The upper bound is $BiggestJN$.

B.13.6 JumpingDivisor

- Headline: Show or not the jumping divisors.
- Used in:
  - MultiplierIdeals: JumpingDivisor.
- Description: Default value $true$. Whether to show the jumping divisors associated to each jumping number or not.

B.13.7 UnloadingValue

- Headline: Show the maximum of the unloaded values.
- Used in:
  - Unloading: UnloadingValue.
- Description: Default value $false$. Whether to show the maximum excess in the unloading procedure.

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UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

ARENBERG DOCTORAL SCHOOL
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
SECTION OF ALGEBRA
Celestijnenlaan 200B box 2400
B-3001 Leuven
http://wis.kuleuven.be/algebra