On fuzzy-qualitative descriptions and entropy

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Abstract

This paper models the assessments of a group of experts when evaluating different magnitudes, features or objects by using linguistic descriptions. A new general representation of linguistic descriptions is provided by unifying ordinal and fuzzy perspectives. Fuzzy-qualitative labels are proposed as a generalization of the concept of qualitative labels over a well-ordered set. A lattice structure is established in the set of fuzzy-qualitative labels to enable the introduction of fuzzy-qualitative descriptions as L-fuzzy sets. A theorem is given that characterizes finite fuzzy partitions using fuzzy-qualitative labels, the cores and supports of which are qualitative labels. This theorem leads to a mathematical justification for commonly-used fuzzy partitions of real intervals via trapezoidal fuzzy sets. The information of a fuzzy-qualitative label is defined using a measure of specificity, in order to introduce the entropy of fuzzy-qualitative descriptions.

Key words: L-fuzzy sets, Fuzzy partitions, Measures of information, Information sciences, Qualitative reasoning

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1. Introduction

One of the current challenges in knowledge representation and knowledge-based systems for decision making is the use of qualitative descriptions of variable values. This becomes necessary when numerical measurements of variables are unavailable, or when they are not convenient. In these cases, linguistic descriptions are used to represent uncertainty, as well as different levels of precision [5, 6, 10, 16, 21]. These types of systems have been used widely in engineering, as well as in biological, medical, economic, and social science applications, and recent examples can be found in [12, 19].

Two main areas of linguistic information representation can be found in the literature [18]. Some approaches use fuzzy representations of linguistic descriptions [16, 17]. On the other hand, some approaches use ordinal models and do not make an effective use of membership functions, being based either on 2-tuple modeling [5, 10] or on order-of-magnitude qualitative models [20, 25, 26]. Methodologies involving different levels of precision during linguistic modeling can be found in both main areas. In the case of fuzzy approaches, they usually rely on a hierarchy defined from a fuzzy partition of a real interval by means of triangular or trapezoidal fuzzy numbers [6, 16, 17]. Approaches based on 2-tuple modeling consider a linguistic hierarchy to deal with different levels of precision [5, 10]. Approaches based on absolute order-of-magnitude qualitative models use different levels of precision or abstraction in linguistic modeling by means of qualitative labels that in some cases correspond to sub-intervals coming from a partition of a real interval [25, 26]. Furthermore, the concept of entropy was formalized

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to measure the amount of information both in fuzzy and in ordinal research areas [1, 20].

On the other hand, there are situations where uncertainty applies, not only to the lack of numerical knowledge of the values of a variable, but also to the selection of the linguistic terms describing such values [13, 24]. To manage these situations, new fuzzy models were developed. For example, type-2 fuzzy sets were introduced as fuzzy sets whose membership grades are themselves fuzzy sets [9, 11, 13, 14], and other fuzzy models, such as discrete interval type-2 fuzzy sets and hesitant fuzzy sets, consider a set of possible values when defining the membership of an element [9, 21, 24, 28]. In addition, entropy has been studied in several fuzzy set theory extensions in recent literature, for instance, in hesitant, intuitionistic, type-2 and interval valued fuzzy sets [7, 23, 27, 30].

This paper presents a mathematical contribution to the area of decision making. It models the assessments of a group of experts when evaluating different magnitudes, features or objects by using linguistic descriptions. In addition, it proposes a measure of the amount of information delivered by the different experts in these group decision-making processes. A new general representation of linguistic descriptions is provided by unifying ordinal and fuzzy perspectives.

Fuzzy-qualitative labels are introduced as fuzzy sets over a set S, whose elements can be associated with linguistic terms, by extending the model proposed in [20]. The set of fuzzy-qualitative labels is structured as a lattice, which enables us to introduce fuzzy-qualitative descriptions as L-fuzzy sets [3, 15]. Moreover, fuzzy-qualitative descriptions are an extension of type-2 fuzzy sets, replacing in the secondary domain the unit interval with the set S [9, 13]. Fuzzy-qualitative descriptions can also be considered as fuzzy random variables interpreted under the ontic model [2, 4]. We formally introduce the concept of entropy of a fuzzy-qualitative description Q of a set Λ as the entropy associated with the probability measure induced by Q based on a measure on the power set $\mathcal{P}(\Lambda)$. The concept of entropy is then formalized by means of a Lebesgue integral and a measure of specificity [30]. In the discrete case, where Q has a finite range, this integral becomes a weighted average of the information of the labels, which corresponds to the Shannon self-information entropy of a discrete random variable. This concept allows the measurement of the amount of information given by a fuzzy-qualitative description and a comparison of expert assessments in group decision making [22]. In addition, a theorem is given that characterizes the finite fuzzy partitions of a well-ordered set using fuzzy-qualitative labels, leading to a full mathematical justification for the commonly used fuzzy partitions of real intervals via trapezoidal fuzzy numbers [16, 17].

The remainder of this paper is organized as follows. In Section 2 the concept and structure of the set $\mathcal{L}$ of fuzzy-qualitative labels are introduced. Section 3 provides a characterization of the fuzzy partitions of a well-ordered set under certain conditions. The fuzzy-qualitative descriptions of a set Λ and the concepts of information and entropy are defined in Section 4. Finally, our conclusions and future research directions are presented in Section 5.

2. Fuzzy-qualitative labels over a well-ordered set $\mathcal{S}$

In this section, the concept of fuzzy-qualitative labels over a well-ordered set $\mathcal{S}$ is presented. This enables us to introduce fuzzy sets into order-of-magnitude qualitative reasoning [25]. Firstly, we provide a brief summary of some necessary concepts related to crisp qualitative labels introduced in [20].

2.1. Qualitative labels over a well-ordered set [20]

Given a well-ordered set $(\mathcal{S}, \leq)$, its singletons $\{a\}$, $a \in \mathcal{S}$, are considered to be *basic qualitative labels* (or *basic labels*) over $\mathcal{S}$. The *qualitative labels* (or *labels*) over $\mathcal{S}$ are the intervals $[a, b) = \{x \in \mathcal{S} \,|\, a \leq x < b\}$, for all $a, b \in \mathcal{S}$ with $a < b$, together with the intervals $[a, \rightarrow) = \{x \in \mathcal{S} \,|\, a \leq x\}$, for all $a \in \mathcal{S}$. In particular, the entire set $\mathcal{S} = [p, \rightarrow)$, where p is the least element of $\mathcal{S}$, is a label, and the basic labels are labels: $\{a\} = [a, s(a))$, where $s(a)$ is the successor of a, except in the case in which a is the last element of $\mathcal{S}$, if it exists, and then $\{a\} = [a, \rightarrow)$. In general, the label $\mathcal{S}$ is denoted by the symbol ?, which is referred to as the *unknown label*.

The set $\mathbb{L}^*$ of all the qualitative labels over $\mathcal{S}$ is named the *order-of-magnitude space over* $\mathcal{S}$:

$$\mathbb{L}^* = \{[a, b) \mid a, b \in \mathcal{S},\ a < b\} \cup \{[a, \rightarrow) \mid a \in \mathcal{S}\}.$$

Note that $\mathbb{L}^* \subseteq \mathcal{P}(\mathcal{S})$, where $\mathcal{P}(\mathcal{S})$ is the power set of $\mathcal{S}$.

The set $\mathbb{L} = \mathbb{L}^* \cup \{\emptyset\}$ is named *the extended set* $\mathbb{L} \subseteq \mathcal{P}(\mathcal{S})$ *of qualitative labels over* $\mathcal{S}$, and $(\mathbb{L}, \sqcup, \cap)$ is a lattice, with the mix operation $\sqcup$ and the set intersection $\cap$.

2.2. Fuzzy-qualitative labels

This subsection presents a formal generalization of the order-of-magnitude space $\mathbb{L}^*$ over a well-ordered set $\mathcal{S}$ to a fuzzy framework. Fuzzy qualitative labels are defined as fuzzy sets with core qualitative labels, as follows.

Definition 1. A *fuzzy-qualitative label over* $\mathcal{S}$ is a fuzzy set $A \in \mathcal{F}(\mathcal{S}) = [0,1]^{\mathcal{S}}$ such that $Core(A) \in \mathbb{L}^*$.

In this manner, $Core(A) = [a, b)$ or $Core(A) = [a, \rightarrow)$, for some $a, b \in \mathcal{S}$, $a < b$. In other words, the fuzzy-qualitative labels are the fuzzy sets on $\mathcal{S}$ for which the set of elements that belong to them with membership value equal to 1 is a qualitative label of $\mathbb{L}^*$.

Let us recall the definition of a fuzzy singleton as a fuzzy set whose membership function assigns membership equal to 1 to only one element, and membership 0 to the rest of the elements of the universe. In this way, fuzzy singletons are fuzzy-qualitative labels, but it would make no sense to call them basic fuzzy-qualitative labels because their unions do not generate the entire set of fuzzy-qualitative labels.

Definition 2. *The set* $\mathcal{L}^* \subseteq \mathcal{F}(\mathcal{S})$ *of all the fuzzy-qualitative labels over* $\mathcal{S}$ is called *the fuzzy order-of-magnitude space over* $\mathcal{S}$:

$$\mathcal{L}^* = \{A \in \mathcal{F}(\mathcal{S}) \mid Core(A) \in \mathbb{L}^*\}.$$

Proposition 1 $\mathbb{L}^* \subseteq \mathcal{L}^*$.

Proof. The proof is straightforward.

The next definition extends $\mathcal{L}^*$ to include the fuzzy empty set $\emptyset$. This will enable the definition of a lattice $\mathcal{L}$ from the set $\mathcal{L}^*$, which will be used in Section 4 to introduce the $\mathcal{L}$-fuzzy sets.

Definition 3. The *extended set* $\mathcal{L} \subseteq \mathcal{F}(\mathcal{S})$ *of fuzzy-qualitative labels over* $\mathcal{S}$ is:

$$\mathcal{L} = \mathcal{L}^* \cup \{\emptyset\}.$$

Hence, the fuzzy-qualitative labels of $\mathcal{L}$ are the elements of $\mathcal{L}^* = \mathcal{L} - \{\emptyset\}$. In addition, from Proposition 1, it holds that $\mathbb{L} \subseteq \mathcal{L}$.

