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N -sided Patches with B-Spline boundaries

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Abstract

We present a method to construct a patch of parametric surface of degree $k + 1$ that fills an n -sided hole, with n bigger than 2, and whose boundary coincides with a B-Spline, thus, the resulting patch can be easily connected with given B-Spline surfaces with fixed continuity conditions.

The method is based on the generic approach by the same authors to construct free form surfaces, which gives a family of practical schemes to design surfaces from an arbitrary given mesh, using the differentiable manifold theory.

The proposal uses a star shaped mesh which describes a generic n -hole and a surface in a neighborhood of the hole. From this mesh, a set of charts is defined, one associated to each vertex or face of the mesh, depending on the input parameter k . A basis function and a control point is defined from each chart, and the surface is obtained as a baricentric combination of the control points using the defined basis functions. The main advantages of the method are the following: arbitrary order k continuity conditions can be imposed; the involved hole can have an arbitrary number of sides and arbitrary shape (convex or not); the simplicity of the construction process gives an easy and flexible method; and finally, the surface near the boundary is a B-Spline with piecewise uniform knot sequences and whose control points are vertices of the given mesh. Implementation details to evaluate a surface point are given, showing that the de Boor algorithm can be exploited for efficiency.

1 Introduction

The representation of free form surfaces is a very important topic in Computer Graphics. Generally, the surfaces are defined by an assembly of rectangular patches. However in some cases, rectangular patches leave some n -sided holes. Many methods have been proposed for constructing non-four-sided patches (see [7, 5, 10, 4, 11, 8], among others). Overviews of these methods can be found in [9] and [6].

Different versions of the problem are treated in the existing literature. In some cases, given n patches of a specific class the hole has to be filled, whereas in others, the construction of arbitrary n -sided patches is the main goal of the work. The existing methods for constructing n -sided patches can be classified into recursive subdivision, surface splitting, data blending and control point schemes.

The method presented in this paper can be classified among the control point schemes. It is a fully automatic method for easily obtaining an order k continuous surface from a control mesh that surrounds a hole with n sides, the parameters $n > 2$ and $k \geq 0$ being arbitrary integers. Since it is a baricentric method, the resulting surface will approximate the input mesh both topologically and geometrically. Besides, the method allows some degrees of freedom for two reasons: first, it supplies a scalar value f , which can be varied between some limits, determining in some way the influence of the control points on the central part of the patch; and second, the construction allows additional layers of interior control points to shape the patch without losing the smooth connection in the boundaries. Furthermore, the boundary sides of the patch coincide with a tensor product B-Spline with known knot sequences and whose control points are vertices of the input mesh.

The rest of the paper is organized as follows: Section 2 gives an overview of the process followed to define the n -sided patches; Section 3 sets the notation for and describes the mesh configuration in both image and parametric spaces; Section 4 presents the different steps followed to define the surface of the patch; in Section 5 we describe how to smoothly connect a patch with a given tensor product patch using the property that the each boundary of a n -sided patch coincides with a B-spline; Section 6 gives an algorithm to evaluate any surface point given its parametric coordinates; in Section 7 several examples of n -sided patches are shown, and some questions concerning our proposal are discussed; finally, in Section 8 conclusions and some future work is given.

2 Overview

In this section we provide a general overview of the scheme proposed to construct n -sided patches. The scheme follows the theoretical approach presented by the

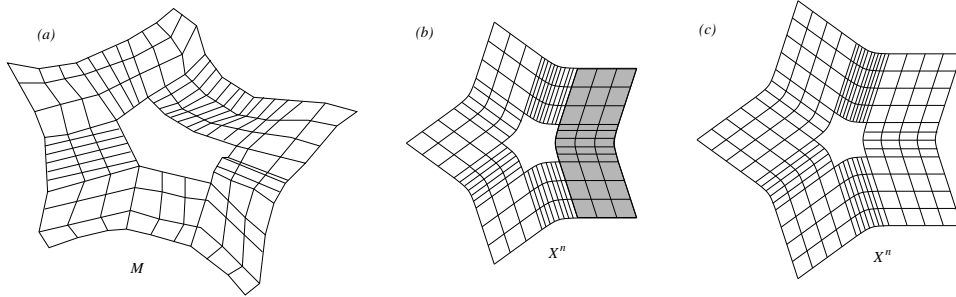


Figure 1: (a) Mesh defining a 5-sided hole. (b) The corresponding configuration of a 5-sided regular patch in parametric space for the case k is even. (c) The parametric space for the same case and odd k . The shaded part in (b) corresponds to one of the sides X_m .

same authors in [3]. The construction process is divided in several steps that we enumerate and briefly describe below.

The patch configuration. The patch configuration is determined by a control mesh, denoted by M , which is made of regular layers of mesh around a central n -sided face (see Figure 1(a)), thus the resulting patches are n -sided patches. This configuration provides a family of patches that can be used in many situations, specially to fill n -sided holes.

To easily connect the resulting patch with existing surfaces, the definition is made in such a way that near each boundary curve of the patch, it describes a tensorial product B-spline surface.

We point out later in Section 7 that this configuration for the regular n -sided patch is not the most general case, but this choice simplifies the definition of our n -sided patches. Later modifications, made to ensure a smooth connection with general B-Spline neighboring patches, can be done with minor modifications.

