

Celebrity Games

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Abstract

We introduce *Celebrity games*, a new model of network creation games. In this model players have weights (being W the sum of all the player's weights) and there is a critical distance β as well as a link cost α . The cost incurred by a player depends on the cost of establishing links to other players and on the sum of the weights of those players that remain farther than the critical distance. Intuitively, the aim of any player is to be relatively close (at a distance less than β) from the rest of players, mainly of those having high weights. The main features of celebrity games are that: computing the best response of a player is NP-hard if $\beta > 1$ and polynomial time solvable otherwise; they always have a pure Nash equilibrium; the family of celebrity games having a connected Nash equilibrium is characterized (the so called *star celebrity games*) and bounds on the diameter of the resulting equilibrium graphs are given; a special case of star celebrity games share its set of Nash equilibrium profiles with the MaxBD games with uniform bounded distance β introduced in (Bilò et al., 2012). Moreover, we analyze the Price of Anarchy (PoA) and of Stability (PoS) of celebrity games and give several bounds. These are that: for non-star celebrity games $\text{PoA} = \text{PoS} = \max\{1, W/\alpha\}$; for star celebrity games $\text{PoS} = 1$ and $\text{PoA} = O(\min\{n/\beta, W\alpha\})$ but if the Nash Equilibrium is a tree then the PoA is $O(1)$; finally, when $\beta = 1$ the PoA is at most 2. The upper bounds on the PoA are complemented with some lower bounds for $\beta = 2$.

Keywords: Network creation games, Nash equilibrium, Price of Anarchy, diameter

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1. Introduction

The global growth of Internet and social networks usage has been accompanied by an increasing interest to model theoretically their creation as well as their behavior. In particular, network creation games (NCG) aim to model Social Networks and Internet by simulating the creation of a decentralized and non-cooperative communication network among n players (the network nodes).

From the seminal paper (Fabrikant et al., 2003) several proposals have been made in the area of NCG. In the original model, the goal of each player is to have, in the resulting network, all the other nodes as close as possible while buying as few links as possible (Fabrikant et al., 2003). Several assumptions are made: all the players have the same interest (all-to-all communication pattern with identical weights); the cost of being disconnected is infinite; and the edges paid by one node can be used by others. Formally, a game Γ in this model is defined as a tuple $\Gamma = \langle V, \alpha \rangle$, where V is the set of n nodes and α the cost of establishing a link. A strategy for player $u \in V$ is a subset $S_u \subseteq V - \{u\}$, the set of players for which player u pays for establishing a link. The n players and their joint strategy choices $S = (S_u)_{u \in V}$ create an undirected graph $G[S]$. The cost function for each node u under strategy S is defined by $c_u(S) = \alpha|S_u| + \sum_{v \in V} d_{G[S]}(u, v)$ where $d_{G[S]}(u, v)$ is the distance between nodes u and v in graph $G[S]$. Because of the summation in the cost function this model is informally known as the *Sum game* model. By changing the cost function to $c_u(S) = \alpha|s_u| + \max\{d_{G[S]}(u, v) | v \in V\}$ as proposed in Demaine et al. (2012) one obtains the *Max game* model.

From here on several versions and variants have been considered. Instead of buying links unilaterally, Corbo and Parkes (2005) proposed the possibility of having links formed by bilateral contracting: both endpoints must agree before creating a link between them and the two players share (half-half) the cost of establishing the link. NCG models can be cooperative –a possibility introduced by Albers et al. (2006)– and therefore any node can purchase any amount of any link in the resulting graph, and a link can be created when its cost is covered by a set of players. The model studied in Bilò et al. (2015b) (see also Bilò et al. (2012)) considers the notion of bounded distance per player and propose two variants: the MaxBD game and the SumBD game, corresponding to the original Max and Sum cost models respectively. The cost in those games depends on whether the player’s eccentricity is smaller or equal than the associated bounded distance. In that case a player pays the number of established links, otherwise its cost is infinite. For further variants we refer the interested reader to (Demaine et al., 2012; Leonardi and Sankowski, 2007; Brandes et al., 2008; Demaine et al., 2009; Lenzner, 2011; Alon et al., 2013, 2014; Bilò et al., 2015a; Nikolettseas et al., 2015; Cord-Landwehr and Lenzner, 2015; Ehsani et al., 2015) among others.

We introduce *celebrity games* a NCG where players have different weights and share a common distance bound. As far as we understand, not all the nodes in Internet based networks have the same importance. It is though natural to consider players with different relevance weights. In such a setting, the cost of being far (even if connected) from important nodes (the ones with high weight)

should be higher than the cost of having them close. Intuitively, the goal of each player in celebrity games is to buy as few links as possible in order to have the high-weighted nodes (or groups of nodes) closer to the given critical distance. Observe that if the cost of establishing links is higher than the benefit of having close a node (or set of nodes), players might rather prefer to stay either far or even disconnected from it.

Our aim is to study the combined effect of having players with different weights that share a common bounded distance. Although heterogeneous players have been considered recently in the context of NCG under bilateral contracting (Meirom et al., 2014; Álvarez et al., 2015), and Bilò et al. (2015b) consider the notion of bounded distance, to the best of our knowledge this is the first model that studies how a common critical distance, different weights, and a link cost, altogether affect the individual preferences of the players.

In our model the cost of a player has two components. The first one is the cost of the links established by the node. The second one is the sum of the weights of those nodes that are farther away than the critical distance. More specifically, the parameters of a celebrity game are: a weight to each player; a cost for establishing a link; and a critical distance. Formally, a celebrity game is defined by $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, where V is a set of nodes with weights $(w_u)_{u \in V}$, α is the cost of establishing a link and, β establishes the desirable distance bound. Celebrity games include the MaxBD games introduced in (Bilò et al., 2015b) (see Section 5 for the details). They capture not only the cases in which players are indistinguishable but those cases where the players may have different weights affecting differently the costs of the other players.

We analyze the structural properties of the Nash equilibrium (NE) graphs of celebrity games and their quality with respect to the optimal strategies under the usual social cost. To do so we address the cases $\beta = 1$ and $\beta > 1$ separately. Notice that, for $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link or w_v (the weight of the non-adjacent node v) while, for $\beta > 1$, every player u has to choose for each non-edge (u, v) between buying the link (u, v) and paying α minus the sum of the weights of those nodes whose distance to u will become less or equal than the critical distance β or paying the sum of the weights of the nodes with distance to u greater than β .

For the general case $\beta > 1$ our results can be summarized as follows:

- Computing a best response for a player is NP-hard
- The optimal social cost of a celebrity game Γ depends on the relation between the total sum of the weights W and the cost α of buying a link: $\text{OPT}(\Gamma) = \min\{\alpha, W\}(n - 1)$. Nevertheless, pure NE always exist and NE graphs are either connected or a set of isolated nodes. Again, the relationship between the cost of establishing a link and the weight of the nodes leads to different types of NE.
- We use the term *celebrity* for a node whose weight is strictly greater than the cost of establishing a link. Having at least one celebrity guarantees that all NE graphs are connected, although there are celebrity games without

celebrities that still have connected NE graphs. In those games having a connected NE graph, a star tree is always a NE graph. We called this subfamily of celebrity games *star celebrity games*.

- For star celebrity games, we obtain a general upper bound of $2\beta + 1$ for the diameter of NE graphs. In particular, if G is a NE tree we show that $diam(G) \leq \beta + 1$, otherwise $\beta/2 < diam(G) \leq 2\beta + 1$. The upper bound can be improved by considering the relationship between α and the maximum and minimum weights, w_{max} and w_{min} , respectively. So, if $w_{min} \leq \alpha < w_{max}$, then $diam(G) \leq 2\beta$. On the contrary, if $\alpha < w_{min}$, then $diam(G) \leq \beta$.
- For star celebrity games with $\alpha < w_{min}$, we show that the set of NE strategy profiles coincides with the set of NE strategy profiles of a MaxBD game with uniform bounded distance β .
- We find several bounds on the Price of Anarchy (PoA) and of stability (PoS). For non-star celebrity games $PoS = PoA = \max\{1, W/\alpha\}$. For star celebrity games the PoS is 1 and we obtain a general upper bound of $O(\min\{n/\beta, W/\alpha\})$ for the PoA. We also show particular games on n players having $PoA = \Omega(n)$, for $\beta = 2$. To complement those results we prove that the PoA on NE trees is constant (special cases like trees are also considered in the literature, see for instance (Alon et al., 2013, 2014; Ehsani et al., 2015)).

Finally, for the particular case $\beta = 1$, we show that computing a best response for a player is polynomial time solvable and that the PoA is at most 2.

The paper is organized as follows. In Section 2 we introduce the basic definitions and the celebrity games model. We also show that computing a best response is NP-hard. In Section 3 we set the fundamental properties of NE and optimal graphs. We characterize star celebrity games and we provide the first bounds for the PoA and the PoS. Section 4 is devoted to the study of the diameter of NE graphs. Section 5 studies the relation between the MaxBD game model and the celebrity game model. In Section 6 we derive the bounds for the PoA. In Section 7 we give the upper bound of the PoA over NE trees and in Section 8 we study the case $\beta = 1$. Finally, we state some conclusions and open problems in Section 9.

2. The Model

In this section we introduce celebrity games and we analyze the complexity of computing a best response. Let us start with some definitions. We use standard notation for graphs and strategic games. All the graphs in the paper are undirected unless explicitly said otherwise. Given a graph $G = (V, E)$ and $u, v \in V$, $d_G(u, v)$ denotes the *distance* between u and v in G , i.e., the length of the shortest path from u to v . The *diameter* (or *eccentricity*) of a vertex $u \in V$ is $diam(u) = \max_{v \in V} d_G(u, v)$ and the *diameter* of G is $diam(G) =$

$\max_{v \in V} \text{diam}(v)$. An *orientation* of an undirected graph is an assignment of a direction to every edge of the graph, turning it into a directed graph. A *bridge* is an edge whose deletion increases the number of connected components of the graph. For a weighted set $(V, (w_u)_{u \in V})$ we extend the weight function to subsets in the usual way. For $U \subseteq V$, $w(U) = \sum_{u \in U} w_u$. Furthermore, we set $W = w(V)$, $w_{max} = \max_{u \in V} w_u$ and $w_{min} = \min_{u \in V} w_u$.