Table 1: Fuzzy sets associated with the candidate's assessments

$\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$ with $a_1 < a_2 < a_3 < a_4 < a_5$	
$A_1 \notin \mathcal{L}^*$	$A_2 \in \mathcal{L}^*$
$A_1 = \{1/a_1, 0.5/a_2, 1/a_4\}$	$A_2 = \{0.2/a_2, 1/a_3, 1/a_4, 0.3/a_5\}$

Example A 1. Let us consider a candidate who applies for a job position in a company, where the fuzzy sets $A_1, A_2 \in \mathcal{F}(\mathcal{S})$ given in Table 1 are two different assessments of the candidate's curriculum vitae.

Only A_2 is a fuzzy-qualitative label because the cores of A_1, A_2 are $\{a_1, a_4\} \notin \mathbb{L}^*, \{a_3, a_4\} = [a_3, a_5) \in \mathbb{L}^*$, respectively. If the singletons $\{a_i\} \in \mathbb{L}^*$ (or the fuzzy singletons $\{1/a_i\} \in \mathcal{L}^*$), $i = 1, \ldots, 5$ represent the terms *very slightly convincing, slightly convincing, average, quite convincing, very convincing*, respectively, then the fuzzy-qualitative label A_2 could represent the term *moderately convincing*.

Example B 1. Let $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, with $a_i < a_j$ if $i < j$, where a_i are real intervals of the temperature in a room as degrees Celsius (see Table 2).

Table 2: Discretization associated with $\mathcal{S}$

	a_1	a_2	a_3	a_4	a_5	a_6
Temperature	$[0, 7)$	$[7, 14)$	$[14, 21)$	$[21, 28)$	$[28, 35)$	$[35, 42]$

An example with two fuzzy-qualitative labels A_1, A_2 is given in Table 3, where two possible linguistic terms are associated.

Table 3: Fuzzy-qualitative labels

$A_1 = \{1/a_1, 0.6/a_2, 0.1/a_3\}$	*cold*
$A_2 = \{0.8/a_2, 1/a_3, 0.3/a_4\}$	*warm*

The examples provided above show that the elements of the set $\mathcal{S}$ can have a representation in the real line or not. In Example A 1, the elements a_i

of the set $\mathcal{S}$ have no mathematical semantics. On the contrary, in Example B 1, the elements a_i are intervals of the real line.

Remark 1. It is frequent the finite case where $\mathcal{S} = \{a_1, \ldots, a_n\}$, with $a_i < a_j$ if $i < j$, and each a_i is an interval of the real line $\mathbb{R}$:

$$\mathcal{S} = \{a_1 = [b_1, b_2), a_2 = [b_2, b_3), \ldots, a_{n-1} = [b_{n-1}, b_n), a_n = [b_n, b_{n+1}]\},$$

where $b_1 < b_2 < \cdots < b_n < b_{n+1}$ gives a partition of the real interval $[b_1, b_{n+1}] \subset \mathbb{R}$.

Since $\mathbb{R}$ is not a well-ordered set, whenever we require that the reference set is $\mathbb{R}$ or an interval of $\mathbb{R}$, a discretization in n subintervals will be necessary to apply the formal methodology presented in this study. The elements of $\mathcal{S}$ are then the subintervals of the discretization. In practice, when working with fuzzy sets over a real interval, usually a discretization of the interval is implicitly considered. The granularity n associated with the discretization is determined by the precision required in each real problem.

2.3. Structure of $\mathcal{L}$ and $\mathcal{L}^*$

In this subsection we introduce two operations on the extended set $\mathcal{L}$ of fuzzy-qualitative labels: the *mix*, $\sqcup$, and the *common,* $\sqcap$, in order to obtain a lattice structure in $\mathcal{L}$. This lattice structure enables us to introduce fuzzy-qualitative descriptions as $\mathcal{L}$-fuzzy sets in Section 4.

Although the standard union $\cup$ is not an operation in $\mathcal{L}$, because the standard union of two fuzzy-qualitative labels is a fuzzy-qualitative label if and only if its core belongs to $\mathbb{L}^*$, the axioms of the definition of the union in $\mathcal{F}(\mathcal{S})$ are used to introduce the definition of the mix operation $\sqcup$ (the same symbol as that used in [20]).

Definition 4. Given $A_1, A_2 \in \mathcal{L}$, the *mix* $A_1 \sqcup A_2$ of A_1 and A_2 is defined as:

1. $A_1 \sqcup A_2 \in \mathcal{L}$;
2. $A_1 \subseteq A_1 \sqcup A_2$, $A_2 \subseteq A_1 \sqcup A_2$;
3. $A_1 \subseteq A', A_2 \subseteq A', A' \in \mathcal{L} \Rightarrow A_1 \sqcup A_2 \subseteq A'$.

In this manner, $A_1 \sqcup A_2$ is the least element of $\mathcal{L}$ (based on the inclusion relation $\subseteq$ in the set $\mathcal{F}(\mathcal{S})$) that contains A_1 and A_2.

The uniqueness of $A_1 \sqcup A_2$ is deduced immediately from the axioms. Regarding the existence of the element $A_1 \sqcup A_2$, $\emptyset \sqcup A = A \ \forall A \in \mathcal{L}$, and if $A_1, A_2 \in \mathcal{L}^*$, the following fuzzy set, defined for any $a \in \mathcal{S}$:

$$(A_1 \sqcup A_2)(a) = \begin{cases} 1 \quad \text{if} \quad a \in Core(A_1) \sqcup Core(A_2); \\ max\{A_1(a), A_2(a)\} = (A_1 \cup A_2)(a) \quad \text{otherwise,} \end{cases}$$

satisfies the three conditions of Definition 4.

In addition, from Definition 4, it is clear that $Core(A_1 \sqcup A_2) = Core(A_1) \sqcup Core(A_2)$ and the operation $\sqcup$ in $\mathcal{L}$, when restricted to the classical qualitative labels of $\mathbb{L}$, coincides with the mix $\sqcup$ in $\mathbb{L}$. Indeed, if $A_1, A_2 \in \mathbb{L}$, it is sufficient to note that:

$$(A_1 \sqcup A_2)(a) = \begin{cases} 1 \quad \text{if} \quad a \in Core(A_1) \sqcup Core(A_2) = Core(A_1 \sqcup A_2), \\ 0 \quad \text{otherwise.} \end{cases}$$

Remark 2. In general, for all $A_1, A_2 \in \mathcal{L}$, it holds that $A_1 \cup A_2 \subseteq A_1 \sqcup A_2$. A sufficient condition for the equality is $Core(A_1) \cap Core(A_2) \neq \emptyset$ because, in this case, we have $Core(A_1) \sqcup Core(A_2) = Core(A_1) \cup Core(A_2) = Core(A_1 \cup A_2)$. However, this is not a necessary condition because, for example, in $\mathbb{L}$ we have $[a,b) \cup [b,c) = [a,c) = [a,b) \sqcup [b,c)$ but $[a,b) \cap [b,c) = \emptyset$. It can be easily seen that a necessary and sufficient condition for the equality is that either $Core(A_1) \cap Core(A_2) \neq \emptyset$ or these cores are of the form $[a,b),\ [b,c)$ or $[a,b),\ [b,\rightarrow)$.

The standard intersection $\cap$ is not an operation in $\mathcal{L}$. A counterexample is obtained by considering $a_1, a_2 \in \mathcal{S}$ and the elements $\{1/a_1, 0.5/a_2\}, \{1/a_2\} \in \mathcal{L}$ since $\{1/a_1, 0.5/a_2\} \cap \{1/a_2\} = \{0.5/a_2\}$ and $\{0.5/a_2\} \notin \mathcal{L}$, because $Core(\{0.5/a_2\}) = \emptyset \notin \mathbb{L}^*$. Similarly, as mentioned above, the axioms of the definition of the intersection in $\mathcal{F}(\mathcal{S})$ are used to introduce the definition of the common operation.

Definition 5. Given $A_1, A_2 \in \mathcal{L}$, the *common* $A_1 \sqcap A_2$ of A_1 and A_2 is defined as:

1. $A_1 \sqcap A_2 \in \mathcal{L}$;
2. $A_1 \sqcap A_2 \subseteq A_1,\ A_1 \sqcap A_2 \subseteq A_2$;
3. $A' \subseteq A_1, A' \subseteq A_2, A' \in \mathcal{L} \Rightarrow A' \subseteq A_1 \sqcap A_2$.

In this manner, $A_1 \sqcap A_2$ is the greatest element of $\mathcal{L}$ (based on the inclusion relation $\subseteq$ in the set $\mathcal{F}(\mathcal{S})$) that is contained in both A_1 and A_2.

The uniqueness of $A_1 \sqcap A_2$ is deduced immediately from the axioms. Regarding the existence of $A_1 \sqcap A_2$, if $A_1, A_2 \in \mathcal{L}$, the following fuzzy set:

$$A_1 \sqcap A_2 = \begin{cases} A_1 \cap A_2 & \text{if} \quad Core(A_1) \cap Core(A_2) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

satisfies the three conditions of Definition 5. Indeed, for the case $Core(A_1) \cap Core(A_2) \neq \emptyset$, it holds that $Core(A_1 \cap A_2) = Core(A_1) \cap Core(A_2) \in \mathbb{L}^*$, thus $A_1 \sqcap A_2 = A_1 \cap A_2 \in \mathcal{L}$ and the intersection satisfies 1, 2, 3. For the case $Core(A_1) \cap Core(A_2) = \emptyset$, any $A' \in \mathcal{L}$ contained in both A_1, A_2 will have an empty core, so $A' = \emptyset$.

In addition, given that $Core(A_1 \cap A_2) = Core(A_1) \cap Core(A_2)$, from Definition 5 we deduce that $Core(A_1 \sqcap A_2) = Core(A_1) \cap Core(A_2)$. On the other hand, the operation $\sqcap$ in $\mathcal{L}$, when restricted to $\mathbb{L}$, coincides with the common $\cap$ in $\mathbb{L}$. Indeed, let $A_1, A_2 \in \mathbb{L}$, if $Core(A_1) \cap Core(A_2) \neq \emptyset$, then $A_1 \sqcap A_2 = A_1 \cap A_2$, and if $Core(A_1) \cap Core(A_2) = \emptyset$, then $A_1 \sqcap A_2 = \emptyset = Core(A_1) \cap Core(A_2) = A_1 \cap A_2$.