The parameter spaces. Since the n -sided patch configuration is not regular, we need to define a special parameter space to generate the surface. To do this we define a *regular* n -sided mesh X in \mathbb{R}^2 (see Figure 1(b)), and then we give a manifold structure to this set, that is, we construct a set of charts and transition functions that define the chart overlapping. The parameter space of X is denoted by $\mathcal{M}(X)$,

To simplify the definition process of the charts, mesh X is split into *sides*, and each side m is then deformed into a regular auxiliar mesh E_m . The auxiliary parameter space of a side m is referred as $\mathcal{M}(E_m)$.

The surface generation. To generate the surface we use a baricentric scheme, that is, a basis function is defined for each chart, and a control point, which is a vertex of the original mesh M , is associated to each chart. The surface points are then baricentric combinations of these control points averaged by the basis functions values.

3 The n -patch mesh configuration

The n -sided patch is defined from a control mesh M made of s' layers of vertices surrounding a polygonal hole with n sides. The layers are distributed regularly forming rectangular faces in the following way: the interior vertices have degree 4; the outer boundary vertices have degree 3, except for n that have degree 2; and the inner boundary vertices have degree 3, except for the n that have degree 4 (the ones that determine the n -sided hole). The numbers of degree-three inner boundary vertices between any pair of degree-four vertices is denoted by r_m' , for $m = 0, \dots, n - 1$. Figure 1(a) represents an example of a 5-sided patch with $s' = 3$ and $r_0' = 5, r_1' = 10, r_2' = 4, r_3' = 6, r_4' = 3$.

The parameters s' and k , that is, the number of regular layers and the surface continuity are related (but not restrictive). For a large continuity order, the number of regular layers must increase in a way that the *irregular* part of the n -patch (that is, the part of the surface filling the hole) is isolated from the boundary, ensuring that the generation of this part of the patch does not perturb the behavior of the surface near the boundary.

The described configuration M always determines a planar mesh X in \mathbb{R}^2 , called a *regular n -sided patch*. In the definition of X , two cases are distinguished, depending on the parity of the parameter k . This distinction is made to achieve B-splines surfaces in the boundaries of the patch, for any k value. Figure 1(b) shows the configuration of the 5-sided regular patch built from the mesh M in Figure 1(a) for the case k is even, and Figure 1(c) shows the configuration when k is odd.

The mesh X is a plane regular mesh, describing a continuous n -sided hole, with s layers of regular vertices surrounding it. The topology of X also depends on the parameters r_m , for $m = 0, \dots, n - 1$, defined similarly to the case of the mesh M . An additional scalar value $f > 0$, called the *aperture factor*, determines the size of the central face of X (the hole in parametric space).

Next, we accurately describe the construction of X for the even case and the odd case is treated at the end of this section.

When k is an even value, s coincides with s' and $r_m = r_m', \forall m$. Mesh vertices of X are defined making use of the symmetry of the construction: the whole star-shaped mesh is divided into n submeshes, called *sides* of the star and denoted by

X_0, \dots, X_{n-1} (see Figure 1(b)). These sides are not disjoint subsets, but X_m and X_{m+1} share vertices and faces. Since this is a circular division, from now on operations involving indices of the sides are considered to work modulo n . It is also useful to consider the representation of X_m in M , denoted by M_m .

We distinguish three parts in X_m : the submeshes L_m , L_{m-1} and C_m . Submeshes L_m and L_{m-1} are determined by $s \times s$ layers of regular mesh, and submesh C_m is determined by $s \times r_m$ layers of regular mesh, where r_m is the number of degree 3 inner boundary vertices between the two inner vertices of degree 4 in the m -th side of X (see Figure 2 for these notations). Notice that vertices belonging to two consecutive sides X_m and X_{m+1} have two different labels associated. Furthermore, labels for vertices in L_m submeshes depend on the side m it is taken (a submesh labelling should be rotated 90 degrees to get the appropriate labelling for the next side $m + 1$).

First of all we label the vertices of L_{m-1} . Bottom left vertex is labeled as l_{00}^{m-1} , and then, following a right-up order we label the rest of vertices as l_{ij}^{m-1} for $i = 0, \dots, s$ and $j = 0, \dots, s$. Vertices of C_m are also labeled from left to right and from bottom to top. Bottom left vertex is c_{00}^m , and the rest of vertices are c_{ij}^m for $i = 0, \dots, s$ and $j = 0, \dots, r_m$. Finally, vertices of L_m are labeled in the same way, l_{00}^m being the bottom left vertex and l_{ss}^m the upper right one. Notice that the upper vertex row of L_{m-1} coincides with the lower row of C_m , and that the upper row of C_m coincides with the lower row of L_m ; in fact, the vertices of these two rows are those in a side that have double labelling (see Figure 2).

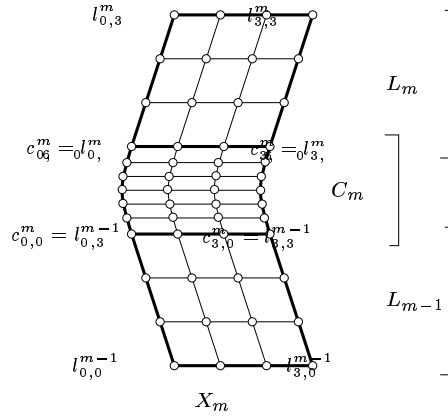


Figure 2: Notations for a side X_m of a 5-sided patch with $s = 3$, $r_m = 6$ and even k .