Definition 1. A celebrity game Γ is a tuple $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ where: $V = \{1, \dots, n\}$ is the set of players, for each player $u \in V$, $w_u > 0$ is the weight of player u , $\alpha > 0$ is the cost of establishing a link, and β , $1 \leq \beta \leq n - 1$, is the critical distance.

A strategy for player u in Γ is a subset $S_u \subseteq V - \{u\}$, the set of players for which player u pays for establishing a direct link. A strategy profile for Γ is a tuple $S = (S_1, \dots, S_n)$ that assigns a strategy to each player. Every strategy profile S has associated an outcome graph, the undirected graph defined by $G[S] = (V, \{\{u, v\} | u \in S_v \vee v \in S_u\})$.

We denote by $c_u(S) = \alpha |S_u| + \sum_{\{v | d_{G[S]}(u, v) > \beta\}} w_v$ the cost of player u in the strategy profile S . And, as usual, the social cost of a strategy profile S in Γ is defined as $C(S) = \sum_{u \in V} c_u(S)$.

Observe that, even though a link might be established by only one of the two players, we assume that once a link is established it can be used in both directions. Note also that players may have different weights. The player's cost function has two components: the cost of establishing links and the sum of the weights of those players who are farther away than the critical distance β . In our model links have uniform length therefore w.l.o.g β is an integer. In what follows we assume that, for a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, the parameters verify the required conditions. Furthermore, unless specifically stated, we assume $\beta > 1$, the case $\beta = 1$ will be analyzed in Section 8. We use the following notation $n = |V|$, $\mathcal{S}(u)$ is the set of strategies for player u and $\mathcal{S}(\Gamma)$ is the set of strategy profiles of Γ . For a strategy profile $S \in \mathcal{S}(\Gamma)$ and a strategy $S'_u \in \mathcal{S}(u)$, for player u , (S_{-u}, S'_u) represents the strategy profile in which S_u is replaced by S'_u while the strategies of the other players remain unchanged. The *cost difference* $\Delta(S_{-u}, S'_u)$ is defined as $\Delta(S_{-u}, S'_u) = c_u(S_{-u}, S'_u) - c_u(S)$. Observe that, if $\Delta(S_{-u}, S'_u) < 0$, then player u has an incentive to deviate from S_u and select S'_u . A *best response* to $S \in \mathcal{S}(\Gamma)$ for player u is a strategy $S'_u \in \mathcal{S}(u)$ minimizing $\Delta(S_{-u}, S'_u)$.

Let us recall the definition of Nash equilibrium.

Definition 2. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. A strategy profile $S \in \mathcal{S}(\Gamma)$ is a Nash equilibrium of Γ if no player has an incentive to deviate from his strategy. Formally, for each player u and each strategy $S'_u \in \mathcal{S}(u)$, $\Delta(S_{-u}, S'_u) \geq 0$.

We denote by $\text{NE}(\Gamma)$ the set of Nash equilibria of a game Γ and we use the term NE to refer to a strategy profile $S \in \text{NE}(\Gamma)$. We say that a graph G is a NE graph of Γ if there is $S \in \text{NE}(\Gamma)$ so that $G = G[S]$. We will drop the explicit

reference to Γ whenever Γ is clear from the context. It is worth observing that, for $S \in \text{NE}(\Gamma)$, it never happens that $v \in S_u$ and $u \in S_v$, for any $u, v \in V$. Thus, if G is the outcome of a NE S , S corresponds to an orientation of the edges in G . Furthermore, a NE graph G can be the outcome of several strategy profiles but not all the orientations of a NE graph G are NE.

Let $\text{opt}(\Gamma) = \min_{S \in \mathcal{S}(\Gamma)} C(S)$ be the minimum value of the social cost. We use the term OPT strategy profile to refer to one strategy profile with optimal social cost.

Observe that, when in a strategy profile S , two players u and v are such that $u \in S_v$ and $v \in S_u$, the social cost is higher than when only one of them is paying for the connection $\{u, v\}$ and therefore, as for NE, this does not happen in an OPT strategy profile. In the following, as we are interested in NE and OPT strategies, among all the possible strategy profiles having the same outcome graph, we only consider those corresponding to orientations of the outcome graph. In this sense the social cost depends only on the outcome graph, the weights and the parameters. Thus, we can express the social cost of a strategy profile as a function of the outcome graph G as follows

$$C(G) = \alpha|E(G)| + \sum_{u \in V} \sum_{\{v | d_G(u,v) > \beta\}} w_v = \alpha|E(G)| + \sum_{\{(u,v) | u < v \text{ and } d_G(u,v) > \beta\}} (w_u + w_v).$$

We make use of three particular outcome graphs on n vertexes: K_n , the complete graph; I_n , the independent set; and ST_n the star graph, i.e., a tree in which one of the vertexes, the *central* one, has a direct link to all the other $n-1$ vertexes. For those graphs, we have the following values of the social cost. For $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, with $|V| = n$, $C(K_n) = \alpha n(n-1)/2$, $C(I_n) = W(n-1)$, for $\beta \geq 1$. Furthermore, $C(ST_n) = \alpha(n-1)$, for $1 < \beta \leq n-1$, and $C(ST_n) = \alpha(n-1) + (n-2)(W - w_c)$ where c is the central vertex, for $\beta = 1$.

We define the PoA and the PoS as usual.

Definition 3. Let Γ be a celebrity game. The Price of Anarchy of Γ is defined as $\text{PoA}(\Gamma) = \max_{S \in \text{NE}(\Gamma)} C(S) / \text{opt}(\Gamma)$ and the Price of Stability of Γ as $\text{PoS}(\Gamma) = \min_{S \in \text{NE}(\Gamma)} C(S) / \text{opt}(\Gamma)$.

Our first result shows that computing a best response in celebrity games is NP-hard by a reduction from the minimum dominating set problem. The problem becomes tractable for $\beta = 1$ as we show in Section 8.

Proposition 1. Computing a best response for a player to a strategy profile in a celebrity game is NP-hard, even when $\beta = 2$ and restricted to the cases in which all players except possibly one have weights bigger than α .

Proof. We provide a reduction from the problem of computing a dominating set of minimum size which is a classical NP-hard problem. Recall that a dominating set of a graph $G = (V, E)$ is a set $U \subset V$ such that any vertex $u \in V$ is in U or has a neighbor in U .

Let $G = (V, E)$ be a graph, we associate to G and u a celebrity game $\Gamma = \langle V', (w_v)_{v \in V'}, \alpha, \beta \rangle$, and a strategy profile S as follows:

- The set of players is $V' = V \cup \{u\}$, where u is a new player (i.e. $u \notin V$).
- $\beta = 2$, $\alpha = 1.5$,
- for every $v \in V$, $w_v = 2$.
- The strategy profile S is obtained from an orientation of the edges in G setting $S_u = \emptyset$. Observe that by construction $G[S]$ is the disjoint union of G with the isolated vertex u .

Finally, set u to be the player for which we want to compute the best response to S . Observe that the weight of u has not been defined yet.

Let $D \subseteq V$ be a strategy for player u . Notice that, if D is a dominating set of G , then $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x) > 2} 2 = \alpha|D|$. If D is not a dominating set of G , $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x) > 2} 2 > \alpha(|D| + |\{x \in V | d(u,x) > 2\}|)$. Then, $D \cup \{x \in V | d(u,x) > 2\}$ is a better response than D and furthermore it is a dominating set. Hence, the best response of player u is a dominating set D of G of minimum size. To conclude the proof just notice that the described reduction is polynomial time computable and that we did not make any assumption on the weight of the node u . \square

3. Social Optimum and Nash equilibrium

We analyze here the main properties of OPT and NE strategy profiles in celebrity games. We start analyzing the cost of optimal graphs for the social cost. Then we characterize the family of star celebrity games having a connected NE graph. Finally, we provide exact bounds on the PoA and the PoS in some particular cases.

Proposition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We have that $opt(\Gamma) = \min\{\alpha, W\}(n-1)$.*

Proof. Let $S \in \text{OPT}(\Gamma)$, and let $G = G[S]$ with connected components G_1, \dots, G_r , $V_i = V(G_i)$, $k_i = |V_i|$, and $W_i = w(V_i)$, for $1 \leq i \leq r$. Observe that the social cost of a disconnected graph can be expressed as the sum of the social cost of its connected components. Each connected component must be a tree of diameter at most β , otherwise a strategy profile with smaller social cost could be obtained by replacing the connections on V_i by such a tree. We can assume w.l.o.g. that, for $1 \leq i \leq r$, the i -th connected component is a star graph ST_{k_i} of k_i vertexes. Since $C(ST_{k_i}) = \alpha(k_i - 1)$ we have that

$$C(G) = \sum_{i=1}^r \alpha(k_i - 1) + \sum_{i=1}^r k_i(W - W_i) = \alpha(n - r) + nW - \sum_{i=1}^r k_i W_i.$$

As $1 \leq k_i \leq n - (r - 1)$, we have $W \leq \sum_{i=1}^r k_i W_i \leq (n - r + 1)W$. Therefore, $\alpha(n - r) + (r - 1)W \leq C(G)$. We consider two cases.

Case 1: $\alpha \geq W$. We have $W(n - 1) \leq C(G)$. Since $C(I_n) = W(n - 1) \leq C(G)$ and G is an optimal graph, then $C(G) = W(n - 1)$.