Remark 3. In general, for all $A_1, A_2 \in \mathcal{L}$, it holds that $A_1 \sqcap A_2 \subseteq A_1 \cap A_2$. It can be easily seen that a necessary and sufficient condition for the equality is that either $Core(A_1) \cap Core(A_2) \neq \emptyset$ or $Support(A_1) \cap Support(A_2) = \emptyset$.

Example A 2. Following Example A 1, and given the fuzzy assessments in Table 4, the mix and the common of both assessments are $A_2 \sqcup A_3 = \{0.5/a_2, 1/a_3, 1/a_4, 0.3/a_5\} \in \mathcal{L}^*$, and $A_2 \sqcap A_3 = \{0.2/a_2, 1/a_3, 0.3/a_4\} \in \mathcal{L}^*$.

Table 4: Fuzzy sets associated with the candidate assessments

$\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$	
$A_2 = \{0.2/a_2, 1/a_3, 1/a_4, 0.3/a_5\}$	$A_3 = \{0.5/a_2, 1/a_3, 0.3/a_4\}$

The following theorem establishes the algebraic structure of the extended set $\mathcal{L}$ with the mix and common operations.

Theorem 1 $(\mathcal{L}, \sqcup, \sqcap)$ *is a lattice.*

Proof. The two operations $\sqcup$ and $\sqcap$ are idempotent, commutative, and satisfy the absorption law. They are associative due to their definitions: fuzzy sets $A_1 \sqcup (A_2 \sqcup A_3)$ and $(A_1 \sqcup A_2) \sqcup A_3$ are both the least element of $\mathcal{L}$ that contains A_1, A_2, A_3; $A_1 \sqcap (A_2 \sqcap A_3)$ and $(A_1 \sqcap A_2) \sqcap A_3$ are both the greatest element of $\mathcal{L}$ contained in A_1, A_2, A_3. $\square$

The lattice $(\mathcal{L}, \sqcup, \sqcap)$ is generally not distributive because it is not in $\mathbb{L}$ [20].

The partial order $\preceq$ induced in the lattice $\mathcal{L}$ by the two operations $\sqcup$, $\sqcap$ is the inverse subset inclusion $\supseteq$ in $\mathcal{F}(\mathcal{S})$. In effect:

$$A_1 \preceq A_2 \Leftrightarrow A_1 \sqcup A_2 = A_1 \Leftrightarrow A_1 \sqcap A_2 = A_2 \Leftrightarrow A_1 \supseteq A_2,$$

the last equivalence due to the definition of $A_1 \sqcap A_2$. Note that $A_1 \sqcup A_2 \preceq A_1, A_2 \preceq A_1 \sqcap A_2$ for any $A_1, A_2 \in \mathcal{L}$.
The least element in the poset $(\mathcal{L}, \preceq)$ is the set $0_{\mathcal{L}} = \mathcal{S}$ because, for all $A \in \mathcal{L}$, $A(a) \leq 1 = \mathcal{S}(a)$ for all $a \in \mathcal{S}$. In the following, this is denoted by the symbol ? (as in $\mathbb{L}$), which is referred to as the *"unknown"* fuzzy-qualitative label. The greatest element is $1_{\mathcal{L}} = \emptyset$ because $\emptyset \subseteq A$ for all $A \in \mathcal{L}$, and it is not a fuzzy-qualitative label, $\emptyset \notin \mathcal{L}^*$.

The set $\mathcal{L}^* = \mathcal{L} - \{\emptyset\} = \mathcal{L} - \{1_{\mathcal{L}}\}$, with the operation $\sqcup$ and the partial order $\preceq$, is a meet-semilattice, i.e., a poset where the infimum exists for every pair of elements. In this semilattice, the fuzzy singletons $\{1/a\}$, with $a \in \mathcal{S}$, are maximal elements because $[A \in \mathcal{L}^*, \{1/a\} \preceq A] \Rightarrow A = \{1/a\}$, and they are the only ones (that can be easily seen). The smallest element is $0_{\mathcal{L}^*} = 0_{\mathcal{L}} =?$, and if the set $\mathcal{S}$ has at least two elements, then it is obvious that there is no greatest element.

3. Fuzzy partitions of $\mathcal{S}$ using fuzzy-qualitative labels

Let us suppose that the diverse states of a variable are associated with linguistic terms, which are themselves associated with crisp qualitative labels $E_1, \ldots, E_k \in \mathbb{L}^*$, that are mutually exclusive and collectively exhaustive, i.e., such that $\{E_1, \ldots, E_k\}$ is a crisp partition of $\mathcal{S}$. Thus, $E_i \cap E_j = \emptyset$ if $i \neq j$, i.e., two linguistic terms cannot be associated with the same state of the variable, and the *"unknown" label* ? is $\mathcal{S} = E_1 \cup \cdots \cup E_k$, i.e., label ? represents a linguistic term that can be associated with all the states of the variable. The generalization of crisp partitions in the case of fuzzy-qualitative labels corresponds to fuzzy partitions of the set $\mathcal{S}$, as in Ruspini's approach.

This section provides a characterization of fuzzy partitions $\{A_1, \ldots, A_k\}$ of the set $\mathcal{S}$, being $A_1, \ldots, A_k \in \mathcal{L}^*$, such that their supports are qualitative labels that belong to $\mathbb{L}^*$. This restriction $Support(A_i) \in \mathbb{L}^*$ for all $i = 1, \ldots, k$, (the supports of the fuzzy-qualitative labels A_i have to be intervals $[a, b)$, or $[a \rightarrow)$ of $\mathcal{S}$,) is imposed to reflect the conditions of mutual exclusivity and collective exhaustivity.

The next definitions establish the framework where these fuzzy partitions of $\mathcal{S}$ are characterized.

Definition 6. $\mathcal{L}_s^* = \{A \in \mathcal{L}^* \mid Support(A) \in \mathbb{L}^*\} = \{A \in \mathcal{F}(\mathcal{S}) \mid Core(A) \in \mathbb{L}^* \text{ and } Support(A) \in \mathbb{L}^*\}$.

To obtain a new lattice structure, as done in Subsection 2.3, the following extension is considered:

Definition 7. $\mathcal{L}_s = \mathcal{L}_s^* \cup \{\emptyset\}$.

Proposition 2 $(\mathcal{L}_s, \sqcup, \sqcap)$ *is a sublattice of* $(\mathcal{L}, \sqcup, \sqcap)$.

Proof. Let $A_1, A_2 \in \mathcal{L}_s$. If A_1 or A_2 are $\emptyset$, then $A_1 \sqcup A_2, A_1 \sqcap A_2 \in \mathcal{L}_s$. Suppose, then, that $A_1, A_2 \in \mathcal{L}_s^*$.

1. Let us prove that $Support(A_1 \sqcup A_2) = Support(A_1) \sqcup Support(A_2)$, which implies that $A_1 \sqcup A_2 \in \mathcal{L}_s$ because $\sqcup$ is an operation in $\mathbb{L}$.
   Indeed, by definition 4, it holds directly that:
   $Support(A_1 \sqcup A_2) = (Core(A_1) \sqcup Core(A_2)) \cup Support(A_1 \cup A_2) = (Core(A_1) \sqcup Core(A_2)) \cup (Support(A_1) \cup Support(A_2))$.
   On the other hand,
   $Support(A_1) \sqcup Support(A_2) = (Core(A_1) \sqcup Core(A_2)) \cup Support(A_1) \cup Support(A_2)$, because $Support(A_1)$ and $Support(A_2)$ are intervals of $\mathbb{L}^*$ that contain $Core(A_1)$ and $Core(A_2)$, respectively.
2. It remains to be seen that $A_1 \sqcap A_2 \in \mathcal{L}_s$.
   Indeed, if $Core(A_1) \cap Core(A_2) = \emptyset$, then $A_1 \sqcap A_2 = \emptyset \in \mathcal{L}_s$.
   If $Core(A_1) \cap Core(A_2) \neq \emptyset$, then it holds directly by definition 5 that:
   $Support(A_1 \sqcap A_2) = Support(A_1 \cap A_2) = Support(A_1) \cap Support(A_2)$, and therefore $A_1 \sqcap A_2 \in \mathcal{L}_s$ because $\cap$ is an operation in $\mathbb{L}$. □

The following lemma characterizes the crisp partitions of the set $\mathcal{S}$ using qualitative labels that belong to $\mathbb{L}^*$.

Lemma 1 *Given $E_1, \dots, E_k \in \mathbb{L}^*,\ k \geq 2, \{E_1, \dots, E_k\}$ is a partition of the set $\mathcal{S} = [p, \rightarrow)$ if and only if they can be sorted in a manner such that:*
$E_1 = [p, a_2), E_2 = [a_2, a_3), \dots, E_{k-1} = [a_{k-1}, a_k), E_k = [a_k, \rightarrow)$, for some $a_2, a_3, \dots, a_k \in \mathcal{S}$ with $p < a_2 < a_3 < \dots < a_k$ (see Figure 1).

Proof. The proof is straightforward.

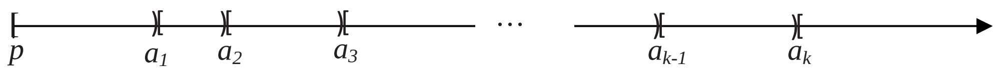

Figure 1: Crisp partition of $\mathcal{S}$ using the labels of $\mathbb{L}$.

The next theorem generalizes Lemma 1 and gives two characterizations of the fuzzy partitions of the set $\mathcal{S}$, in the sense of Ruspini's approach, by using the fuzzy-qualitative labels of $\mathcal{L}_s^*$.

Theorem 2 *Let us consider a finite set of fuzzy-qualitative labels $A_1, \dots, A_k \in \mathcal{L}_s^*,\ k \geq 2$. The following three conditions are equivalent:*

(a) $\{A_1, \dots, A_k\}$ is a fuzzy partition of the set $\mathcal{S} = [p, \rightarrow)$, i.e., $\sum_{i=1}^{k} A_i(a) = 1\ \forall a \in \mathcal{S}$.

