A normal form theorem for integrable systems on contact manifolds

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Abstract. We present a normal form theorem for singular integrable systems on contact manifolds.

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1. Introduction

The theorem of Darboux is probably the first normal form theorem in symplectic geometry. This theorem has its analogue in contact geometry. Normal forms let us reduce the study of our system to model-like situations in which the computations are simplified. However the theorem of Darboux is local and does not take into account additional geometrical structures on the manifold. In this paper we review some normal forms results for integrable systems on symplectic manifolds and find an application to study normal forms for the contact analogous situation.

2. Completely integrable Hamiltonian systems on a symplectic manifold

We consider a completely integrable Hamiltonian system on a symplectic manifold $(M, \omega)$. It is given by a moment map $F = (f_1, \ldots, f_n)$. The condition $\{f_i, f_j\} = 0, \forall i, j$ implies that the distribution generated by the Hamiltonian
vector fields \( X_f \) is involutive. We denote this foliation by \( \mathcal{F} \). This foliation has Lagrangian regular orbits and isotropic singular ones. A natural question in this situation arises: Can we find a classification theorem for completely integrable Hamiltonian systems in a neighbourhood of an orbit \( L \)?

In the case \( L \) is a regular compact orbit, the theorem of Liouville-Mineur-Arnold for integrable systems gives a positive answer to this question. The existence of action-angle coordinates in a neighbourhood of \( L \) implies that the completely integrable system is equivalent in a neighbourhood of \( L \) to the completely integrable system determined by the action functions and the Darboux symplectic form.

If \( L \) is singular the problem remains unsolved in general. In the case the orbits of the integrable system are compact and \( L \) is a singular nondegenerate orbit, the answer is given by the following theorem due to the author and Nguyen Tien Zung:

**Theorem 1** (Miranda, Nguyen Tien Zung [11]). *Under the hypotheses mentioned above, the completely integrable Hamiltonian system is symplectically equivalent to the linearized integrable Hamiltonian system with the Darboux symplectic form. In the case there exists a symplectic action of a compact Lie group \( G \) preserving the system, this equivalence can be established in a \( G \)-equivariant way.*

**Remarks.**

1) In the case \( \dim L = 0 \) and \( L \) is nondegenerate Eliasson ([4], [5]) established local linear models for the singularity and provided a complete proof for the symplectic equivalence with the linear model in any dimension in the completely elliptic case.

2) Details of Eliasson’s proof have been recently clarified by the author and Vu Ngoc San in [12]. In [10] the author provided a complete proof of Eliasson’s result in cases other than elliptic. This proof uses a generalization of the Morse Isochore lemma and Moser’s path method for foliations to achieve a symplectically orthogonal decomposition into 2 and 4-dimensional cells depending on the Williamson type of the singularity.

3) This result generalizes previous partial results for nondegenerate compact singular orbits of rank greater than 0. In particular it generalizes the result of Eliasson ([5]) in the case the orbits are of completely elliptic type. It also generalizes the results of Colin de Verdière and San Vu Ngoc ([2]) and Currás-Bosch and Miranda ([3]) in the case \( \dim M = 4 \) and \( \dim L = 1 \).
3. Integrable systems on contact manifolds

The aim of this section is to present an analogue to the linearization result for singular integrable Hamiltonian systems stated but in the case of singular integrable system in contact manifolds.

Consider a contact manifold $M^{2n+1}$ together with a contact form. We assume that the Reeb vector field associated to $\alpha$ coincides with the infinitesimal generator of an $S^1$ action. We assume further that there exists $n$-first integrals of the Reeb vector field which commute with respect to the Jacobi bracket. Then there are two foliations naturally attached to the situation. On the one hand, we can consider the foliation associated to the distribution generated by the contact vector fields. We call this foliation $\mathcal{F}'$. On the other hand, we can consider a foliation $\mathcal{F}$ given by the horizontal parts of the contact vector fields. The functions determining the contact vector fields may have singularities. We will always assume that those singularities are of non-degenerate type. Observe that $\mathcal{F}'$ is nothing but the enlarged foliation determined by the foliation $\mathcal{F}$ and the Reeb vector field.

Let $\alpha'$ be another contact form in a neighbourhood of a compact orbit $O$ of $\mathcal{F}'$ for which $\mathcal{F}$ is generically Legendrian and such that the Reeb vector field with respect to $\alpha'$ coincides with the Reeb vector field associated to $\alpha$. A natural question is to know if $\alpha$ is equivalent to $\alpha'$. This entails naturally the study of the existence of normal forms for $\alpha$ in a neighbourhood of $O$ preserving the foliation $\mathcal{F}$.

The condition that the Reeb vector field is an infinitesimal generator of an $S^1$-action is fulfilled in many examples present in contact geometry. For instance, model contact structures for a transverse knot can be obtained by considering contact forms satisfying this condition (see for example [6]). Furthermore, as proved in [7], a contact form whose Reeb flow generates a torus action is “stable” in the sense that the Reeb flow of any $C^2$-close contact form has at least one periodic orbit.

The problem of determining normal forms for foliations related to Legendrian foliations has its own story. P. Libermann in [8] established a local equivalence theorem for $\alpha$-regular foliations. Loosely speaking, those foliations are regular foliations containing the Reeb vector field and a Legendrian foliation. The problem of classifying contact structures which are invariant under a Lie group was considered by Lutz in [9]. The foliations studied by Libermann and Lutz are regular. The singular counterpart to the result of Lutz was proved by Banyaga and Molino in [1] but for contact forms. Namely, Banyaga and Molino study the problem of finding normal forms under the additional assumption of transversal ellipticity. The assumption of transversal ellipticity
allows to relate the foliation $\mathcal{F}'$ of generic dimension $(n+1)$ with the foliation given by the orbits of a torus action.