Case 2: $\alpha < W$. Now $\alpha(n-1) \leq C(G)$. As $C(ST_n) = \alpha(n-1) \leq C(G)$, the optimal graph G has a social cost $C(G) = \alpha(n-1)$. We conclude that $\text{OPT} = \min\{\alpha, W\}(n-1)$. \square

Now we turn our attention to the study of the NE graph topologies showing that any NE graph is either an independent set or a connected graph.

Proposition 3. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. Every NE graph of Γ is either connected or the graph I_n , where $n = |V|$.*

Proof. If $n \leq 2$ the proposition follows immediately. When $n > 2$, let us suppose that there is a NE S such that the graph $G = G[S]$ is not connected and different from I_n . In this case G is composed of at least two different connected components G_1 and G_2 . Furthermore, as $G \neq I_n$, we can assume that $|V(G_1)| > 1$ as at least one of the connected components contains two vertexes connected by an edge. Let $u \in V(G_1)$ be such that $S_u \neq \emptyset$. Let $x \in S_u$ and $v \in V(G_2)$. Let us consider the strategies $S'_u = S_u \setminus \{x\}$ and $S'_v = S_v \cup \{x\}$. As S is a NE we know that $\Delta(S_{-u}, S'_u) \geq 0$. Let $G' = G[S_{-v}, S'_v]$, observe that $d_{G'}(v, u) = 2 \leq \beta$, therefore $\Delta(S_{-v}, S'_v) \leq -\Delta(S_{-u}, S'_u) - w_u < 0$. This contradicts the hypothesis that S is a NE. \square

Next we study the conditions under which particular topologies are NE graphs. Those results prove that celebrity games always have a NE.

Proposition 4. *Every celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ has a NE. Furthermore, when $\alpha \geq w_{max}$, I_n is a NE graph, otherwise ST_n is a NE graph but I_n is not, where $n = |V|$.*

Proof. When $\alpha \geq w_{max}$ let us show that I_n is a NE graph. Observe that $G = I_n$ is the outcome of a unique strategy profile S in which $S_u = \emptyset$, for any $u \in V$. Let us consider a player u and a strategy $S'_u \neq \emptyset$. The cost difference of player u is then $\Delta(S_{-u}, S'_u) = \alpha|S'_u| - \sum_{v \in S'_u} w_v = \sum_{v \in S'_u} (\alpha - w_v) \geq 0$. Therefore player u has no incentive to deviate from S_u and I_n is a NE graph.

When $\alpha < w_{max}$, let u be a vertex with $w_u = w_{max}$ and let ST_n be a star graph with n vertexes in which the center is u , let us show that ST_n is a NE graph. Consider the strategy profile S in which $S_u = \emptyset$ and $S_v = \{u\}$, for any $v \in V$ different from u . Observe that the center u is a vertex with maximum weight. As $\beta > 1$ no player will get a cost decrease by connecting to more players. Furthermore, for $u \neq v$, $w_v + \alpha < w_v + w_{max} < W$. Thus $\alpha < W - w_v$ and v will not get any benefit by deleting the actual connection. The only remaining possibility is to reconnect to another vertex, but in such a case the cost cannot decrease. Therefore, ST_n is a NE graph. Notice that in this case I_n can not be a NE, as every player u has incentive to connect with any other player v such that $w_v = w_{max}$. \square

To conclude the study of NE we characterize the celebrity games where I_n is the unique NE graph. The negated condition characterizes those games in which ST_n is a NE.

Proposition 5. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game on n players with $\alpha \geq w_{max}$. If there is more than one vertex $u \in V$ with $\alpha > W - w_u$, then I_n is the unique NE graph of Γ , otherwise ST_n is a NE graph of Γ .*

Proof. Assume that, for two vertexes $u \neq v$, $\alpha > W - w_u$ and $\alpha > W - w_v$, and that there exists a NE graph $G = G[S]$ different from I_n . By Proposition 3, G is connected. Therefore, it has at least $n - 1$ edges. Since, $\alpha > W - w_u$ and $\alpha > W - w_v$, we have that $S_u = S_v = \emptyset$, otherwise S would not be a NE. Therefore, there must be a vertex, $z \neq u, v$ such that $|S_z| \geq 2$. Let $x, y \in S_z$ and consider the strategy $S'_z = S_z \setminus \{x, y\}$. Then, $\Delta(S_{-z}, S'_z) \leq -2\alpha + W - w_z$. As G is a NE graph and we have that $2\alpha > W - w_u + W - w_v$, we conclude that $W - w_z \geq 2\alpha > W - w_u + W - w_v$. Hence, $W < w_u + w_v - w_z < w_u + w_v$, which is impossible. In the case that there is at most one vertex u with $\alpha > W - w_u$, the strategy profile S , where $S_u = \emptyset$, and $S_v = \{u\}$, for all $v \neq u$, is a NE. Furthermore $G[S] = ST_n$. \square

Corollary 1. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game on n players. I_n is the unique NE graph of Γ if and only if $\alpha \geq w_{max}$ and there is more than one vertex $u \in V$ such that $\alpha > W - w_u$.*

Observe that in our model it is preferable to be an isolated node than to pay a huge amount for establishing a link. In fact, in a NE graph either all nodes are isolated, or the graph is connected. Hence, selecting an appropriate price per link is a key fact to guarantee the connectivity of the equilibrium graphs.

Finally, using this characterization, we can formally define the subfamily of celebrity games that have always a connected NE graph. Those games have ST_n as a NE graph.

Definition 4. $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ is a star celebrity game if Γ has a NE graph that is connected.

Corollary 2. *For a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, the following statements are equivalent.*

- Γ is a star celebrity game.
- Either $\alpha < w_{max}$ or $\alpha \geq w_{max}$ and there is at most one $u \in V$ for which $\alpha > W - w_u$.
- ST_n is a NE graph of Γ .

Putting all together we can compute the PoS and, in some cases, the PoA.

Theorem 1. *Let Γ be a celebrity game. Then we have.*

- If Γ is a star celebrity game, $PoS(\Gamma) = 1$.
- If Γ is not a star celebrity game and $\alpha \geq W$, then $PoS(\Gamma) = PoA(\Gamma) = 1$.
- If Γ is not a star celebrity game and $\alpha < W$, then $PoS(\Gamma) = PoA(\Gamma) = W/\alpha > 1$.

Proof. From Proposition 2, we have that $\text{opt}(\Gamma) = W(n-1)$ if $\alpha \geq W$ and $\text{opt}(\Gamma) = \alpha(n-1)$, otherwise. When Γ is a star celebrity game, by Corollary 2 we know that ST_n is a NE graph. Let us see that in star celebrity games it can only occur that $\alpha < W$. If $\alpha < w_{max}$, clearly $\alpha < W$. If $\alpha \geq w_{max}$, by Corollary 2 there is at most one $u \in V$ for which $\alpha > W - w_u$. Assuming that $w_{u_1} \leq \dots \leq w_{u_{n-1}} \leq w_{u_n}$, we have that $W > W - w_{u_1} \geq \dots \geq W - w_{u_{n-1}} \geq W - w_{u_n}$, and then $W - w_{u_{n-1}} \geq \alpha$. Hence, $\text{PoS}(\Gamma) = 1$.

When Γ is not a star celebrity game, I_n is the unique NE graph. Thus, when $\alpha \geq W$ we have, $\text{PoS}(\Gamma) = \text{PoA}(\Gamma) = 1$ and, when $\alpha < W$ we have, $\text{PoS}(\Gamma) = \text{PoA}(\Gamma) = W/\alpha > 1$. \square

4. Critical distance and diameter in Nash equilibrium graphs

In this section we analyze the diameter of NE graphs and its relationship with the parameters defining the game. We are interested only in games in which NE graphs with finite diameter exist, thus we only consider star celebrity games. In stating the characterization, nodes with a high weight with respect to the link cost play a fundamental role and it is worth to give them a name.

Definition 5. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We say that a vertex $u \in V$ is a celebrity if $\alpha < w_u$.

Given a celebrity u , any other node v with $d(u, v) > \beta$ has an incentive to pay for connecting to u . Thus, in any NE graph G , every celebrity node u satisfies that $\text{diam}(u) \leq \beta$.

In some of the proofs of the following results we refer to a set of *critical nodes* $z \in V$ of a graph $G = (V, E)$ with respect to a node u and an edge $\{x, y\}$. Critical is used in the sense that as all the shortest paths from u to z use $\{x, y\}$, removing the edge $\{x, y\}$ results in an increase of the distance from u to z . We use the notation

$$A_{\{x, y\}}^G(u) = \{z \in V \mid \text{all the shortest paths in } G \text{ from } u \text{ to } z \text{ use the edge } \{x, y\}\}$$

We drop the explicit reference to G whenever G is clear from the context.

Proposition 6. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. If G is a NE graph of Γ , then $\text{diam}(G) \leq 2\beta + 1$.

Proof. Let S be a NE of Γ such that $G = G[S]$. Assume that $\text{diam}(G) \geq 2\beta + 2$. Then, there are two nodes $u, v \in V$ such that $d(u, v) = 2\beta + 2$. Consider a shortest path from u to v , $u = u_0, u_1, \dots, u_{2\beta+1}, u_{2\beta+2} = v$.

Let $A_u = \{x \in V \mid d(u, x) \leq \beta\}$ and let $A_{u_1} = \{x \in V \mid d(u_1, x) \leq \beta\}$. Let us show that if a node $x \in A_u \cup A_{u_1}$, then $d(x, v) > \beta$. If $x \in A_u$ then $d(x, v) > \beta$, otherwise $d(u, v) \leq d(u, x) + d(x, v) \leq 2\beta$ contradicting the fact that $d(u, v) = 2\beta + 2$. Moreover, if $x \in A_{u_1}$ then $d(x, v) > \beta$, otherwise $d(u, v) \leq 1 + d(u_1, x) + d(x, v) \leq 2\beta + 1$ which also contradicts the fact that $d(u, v) = 2\beta + 2$.