(b) The following conditions are satisfied:

1. *$Core(A_1), Core(A_2), \dots, Core(A_k)$ are pairwise disjoint.*
2. *These cores can be sorted in such a manner that (see Figure 2):*
   (i) For all $i = 1, \dots, k-1$, $A_i(a) + A_{i+1}(a) = 1\ \forall a \in \big(Core(A_i) \sqcup Core(A_{i+1})\big) - \big(Core(A_i) \cup Core(A_{i+1})\big)$,
   (ii) $Core(A_1) = [p, b_1), Core(A_k) = [a_k, \rightarrow)$, for some $b_1, a_k \in \mathcal{S}$, with $p < b_1 \leq a_k$.

(c) Labels $A_1, \dots, A_k$ can be sorted so that the standard fuzzy complement of each is the standard union of the mix of the previous labels and the mix of the subsequent labels:
$A_i^c = (A_1 \sqcup \dots \sqcup A_{i-1}) \cup (A_{i+1} \sqcup \dots \sqcup A_k) \quad \forall i = 2, \dots, k-1,$
and $A_1^c = A_2 \sqcup \dots \sqcup A_k,\ A_k^c = A_1 \sqcup \dots \sqcup A_{k-1}$.

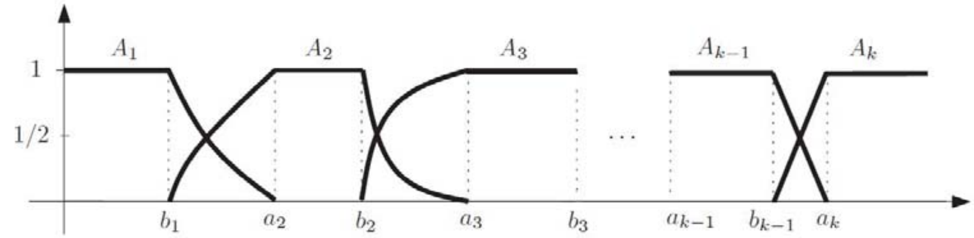

Figure 2: Fuzzy partition of $\mathcal{S}$ using the labels of $\mathcal{L}_s^*$

Proof. $(a) \Rightarrow (b)$
Let $A_1, \ldots, A_k \in \mathcal{L}_s^*$ such that $\{A_1, \ldots, A_k\}$ is a fuzzy partition of the set $\mathcal{S}$.

1. $Core(A_i) \cap Support(A_j) = \emptyset$ if $i \neq j$ because:
   $a \in Core(A_i) \Rightarrow A_i(a) = 1 \Rightarrow A_j(a) = 0 \ \forall j \neq i \Rightarrow a \notin Support(A_j)$ $\forall j \neq i$. Therefore, $Core(A_1), \ldots, Core(A_k)$ are pairwise disjoint because $Core(A_j) \subseteq Support(A_j) \ \forall j$.
2. Thus, according to Lemma 1, the cores can be sorted in such a manner that:
   $Core(A_1) = [a_1, b_1), Core(A_2) = [a_2, b_2), \ldots, Core(A_{k-1}) = [a_{k-1}, b_{k-1})$, $Core(A_k) = [a_k, b_k)$ (or $[a_k, \rightarrow)$), for some $a_1, \ldots, a_k \in \mathcal{S}$, $b_1, \ldots, b_k \in \mathcal{S}$ (or $b_1, \ldots, b_{k-1} \in \mathcal{S}$ in the second case), with $p \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \leq \cdots \leq a_{k-1} < b_{k-1} \leq a_k < b_k$ (or $p \leq a_1 < b_1 \leq \cdots < b_{k-1} \leq a_k$ in the second case).
   (i) Let $i \in \{1, \ldots, k-1\}$. Since $Core(A_1), \ldots, Core(A_k)$ are pairwise disjoint and the fact that $Support(A_j) \in \mathbb{L}^* \ \forall j$ (and therefore connected), in the interval $[b_i, a_{i+1})$ (if it is non-empty) only the fuzzy sets A_i, A_{i+1} can have positive degrees of membership. Hence, the hypothesis $\sum_{j=1}^{k} A_j(a) = 1 \ \forall a \in \mathcal{S}$ implies $A_i(a) + A_{i+1}(a) = 1 \ \forall a \in [b_i, a_{i+1})$. This interval can be written as $[b_i, a_{i+1}) = \big(Core(A_i) \sqcup Core(A_{i+1})\big) - \big(Core(A_i) \cup Core(A_{i+1})\big)$.
   In addition, note that for the case $Core(A_i) \sqcup Core(A_{i+1}) = Core(A_i) \cup Core(A_{i+1})$, i.e., $[b_i, a_{i+1}) = \emptyset$, the statement (i) is evidently true.
   (ii) First: $1 = \sum_{i=1}^{k} A_i(a) = A_1(a) \ \forall a \in [p, a_1)$, because the supports of $A_2, \ldots, A_k$ are in $\mathbb{L}^*$; so $p = a_1$ and $Core(A_1) = [p, b_1)$. Analogously, $1 = \sum_{i=1}^{k} A_i(a) = A_k(a) \ \forall a \in [a_k, \rightarrow)$, because the supports of $A_1, \ldots, A_{k-1}$ are in $\mathbb{L}^*$; so $Core(A_k) = [a_k, \rightarrow)$.

Note that conditions 1, 2 lead to $0 < A_i(a) < 1$ and $0 < A_{i+1}(a) < 1$, for any $i = 1, \ldots, k-1$ and for any $a \in [b_i, a_{i+1}) \neq \emptyset$. Thus, we can state that: $Support(A_1) = [p, a_2), Support(A_2) = [b_1, a_3), \ldots, Support(A_{k-1}) = [b_{k-2}, a_k)$ and $Support(A_k) = [b_{k-1}, \rightarrow)$ (see Figure 2).

$(b) \Rightarrow (c)$
It is simple to check that, in each of the intervals: $[p, b_1)$, $[b_1, a_2)$, $[a_2, b_2)$, $[b_2, a_3)$,$\ldots$, $[a_{k-1}, b_{k-1})$,$[b_{k-1}, a_k)$, $[a_k, \rightarrow)$, the fuzzy sets $A_1, \ldots, A_k$ satisfy the conditions of (c) (see Figure 2). For instance, in the case $a \in [b_{i-1}, a_i)$, $i = 2, \ldots, k$, it holds that:

$$\big((A_1 \sqcup \ldots \sqcup A_{i-1}) \cup (A_{i+1} \sqcup \ldots \sqcup A_k)\big)(a) =$$
$$\max\{(A_1 \sqcup \ldots \sqcup A_{i-1})(a), (A_{i+1} \sqcup \ldots \sqcup A_k)(a)\} =$$
$$\max\{A_{i-1}(a), 0\} = A_{i-1}(a) = 1 - A_i(c) = A_i^c(a).$$

$(c) \Rightarrow (a)$
We will prove $(c) \Rightarrow (a)$ through some part of (b), because it is difficult to go directly from (c) to (a) without using the explicit description of the fuzzy sets in (b).
First, the assumptions of (c) lead to $Core(A_i) \cap Support(A_j) = \emptyset$ if $i \neq j$.
Indeed, on the one hand, if $i = 2, \ldots, k-1$, it holds that:

$$A_i(a) = 1 \Rightarrow \Big\{(A_1 \sqcup \ldots \sqcup A_{i-1})(a) = 0 \text{ and } (A_{i+1} \sqcup \ldots \sqcup A_k)(a) = 0\Big\}$$
$\Rightarrow A_j(a) = 0\, \forall j \neq i$ given that the mix contains its members.

So, $Core(A_i) \cap Support(A_j) = \emptyset$ if $j \neq i$. Analogously, with respect to A_1 and A_k: $Core(A_1) \cap Support(A_j) = \emptyset\ \forall j = 2, \ldots, k$ and $Core(A_k) \cap Support(A_j) = \emptyset\ \forall j = 1, \ldots, k-1$. The same reasoning as that found in items 1 and 2 of the proof $(a) \Rightarrow (b)$ leads to: $Core(A_1) = [a_1, b_1), Core(A_2) = [a_2, b_2), \ldots, Core(A_{k-1}) = [a_{k-1}, b_{k-1})$, $Core(A_k) = [a_k, b_k)$ (or $[a_k, \rightarrow)$).
On the other hand, $A_1^c = A_2 \sqcup \ldots \sqcup A_k$ and the fact that $Support(A_j) \in \mathbb{L}^*\, \forall j$ readily imply that $Core(A_1) = [p, b_1)$, and the same applies to $Core(A_k) = [a_k, \rightarrow)$ from $A_k^c = A_1 \sqcup \ldots \sqcup A_{k-1}$. In addition, given that $Support(A_j) \in \mathbb{L}^*\, \forall j$, in the interval $[b_i, a_{i+1})$ (if it is non-empty), for $i = 1, \ldots, k-1$, only the fuzzy sets A_i, A_{i+1} can have positive degrees of membership.
Now, it only remains to check that $\sum_{i=1}^{k} A_i(a) = 1\ \forall a \in \mathcal{S}$ in each of the intervals $[p, b_1), [b_1, a_2), [a_2, b_2), [b_2, a_3), \ldots, [a_{k-1}, b_{k-1}), [b_{k-1}, a_k), [a_k, \rightarrow)$, which follows in a straightforward manner from the assumptions of (c).
This concludes the proof of the theorem. $\square$

Remark 4. Note that condition $Support(A_j) \in \mathbb{L}^* \, \forall j$ is necessary for Theorem 2 to be true. A counterexample where $Support(A_j) \notin \mathbb{L}^*$ for some j and a fuzzy partition of $\mathcal{S}$ does not satisfy condition (c) of Theorem 2 is as follows: $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{1/a_1, \frac{1}{3}/a_4\}, A_2 = \{1/a_2, \frac{1}{3}/a_4\}, A_3 = \{1/a_3, \frac{1}{3}/a_4\}$.

The following corollary provides a third characterization of the fuzzy partitions of $\mathcal{S}$ using fuzzy-qualitative labels $A_1, \ldots, A_k \in \mathcal{L}_s^*$:

Corollary 1 *The following statement is also equivalent to conditions* $(a), (b), (c)$ *in Theorem 2:*

Labels $A_1, \ldots, A_k$ *can be sorted so that:*
$(A_i \sqcup A_j)^c = (A_0 \sqcup \ldots \sqcup A_{i-1}) \cup (A_{j+1} \sqcup \ldots \sqcup A_{k+1}) = (A_0 \sqcup A_{i-1}) \cup (A_{j+1} \sqcup A_{k+1})$, *for all* $1 \leq i \leq j \leq k$, *assuming that* $A_0 = A_{k+1} = \emptyset$.