To set the coordinates of vertices in X_m , we first define the angle $\alpha = 2\pi/n$, and vectors $\vec{v}_m = (\cos(\alpha m), \sin(\alpha m))$ for $m = 0, \dots, n - 1$. Then, vertices

$l_{ij}^{m-1} \in L_{m-1}$, for $i = 0, \dots, s, j = 0, \dots, s$ and $m = 0, \dots, n-1$ have coordinates

$$l_{ij}^{m-1} = i\vec{v}_m - j\vec{v}_{m-1} + (\vec{v}_{m-1} + \vec{v}_m)f/2 \quad (1)$$

Vertices $l_{ij}^m \in L_m$ with indices $i = 0, \dots, s, j = 0, \dots, s$ and $m = 0, \dots, n-1$ have coordinates

$$l_{ij}^m = i\vec{v}_m + j\vec{v}_{m+1} + (\vec{v}_m + \vec{v}_{m+1})f/2 \quad (2)$$

Finally, coordinates of vertices in C_m , are set. In this case, we first place the set of vertices of C_0 , and then the rest of vertices for faces C_1, \dots, C_{n-1} are defined using the same equations and applying a rotation to them.

To achieve this, we must define the curve $\mu(t) : [-1, 1] \rightarrow \mathbb{R}^2$ joining the segments $\overline{l_{0s}^{-1}l_{00}^{-1}}$ and $\overline{l_{00}^0l_{0s}^0}$, with C^k continuity. This curve μ therefore fulfills:

$$\text{Endpoint interpolation: } \mu(-1) = l_{0s}^{-1} \text{ and } \mu(1) = l_{00}^0 \quad (3)$$

$$\text{Tangency: } \mu'(-1) = \vec{v}_{-1} \text{ and } \mu'(1) = \vec{v}_1 \quad (4)$$

$$\forall i, 2 \leq i \leq k, \quad \mu^{(i)}(-1) = \mu^{(i)}(1) = 0 \quad (5)$$

Then, vertices c_{ij}^m , for $i = 0, \dots, s, j = 0, \dots, r_m, m = 0$ are defined as

$$c_{ij}^m = \mu(t_j) + i\vec{v}_m \quad (6)$$

with increasing parameter values $-1 = t_s < t_{s+1} < \dots < t_{s+r_m} = 1$ such that the vertices c_{ij} for $j = s, \dots, s + r_m$ are equally separated vertically.

As stated, vertices for the rest of sides C_m for $m = 1, \dots, n-1$ are defined following equation (6) and applying a counterclockwise rotation of αm to them.

As it has been described, the topology of X in the case k is even coincides with that of M . For the case k is odd, mesh M is the dual graph of X (if neither the central face nor the outer unbounded face are not considered). This is accomplished by considering a regular n -sided patch just being described case but taking $s = s' + 1$ and $r_m = r_m' - 1, \forall m$ (see Figure 1(c)). In this way, for k even each vertex $v \in X$ has its image vertex $V \in M$, and in the odd case, each face in X has its corresponding vertex in M .

4 The C^k patch

To construct the n -sided patch we use the regular n -sided patch configuration as parameter space. The applied process follows [3] and is divided in several steps, that we detail in next subsections.

Before, let us define the bijective map F that transforms the side M_0 into a geometrically more simple shape in \mathbb{R}^2 (a rectangular mesh), denoted by E_0 (see Figure 3). This map allows us not only to work in an auxiliary simpler parametric space, which simplifies both the definition process and the implementation, but also to better understand the relationship between our spline scheme and usual B-splines.

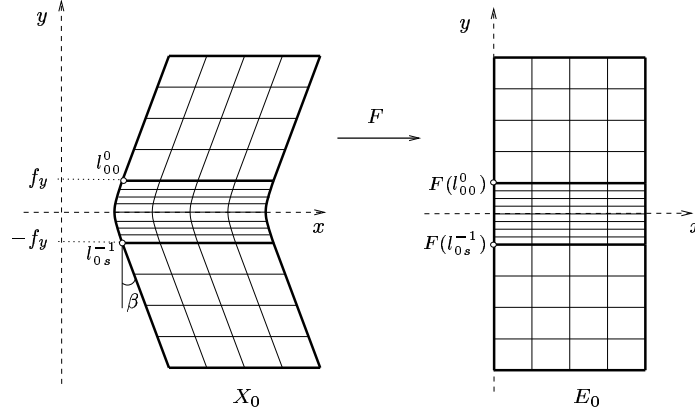


Figure 3: Mapping from $\mathcal{M}(X_0)$ to $\mathcal{M}(E_0)$.

The subset X_0 is the drag of a C^k curve, the left boundary of X_0 , along a horizontal straight line. This C^k curve is in fact composed of three parts, the curve μ and the segments $\overline{l_{00}^{-1}, l_{0s}^{-1}}$ and $\overline{l_{00}^0, l_{0s}^0}$. Let us call $\tilde{\mu}(y)$ the horizontal distance between a point (x, y) on the left boundary of X_0 and the vertical coordinate axis $x = 0$.

Then the C^k -continuous map F is defined as

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ (x, y) \rightarrow (x - \tilde{\mu}(y), y)$$

Thus, F transforms the mesh X_0 into an axe-aligned rectangular mesh, E_0 .

Notice that the inverse function F^{-1} maps a point (x, y) to $(x + \tilde{\mu}(y), y)$. The rest of sides E_i for $i = 1, \dots, n - 1$ can also be transformed into a rectangular shape by applying a clockwise rotation of $i\alpha$ to them followed by the map F just being defined.

4.1 The parameter space

We use X as parameter space. First we define a k -differentiable manifold structure on X , defining the set of charts and the family of transition functions.