Consider the edge $\{u, u_1\}$. Then, either $u_1 \in S_u$ or $u \in S_{u_1}$. In the case that $u_1 \in S_u$, let $S'_u = S_u \setminus \{u_1\}$ and $S'_v = S_v \cup \{u_1\}$. Observe that,

$$\Delta(S_{-u}, S'_u) \leq -\alpha + w(A_{\{u, u_1\}}(u) \cap A_u)$$

By the previous remark about distances, we know that all the vertexes $x \in A_{\{u, u_1\}}(u) \cap A_u$ verify $d(x, v) > \beta$, but after adding $\{v, u_1\}$ all of them and u become at distance $\leq \beta$ from v , therefore

$$\Delta(S_{-v}, S'_v) \leq \alpha - w_u - w(A_{\{u, u_1\}}(u) \cap A_u).$$

Hence, $\Delta(S_{-u}, S'_u) + \Delta(S_{-v}, S'_v) \leq -w_u < 0$. Therefore, either $\Delta(S_{-u}, S'_u) < 0$ or $\Delta(S_{-v}, S'_v) < 0$ and then S can not be a NE.

The case $u \in S_{u_1}$, follows in a similar way by interchanging the roles of u and u_1 . \square

The previous result can be refined to get better bounds on the diameter when all the nodes are celebrities or when at least one of the nodes is a celebrity.

Property 1. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a NE graph of Γ , then

- if $w_{min} \leq \alpha < w_{max}$, $diam(G) \leq 2\beta$ and,
- if $\alpha < w_{min}$, $diam(G) \leq \beta$.

Proof. When $w_{min} \leq \alpha < w_{max}$, there is a celebrity $u \in V$ with $w_u > \alpha$. We know that $diam(u) \leq \beta$. Let x and z be any two different nodes of G , then $d(x, u) \leq \beta$ and $d(z, u) \leq \beta$. Therefore, $d(x, z) \leq d(x, u) + d(z, u) \leq 2\beta$ and the claim follows. When $\alpha < w_{min}$, each $u \in V$ is a celebrity, thus $diam(u) \leq \beta$. Therefore $diam(G) \leq \beta$. \square

For NE trees we have a trivial lower bound of 2 on the diameter as a star is a NE graph. For non-tree NE graphs we provide a lower bound on the diameter. We first prove a technical result.

Lemma 1. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. In a NE graph of Γ containing at least one cycle, if u is a node of a cycle and $diam(u) \leq \beta - k$, for some $k \geq 1$, then the length of any cycle containing u is bigger than $2k + 2$.

Proof. Let us suppose that S is a NE and that $G = G[S]$ contains a cycle C through a node u such that $diam(u) \leq \beta - k$, for some $k \geq 1$. Assume that C is the shortest cycle containing u and that the length ℓ of C verifies $\ell \leq 2k + 2$. We split the proof in two cases, depending on the parity of ℓ .

Case 1: C has odd length, $\ell = 2i + 1$. Let v_1, v_2 be the two vertexes in C that are at distance i of u in C , as C is of minimal length $d(u, v_1) = d(u, v_2) = i$. By our hypothesis, $2i + 1 \leq 2k + 2$ and thus $i \leq k$. Assume w.l.o.g. that $v_2 \in S_{v_1}$ and consider the strategy $S'_{v_1} = S_{v_1} \setminus \{v_2\}$. Let $G' = G[S_{-v_1}, S'_{v_1}]$. Notice that $d_{G'}(v_2, u) = i$. Therefore, $diam_{G'}(v_1) \leq k + \beta - k = \beta$, by selecting a path going through u , so $\Delta(S_{-v_1}, S'_{v_1}) \leq -\alpha < 0$ and G would not be a NE graph.

Case 2: C has even length, $\ell = 2i$. Let v be the antipodal vertex to u , at distance i from u in C and let v_1, v_2 be the two vertexes in C that are at distance $i - 1$ of u in C . By our hypothesis, $2i \leq 2k + 2$ and thus $i - 1 \leq k$. If $v \in S_{v_1}$, consider the strategy $S'_{v_1} = S_{v_1} \setminus \{v\}$. Using the same arguments as in Case 1 and the fact that the distance from v_1 to u in C is $\leq k$, we conclude that S is not a NE. The same happens when $v \in S_{v_2}$. It remains to consider the case in which $v_1, v_2 \in S_v$. Consider the strategy $S'_v = (S_v \cup \{u\}) \setminus \{v_1, v_2\}$. Now all shortest paths in G from v passing through v_1 or v_2 can be rerouted through u with an increment in length of at most $i - 1 \leq k$. Therefore, $diam_{G'}(v) \leq 1 + \beta - k \leq \beta$. Thus $\Delta(S_{-v}, S'_v) \leq -\alpha < 0$ and G would not be a NE graph. \square

Proposition 7. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a NE graph of Γ . If G is not a tree, then $diam(G) > \beta/2$.*

Proof. Let G be a NE graph containing at least one cycle. If $diam(G) \geq \beta$, the claim holds. Assume that $diam(G) \leq \beta - 1$. We know that the length of the shortest cycle C is $\leq 2 diam(G) + 1$. Let u be any node of C . Then, we have $diam(u) \leq diam(G) = \beta - (\beta - diam(G))$. By Lemma 1, $2 diam(G) + 1 > 2(\beta - diam(G)) + 2$. The last inequality implies $diam(G) > \beta/2$. \square

5. MaxBD network creation games versus celebrity games

In this section we show that MaxBD games are equivalent to celebrity games where all players are celebrities. Let us formalize the definition of MaxBD game taken from (Bilò et al., 2015b).

A *MaxBD game* Γ is defined by a tuple $\langle V, D \rangle$ where $V = \{1, \dots, n\}$ is the set of players and D , $1 \leq D \leq n - 1$, is an integer representing the bound on the diameter of each node $v \in V$. Concepts like strategy of a player, strategy profile, and outcome graph are defined as in the celebrity game model. The *cost* of player u in the strategy profile S is $c_u^{\text{MaxBD}}(S) = |S_u|$, if $diam_{G[S]}(u) \leq D$; $c_u^{\text{MaxBD}}(S) = +\infty$, otherwise. The social cost of S is $C^{\text{MaxBD}}(S) = \sum_{u \in V} c_u^{\text{MaxBD}}(S)$. Notice that by the definition of MaxBD game, any strategy profile S that is either a NE or a social OPT satisfies $diam_{G[S]}(u) \leq D$ and, therefore $C(S) = \alpha C^{\text{MaxBD}}(S)$.

In the following we show how a MaxBD game can be translated, preserving NE, to different instances of celebrity games. A MaxBD game can be seen as a celebrity game in which the weights of each one of the players are large enough so that buying a link is more suitable than having an eccentricity greater than the given distance bound. On the other hand, we show that every celebrity game with $\alpha < w_{\min}$ corresponds to a MaxBD game, again preserving NE.

Proposition 8. *Let V be a set of players and $\beta > 1$. Let $\Gamma = \langle V, \beta \rangle$ be a MaxBD game and $\Gamma' = \langle V, (w_v)_{v \in V}, \alpha, \beta \rangle$ be a celebrity game where $\alpha < w_{\min}$. Then, $\text{NE}(\Gamma) = \text{NE}(\Gamma')$.*

Proof. Let us prove first that $\text{NE}(\Gamma) \supseteq \text{NE}(\Gamma')$. Assume that $S \in \text{NE}(\Gamma')$. By Property 1, $diam_{G[S]}(u) \leq \beta$ and this implies that $c_u(S) = \alpha |S_u|$. Let us suppose that S is not a NE for Γ . Then there exists a player $u \in V$ and a

strategy S'_u such that $c_u^{\text{MaxBD}}(S') < c_u^{\text{MaxBD}}(S) = |S_u|$, where $S' = (S_{-u}, S'_u)$. Hence, the only possibility is that $\text{diam}_{G[S']}(u) \leq \beta$ and $|S'_u| < |S_u|$. Therefore $c_u(S') < c_u(S)$ contradicting the fact that $S \in \text{NE}(\Gamma')$.

It remains to show that $\text{NE}(\Gamma) \subseteq \text{NE}(\Gamma')$. Let $S \in \text{NE}(\Gamma)$. We know that $\text{diam}(G[S]) \leq \beta$ and $c_u^{\text{MaxBD}}(S) = |S_u|$, for $u \in V$. For Γ' , we have that $c_u(S) = \alpha|S_u|$. Now let us assume that S is not a NE of Γ' . Then, there exists $u \in V$ and a strategy S'_u such that $c_u(S) > c_u(S')$, where $S' = (S_{-u}, S'_u)$. Since $w_v > \alpha$, then we have that $c_u(S') = \alpha|S'_u| + \sum_{\{v | d_{G[S']}(u,v) > \beta\}} w_v \geq \alpha(|S'_u| + |\{v | d_{G[S']}(u,v) > \beta\}|)$. Consider the strategy profile $S'' = (S_{-u}, S''_u)$, where $S''_u = S'_u \cup \{v | d_{G[S']}(u,v) > \beta\}$. We have $\text{diam}_{G[S'']}(u) \leq \beta$. Thus, $c_u(S'') = \alpha|S''_u|$. Combining the inequalities $c_u(S) = \alpha|S_u| > c_u(S'') = \alpha|S''_u|$. Then, $|S_u| > |S''_u|$ contradicting the fact that $S \in \text{NE}(\Gamma)$. \square

The previous correspondences allow us to get a relationship on the PoA and the PoS,

Corollary 3. *Let V be a set of players and $\beta > 1$. Let $\Gamma = \langle V, \beta \rangle$ be a MaxBD game and let $\Gamma' = \langle V, (w_v)_{v \in V}, \alpha, \beta \rangle$ be a celebrity game where $\alpha < w_{\min}$. Then,*

- $\text{PoS}(\Gamma) = \text{PoS}(\Gamma') = 1$,
- $\text{PoA}(\Gamma) = \text{PoA}(\Gamma')$.