Proof. The proof is straightforward (see Figure 2). □

Remark 5. Theorem 2 and Corollary 1 provide three characterizations of fuzzy partitions, in the sense of Ruspini's approach, by means of fuzzy qualitative labels in $\mathcal{L}_s^*$.

When supposing a suitable discretization of an interval of $\mathbb{R}$ as explained in Remark 1, the fuzzy-qualitative labels in $\mathcal{L}_s^*$ can be considered trapezoidal fuzzy sets over the given interval. Regarding this perspective, Theorem 2 can be reformulated in the case of fuzzy partitions of a real interval by means of trapezoidal fuzzy sets placed as in Figure 2, which are the most commonly-used partitions in the literature of fuzzy sets:

Theorem 3 *Finite fuzzy partitions, in the sense of Ruspini, of a real interval by means of trapezoidal fuzzy numbers are characterized as in Theorem 2 and Corollary 1.*

Proof. If the set $\mathcal{S} = [p, \to)$ is replaced by $[p, b) \subseteq \mathbb{R}$ o $[p, +\infty) \subseteq \mathbb{R}$, and all the intervals $[a_i, b_j)$ are considered as real intervals, it is straightforward to check that all steps of the proof are valid. □

Table 5: Differences between the crisp and fuzzy cases

$E_1, \ldots, E_k \in \mathbb{L}^*$	$A_1, \ldots, A_k \in \mathcal{L}_s^*$
$\{E_1, \ldots, E_k\}$ partition of $\mathcal{S}$	$\{A_1, \ldots, A_k\}$ fuzzy partition of $\mathcal{S}$
equivalent to: $E_i \cap E_j = \emptyset$ if $i \neq j$, $\mathcal{S} = E_1 \cup \cdots \cup E_k$	not equivalent to: $A_i \sqcap A_j = \emptyset$ if $i \neq j$, $\mathcal{S} = A_1 \sqcup \ldots \sqcup A_k$
equivalent to: $E_i^c = \bigcup_{j \neq i} E_j \ \forall i = 1, \ldots, k$	not equivalent to: $A_i^c = \bigcup_{j \neq i} A_j \ \forall i = 1, \ldots, k$

Since $\mathbb{L}^* \subset \mathcal{L}_s^*$, Theorem 2 is applicable to crisp qualitative labels, but there are some differences between the crisp and fuzzy cases, which are presented in Table 5.
Indeed, it is clear that fuzzy partitions satisfy $A_i \sqcap A_j = \emptyset$ if $i \neq j$ and $\mathcal{S} = A_1 \sqcup \ldots \sqcup A_k$ (see Figure 2), but these two conditions do not characterize fuzzy partitions because the converse is not true. A counterexample is given by: $k = 2$, $\mathcal{S} = \{a_1, a_2, a_3\}$, $A_1 = \{1/a_1, 0.5/a_2\}$, $A_2 = \{1/a_3\}$. It holds that $A_1 \sqcap A_2 = \emptyset$ and $\mathcal{S} = A_1 \sqcup A_2$, but $\{A_1, A_2\}$ is not a fuzzy partition of $\mathcal{S}$ since $A_1(a_2) + A_2(a_2) = 0.5 \neq 1$. Note that condition (c) of Theorem 2 cannot be replaced by $A_i^c = \bigcup_{j \neq i} A_j \ \forall i = 1, \ldots, k$, which is highlighted in the next example.

Example A 3. Following Examples A 1 and A 2, let us consider now that the linguistic terms {*very slightly convincing, slightly convincing, average, quite convincing, very convincing*} represent at the same time a crisp partition $\{E_1, \ldots, E_k\}$ and a fuzzy partition $\{A_1, \ldots, A_k\}$ of a well-ordered set $\mathcal{S}$. Table 6 shows the differences in the complement of a label between both cases. In the case of fuzzy partitions, the complement of a fuzzy-qualitative label cannot be expressed using only the fuzzy union $\cup$, and requires the operation $\sqcup$.

Finally, note that, although in the previous example a fuzzy partition of $\mathcal{S}$ was associated with a set of linguistic terms, a set of linguistic terms does not always correspond to a fuzzy partition of a well-ordered set $\mathcal{S}$. For example, the terms *very slightly convincing, convincing, very convincing* could correspond to the fuzzy-qualitative labels $A_1, A_4 \sqcup A_5, A_5$, respectively, and they fulfill neither the condition of being mutually exclusive nor the condition of being collectively exhaustive.

Table 6: Differences in the complement between the crisp and fuzzy cases

Crisp partition of $\mathcal{S}$ by means of $E_i \in \mathbb{L}^*$	E_1 *very slightly convincing*	E_2 *slightly convincing*	E_3 *average*	E_4 *quite convincing*	E_5 *very convincing*
$E_4^c = E_1 \cup E_2 \cup E_3 \cup E_5$ *not quite convincing*					
Fuzzy partition of $\mathcal{S}$ by means of $A_i \in \mathcal{L}_s^*$	A_1 *very slightly convincing*	A_2 *slightly convincing*	A_3 *average*	A_4 *quite convincing*	A_5 *very convincing*
$A_4^c = (A_1 \sqcup A_2 \sqcup A_3) \cup A_5 \neq A_1 \cup A_2 \cup A_3 \cup A_5$ *not quite convincing*					

4. Fuzzy-qualitative descriptions and entropy

Let us consider a decision-making problem where experts use a set of order-of-magnitude linguistic terms $\mathcal{S} = \{E_1, \ldots, E_k\}$ to describe a feature. Experts are supposed to give their assessments by using qualitative labels of type $[E_i, E_{i+h}] = \{E_i, \ldots, E_{i+h}\}$, i.e., expert assessments may correspond to some consecutive linguistic terms. However, it is not unrealistic that experts may wish (or may be required to) tune a little more by using degrees of membership in $[0, 1]$ for each linguistic term. That is to say, expert assessments can move from labels of type $[E_i, E_{i+h}] = \{0/E_1, \ldots, 0/E_{i-1}, 1/E_i, \ldots, 1/E_{i+h}, 0/E_{i+h+1}, \ldots, 0/E_k\}$ to labels of type $\{d_1/E_1, \ldots, d_{i-1}/E_{i-1}, 1/E_i, \ldots, 1/E_{i+h}, d_{i+h+1}/E_{i+h+1}, \ldots, d_k/E_k\}$, with $d_j \in [0, 1)$, which corresponds to a fuzzy-qualitative label, i.e., a fuzzy set over $\mathcal{S}$ with core $[E_i, E_{i+h}]$, as defined in Section 2. The following subsections provide the mathematical structure for this type of assessment process.

4.1. Fuzzy-qualitative descriptions of a set

The $\mathcal{L}$-fuzzy sets on Λ assign a fuzzy-qualitative label from the extended set $\mathcal{L}$ to each element of a set Λ, whose elements are to be assessed:

Definition 8. The *set* $\mathbf{Q}$ *of* $\mathcal{L}$*-fuzzy sets on* Λ is:

$$\mathbf{Q} = \mathcal{L}^{\Lambda} = \{Q \mid Q : \Lambda \to \mathcal{L}\}.$$

The set Λ can be interpreted as a set of magnitudes, features, or objects. For example, $\Lambda = \{T(t) \,|\, t \in [t_0, t_1]\}$ where $T(t)$ is the room temperature at a given instant t in the period of time $[t_0, t_1]$. Another example is $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$, where $\lambda_1, \ldots, \lambda_m$ are the merits to be considered by an evaluator in a recruitment process or the projects that need to be rated based on their quality.

The label $Q(\lambda)$ *describes fuzzy-qualitatively* the element $\lambda \in \Lambda$, for some $\lambda \in \Lambda$, when $Q(\lambda)$ is a fuzzy-qualitative label of $\mathcal{L}$ i.e., $Q(\lambda) \in \mathcal{L}^* = \mathcal{L} - \{\emptyset\}$. Every $Q \in \mathcal{L}^\Lambda$ such that $Q(\lambda)$ is a fuzzy-qualitative label for all $\lambda \in \Lambda$ may be interpreted as an expert who assigns a fuzzy-qualitative label in $\mathcal{L}^*$ to each element of Λ.

Definition 9. A *fuzzy-qualitative description of the set* Λ *by* $\mathcal{L}$ is an $\mathcal{L}$-fuzzy set on Λ such that for all $\lambda \in \Lambda$, $Q(\lambda) \in \mathcal{L}^*$.

Definition 10. The *set of fuzzy-qualitative descriptions of* Λ *by* $\mathcal{L}$ is:

$$\mathbf{Q}^* = \{Q \in \mathbf{Q} \mid Q(\Lambda) \subseteq \mathcal{L}^*\}.$$

Remark 6. Let us highlight two interpretations of fuzzy-qualitative descriptions of a set Λ.

On the one hand, from Definition 9, fuzzy-qualitative descriptions are elements of $\left([0,1]^{\mathcal{S}}\right)^{\Lambda}$. A type-2 fuzzy set over Λ is an element in $\left([0,1]^{[0,1]}\right)^{\Lambda}$[9, 13]. Therefore, fuzzy-qualitative descriptions can be considered a generalized type of type-2 fuzzy sets, i.e., fuzzy sets whose membership grades are themselves fuzzy sets over $\mathcal{S}$.

Moreover, $\mathbf{Q} = \mathcal{L}^\Lambda \subseteq ([0,1]^{\mathcal{S}})^\Lambda = [0,1]^{\Lambda \times \mathcal{S}}$. Thus, any $\mathcal{L}$-fuzzy set on Λ is actually an ordinary fuzzy set on $\Lambda \times \mathcal{S}$. In this sense, a fuzzy-qualitative description of the set Λ by $\mathcal{L}$ is a fuzzy set Q on $\Lambda \times \mathcal{S}$ such that for all $\lambda \in \Lambda$, there exists an interval $[a, b)$ or $[a, \rightarrow)$ in $\mathcal{S}$ such that $Q(\lambda, x) = 1$, for all x that belong to this interval.