4.1.1 The charts

We construct a set of charts satisfying the following properties: First, they are geometrically simple subsets of \mathbb{R}^2 , which is an important property since we use them as support for the basis functions. Second, each point inside X belongs to a minimum number of charts, which grants the smoothness of the resulting surface. And third, charts near the boundary of the patch coincide with the ones that would be obtained for a B-spline with piecewise uniform knot distribution (this last property will be formalized in Section 5).

Depending on the parity of k we will define a chart for each vertex, when k is even, and a chart for each face of X (except for the n -sided hole face), when k is odd. To simplify the exposition we only consider the submesh X_0 , and we define the charts that are associated to the vertices. The charts for the rest of sides are defined symmetrically.

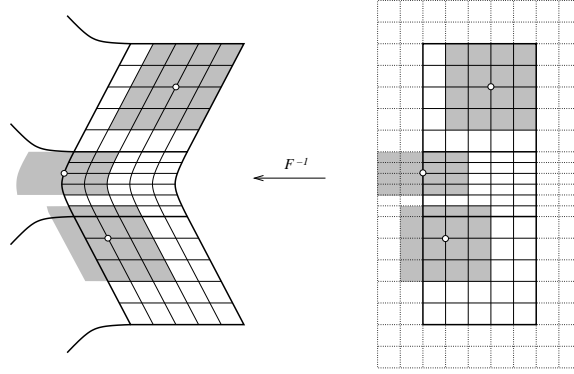


Figure 4: Charts definition process for the case they are centered on vertices ($k = 2$, $s = 5$, $r_m = 6$). Three different cases are represented.

To define the charts we work in the space $\mathcal{M}(E_0)$. Let $h = \lfloor k/2 \rfloor + 1$. First, h rectangular mesh layers are added to the four boundaries of E_0 , giving place to the extended mesh E_0^+ (see Figure 4). These auxiliary layers are built following the spacing between vertices of the mesh E_0 (see Figure 4).

Sets z_{ij}^m are defined to be the rectangles in E_0^+ that cover $(k+2) \times (k+2)$ mesh layers. There exists a z_{ij}^m set centered either on each vertex $e_{ij} \in E_0$ when k is even, or on each face $f_{ij} \in E_0$ when k is odd.

More formally, let e_{ij} for $i = -h, \dots, s+h$, $j = -h, \dots, 2s+r_m+h$ be the vertices of E_0^+ (labeled from bottom left to right top), and f_{ij} be the faces of E_0^+ such that f_{ij} has e_{ij} as lower left vertex. The sets z_{ij}^m are defined to be the rectangles from $e_{i-h-1, j-h-1}$ to $e_{i+k-h+1, j+k-h+1}$.

Then, charts \mathcal{C}_{ij}^m for $i = 0, \dots, s$ and $j = 0, \dots, 2s + r_m$ are defined as $F^{-1}(z_{ij}^m)$.

Notice that charts in $\mathcal{M}(X)$ have a relatively simple shape, since they are a sweep of a curve along a straight segment. Figure 4 shows three different examples of charts.

Following this construction, a chart is defined for every element (vertex or face, depending on the parity of k) of X_0 . Since consecutive sides X_i and X_{i+1} have elements in common, if the above definition was used for the rest of sides, some elements of the regular n -sided patch would have two charts attached. In order to obtain a B-spline surface near the boundaries of the patch, duplicity of charts have to be avoided. In particular, for each side X_m , the following charts have to be discarded (see Figure 5):

$$\text{charts } \mathcal{C}_{ij}^m \text{ such that } j \geq s + r_m + h \text{ and } i < h, \text{ and} \quad (7)$$

$$\text{charts } \mathcal{C}_{ij}^m \text{ such that } j < s - h. \quad (8)$$

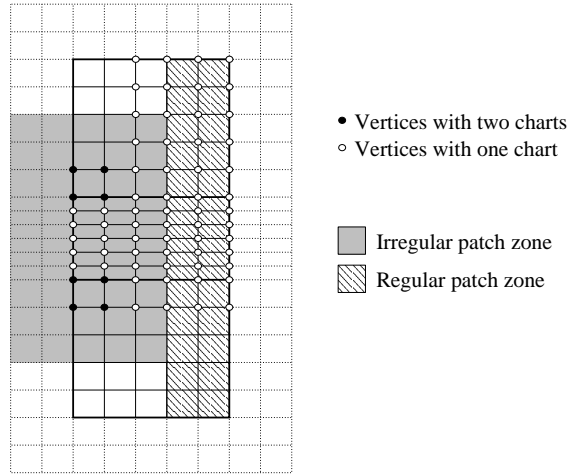


Figure 5: Vertices of a side E_m for which charts are to be considered are marked ($k = 2, s = 5, r_m = 6$), distinguishing between vertices with one chart associated and vertices with two charts associated. The regular patch zone Z_R and the irregular central part are also represented.

In this way, only some elements of X_m that are close to the hole of the n -sided patch have two charts associated (black vertices in Figure 5). Furthermore, the definition for charts of L_m close to the vertex boundaries of the patch is the same regardless on the side (m or $m + 1$) it is considered for defining them. In fact, these are the charts with rhombic shape in $\mathcal{M}(X)$ (for example, upper chart in Figure 4).

This definition associates a vertex or face of X to each chart, depending on the parity of k , and therefore each chart has a vertex of the original mesh M associated. Remark, however, that some of these mesh vertices will have two (different) charts associated to it

4.1.2 Transition functions

Following [3], a family of transition functions is constructed to define the overlapping between the charts, but in the actual situation all the charts are defined in the regular n -sided patch, so the overlapping between charts is yet defined. Thus we define the transition functions as the identity map in \mathbb{R}^2 .