Proof. We know by Proposition 4 that the star tree is a social optimum as well as a NE for celebrity games when $\alpha < w_{\min}$. The same occurs for MaxBD games as it was shown in Theorem 3.3 of (Bilò et al., 2012). Hence, $\text{PoS}(\Gamma) = \text{PoS}(\Gamma') = 1$.

For the celebrity game Γ' , we have that

$$\text{PoA}(\Gamma) = \frac{\alpha \max_{S \in \text{NE}(\Gamma)} \{|E(G[S])|\}}{\alpha(n-1)} = \frac{\max_{S \in \text{NE}(\Gamma)} \{|E(G[S])|\}}{(n-1)}.$$

By Proposition 8, $\text{NE}(\Gamma) = \text{NE}(\Gamma')$. Thus NE of Γ' have diameter $\leq \beta$ and then we can conclude that $\text{PoA}(\Gamma) = \text{PoA}(\Gamma')$. \square

Hence, the upper bound on the PoA of MaxBD games shown in (Bilò et al., 2015b) is also an upper bound for celebrity games. In the subsequent sections we consider the general case where the assumption $\alpha < w_{\min}$ is not required.

We have considered here only the uniform version of the MaxBD games in which the eccentricity bound is equal for all the nodes. Bilò et al. (2015b) considers also a non uniform version in which each node has a different eccentricity requirement. It is easy to extend Proposition 8 to show that the set of NE is preserved provided that the eccentricity bounds are the same in both games and $\alpha < w_{\min}$. Therefore, non-uniform celebrity games have unbounded PoA, as it was shown for the non-uniform MaxBD games in (Bilò et al., 2015b).

6. Bounding the price of anarchy

We provide here bounds on the contribution of the edges and the weights to the social cost of NE graphs. Those bounds allow us to provide a bound on the PoA. Our next result establishes an upper bound on the PoA in terms of W and α .

Lemma 2. *For a star celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, $PoA(\Gamma) \leq W/\alpha$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. As S is a NE, no player has an incentive to deviate from S . Thus, for any $u \in V$,

$$0 \leq \Delta(S_{-u}, \emptyset) \leq -\alpha|S_u| + w(\{v \mid d(u, v) \leq \beta\}) - w_u.$$

Summing up, for all $u \in V$, we have

$$0 \leq \sum_{u \in V} (-\alpha|S_u| + \sum_{\{v \mid d(u, v) \leq \beta\}} w_v - w_u) = -\alpha|E| + \sum_{u \in V} \sum_{\{v \mid d(u, v) \leq \beta\}} w_v - W.$$

Therefore,

$$\begin{aligned} C(G) &= \alpha|E| + \sum_{u \in V} \sum_{\{v \mid d(u, v) > \beta\}} w_v \\ &\leq \sum_{u \in V} \left(\sum_{\{v \mid d(u, v) \leq \beta\}} w_v + \sum_{\{v \mid d(u, v) > \beta\}} w_v \right) - W = (n-1)W. \end{aligned}$$

$$\text{Hence, } PoA(\Gamma) \leq \frac{(n-1)W}{\alpha(n-1)} = \frac{W}{\alpha}. \quad \square$$

Using the previous lemma we can get an $O(n)$ upper bound on the PoA of star celebrity games. Let us see that this upper bound can be improved by bounding the weight component and the link component of the social cost, separately.

Define the *weight component* of the social cost, for a critical distance β , $W(G, \beta)$, as

$$W(G, \beta) = \sum_{u \in V(G)} \sum_{\{v \mid d(u, v) > \beta\}} w_v = \sum_{\{\{u, v\} \mid d(u, v) > \beta\}} (w_u + w_v).$$

Lemma 3. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. In a NE graph G , $W(G, \beta) = O(\alpha n^2/\beta)$.*

Proof. Let S be a NE and $G = G[S]$ be a NE graph. Let $u \in V$ and let $b = \text{diam}(u)$. Recall that, by Proposition 6, $b \leq 2\beta + 1$. We have three cases.

Case 1: $b < \beta$. For any node $v \in V \setminus \{u\}$ consider the strategy $S'_v = S_v \cup \{u\}$, and let $G' = G[S_{-v}, S'_v]$. By connecting to u we have $\text{diam}_{G'}(v) \leq \beta$ and, as S is a NE, we have

$$\Delta(S_{-v}, S'_v) = \alpha - \sum_{\{x \mid d_G(x, v) > \beta\}} w_x \geq 0.$$

Therefore we have

$$\sum_{\{x|d_G(x,v)>\beta\}} w_x \leq \alpha.$$

As $b < \beta$ we conclude that

$$W(G, \beta) \leq n\alpha.$$

Since $1 < \beta \leq n - 1$, we get $n/\beta \leq \alpha n^2/\beta$.

Case 2: $b \geq \beta$ and $b \geq 6$. For $1 \leq i \leq b$, consider the set $A_i(u) = \{v \mid d(u, v) = i\}$ and the sets

$$\begin{aligned} C_1 &= \{v \in V \mid 1 \leq d(u, v) \leq b/3\} = \cup_{1 \leq i \leq b/3} A_i(u), \\ C_2 &= \{v \in V \mid b/3 < d(u, v) \leq 2b/3\} = \cup_{b/3 < j \leq 2b/3} A_j(u), \\ C_3 &= \{v \in V \mid 2b/3 < d(u, v) \leq b\} = \cup_{2b/3 < k \leq b} A_k(u). \end{aligned}$$

As $b = \text{diam}(u)$, $A_\ell(u) \neq \emptyset$, $1 \leq \ell \leq b$, and all those sets constitute a partition of $V \setminus \{u\}$. As $b \geq 6$, for each ℓ , $1 \leq \ell \leq 3$, C_ℓ contains vertexes at a $b/3 \geq 2$ different distances. Therefore, for $1 \leq \ell \leq 3$, it must exist i_ℓ such that $A_{i_\ell}(u) \subseteq C_\ell$ and $|A_{i_\ell}(u)| \leq 3n/b$, otherwise the total number of elements in C_ℓ would be bigger than n .

For any $v \in V$, let $S'_v = (S_v \cup A_{i_1}(u) \cup A_{i_2}(u) \cup A_{i_3}(u)) \setminus \{v\}$ and let $G' = G[S_{-v}, S'_v]$. Since $b \leq 2\beta + 1$, we have that $b/3 < \beta$. Hence, by construction, $\text{diam}_{G'}(v) \leq \beta$. Therefore, as S is a NE, we have

$$0 \leq \Delta(S_{-v}, S'_v) \leq \frac{9n\alpha}{\beta} - \sum_{\{x|d_G(x,v)>\beta\}} w_x.$$

Thus,

$$\sum_{\{x|d_G(x,v)>\beta\}} w_x \leq \frac{9n\alpha}{\beta} \text{ and } W(G, \beta) \leq \frac{9n^2\alpha}{\beta}.$$

Case 3: $b \geq \beta$ and $b \leq 6$. Consider the sets $A_i(u) = \{v \mid d(u, v) = i\}$, $0 \leq i \leq b$, and the sets $C_0 = \{v \in V \mid d(u, v) \text{ is even}\}$ and $C_1 = V \setminus C_0$. Both sets are non-empty and one of them must have $\leq n/2$ vertexes. By connecting to all the vertexes in the smaller of those sets the diameter of the resulting graph is 2. Therefore, using a similar argument as in case 2, we get

$$W(G, \beta) \leq \frac{n^2\alpha}{2},$$

which is $O(n^2/\beta)$ as $\beta < 6$. Which concludes the proof. \square

Our next result provides a bound for the number of edges in a NE graph.

Lemma 4. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. In a NE graph G , $|E(G)| \leq n - 1 + \frac{3n^2}{\beta}$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. Let u be a node in V . For any $v \in S_u$, recall that $A_{\{u,v\}}(u)$ denotes the set of nodes z such that all shortest paths from u to z use the edge $\{u, v\}$. Observe that $v \in A_{\{u,v\}}(u)$ and that, for $v, v' \in S_u$ with $v \neq v'$, $A_{\{u,v\}}(u) \cap A_{\{u,v'\}}(u) = \emptyset$.

Let $B(G)$ be the set of bridges of G , recall that $|B(G)| \leq n-1$. For $u \in V$, let $\overline{B}(u) = \{x \in S_u \mid \{u, x\} \notin B(G)\}$. Observe that $|E| = |B(G)| + \sum_{u \in V} |\overline{B}(u)|$.

Let us show that for any $v \in S_u$ such that $\{u, v\}$ is not a bridge, there exists $z \in A_{\{u,v\}}(u)$ such that $d(u, z) > \beta/3$.

Let us suppose that $\{u, v\}$ is not a bridge and that, for every $z \in A_{\{u,v\}}(u)$ $d(u, z) \leq \beta/3$. In such a case there must be some edge $\{x, y\}$ with $x \notin A_{\{u,v\}}(u)$ and $y \in A_{\{u,v\}}(u)$. Furthermore, we can select x so that $x \neq u$ and such that there is a shortest path P from u to x using only vertexes in $V \setminus A_{\{u,v\}}(u)$. Observe that $d(u, x) \leq d(u, y) + 1$. Furthermore, for $z \in A_{\{u,v\}}(u)$, there exists a path from u to z that follows P from u to x , the edge $\{x, y\}$, a shortest path from y to v (part of a shortest path to u through $A_{\{u,v\}}(u)$), and a shortest path from v to z (through $A_{\{u,v\}}(u)$). Notice that $d(u, x) \leq \beta/3 + 1$, $d(y, v) \leq \beta/3 - 1$, and $d(v, z) \leq \beta/3 - 1$. Hence, there is a path from u to z of distance $\leq (\beta/3 + 1) + 1 + (\beta/3 - 1) + (\beta/3 - 1) = \beta$ which does not use $\{u, v\}$. Thus, u has incentive to remove $\{u, v\}$ since $\Delta(S_{-u}, S_u \setminus \{v\}) = -\alpha < 0$, which contradicts the fact that S is a NE.