4.2. Structure of $\mathbf{Q}$ *and* $\mathbf{Q}^*$

The set $\mathbf{Q}$ of $\mathcal{L}$-fuzzy sets on Λ inherits the operations $\sqcup$ and $\sqcap$ from the lattice $\mathcal{L}$ (the same symbols are used), where the operations are extended in a pointwise manner.

Definition 11. The *mix* operation $Q \sqcup Q'$ and the *common* operation $Q \sqcap Q'$ are defined as follows for all $\lambda \in \Lambda$:

$$(Q \sqcup Q')(\lambda) = Q(\lambda) \sqcup Q'(\lambda) \text{ and } (Q \sqcap Q')(\lambda) = Q(\lambda) \sqcap Q'(\lambda).$$

As described in [20], for the case of $Q, Q' \in \mathbf{Q}^*$, the operation $\sqcup$ mixes the two fuzzy-qualitative descriptions in a new fuzzy-qualitative description, including both opinions regarding each element of Λ. In this manner, $Q \sqcup Q'$ assigns to each element of Λ the fuzzy set whose core corresponds to the connected union of both cores and the maximum of the membership function values is considered outside. The operation $\sqcap$ takes that which is common between the two fuzzy-qualitative descriptions in the case where the cores are not disjoint. In this manner, when the cores of $Q(\lambda)$ and $Q'(\lambda)$ are not disjoint, $Q \sqcap Q'$ assigns to λ the fuzzy set for which the membership function is the minimum of both membership functions.

The algebraic structure of $\mathcal{L}$ is transferred to the set of $\mathcal{L}$-fuzzy sets on a specified set. Thus, based on Theorem 1, we obtain the following:

Theorem 4 $(\mathbf{Q}, \sqcup, \sqcap)$ *is a lattice.*

Proof. Idempotence, commutativity, associativity, and absorption all hold for $\mathbf{Q}$ because of the pointwise extension from $\mathcal{L}$. $\square$

In general, the lattice $(\mathbf{Q}, \sqcup, \sqcap)$ is not distributive because $(\mathcal{L}, \sqcup, \sqcap)$ is not distributive.

The partial order $\preceq$ (the same symbol as that used for $\mathcal{L}$) induced by the two operations $\sqcup$ and $\sqcap$ in $\mathbf{Q}$ is the pointwise extension of the order $\preceq$ in $\mathcal{L}$: $Q \preceq Q' \Leftrightarrow Q \sqcup Q' = Q \Leftrightarrow Q \sqcap Q' = Q' \Leftrightarrow Q(\lambda) \sqcap Q'(\lambda) = Q'(\lambda)\ \forall \lambda \in \Lambda \Leftrightarrow Q'(\lambda) \subseteq Q(\lambda)\ \forall \lambda \in \Lambda \Leftrightarrow Q(\lambda) \preceq Q'(\lambda)\ \forall \lambda \in \Lambda$.

Moreover, it holds that $Q \sqcup Q' \preceq Q, Q' \preceq Q \sqcap Q'$ for any fuzzy-qualitative descriptions Q, Q' in $\mathbf{Q}$. In addition, the least element $0_{\mathbf{Q}}$ of the poset $(\mathbf{Q}, \preceq)$ is the fuzzy-qualitative description that maps every $\lambda \in \Lambda$ to $0_{\mathcal{L}} = ? \in \mathcal{L}$, and the greatest element $1_{\mathbf{Q}}$ is the $\mathcal{L}$-fuzzy set that maps every $\lambda \in \Lambda$ to the greatest element $1_{\mathcal{L}} = \emptyset \in \mathcal{L}$.

Given any $Q, Q' \in \mathbf{Q}^*$, it holds that $Q \sqcup Q' \in \mathbf{Q}^*$: for any $\lambda \in \Lambda$, $Q(\lambda), Q'(\lambda) \in \mathcal{L}^* \Rightarrow Q(\lambda) \sqcup Q'(\lambda) \in \mathcal{L}^*$. Hence, $\mathbf{Q}^*$ is a meet-semilattice with the operation $\sqcup$ and partial order $\preceq$. In this semilattice, the least element is clearly $0_{\mathbf{Q}^*} = 0_{\mathbf{Q}}$.

Proposition 3 *If the set $\mathcal{S}$ has at least two elements, then there is no greatest element in* $(\mathbf{Q}^*, \sqcup)$.

Proof. Suppose that $Q' \in \mathbf{Q}^*$ is the greatest element of $\mathbf{Q}^*$. We have: $Q \preceq Q' \, \forall Q \in \mathbf{Q}^* \Rightarrow Q(\lambda) \preceq Q'(\lambda) \, \forall Q \in \mathbf{Q}^* \, \forall \lambda \in \Lambda \Rightarrow Q'(\lambda) \subseteq Q(\lambda) \, \forall Q \in \mathbf{Q}^* \, \forall \lambda \in \Lambda$. In particular, this implies that $Q'(\lambda)$, for every $\lambda \in \Lambda$, is contained in all the fuzzy singletons $\{1/a\}$, $a \in \mathcal{S}$. Since $\mathcal{S}$ has at least two elements, we deduce immediately that $Q'(\lambda)(a) = 0 \; \forall a \in \mathcal{S}$. This leads to $Q'(\lambda) = \emptyset \; \forall \lambda \in \Lambda$, which is absurd since $Q' \in \mathbf{Q}^*$. □

We present two examples of fuzzy-qualitative descriptions of a set Λ, one where Λ is finite and the other where Λ is infinite.

Example A 4. Following Examples A 1, A 2 and A 3, let us consider a candidate who applies for a job in a company. Let us suppose that the hiring committee is composed of two experts. The experts evaluate the following features: λ_1 = curriculum vitae, λ_2 = an interview, and λ_3 = the salary requested by the applicant, which are the elements of the set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$. Let $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$, and consider the linguistic terms associated with the singletons $\{1/a_1\}, \{1/a_2\}, \{1/a_3\}, \{1/a_4\}, \{1/a_5\}$, respectively: *very slightly convincing, slightly convincing, average, quite convincing, very convincing.*
The experts use the following labels:
$A_1 = \{1/a_1, 0.7/a_2, 0.3/a_3\}$, $B_1 = \{1/a_1, 0.5/a_2\}$
$A_2 = \{0.5/a_1, 1/a_2, 1/a_3\}$, $B_2 = \{0.3/a_1, 1/a_2, 0.6/a_3\}$
$A_3 = \{0.1/a_2, 0,7/a_3, 1/a_4\}$, $B_3 = \{0,7/a_3, 1/a_4, 0,7/a_5\}$
$A_4 = \{0.5/a_3, 1/a_4, 1/a_5\}$, $B_4 = \{0.3/a_3, 0.7/a_4, 1/a_5\}$
Table 7 summarizes the evaluation process, as well as the mix and the common of the two fuzzy-qualitative descriptions.

Table 7: Fuzzy-qualitative descriptions that correspond to the expert assessments

expert	λ_1	λ_2	λ_3
Q_1	A_4	A_3	A_2
Q_2	B_1	B_3	B_2
$Q_1 \sqcup Q_2$	$A_4 \sqcup B_1$ $\{1/a_1, 1/a_2, 1/a_3, 1/a_4, 1/a_5\}$	$A_3 \sqcup B_3$ $\{0.1/a_2, 0.7/a_3, 1/a_4, 0.7/a_5\}$	$A_2 \sqcup B_2$ $\{0.5/a_1, 1/a_2, 1/a_3\}$
$Q_1 \sqcap Q_2$	$A_4 \sqcap B_1$ $\emptyset$	$A_3 \sqcap B_3$ $\{0.7/a_3, 1/a_4\}$	$A_2 \sqcap B_2$ $\{0.3/a_1, 1/a_2, 0.6/a_3\}$

Example B 2. Following Example B 1, now let us suppose that during a 60-minute period an evaluator provides qualitative descriptions of the temperature in a room to adjust a heat pump using fuzzy-qualitative labels of $\mathcal{L}$, with $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, where a_i are the real intervals of the temperature (degrees Celsius) in Table 2.

Let us consider the fuzzy-qualitative labels A_1, A_2, A_3 and the corresponding linguistic terms in Table 8.

Table 8: Fuzzy-qualitative labels Example B 2

$A_1 = \{1/a_1, 0.6/a_2, 0.1/a_3\}$	*cold*
$A_2 = \{0.8/a_2, 1/a_3, 0.3/a_4\}$	*warm*
$A_3 = \{0.1/a_3, 0.5/a_4, 1/a_5, 1/a_6\}$	*extremely hot*

Let $\Lambda = \{T(t) \,|\, t \in [0, 60]\}$ be the set of temperatures that need to be described. An example of a $\mathcal{L}$-fuzzy set on Λ is the fuzzy-qualitative description $Q : \Lambda \to \mathcal{L}$ of Λ given by the evaluator as:

$$Q(T(t)) = \begin{cases} A_1 = \{1/a_1, 0.6/a_2, 0.1/a_3\}, & \text{if } t \in [0, 15); \\ A_1 \sqcup A_2 = \{1/a_1, 1/a_2, 1/a_3, 0.3/a_4\}, & \text{if } t \in [15, 30); \\ A_2 = \{0.8/a_2, 1/a_3, 0.3/a_4\}, & \text{if } t \in [30, 45); \\ A_2 \sqcup A_3 = \{0.8/a_2, 1/a_3, 1/a_4\, 1/a_5, 1/a_6\}, & \text{if } t \in [45, 60]. \end{cases}$$

4.3. Entropy of a fuzzy-qualitative description

In this subsection, we introduce the entropy of a fuzzy-qualitative description $Q : \Lambda \to \mathcal{L}$, with $Q(\Lambda) \subseteq \mathcal{L}^*$, using a measure of specifity $Sp : [0,1]^{\mathcal{S}} \to [0,1]$ on the set of fuzzy-qualitative labels $\mathcal{L}^*$ and a normalized additive measure $\overline{\mu} : \mathcal{P}(\Lambda) \to [0,1]$ on the power set $\mathcal{P}(\Lambda)$.