4.2 Surface generation

Once the parameter space is defined, next step is to construct the surface, that is, the n -sided patch. First we define a basis function associated to each chart, in fact, with support in the chart. We also define a control point associated to each chart. Then we use a baricentric scheme to construct the surface.

4.2.1 Basis functions

In this section we construct a family of functions with the following properties. A function with support in each chart is defined. Moreover the family of functions determines a partition of the unity, that is, for each point in the parameter space the sum of the function values is equal to one.

The definition of map F in the space $\mathcal{M}(E_m)$ makes easy to define a basis function for each chart. Recall that each chart \mathcal{C} is the anti-image of a rectangular zone z_{ij}^m in $\mathcal{M}(E_m)$. Then, we define a tensorial product B-spline basis function on this rectangle, denoted by $N_{\mathcal{C}}^k$, using as knots the coordinates x and y of the vertices in E_m^+ .

However, recall that while the majority of elements have a unique chart associated, some of them have two (see Section 4.1.1). Using a baricentric scheme would cause these elements to have double influence on the shape of the surface. Although in principle this is not a theoretical problem, in practice one expects all the control points to shape the surface in an homogeneous way. Therefore, we define a normalizing factor $w_{\mathcal{C}}$ associated to each chart \mathcal{C} . For double association charts, $w_{\mathcal{C}} = 1/2$, and for the rest of charts $w_{\mathcal{C}} = 1$.

Since the overlapping of the charts do not follow exactly the overlapping that arise in a tensorial product B-spline scheme (in the irregular part of the patch),

we can not ensure that these functions determine a partition of the unity, so it is necessary to normalize them.

Therefore we define the basis functions as

$$\varphi_{\mathcal{C}}^k = \frac{w_{\mathcal{C}} N_{\mathcal{C}}^k}{\sum_{\mathcal{A} \in CS} w_{\mathcal{A}} N_{\mathcal{A}}^k}$$

where CS is the set of charts associated to elements in X .

4.2.2 Control points

We associate a control point to each chart. In Section 4.1.1 we point out that each chart is associated to a vertex or face of X , and since there is a one-to-one correspondence between elements of X and vertices of the control mesh M , a control point is associated to each chart. The control point associated to the chart \mathcal{C} is denoted by $P_{\mathcal{C}}$.

4.2.3 The surface

Finally the surface $S : \mathcal{M}(X) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined as follows. Each chart \mathcal{C} is associated to a control point $P_{\mathcal{C}}$ and to a basis function $\varphi_{\mathcal{C}}^k$, thus we can compute the image of a parameter point u as

$$S(u) = \sum_{\mathcal{C}} P_{\mathcal{C}} \varphi_{\mathcal{C}}^k(u).$$

Since this construction follows the scheme defined in [3], we can ensure that the surface has the following properties:

1. It is a k -differentiable surface.
2. Affine invariance property.
3. Local and global containment in the convex hull of the control mesh.
4. The computation of the manifold $\mathcal{M}(X)$ which is the most expensive step, only depends on the topology of the mesh and parameters f and k . Therefore, it can be performed only once, even if the position of the control points is modified.

5 B-spline boundaries

We have defined a n -sided patch as an independent patch, but in practical situations this patch must be smoothly connected to existing surfaces. There are several possible situations, but the simplicity of our proposal resides in the fact that the boundaries of the n -patch coincide with B-splines, as the following theorem announces. Thus, the conditions for connecting a n -sided patch with a given B-spline rectangular patch are analogous to the ones obtained to connect two B-spline patches.

Theorem 1. *If $s > k$ then each boundary m of the n -sided patch coincides with a B-spline of degree $k+1$ defined by some control points of M_m and knot distribution given by the vertex coordinates of E_m^+ .*

Being more precise, given a side m , the regular part of the patch corresponds to a region Z_R in parametric space that covers a vertical stripe of $s - k - 1$ mesh layers (see Figure 5). That is, the regular patch zone of a side m is the image $S(F^{-1}(Z_R))$ of the rectangle $Z_R \in \mathcal{M}(E_m)$ with extreme vertices $e_{0,k+1}$ and $e_{2s+m_k,s}$.

Proof. The proof for the above Theorem is based on the proof of Theorem 1 in [1], which states that any tensor product B-spline scheme can be obtained applying the generic scheme which is also the one used in the present paper. The construction of [1] for a regular mesh and the restriction of the proposed method to Z_R are the same.

We can ensure that Theorem 1 in [1] can be applied to the zone Z_R because:

1. Charts implied in the definition of $S(F^{-1}(Z_R))$ are not associated to elements with duplicate charts.
2. Since $s > k$, charts with non-empty intersection with Z_R are either charts associated to non-discarded vertices of E_m , or discarded charts associated to vertices in E_{m-1} whose definition coincides if side m is considered.