The results that we present here and whose proof is contained in [10] pretend to extend these results for foliations which are related in the same sense to foliations with generical $(n+1)$-dimensional leaves but which are not necessarily identified with the orbits of a torus action. All our study of the problem is done in a neighbourhood of a compact singular orbit.

The linear model for the contact setting

Let $(M^{2n+1}, \alpha)$ be a contact pair and let $Z$ be its Reeb vector field. We assume that $Z$ coincides with the infinitesimal generator of an $S^1$ action. We also assume that there are $n$ first integrals $f_1, \ldots, f_n$ of $Z$ which are generically independent and which are pairwise in involution with respect to the Jacobi bracket associated to $\alpha$. Let $O$ be the orbit of the foliation $\mathcal{F}'$ through a point $p$ in $M^{2n+1}$. We will assume that $O$ is diffeomorphic to a torus of dimension $k+1$ and that the singularity is non-degenerate in the Morse-Bott sense along $O$.

In [10] it is proven that there exists a diffeomorphism from a neighbourhood of $O$ to a model manifold $M_0^{2n+1}$ taking the foliation $\mathcal{F}'$ to a linear foliation in the model manifold with a finite group attached to it and taking the initial contact form to the Darboux contact form.

Theorem 2 (Miranda [10]).
There exist coordinates $(\theta_0, \ldots, \theta_k, p_1, \ldots, p_k, x_1, y_1, \ldots, x_{n-k}, y_{n-k})$ in a finite covering of a tubular neighbourhood of $O$ such that,

- The Reeb vector field is $Z = \frac{\partial}{\partial \theta_0}$.
- There exists a triple of natural numbers $(k_e, k_h, k_f)$ with $k_e + k_h + 2k_f = n - k$ and such that the first integrals $f_i$ are of the following type, $f_i = p_i, \ 1 \leq i \leq k$, and

  \[
  f_{i+k} = x_i^2 + y_i^2 \quad \text{for} \quad 1 \leq i \leq k_e,
  
  f_{i+k} = x_i y_i \quad \text{for} \quad k_e + 1 \leq i \leq k_e + k_h,
  
  f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \quad \text{and}
  
  f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f
  
- The foliation $\mathcal{F}$ is given by the orbits of $D = \langle Y_1, \ldots, Y_n \rangle$ where $Y_i = X_i - f_i Z$ being $X_i$ the contact vector field of $f_i$ with respect to the contact form $\alpha_0 = \theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$. 

The model manifold is the manifold $M^{2n+1}_0 = \mathbb{T}^{k+1} \times U^k \times V^{2(n-k)}$, where $U^k$ and $V^{2(n-k)}$ are $k$-dimensional and $2(n-k)$ dimensional disks respectively endowed with coordinates $(\theta_0, \ldots, \theta_k)$ on $\mathbb{T}^{k+1}$, $(p_1, \ldots, p_k)$ on $U^k$ and $(x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-k})$ on $V^{2(n-k)}$. The linear model for the foliation $\mathcal{F}'$ is the foliation expressed in the coordinates provided by the theorem together with a finite group attached to the finite covering. The different smooth sub-models corresponding to the model manifold $M^{2n+1}_0$ are labeled by a finite group which acts in a contact fashion and preserves the foliation in the model manifold. This is the only differentiable invariant. In fact, this finite group comes from the isotropy group of an associated Hamiltonian action. In the symplectic case this finite group was already introduced in [13].

Contact equivalence in the model manifold

**Theorem 3** (Miranda [10]). Let $\alpha$ be a contact form on the model manifold $M^{2n+1}_0$ for which $\mathcal{F}$ is a generically Legendrian foliation and such that the Reeb vector field is $\frac{\partial}{\partial \theta_0}$. Then there exists a diffeomorphism $\phi$ defined in a neighbourhood of the singular orbit $O = (\theta_0, \ldots, \theta_k, 0, \ldots, 0)$ preserving $\mathcal{F}'$ and taking $\alpha$ to $\alpha_0$.

The $G$-equivariant result

Consider a compact Lie group $G$ acting on contact model manifold in such a way that preserves the $n$ first integrals of the Reeb vector field and preserves the contact form as well. In [10] we prove that there exists a diffeomorphism in a neighbourhood of $O$ preserving the $n$ first integrals, preserving the contact form and linearizing the action of the group. Namely we prove,

**Theorem 4** (Miranda [10]). There exists a diffeomorphism $\phi$, preserving $F = (f_1, \ldots, f_n)$ defined in a tubular neighbourhood of $O$ such that $\phi^*(\alpha_0) = \alpha_0$ and such that $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$, being $\rho_g^{(1)}$ the linearization of $\rho_g$.

Contact linearization

Applying this $G$-equivariant version to the particular case of the finite group attached to the finite covering, we obtain as a consequence the following contact linearization result:

**Theorem 5** (Miranda [10]). Let $\alpha$ be a contact form for which $\mathcal{F}$ is generically Legendrian and such that $Z$ is the Reeb vector field then there exists a diffeomorphism defined in a neighbourhood of $O$ taking $\mathcal{F}'$ to the linear foliation, the orbit $O$ to the torus $\{x_i = 0, y_i = 0, p_i = 0\}$ and taking the contact form to the Darboux contact form $\alpha_0$. 
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References