A fuzzy-qualitative description can be considered as a fuzzy random variable, i.e., a mapping defined on a probability space whose values are fuzzy sets under some measurability conditions, interpreted according to the ontic model [2, 4]. We define the entropy of a fuzzy-qualitative description Q of a set Λ as the entropy associated with the probability measure $\overline{\mu} \circ Q^{-1}$ induced by Q based on the normalized additive measure $\overline{\mu}$ on the power set $\mathcal{P}(\Lambda)$. The proposed entropy is then formalized based on a Lebesgue integral and a measure of specificity [30]. In the discrete case, where Q has a finite range, this integral becomes a weighted average of the information of the fuzzy-qualitative labels used by Q, corresponding to the Shannon self-information entropy of a discrete random variable. The chosen definition of entropy based on Shannon requires the ontic versus the epistemic interpretation of fuzzy sets [2, 4].

4.3.1. Information of a fuzzy-qualitative label

The choice of an additive measure μ to quantify the amount of information in a crisp qualitative label, $I(E) = -\log \mu(E)$, led to a formula of the entropy of a crisp qualitative description that was inspired by the Shannon entropy, which uses a probability [20]. However, $I(E) = -\log m(E)$, with m as a generic fuzzy measure (monotone measure) or a capacity (normal monotone measure), that is not necessarily additive [8], could also have been considered. These types of measures are more generic and useful in cases where, for example, the measure of the union of two fuzzy sets does not have to be exactly the sum of the two measures.

In the case of the fuzzy-qualitative labels $A \in \mathcal{L}$, we cannot consider the information $I(A) = -\log m(A)$ with m any fuzzy measure because $\mathcal{L} \subseteq [0,1]^{\mathcal{S}}$ does not satisfy the required properties of a measurable space. Therefore, we must resort to measures on fuzzy sets. In particular, we consider the measures of specificity introduced by Yager [29], which are measures of the utility of the information contained in a fuzzy set.

In the following, let Sp be a *measure of specificity*, i.e., $Sp : \mathcal{F}(\mathcal{S}) = [0,1]^{\mathcal{S}} \to [0,1]$ a function that satisfies the following three conditions: $Sp(A) = 1$ if and only if A is a singleton, $Sp(\emptyset) = 0$, and if A, B are normal sets, with $A \subseteq B$, then $Sp(A) \geq Sp(B)$.

Definition 12. Given a fuzzy-qualitative label $A \in \mathcal{L}^* \subseteq \mathcal{F}(\mathcal{S})$, the *information of A given by the measure of specificity Sp on $\mathcal{L} \subseteq \mathcal{F}(\mathcal{S})$* is $I(A) = Sp(A)$.

Thus, for each fuzzy-qualitative label A, we associate a number $I(A) \in [0,1]$ that is defined using a measure of specificity, which is considered to be a measure of the amount of information contained in or provided by the label A.

The next definition introduces the concept of *being more precise than* in the set $\mathcal{L}$.

Definition 13. Given A_1 and A_2 in $\mathcal{L}$, we say that A_1 is *more precise or equal than* A_2 if $A_1 \subseteq A_2$.

In this manner, A_1 being more precise or equal than A_2 implies that $Core(A_1) \subseteq Core(A_2)$, thus this concept generalizes the same concept in $\mathbb{L}$. Since fuzzy-qualitative labels are normal fuzzy sets, given any $A_1, A_2 \in \mathcal{L}^*$, if $A_1 \subseteq A_2$, then $I(A_1) \geq I(A_2)$, i.e., if the label A_1 is more precise than the label A_2, then A_1 provides more information than A_2.

The fuzzy-qualitative label with the lowest value of the information $I(A)$ is the "unknown" label $? = \mathcal{S}$, because for any $A \in \mathcal{L}^*$, $A \subseteq \mathcal{S} = ? \Rightarrow Sp(A) \geq Sp(?)$. The greatest value of the information $I(A)$, which is equal to 1, is provided by the fuzzy singletons $\{1/a\}$, with $a \in \mathcal{S}$.

4.3.2. Entropy in the continuous case

Below the entropy $H(Q)$ of a fuzzy-qualitative description Q is defined and interpreted as a measure of the amount of information emitted by Q. The advantage of defining the entropy of an $\mathcal{L}$-fuzzy set as a Lebesgue integral is to unify the discrete and continuous cases.

First, we formalize the concept of the entropy of a fuzzy-qualitative description $Q : \Lambda \to \mathcal{L}$ of Λ in the continuous case. Let $Sp : [0,1]^{\mathcal{S}} \to [0,1]$ be a measure of specificity, and let $\overline{\mu} : \mathcal{P}(\Lambda) \to [0,1]$ be a normalized additive measure on the power set $\mathcal{P}(\Lambda)$, i.e., a measure on the measurable space $(\Lambda, \mathcal{P}(\Lambda))$ such that $\overline{\mu}(\Lambda) = 1$.

The composition function $\Lambda \xrightarrow{Q} \mathcal{L} \xrightarrow{Sp_{|\mathcal{L}}} [0,1] \hookrightarrow [0,+\infty]$ (we denote this as $Sp \circ Q : \Lambda \to [0,+\infty]$) is a measurable map with respect to the σ-algebra $\mathcal{P}(\Lambda)$ and the Borel algebra $\mathcal{B}([0,+\infty])$ because the set $(Sp \circ Q)^{-1}(B)$ $\forall B \in \mathcal{B}([0,+\infty])$ is a subset of Λ. This justifies the following:

Definition 14. Given the measure space $(\Lambda, \mathcal{P}(\Lambda), \overline{\mu})$ with normalized measure $\overline{\mu}$, and the extended set of fuzzy-qualitative labels $\mathcal{L}$ over $\mathcal{S}$ provided with a measure of specificity Sp, the *entropy of an $\mathcal{L}$-fuzzy set* $Q \in \mathbf{Q}^*$ is the Lebesgue integral of the measurable map $Sp \circ Q$ with respect to $\overline{\mu}$:

$$H(Q) = \int_{\Lambda} (Sp \circ Q)\, d\overline{\mu}. \tag{1}$$

The function $H : \mathbf{Q}^* \to [0,+\infty]$ is called the *entropy function*.

Theorem 5 *The entropy function H is an isotone mapping between the posets $(\mathbf{Q}^*, \preceq)$ and $([0,1], \leq)$.*

Proof. For any $Q \in \mathbf{Q}^*$, we have $H(Q) \leq 1$, because $Sp(Q(\lambda)) \leq 1 \ \forall\lambda \in \Lambda$ and therefore $H(Q) = \int_\Lambda (Sp \circ Q)\, d\overline{\mu} \leq \int_\Lambda 1\, d\overline{\mu} = \overline{\mu}(\Lambda) = 1$.
Regarding monotonicity, for any $Q, Q' \in \mathbf{Q}^*$ it holds that:

$Q \preceq Q' \Rightarrow Q(\lambda) \preceq Q'(\lambda) \, \forall\lambda \in \Lambda \Rightarrow Q'(\lambda) \subseteq Q(\lambda) \, \forall\lambda \in \Lambda$, and, since $Q'(\lambda), Q(\lambda)$ are normal fuzzy sets, this implies that $Sp(Q(\lambda)) \leq Sp(Q'(\lambda)) \, \forall\lambda \in \Lambda$. The monotonicity of the Lebesgue integral leads to:
$H(Q) = \int_\Lambda (Sp \circ Q)\, d\overline{\mu} \leq \int_\Lambda (Sp \circ Q')\, d\overline{\mu} = H(Q')$. □

If Q and Q' are fuzzy-qualitative descriptions of Λ and, for every $\lambda \in \Lambda$, the label $Q'(\lambda)$ is more precise than the label $Q(\lambda)$, then, from Theorem 5, the entropy of Q' is greater than or equal to that of Q, and Q' provides at least as much information as Q.

The entropy of the the constant $\mathcal{L}$-fuzzy sets is given by the following:

Proposition 4 *If $Q(\lambda) = A \ \forall\lambda \in \Lambda$ for some $A \in \mathcal{L}^*$, then the entropy of Q is then $Sp(A)$.*

Proof. The proof is straightforward.

Proposition 5 *The least entropy, which is equal to $Sp(?)$, is provided by the least element $0_\mathbf{Q}$ of the poset $(\mathbf{Q}^*, \preceq)$, which corresponds to the case when all the elements of Λ are described by the "unknown" label ? The greatest entropy, which is equal to 1, occurs when all the labels that describe the elements of Λ are singletons.*

Proof. $0_\mathbf{Q} \preceq Q \, \forall Q \in \mathbf{Q} \Rightarrow H(0_\mathbf{Q}) \leq H(Q) \, \forall Q \in \mathbf{Q}$, and by Proposition 4, $H(0_\mathbf{Q}) = Sp(?)$.
On the other hand, if $Q(\lambda)$ is a singleton for all $\lambda \in \Lambda$, then $(Sp \circ Q)(\lambda) = 1$ for all $\lambda \in \Lambda$ and therefore $H(Q) = \int_\Lambda (Sp \circ Q)\, d\overline{\mu} = \int_\Lambda 1\, d\overline{\mu} = 1$. □

It is not surprising that the minimum entropy is $H(0_\mathbf{Q}) = Sp(?)$ since the least amount of information is provided by a qualitative description of a set Λ when all the elements of Λ are described by the "unknown" label ?, which is the least precise label. By contrast, the greatest amount of information is obtained when all the elements of Λ are described by singletons, which are the most precise labels.

4.3.3. Entropy in the discrete case

Consider the discrete case $Q(\Lambda) = \{A_1, \ldots, A_r\} \subseteq \mathcal{L}^*$, where only a finite number of labels $A_1, \ldots, A_r$ are used to describe the elements of the set Λ, and let $\{\Lambda_1 = Q^{-1}(\{A_1\}), \ldots, \Lambda_r = Q^{-1}(\{A_r\})\}$ be the partition of Λ induced by Q. In this case, the Lebesgue integral of Formula (1) becomes a weighted average, which is proven by the following:

Theorem 6 *The entropy of the $\mathcal{L}$-fuzzy set Q with the finite range $\{A_1, \ldots, A_r\}$ is given by:*

$$H(Q) = \sum_{i=1}^{r} Sp(A_i) \cdot \overline{\mu}(Q^{-1}(\{A_i\})). \tag{2}$$

Proof. The map $Sp \circ Q$ can be expressed as follows:

$$Sp \circ Q = \sum_{i=1}^{r} Sp(A_i) \cdot 1_{\Lambda_i},$$

where $1_{\Lambda_i} : \Lambda \to \{0, 1\}$ is the characteristic function of Λ_i. Using the linearity of the Lebesgue integral, the entropy of Q in this case is given by:

$$H(Q) = \int_{\Lambda} (Sp \circ Q)\, d\overline{\mu} = \int_{\Lambda} \left(\sum_{i=1}^{r} Sp(A_i) \cdot 1_{\Lambda_i} \right) d\overline{\mu} = \sum_{i=1}^{r} Sp(A_i) \cdot \int_{\Lambda} 1_{\Lambda_i}\, d\overline{\mu} =$$

$$= \sum_{i=1}^{r} Sp(A_i) \cdot \overline{\mu}(\Lambda_i) = \sum_{i=1}^{r} Sp(A_i) \cdot \overline{\mu}(Q^{-1}(\{A_i\})). \;\square$$

The entropy of a qualitative description Q of a set Λ with a finite range is expressed in Formula (2) as a weighted average of the information of the labels used by Q in the description of Λ, where the weights are the measures of the subsets of the elements of Λ that are described by the same label. In the discrete case, note that the entropy $H(Q)$ is analogous to the Shannon self-information entropy of a discrete random variable.