Thus, the definition for charts and control points for the portion of the surface we are considering is the same than the one for the B-spline case. \square

6 Implementation

6.1 Surface evaluation

In this section we detail how to compute a point on the surface given its parametric coordinates $p = (x, y)$. Following the definition and the algorithm given in [2],

the image point $S(p)$ is evaluated summing up the terms of each chart c_i such that $p \in c_i$, and then normalizing the result. However, remark that our proposal differs from that in [2] in that some elements have two charts associated, thus the normalization has to be performed in a slightly different way. The sum of basis functions is performed for each side M_m , traversing the charts associated to the mesh vertices in the spaces E_m . This results in the following algorithm, which given a point $p \in \mathcal{M}(X)$ computes its image point $S(p)$:

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( $Sx, Sy, Sz, D$ ) := (0, 0, 0, 0)
For  $m$  in  $[0..n - 1]$  do
   $p := Rotate(p, -\alpha * m)$ 
   $q := F(p)$ 
   $AddChartsOfSide(m, q, Sx, Sy, Sz, D)$ 
endFor
 $S(p) := (Sx/D, Sy/D, Sz/D)$ 

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The rotation followed with the map F projects the point p from $\mathcal{M}(X)$ to the parameter space of E_m . The core of the algorithm is thus the procedure *AddChartsOfSide*, which actualizes the sum for the charts of a side m (the three coordinate components Sx, Sy, Sz and the sum of base functions D , used to normalize the result). The computation of the F mapping is further detailed in next subsection. This procedure is next detailed.

Since we are traversing the sides and elements of consecutive sides are shared, we have to avoid visiting twice shared vertices with a single chart associated (see Section 4.1.1). Thus, for each side charts associated to vertices or faces fulfilling conditions (7) and (8) must not be taken into account.

For simplicity, we work in the spaces $\mathcal{M}(E_m)$, where the meshes and charts have a regular shapes. This rectangular configuration can be exploited to use the usual algorithms developed for the tensor product B-Spline patches, as well as performing bounding-box inclusion tests for improving the efficiency.

In particular, for the surface evaluation the de Boor algorithm for evaluating NURBS can be used in the following way: (1) A fourth term, with unitary value, is added to the control points. This fourth coordinate is used in order that the de Boor algorithm also computes the sum term D ; (2) setting to zero all four coordinates of control points corresponding to discarded charts; and (3), the fourth coordinate is set to $1/2$ for all control points that have two charts associated, following the definition of the surface given in Section 4.2. This trick allows us to easily implement the procedure *AddChartsOfSide* making usage of the de Boor algorithm.

In summary, the point evaluation essentially requires n mappings between the parametric spaces $\mathcal{M}(X)$ and $\mathcal{M}(E_m)$ plus n evaluations of an order $k + 1$ B-spline rectangular patch with $(s + 1) \times (2s + r_m + 1)$ four-dimensional control

points. Since these extended control points are fixed, they can be pre-computed and stored so that multiple point evaluations are efficiently computed.

Notice also that one can improve the efficiency by pre-computing and storing the knot sequences. Besides, the horizontal knot sequence (the one corresponding to direction x) is common to all the $\mathcal{M}(E_m)$ spaces, since horizontal knots are uniformly spaced and the number of knots is the same for all the sides. On the other hand, the vertical knot sequences (the y knot sequences) do depend on the number of layers of the side, r_m . Nevertheless, if all the sides have the same number of layers ($\forall i, j \ r_i = r_j$), not only the vertical knot sequence is unique, but also, by choosing the appropriate aperture factor, the vertical spacing between all vertices in E_m can be forced to be the unit, so that a uniform B-spline scheme is obtained.

As we see, although the theoretical development of the multisided patch is quite complex, the method for point evaluation is rather simple, since efficient known algorithms for B-spline patches can be applied. The only process that remains to be explained is the computation of F that maps the parameter space $\mathcal{M}(X_m)$ to the space of E_m .

6.2 Computing the F mapping

The computation of the F mapping requires to distinguish three cases, depending on the y value. Let $l_{00}^m = (f_x, f_y)$, then if y is greater than f_y or smaller than $-f_y$, a simple linear transformation is needed. Otherwise, that is, if the point lies on the horizontal stripe corresponding to submesh C_m , the mapping is given by the sweep of curve μ along a vertical segment.

Suppose that μ is parameterized in terms of y/f_y . Since it has been imposed that in E_m the vertical spacing between consecutive vertices $c_{i,j}$ has to be uniform, the parameter values t_i such that $\mu(t_i) = c_{i,j}$ are also equally spaced.

Thus, we only need to identify the curve $\mu(t)$, or, better said, since the parameter value t coincides with y/f_y , the problem reduces to defining the horizontal term of the curve, $\mu_x(t)$. The definition for curve μ given in Section 3 gives

$$(3) \ \mu_x(1) = \mu_x(-1) = f_x$$

$$(4) \ \mu_x'(1) = \tan(-\beta), \ \mu_x'(-1) = \tan(\beta)$$

$$(5) \ \mu_x^{(i)}(1) = \mu_x^{(i)}(-1) = 0, \ i = 2, \dots, k$$

where $\beta = \pi/2 - \alpha$. In fact, there is not a unique way to define this curve. In the actual implementation, we choose μ_x to be a polynomial. Thus, taking into

account that the result has to be symmetric with respect to the x axis, the odd degree coefficients of the polynomial must be zero:

$$\mu_x(t) = a_{2k}t^{2k} + a_{2k-2}t^{2k-2} + \dots + a_2t^2 + a_0$$

For example, for $k = 1$, we obtain a parabola,

$$\mu_x(t) = \frac{f_y \tan(\beta)}{2}(-t^2 + 1) + f_x$$

and for $k = 3$,

$$\mu_x(t) = \frac{f_y \tan(\beta)}{16}(-t^6 + 5t^4 - 15t^2 + 11) + f_x$$

Notice also that this results in a quite simple function, and that although it has degree $2k$, it can be lowered to the half with a simple variable change $w = t^2$. Furthermore, the implementation could only consider the curve corresponding to the maximum order admitted.