Therefore, for $v \in \overline{B}(u)$, there exists $z \in A_{\{u,v\}}(u)$ such that $d(u, z) > \beta/3$ and as all the predecessors of z in a shortest path from u belong to $A_{\{u,v\}}(u)$, we have $|A_{\{u,v\}}(u)| > \beta/3$. Observe that $n \geq \sum_{\{v \in S_u \mid v \in \overline{B}(u)\}} |A_{u,v}(u)| \geq |\overline{B}(u)|(\beta/3)$, thus $|\overline{B}(u)| \leq \frac{3n}{\beta}$. Finally, combining the two bounds, we have $|E| = |B(G)| + \sum_{u \in V} |\overline{B}(u)| \leq (n-1) + \frac{3n^2}{\beta}$. \square

Observe that, the previous results jointly with $\text{opt}(\Gamma) = \alpha(n-1)$, leads us to the following upper bound of the PoA.

Theorem 2. *For a star celebrity game Γ , $PoA(\Gamma) = O(\min\{n/\beta, W/\alpha\})$.*

We finalize this section showing a family of star celebrity games having $PoA = \Omega(n)$, for $\beta = 2$.

Lemma 5. *Let $k > 2$, $\alpha > 0$ and let $w = (w_1, \dots, w_k)$ be a positive weight assignment. There is a star celebrity game $\Gamma = \Gamma(k, \alpha, w)$ with $n = 2k$ players and $\beta = 2$ having $PoA(\Gamma) > \frac{3n}{8}$.*

Proof. Consider the game $\Gamma_k = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, where

- $V = \{u_1, \dots, u_k\} \cup \{v_1, \dots, v_k\}$,
- $w(u_i) = \alpha$ and $w(v_i) = w_i$, for $1 \leq i \leq k$,
- $\beta = 2$.

Consider any strategy profile S where, for $1 \leq i \leq k$, $\{u_1, \dots, u_k\} \cap S_{v_i} = \{u_i\}$ and $S_{u_i} = \emptyset$, and such that in $G[S]$ the subgraph induced by $\{v_1, \dots, v_k\}$ is a clique. An example of such a strategy, for $k = 4$, is given in Figure 1.

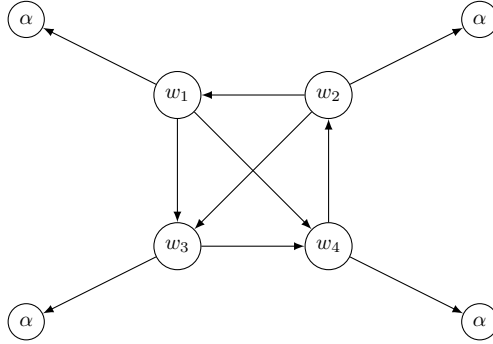


Figure 1: A NE for the game $\Gamma(4, \alpha, w)$.

Observe that there is no vertex in $G[S]$ that is at distance 1 of more than one vertex in $\{u_1, \dots, u_k\}$. Furthermore, any edge (u, v) lies in the unique shortest path from u to a vertex in $\{u_1, \dots, u_k\}$. Therefore S is a NE.

We have $C(G[S]) = \alpha \left(\frac{k(k-1)}{2} + k \right) + \alpha k(k-1) = \alpha(3k(k-1) + 2k)/2$. As a star tree is an OPT graph and $n = 2k$, we conclude that

$$\text{PoA}(\Gamma) = \frac{\alpha \frac{\frac{3n}{2}(\frac{n}{2}-1) + 2\frac{n}{2}}{2}}{\alpha(n-1)} = \frac{3n(n-1) + \frac{1}{3}}{8(n-1)} = \frac{3n}{8} \left(1 + \frac{1}{3(n-1)} \right).$$

□

7. Price of anarchy on Nash equilibrium trees

Now we complement the results of the previous sections by providing a constant upper bound on the PoA when we restrict the NE graphs to be trees. We can find in the literature different models for which the diameter or the PoA can not be proved to be constant on general NE graphs, but they are shown to be constant in the case of NE trees (see for example Alon et al. (2013, 2014); Ehsani et al. (2015)).

In order to get a tighter upper bound for the PoA on NE trees, we first improve the bound on the diameter of NE trees to $\beta + 1$.

Proposition 9. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. If T is a NE tree of Γ , $\text{diam}(T) \leq \beta + 1$.*

Proof. Let T be a tree such that $T = G[S]$ where S is a NE of Γ . Let $d = \text{diam}(T)$ and let $P : u = u_0, u_1, \dots, u_d$ be a diametral path of T . Assume that $d > \beta + 1$. For $1 \leq i < d$, let T_i be the connected subtree containing u_i after removing edges $\{u_{i-1}, u_i\}$ and $\{u_i, u_{i+1}\}$. As P is a diametral path, both u and u_d are leaves in T . Furthermore, T_1 and T_{d-1} are star trees. In general, the distance from the leaves of any T_i to both u and u_d is at most d .

We consider two cases depending on who is paying for the connections to the end points of P .

Case 1: $u \in S_{u_1}$ or $u_d \in S_{u_{d-1}}$. W.l.o.g. assume that $u_d \in S_{u_{d-1}}$. As S is a NE we have $w_{u_d} \geq \alpha$. Consider the strategy $S'_{u_1} = S_{u_1} \cup \{u_{d-1}\}$, then $\Delta(S_{-u_1}, S'_{u_1}) \leq \alpha - w_{u_d} - w_{u_{d-1}} < 0$ and T can not be a NE graph.

Case 2: $u_1 \in S_u$ and $u_{d-1} \in S_{u_d}$. When $\beta \geq 3$. Set $S'_u = S_u - \{u_1\} \cup \{u_2\}$ and $T' = G[(S_{-u}, S'_u)]$. Observe that, for $x \in T_1$, $d_{T'}(u, x) \leq 3 \leq \beta$ and, for $x \notin T_1 \cup \{u\}$, $d_{T'}(u, x) = d_T(u, x) - 1$. Therefore, $\Delta(S_{-u}, S'_u) \leq -w_{u_{\beta+1}} < 0$. Therefore, T is not a NE graph.

The previous argument fails when $\beta = 2$ as there might be $x \in T_1$ with $d_{T'}(u, x) = 3$. From Proposition 6, we know that $d \leq 2\beta + 1 \leq 5$. Let us see that it can not be the case that $d = 4$ or $d = 5$. Let $S'_u = S_u - \{u_1\} \cup \{u_{d-1}\}$ and $S'_{u_d} = S_{u_d} - \{u_{d-1}\} \cup \{u_1\}$. Let $T^1 = G[(S_{-u}, S'_u)]$ and $T^2 = G[(S_{-u_d}, S'_{u_d})]$.

When $d = 4$, for any $x \in T_2$, $d_{T^1}(u, x) = d_T(u, x)$ and $d_{T^2}(u_4, x) = d_T(u_4, x)$. Therefore, we have

$$\Delta(S_{-u}, S'_u) = w(T_1) - w(T_3) - w_{u_4} \text{ and } \Delta(S_{-u_4}, S'_{u_4}) = w(T_3) - w(T_1) - w_u.$$

Thus $\Delta(S_{-u}, S'_u) + \Delta(S_{-u_4}, S'_{u_4}) = -w_u - w_{u_4} < 0$ and one of the two players has an incentive to deviate.

When $d = 5$, we have $\Delta(S_{-u}, S'_u) = w(T_1) + w_{u_2} - w_{u_3} - w(T_4) - w_{u_5}$ and $\Delta(S_{-u_5}, S'_{u_5}) = w(T_4) + w_{u_3} - w_u - w(T_1) - w_{u_2}$. Therefore we have that $\Delta(S_{-u}, S'_u) + \Delta(S_{-u_5}, S'_{u_5}) = -w_u - w_{u_5} < 0$ and one of the two players has an incentive to deviate. \square

We need to prove first an auxiliary result.

Lemma 6. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a NE graph of Γ . If there is $v \in V$ with $\text{diam}_G(v) \leq \beta - 1$, then $W(G, \beta) \leq \alpha(n - 1)$.*

Proof. Let $S \in \text{NE}(\Gamma)$ and let $G = G[S]$. Let $u \in V$, $u \neq v$. If $v \notin S_u$, $\Delta(S_{-u}, S_u \cup \{v\}) \geq \alpha - \sum_{\{x | d_G(u, x) > \beta\}} w_x \geq 0$. But, if $v \in S_u$, $\text{diam}(u) \leq \beta$.

Hence, $\alpha \geq \sum_{\{x | d_G(u, x) > \beta\}} w_x$ and summing over all $u \neq v$ we have that $\alpha(n - 1) \geq W(G, \beta)$. \square

The proof of the upper bound for the PoA on NE trees uses the previous statements and examines the particular cases $\beta = 2, 3$.

Theorem 3. *The PoA on NE trees of a star celebrity game is at most 2.*

Proof. Let T be a NE tree of Γ . From Proposition 9 we have a bound on the diameter, so we know that $\text{diam}(T) \leq \beta + 1$. Since T is a tree, we have that there exists $u \in V$ such that $\text{diam}(u) \leq (\text{diam}(T) + 1)/2 \leq \beta/2 + 1$. If $\beta \geq 4$, then $\text{diam}(u) \leq \beta - 1$. By Lemma 6, $C(T) \leq 2\alpha(n - 1)$. Hence, the PoA of NE trees of Γ is at most 2 for $\beta \geq 4$.

In the case of $\beta = 3$, either $\text{diam}(T) \leq 3$ or $\text{diam}(T) = 4$. In the first case $C(T) = \alpha(n - 1)$ and in the second there is u with $\text{diam}_T(u) = 2 = \beta - 1$ and we can use Lemma 6.