The next example illustrates how the entropy of a fuzzy-qualitative description is computed. The entropy allows the comparison of expert assessments in group decision-making by means of the amount of information provided.

Example A 5. Following Examples A 1, A 2, A 3 and A 4, and using the same data as in Example A 4, let us suppose that the hiring committee want to weight the three features, λ_1 = curriculum vitae, λ_2 = interview, and λ_3

= salary requested, with corresponding weights 3, 1 and 1. We consider the following normalized measure on Λ: $\overline{\mu}(\{\lambda_1\}) = \frac{3}{5}, \overline{\mu}(\{\lambda_2\}) = \frac{1}{5}, \overline{\mu}(\{\lambda_3\}) = \frac{1}{5}$.

In addition, we consider the linear specificity measure [29] given by $Sp(A) = d_1 - \sum_{j=2}^{n} w_j d_j$, where X is a finite set of cardinality n, A is a fuzzy set of X, $d_1 \geq \cdots \geq d_n$ are the corresponding membership degrees ordered from the largest to the smallest, and the weights satisfy $w_i \in [0,1]$ and $\sum_{j=2}^{n} w_j = 1$.

In our case, $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$, $Q_1(\Lambda) = \{A_4, A_3, A_2\}$, $Q_2(\Lambda) = \{B_1, B_3, B_2\}$, and we consider $w_j = \frac{1}{4}$ for $j = 2, \ldots, 5$. Then:

$Sp(A_4) = Sp(\{0.5/a_3, 1/a_4, 1/a_5\}) = 1 - \frac{1}{4}1 - \frac{1}{4}0.5 = 0.625$.

$Sp(A_3) = Sp(\{0.1/a_2, 0, 7/a_3, 1/a_4\}) = 1 - \frac{1}{4}0.7 - \frac{1}{4}0.1 = 0.800$.

$Sp(A_2) = Sp(\{0.5/a_1, 1/a_2, 1/a_3\}) = 1 - \frac{1}{4}1 - \frac{1}{4}0.5 = 0.625$.

$Sp(B_1) = Sp(\{1/a_1, 0.5/a_2\}) = 1 - \frac{1}{4}0.5 = 0.875$.

$Sp(B_3) = Sp(\{0, 7/a_3, 1/a_4, 0, 7/a_5\}) = 1 - \frac{1}{4}0.7 - \frac{1}{4}0.7 = 0.650$.

$Sp(B_2) = Sp(\{0.3/a_1, 1/a_2, 0.6/a_3\}) = 1 - \frac{1}{4}0.6 - \frac{1}{4}0.3 = 0.775$.

By applying Formula (2) to the discrete case we compute the entropy of Q_1 and Q_2 and compare the amount of information provided by both experts of the committee:

$H(Q_1) = Sp(A_4) \cdot \frac{3}{5} + Sp(A_3) \cdot \frac{1}{5} + Sp(A_2) \cdot \frac{1}{5} = 0.660$.

$H(Q_2) = Sp(B_1) \cdot \frac{3}{5} + Sp(B_3) \cdot \frac{1}{5} + Sp(A_2) \cdot \frac{1}{5} = 0.810$.

Expert Q_2 provides more information than expert Q_1 (as expected because the cores of fuzzy-qualitative labels used by Q_2 are more precise than those used by Q_1).

5. Conclusions and future work

This paper provides a new general representation of linguistic descriptions by unifying qualitative and fuzzy perspectives. Fuzzy-qualitative labels are defined as fuzzy sets where the core is a qualitative label. These include cases where the core is either a basic qualitative label or a non-basic label – which enables us to represent different levels of precision. A lattice structure is given to the set of fuzzy-qualitative labels. In accordance with Ruspini's approach, a theorem that characterizes finite fuzzy partitions of a well-ordered set $\mathcal{S}$ using fuzzy-qualitative labels, where the cores and supports are qualitative labels, is proven. The theorem leads to a mathematical justification for the commonly-used fuzzy partitions of real intervals via trapezoidal (or triangular) fuzzy sets. A fuzzy-qualitative description of a set is defined as both an L-fuzzy set and an extension of type-2 fuzzy sets by replacing the secondary domain with a well-ordered set whose elements can be associated with linguistic terms.

In addition, the information of a fuzzy-qualitative label is defined by considering a measure of specificity. Finally, the concept of entropy of a fuzzy-qualitative description of a set is formally introduced using a Lebesgue integral. The amount of information given by fuzzy-qualitative descriptions is measured via the proposed entropy. In this way, expert assessments in group decision-making can be compared by means of the amount of information provided.

The results of this study highlight two main areas for future research. Firstly, given the lattice structure of $\mathcal{L}$, a study of possible distances on $\mathcal{L}^\Lambda$ will be conducted. Secondly, based on the entropy introduced in this study, the development of a consensus model for multi-attribute group decision-making problems that support incomplete or missing information is being considered.

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Appendix A

In this appendix, we have included a table with the different notations used along the paper together with their respective meanings (see Table 9).

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Table 9: Notations

$\mathcal{S}$	well-ordered set $(\mathcal{S}, \leq)$
$\{a\}$	basic qualitative labels over $\mathcal{S}$
$[a,b),\ [a,\rightarrow)$	qualitative labels over $\mathcal{S}$
$\mathbb{L}^* = \{[a,b) \mid a,b \in \mathcal{S},\ a<b\} \cup \{[a,\rightarrow) \mid a \in \mathcal{S}\}$	set of qualitative labels over $\mathcal{S}$
$\mathbb{L} = \mathbb{L}^* \cup \{\emptyset\}$	extended set of $\mathbb{L}^*$
$A \in [0,1]^{\mathcal{S}} = \mathcal{F}(\mathcal{S})$ such that $Core(A) \in \mathbb{L}^*$	fuzzy-qualitative labels over $\mathcal{S}$
$\mathcal{L}^* = \{A \in \mathcal{F}(\mathcal{S}) \mid Core(A) \in \mathbb{L}^*\}$	set of fuzzy-qualitative labels over $\mathcal{S}$
$\mathcal{L} = \mathcal{L}^* \cup \{\emptyset\}$	extended set of $\mathcal{L}^*$
$A_1 \sqcup A_2$	mix of A_1 and A_2
$A_1 \sqcap A_2$	common of A_1 and A_2
$A_1 \preceq A_2 \Leftrightarrow A_1 \supseteq A_2$	the partial order induced in the lattice $\mathcal{L}$
$0_{\mathcal{L}} = \mathcal{S} =? =$ unknown fuzzy-qualitative label	the least element in the poset $(\mathcal{L}, \preceq)$
$1_{\mathcal{L}} = \emptyset$	the greatest element in the poset $(\mathcal{L}, \preceq)$
$\mathcal{L}_s^* = \{A \in \mathcal{L}^* \mid Support(A) \in \mathbb{L}^*\}$	set of fuzzy-qualitative labels with support in $\mathbb{L}^*$
$\mathcal{L}_s = \mathcal{L}_s^* \cup \{\emptyset\}$	extended set of $\mathcal{L}_s^*$
A^c	standard fuzzy complement of A
Λ	set of magnitudes, features or objects to be described
$\mathbf{Q} = \mathcal{L}^{\Lambda} = \{Q \mid Q : \Lambda \rightarrow \mathcal{L}\}$	set of $\mathcal{L}$-fuzzy sets on Λ
$Q : \Lambda \rightarrow \mathcal{L}$, with $Q(\Lambda) \subseteq \mathcal{L}^*$	fuzzy-qualitative description of Λ by $\mathcal{L}$
$\mathbf{Q}^* = \{Q \in \mathbf{Q} \mid Q(\Lambda) \subseteq \mathcal{L}^*\}$	set of fuzzy-qualitative descriptions of Λ by $\mathcal{L}$
$Q \sqcup Q'$	mix of Q and Q'
$Q \sqcap Q'$	common of Q and Q'
$Q \preceq Q' \Leftrightarrow Q(\lambda) \preceq Q'(\lambda)\, \forall \lambda \in \Lambda$	the partial order induced in the lattice $\mathbf{Q}$
$0_{\mathbf{Q}},\ 0_{\mathbf{Q}}(\lambda) = 0_{\mathcal{L}} =? \ \forall \lambda \in \Lambda$	the least element of the poset $(\mathbf{Q}, \preceq)$
$1_{\mathbf{Q}},\ 1_{\mathbf{Q}}(\lambda) = 1_{\mathcal{L}} = \emptyset \ \forall \lambda \in \Lambda$	the greatest element of the poset $(\mathbf{Q}, \preceq)$
$Sp : [0,1]^{\mathcal{S}} \rightarrow [0,1]$	measure of specificity
$I(A) = Sp(A)$	information of A given by Sp
$A_1 \subseteq A_2$	A_1 is more precise or equal than A_2
$\overline{\mu} : \mathcal{P}(\Lambda) \rightarrow [0,1]$	a normalized additive measure on $\mathcal{P}(\Lambda)$
$H(Q) = \int_{\Lambda} (Sp \circ Q)\, d\overline{\mu}$	entropy of an $\mathcal{L}$-fuzzy set $Q \in \mathbf{Q}^*$
$H : \mathbf{Q}^* \rightarrow [0, +\infty]$	the entropy function

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