7 Results and discussion

Snapshots in Figures 6 to 11 show surfaces obtained by applying the method presented in this article from n -sided control meshes. In Figure 6 a 3-sided symmetric mesh is represented, and resulting surfaces computed by applying our method are shown. The different cases represented correspond to patches with continuity of order $k = 0, 1$ and 2 .

Surface in Figure 7 is a mesh defining a 5-hole, the corresponding 5-sided patch and a close-up of the central part of the patch. Different zones of the patch have been represented in alternating colors in order to better appreciate the shape of the resulting surface. Figure 8, shows 5-sided hole and the resulting surface for $k = 3$, Figure 9 represents a 6-sided patch, and Figure 10 shows a 7-sided patch with order 2 continuity.

The surfaces in Figure 10 are C^2 5-sided patches computed using the same control mesh but using a different aperture factors. The region corresponding to the central face in parametric space of the patch is colored in red to see the influence of the f factor. In the bottom right example, since f is too big the number of charts covering the central part is too small, and thus an almost flat zone is produced.

To avoid this undesirable effect the aperture factor f must be bounded. In fact, if the aperture factor is too big it could even happen that some points of the central face of X (the hole) do not belong to any chart, and then the surface will not be defined there. Notice that in a usual tensor product surface each parametric point

is covered by $(k + 2) \times (k + 2)$ charts. This is the minimum number of charts that we require for an 4-sided regular patch. A geometric reasoning shows that, in the general case, an analogous condition can be imposed if $2f \cos(\alpha/2) < \lfloor k/2 + 1 \rfloor$ (this result is obtained by noticing that the most conflictive point is the center of X). In this way, we obtain a patch of a sufficient degree that allows a smooth enough free-form surface.

Another aspect that should be taken into account is that, although near each boundary of a n -sided patch the surface coincides with a B-spline, the knot sequences are not general. In fact, the knot sequence in one direction is uniform and in the other it is piecewise uniform. Thus, connecting a given rectangular B-spline with a n -sided patch can become in some cases not straightforward. Given a tensorial product B-spline patch with non-uniform knot distribution, under some circumstances we can still use the knots of the given surface to construct the regular part of the n -sided patch X . That is, we can define the points of X in a way that satisfies the distribution (distances) between the given knots. In this case the n -sided patch may not be regular (in terms of the given definition), but the process applied in the previous sections can also be performed without important modifications.

8 Conclusions and future work

In summary, we have presented a method that, given a control mesh surrounding an n -sided hole constructs a patch of parametric surface of degree $k + 1$ that fills the hole. The mesh is composed of s' layers of vertices distributed in n sides, and the boundary of the patch being built coincides with a tensor product B-spline surface for each side. Since there are no restrictive conditions for neither n nor k , the proposal allows joining a given tensor product B-spline patches with GC^k .

The method only requires that the number of vertex layers s' is big enough to attain the order k continuity conditions. Thus, additional layers of control points can be placed to shape the central part of the patch (without loosing the order k continuity). In addition, the method also supplies a scalar value f that can be used to have a more accurate control the influence of the control points towards the central part.

In spite of the theoretical difficulty to define the patch, we have seen that the simplicity of the construction process gives an easy method. We have also presented an algorithm to easily evaluate a surface point given its parametric coordinates. Furthermore, as explained the de Boor algorithm can be exploited to efficiently perform this evaluation.

Although our proposal allows an easy GC^k connection between an n -sided patch and a given B-spline rectangular patch, in the paper we have not treated the

case of generating a patch given n rectangular patches surrounding a hole. Future work includes giving the exact coordinates of the control points of an n -sided patch filling a hole like this, as well as analyzing the best number of additional layers of control points and its placement for usual situations in computer aided design (vertex rounding, surface blendings, etc). Another future development consists on managing the special case $n = 0$, that is, filling the hole of a ring-shaped control mesh (for example, blending the apex of a conic surface).

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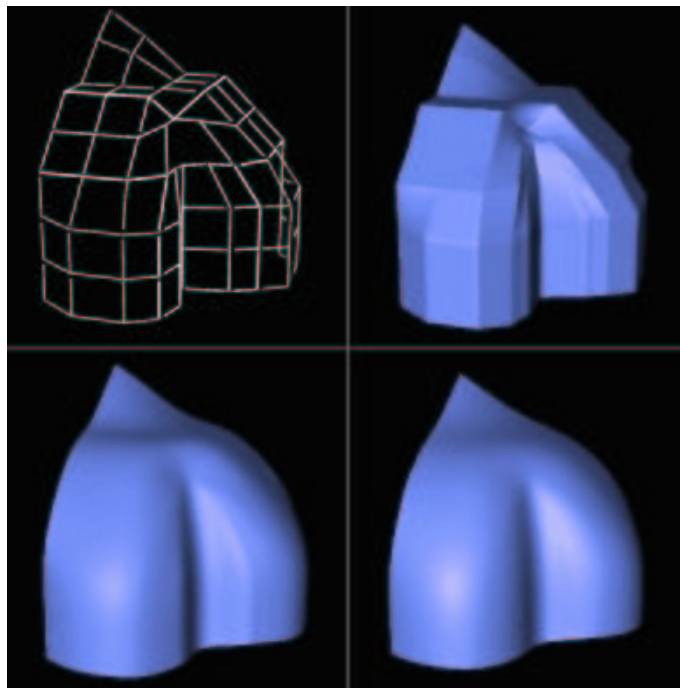


Figure 6: *Top left to bottom right*: Mesh defining a 3-sided hole, and surfaces of order 0, 1 and 2.



Figure 7: A 5-sided patch. Control mesh, resulting patch and close-up of the central zone.