Finally, we consider the case $\beta = 2$. Notice that the unique tree T with diameter 3 is a double star, a graph that is formed by connecting the centers of two stars. Assume that a NE tree T is formed by T_u , a star with center u , and T_v , a star graph with center v , joined by the edge (u, v) . Let L_u (L_v) be the set of leaves in T_u (T_v). As T is a NE graph we have that $w(L_u), w(L_v) \leq \alpha$. Furthermore

$$\begin{aligned} C(T) &= \alpha(n-1) + \sum_{w \in L_u} w(L_v) + \sum_{w \in L_v} w(L_u) \leq \alpha(n-1) + \sum_{w \in L_u} \alpha + \sum_{w \in L_v} \alpha \\ &\leq \alpha(n-1) + \alpha(n-2) \leq 2\alpha(n-1). \end{aligned}$$

□

Note that in a NE tree T , if $\alpha > w_{max}$, for an edge $\{u, v\}$ connecting a leaf u , it must be the case that $v \in S_u$. Then, in the proof of Proposition 9, we only have the case $u_1 \in S_u$. In such case $diam(T) \leq \beta$. Hence, if $\alpha > w_{max}$, the PoA on NE trees is 1.

Corollary 4. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game such that $\alpha > w_{max}$. For any NE tree of Γ , $diam(T) \leq \beta$ and therefore the PoA on NE trees is 1.*

To tighten the upper bound let us analyze the properties of the NE trees with diameter $\beta + 1$

Lemma 7. *Let T be a NE tree of $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ having $diam(T) = \beta + 1$, for some $\beta \geq 3$. Let $P = u, u_1, \dots, u_\beta, v$ be a diametral path in T and let S be a NE such that $T = G[S]$. We have that*

1. u and v are leaves of T .
2. $S_u = S_v = \emptyset$.
3. $w(u) = w(v) = \alpha$.
4. P is the unique diametral path in T .

Proof. Statement 1 follows from the fact that T is a tree with diameter $\beta + 1$.

To prove the second statement, assume that $S_u \neq \emptyset$. As u is a leaf it must be the case that $S_u = \{u_1\}$. Consider the strategy $S'_u = \{u_2\}$. Taking into account that $d_T(u_2, v) = \beta - 1$ and that the tree rooted at u_2 after deleting (u, u_1) and (u_2, u_3) has depth at most 2, we have that $\Delta(S_{-u}, S'_u) \leq -w(v) < 0$. Contradicting the fact that T is a NE tree. A symmetric argument shows that $S_v = \emptyset$.

To prove the third statement we consider two cases.

Case 1: $w(u) > \alpha$. Let $S'_v = \{u_1\}$, then $\Delta(S_{-v}, S'_v) \leq \alpha - w(u) < 0$. Thus T could not be a NE.

Case 2: $w(u) < \alpha$. By 2 we know that $S_u = S_v = \emptyset$, therefore $u \in S_{u_1}$. Taking $S'_{u_1} = S_{u_1} \setminus \{u_1\}$ we have $\Delta(S_{-u_1}, S'_{u_1}) \leq w(u) - \alpha < 0$. Again S could not be a NE.

We conclude that $w(u) = \alpha$. A symmetric argument shows that $w(v) = \alpha$.

To prove the last statement assume that T has two diametral paths with length $\beta+1$. Let u, v, u', v' be four vertexes such that $d(u, v) = d(u', v') = \beta+1$. We consider two cases.

Case 1: the four vertexes are different. Let P be the shortest path from u to v and P' the shortest path from u' to v' . Let us first show that P and P' must share at least one point. Otherwise let y be the vertex in P that is closest to P' and let x be the vertex in P' that is closest to y . By construction P' lies in the subtree rooted at y after removing the edges in P , thus $d(y, x) > 0$. Therefore, $\max\{d(u, y), d(y, v)\} + d(x, y) + \max\{d(u', x), d(x, v')\} > \beta + 1$. Contradicting the fact that T has diameter $\beta + 1$.

Thus P and P' share at least one point. Let $x(y)$ be the vertex common to P and P' that is closer to $u(v)$. If there is only one common point $x = y$. Observe that when $x = y$ it must happen that x is the central point of both paths, that is $\beta + 1$ must be even and $d(u, x) = d(v, x) = d(u', x) = d(v', x) = (\beta + 1)/2$. When $x \neq y$ assume without loss of generality that u' is the vertex in the subtree rooted at x after removing P . In such a case, $d(u', x) = d(u, x) \leq (\beta + 1)/2$ and $d(v, y) = d(v', y) \leq (\beta + 1)/2$ as otherwise the tree will not have diameter $\beta + 1$. Thus $d(u, v') = \beta + 1$. By 2 we know that $S_u = \emptyset$ and by 3 that $w(v) = w(v') = \alpha$. Consider the strategy profile, $S'_u = \{y\}$. We have that $\delta(S_{-u}, S'_u) \leq \alpha - w(v) - w(v') < 0$. Therefore T cannot be a NE.

Case 2: two vertexes are the same. Without loss of generality assume that $u' = u$. Let y be the branching point of the paths from u to v and u to v' . As in the previous case, we have that $d(y, v) = d(y, v') \leq (\beta + 1)/2$. Considering $S'_u = \{y\}$ we have again that $\delta(S_{-u}, S'_u) \leq \alpha - w(v) - w(v') < 0$. Therefore T cannot be a NE.

We conclude that there are only two vertexes at distance $\beta + 1$ in T . \square

Putting all together we get an upper bound on the PoA on NE trees when $\beta \neq 2$.

Theorem 4. *The PoA on NE trees of a star celebrity game with $\beta \geq 3$ and n players is at most $1 + \frac{2}{n-1}$.*

Proof. For NE trees with diameter $\leq \beta$ the social cost is $\alpha(n - 1)$ but for NE with diameter $\beta + 1$, by Lemma 7, the social cost is $\alpha(n - 1) + 2\alpha$. As a star is an optimal graph with social cost $\alpha(n - 1)$ the claim follows. \square

For the case $\beta = 2$ it remains to analyze whether a double star can be a NE for a star celebrity game.

Lemma 8. *Let T be a NE tree of a star celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ let $\beta = 2$. There is no NE tree for Γ with diameter 3 except when $|V| = 4$ and at least two players have weight α .*

Proof. Assume that a double star T is formed by two stars T_u and T_v with centers u and v respectively and the edge $\{u, v\}$. Let L_u (L_v) be the set of leaves in T_u (T_v). Let S be a NE so that $T = G[S]$. As T is a NE we know that

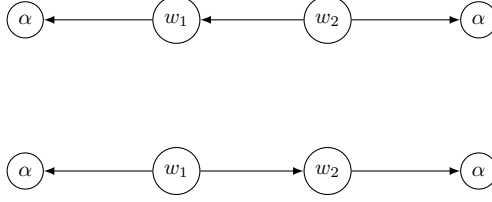


Figure 2: The NE trees with diameter 3

$w(L_u), w(L_v) \leq \alpha$, otherwise by connecting a leaf to the other center their cost will decrease.

Assume that $|L_u|, |L_v| \geq 2$. We have that, for a leaf x , $w(x) < \alpha$. So, $u \in S_x$, for $x \in L_u$, and $v \in S_y$, for $y \in L_v$. Otherwise, u (or v) would benefit by disconnecting to their leaves. For any leaf $x \in L_u$ ($y \in L_v$), consider the strategy $S'_x = \{v\}$ ($S'_y = \{u\}$). For $x \in L_u$, we have $\Delta(S_{-x}, S'_x) = w(L_u) - w(x) - w(L_v) \geq 0$, that is $w(x) \leq w(L_u) - w(L_v)$. For $y \in L_v$, we have $\Delta(S_{-y}, S'_y) = w(L_v) - w(y) - w(L_u) \geq 0$, thus $w(y) \leq w(L_v) - w(L_u)$. Which is impossible as the node weights are positive. Therefore $|L_u| = 1$ or $|L_v| = 1$.

Let us assume w.l.o.g that $L_u = \{x\}$. If $w(x) < \alpha$ and $x \in S_u$, $\Delta(S_u, \emptyset) = w(x) - \alpha < 0$, which is not possible. Therefore, $u \in S_x$. But in such a case $\Delta(S_{-x}, \{v\}) = -w(L_v) < 0$. So, $w(x) = \alpha$.

If $|L_v| > 1$, let $y \in L_v$. As $w(L_v) \leq \alpha$ and $w(y) > 0$, we have $w(y) < \alpha$. Therefore, $v \in S_y$, but then

$$\Delta(S_{-y}, \{u\}) = -w(x) + w(L_v) - w(y) = -\alpha + w(L_v) - w(y) < 0.$$

Contradicting that S is a NE. Thus, $L_v = \{y\}$ and, as for the case $L_u = \{x\}$, we can conclude that $w(y) = \alpha$.

The unique graph satisfying all conditions is a path on 4 vertexes. Furthermore the leaf nodes must have weight α and there are no restrictions for the weights of the internal vertexes. It is easy to see that the unique orientations producing a NE in this particular case are the ones depicted in Figure 2. \square

Theorem 5. *The PoA on NE trees of star celebrity games is $\leq 5/3$ and there are games for which a NE tree has cost $5 \text{opt}/3$.*

Proof. For $\beta \geq 3$ and $n \geq 4$, the PoA on NE trees is at most $1 + \frac{2}{n-1} \leq 5/3$, by Theorem 4. For $\beta \geq 3$ and $n < 4$, all trees have diameter at most β , so the PoA on NE trees is 1. For $\beta = 2$ according to Lemma 8 all NE trees have diameter at most β except for P_3 in some cases. When P_3 is a NE we have that $C(P_3) = 3\alpha + 2\alpha = 5\alpha$, giving the upper bound. As there are games for which P_3 is a NE (see Figure 2), the claim follows. \square

8. Celebrity games for $\beta = 1$

Let us now analyze the case $\beta = 1$. Observe that every player u for each non-adjacent node v pays w_v , and for each adjacent node either α if he has bought the link, or 0, otherwise. Notice that if u establishes the link (u, v) , only the node v will take profit of this decision. Contrasting with this, when $\beta > 1$, if player u pays a new link, then all the nodes that get closer to u but not farther than β , will take advantage of this new link.