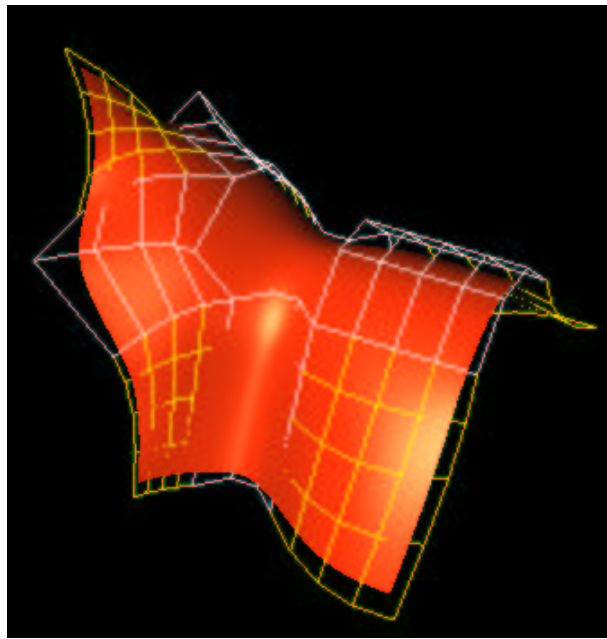


Figure 8: A C^3 5-sided patch and its control mesh.

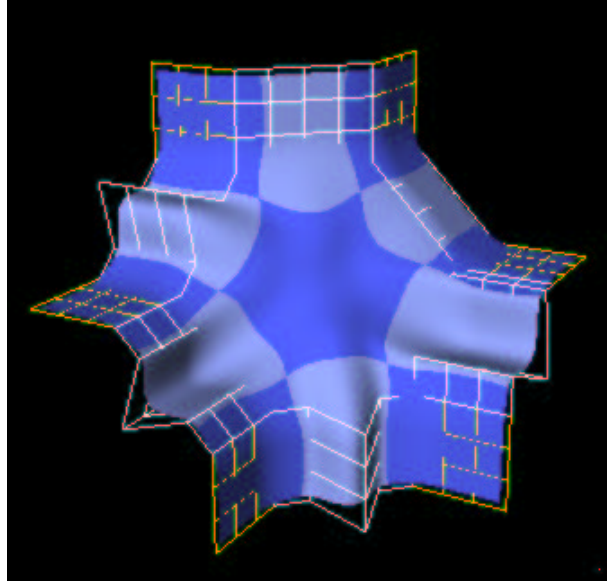


Figure 9: A C^2 6-sided patch and its control mesh. Central (darker) part of the patch corresponds to the hole in parametric space.

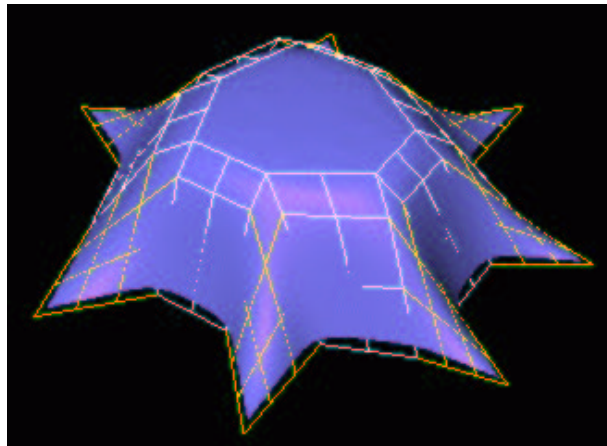


Figure 10: A C^2 7-sided patch and its control mesh.

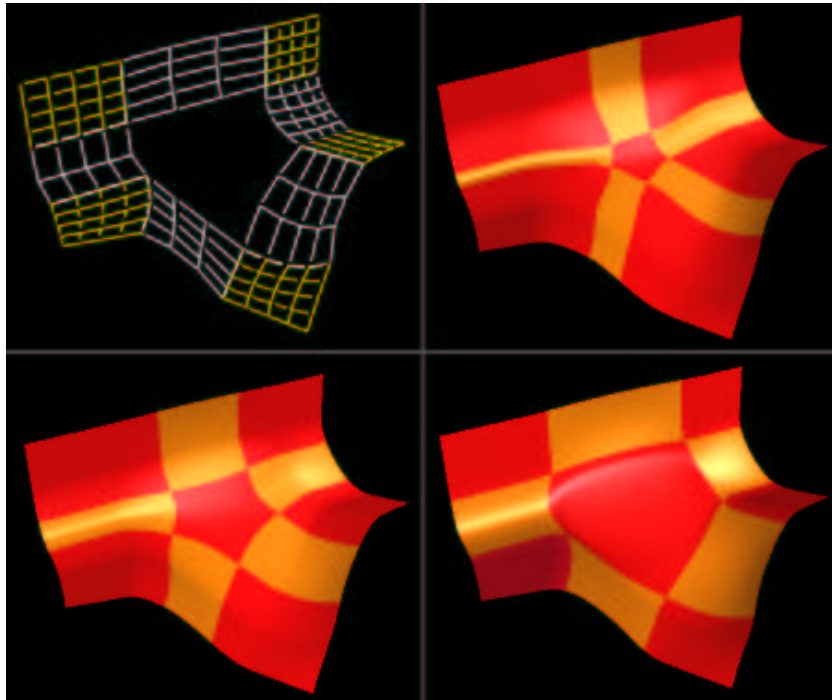


Figure 11: Three 5-sided patches defined using the same control mesh and increasing aperture factors.