This particular behavior allows us to show that computing a best response becomes a tractable problem. Furthermore, the structure of NE and OPT graphs is quite different from the case of $\beta > 1$ and we can obtain a tight bound for the PoA.

Proposition 10. *The problem of computing a best response of a player to a strategy profile in celebrity games is polynomial time solvable when $\beta = 1$.*

Proof. Let S be a strategy profile of $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1 \rangle$ and let $u \in V$. Consider another strategy profile $S' = (S_{-u}, S'_u)$, for some $S'_u \subseteq V \setminus \{u\}$. As $\beta = 1$ we have

$$c_u(S') = \alpha |S'_u| + \sum_{v \notin S'_u} w_v.$$

Note that, when $|S'_u| = k$, the first component of the cost is the same and thus a best response on strategies with k players can be obtained by taking from S'_u the players with the k -th highest weights. Let $S'_u(k)$ be the set of those players and let $W_k = W - w(S'_u(k))$. Thus $c_u(S_{-u}, S'_u(k)) = \alpha k + W_k$. To obtain a best response it is enough to compute the value k for which $c_u((S_{-u}, S'_u(k)))$ is minimum and output $S'_u(k)$. Observe that the overall computation can be performed in polynomial time. \square

In order to show a bound for the PoA we prove first some auxiliary results. When $\beta = 1$ pairs of vertexes at distance bigger than one correspond to pairs of vertexes that are not connected by an edge and such a property does not hold for higher values of β .

Proposition 11. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1 \rangle$ be a celebrity game. If $G = (V, E)$ is a NE graph of Γ , for each $u, v \in V$,*

- *if either $w_u > \alpha$ or $w_v > \alpha$ then $\{u, v\} \in E$,*
- *if both $w_u < \alpha$ and $w_v < \alpha$ then $\{u, v\} \notin E$,*
- *otherwise the edge $\{u, v\}$ might or might not belong to E .*

Proof. Let S be a NE and let $G = G[S] = (V, E)$. Observe that due to the fact that $\beta = 1$, for any player u ,

$$c_u(S) = \alpha |S_u| + \sum_{\{v | v \neq u, \{u, v\} \notin E\}} w_v.$$

The cost is thus expressed in terms of the existence or non existence of a connection between pairs of nodes and thus the strategy can be analyzed considering only deviations in which a single edge is added or removed. We analyze the different cases for players u and v .

Case 1: $w_u > \alpha$. For any player $v \neq u$, if the edge $\{u, v\}$ is not present in G the graph cannot be a NE graph as v improves its cost by connecting to u . For the same reason, if the edge is present either $u \in S_v$ or $v \in S_u$. The latter case $v \in S_u$, can happen only when $w_v > \alpha$. Therefore, the player that is paying for the connection will not obtain any benefit by deviating.

Case 2: $w_u, w_v < \alpha$. If the edge $\{u, v\}$ is present in G the graph cannot be a NE graph as the player establishing the connection improves its cost by removing the connection to the other player. For the same reason, if the edge is not present none of the players will obtain any benefit by deviating and paying for the connection.

Case 3: $w_u, w_v = \alpha$. The cost, for any of the players, of establishing the connection or not is the same. In consequence the edge can or cannot be in a NE graph.

Case 4: $w_u = \alpha$ and $w_v < \alpha$. Player v is indifferent to be or not to be connected to u , but player u in a NE will never include v in its strategy. Observe that again the edge can or cannot exist in a NE graph but, if it exists, it can only be the case that $u \in S_v$. \square

Let us analyze now the structure of the OPT graphs.

Proposition 12. *Let $G = (V, E)$ be a OPT graph of a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1 \rangle$. For any $u, v \in V$, we have*

- if $w_u + w_v < \alpha$ then $\{u, v\} \notin E$,
- if $w_u + w_v > \alpha$ then $\{u, v\} \in E$,
- if $w_u + w_v = \alpha$ then $\{u, v\}$ might or not be an edge in G .

Proof. Let S be a strategy profile and let $G = G[S] = (V, E)$ be an OPT graph. As we have seen before as $\beta = 1$, for any player u ,

$$c_u(S) = \alpha|S_u| + \sum_{\{v|v \neq u, \{u,v\} \notin E\}} w_v,$$

and we get the following expression for the social cost

$$C(G) = \alpha|E| + \sum_{\{u,v|u < v, \{u,v\} \notin E\}} (w_u + w_v).$$

The above expression shows that to minimize the contribution to the cost, an edge $\{u, v\}$ can be present in the graph only if $w_u + w_v \geq \alpha$ and will appear for sure only when $w_u + w_v > \alpha$. Thus the claim follows. \square

From the previous characterizations we can derive a constant upper bound for the price of anarchy when $\beta = 1$.

Theorem 6. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1 \rangle$ be a celebrity game. $PoA(\Gamma) \leq 2$. Furthermore the, ratio among the social cost of the best and the worst NE graphs of Γ is bounded by 2.*

Proof. $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1 \rangle$. Observe that due to the conditions given in Propositions 11 and 12 the social cost of an OPT graph is

$$\sum_{\{\{u,v\} | w_u + w_v \geq \alpha\}} \alpha + \sum_{\{\{u,v\} | w_u + w_v < \alpha\}} (w_u + w_v),$$

and the social cost of a NE graph with minimum number of edges, i.e., one in which all the optional are not present, is at most

$$\begin{aligned} & \sum_{\{\{u,v\} | w_u > \alpha \text{ or } w_v > \alpha\}} \alpha + \sum_{\{\{u,v\} | w_u, w_v \leq \alpha\}} (w_u + w_v) = \\ & = \sum_{\{\{u,v\} | w_u > \alpha \text{ or } w_v > \alpha\}} \alpha \\ & + \sum_{\{\{u,v\} | w_u, w_v \leq \alpha \text{ and } w_u + w_v = \alpha\}} \alpha \\ & + \sum_{\{\{u,v\} | w_u, w_v \leq \alpha \text{ and } w_u + w_v < \alpha\}} (w_u + w_v) \\ & + \sum_{\{\{u,v\} | w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}} (w_u + w_v) \end{aligned}$$

Observe that the difference with the cost of an OPT graph is in the last term

$$D = \{\{u, v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}.$$

Notice that $\{u, v\} \in D$ contributes to the cost of an OPT graph with α and to the cost of a NE graph with $w_u + w_v$. By taking Γ with $w_u = \alpha$, for any $u \in V$, we can maximize the size of D and this leads to the worst possible NE graph. For such a Γ , I_n is a NE graph and we have that $C(I_n) = \sum_{u,v \in V, u < v} (w_u + w_v) = \alpha n(n-1)$. Furthermore, in any OPT graph of Γ , all the edges will be present, thus we have $OPT = \alpha n(n-1)/2$. Thus

$$PoA(\Gamma) \leq \frac{n(n-1)\alpha}{\alpha n(n-1)/2} = 2$$

Observe that when $w_u = \alpha$, for any u , the complete graph is also a NE graph and thus we have that the ratio between the social cost of the worst and the best NE graph is bounded by 2. \square

Observe that when $\alpha < w_{min}$ the unique NE is a complete graph which is also an OPT graph. Taking into account that the relationship among celebrity games and MaxBD games provided in Proposition 8 also holds for $\beta = 1$ we can conclude.

Corollary 5. *For $\beta = 1$, the PoA and the PoS of MaxBD games and celebrity games with $\alpha < w_{min}$ is 1.*

9. Conclusions and Open Problems

We have introduced the celebrity games model aiming to address the creation of networks in a scenario where the nodes or players may have different weights and where the requirement of being close to a global critical distance has to be balanced against the node weights. Our results provide further understanding of the structural properties of stable networks. We have shown that the critical distance affects directly the diameter of the stable networks. For star celebrity games the diameter is $\leq 2\beta + 1$ and, in the case that the NE graph is not a tree, the diameter is $> \beta/2$. Furthermore, this critical distance, jointly with player weights and link establishment cost, have implications on the quality of the NE. We have shown that the PoA of star celebrity games is $O(\min\{n/\beta, W/\alpha\})$ and, for $\beta = 2$, we have found games whose PoA is $\Omega(n)$. In contra-position restricting the NE to be trees the PoA is constant.

We can observe that, as one can expect, enlarging the value of the critical distance improves the quality of equilibria. Furthermore, if the total game weight $W = O(\alpha)$, the PoA is $O(1)$. Corresponding to the intuition that when player's weights are negligible players prefer to be isolated. In contrast, when all the players are celebrities, even though their weights could be very different, players prefer to be closer, and the NE graphs have diameter $\leq \beta$. In this latter case, the upper bound on the PoA obtained in Bilò et al. (2015b) for MaxBD games ameliorates the upper bound of celebrity games.

It still remains open to shorten the gap between the lower and upper bounds on the PoA. Our results are only tight for $\beta = 1$ and $\beta = 2$. The cases where β is constant are of particular interest. In the family of graphs providing the lower bound on the PoA not all the nodes are celebrities, so our result has no implication for MaxBD games.

Further questions of interest are to study natural variations of our framework. Among the many possibilities, we propose to analyze celebrity games under (i) the Max cost model (work in progress), (ii) other definitions of the social cost.

Finally, we have not considered the non uniform version where each player u can have its own critical distance β_u . Bilò et al. (2015b) showed that the PoA of MaxBD game is $\Omega(n)$ even for the non uniform model with only two distance-bound values. As we have mentioned before such a negative result for MaxBD games translates to the celebrity games when all the players are celebrities.

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