Fig. 4.   Test images.

TABLE I
COMPRESSION RATIO PERFORMANCE OF PROPOSED MEP SYSTEM

Image			Entropy		Compression Ratio	
Name	Width	Height	MEP	DPCM	MEP	DPCM
Boat	512	512	4.663	4.444	1.71	1.79
Lenna₁	512	512	4.595	4.526	1.73	1.75
Lenna₂	256	256	4.948	4.822	1.61	1.65
Mandrill	512	512	6.379	6.208	1.25	1.28
Peppers	512	512	5.056	5.215	1.58	1.52

of the two coding schemes would be an important consideration. MEP coding requires only integer operations, while DPCM requires many costly floating point operations, including multiplications, as well. DPCM also requires an additional image prescan to calculate predictor coefficients. In the case where predictor coefficients are not calculated separately for each image being coded, it is expected that the performance of the DPCM scheme will become inferior that of the MEP scheme. It is also possible to apply the DPCM scheme to the MEP subimages separately. With DPCM coding of only the largest MEP subimage, or base of the pyramid, a 0.25% marginal average performance improvement of MEP over DPCM coding was observed for the same set of test images. However, in this case, the computational advantage of MEP coding is lost.

The MEP coding results presented in Table I also appear to represent significant improvements over similar lossless compression results for 8 b/pixel originals. For example, a combination of pyramid coding and vector quantization was used in [2] to achieve 5.862 for a 256×256 region of "Boat." An optimal model based arithmetic coder was used in [7] to achieve 5.1 b/pixel for a 256×256 version of "Lenna" (similar to "Lenna₂"), and a combination of transform coding and vector quantization was used in [6] to achieve 5.067 b/pixel for the 512×512 "Lenna" and 5.262 b/pixel for a 512×512 region of "Boat." Finally, the hierarchical "low complexity" coding method presented in [8] was simulated to achieve 5.040 b/pixel for "Lenna₁" and 4.992 b/pixel for "Boat" using the originals shown in Fig. 4.

IV. CONCLUSION

This correspondence has presented a new image source coding technique that offers significantly improved lossless compression performance over other related techniques. The presented method also offers comparable performance to the widely applied technique of DPCM coding but at a considerably lower computational cost. Since the new method is based on pyramid coding, it is also capable of supporting progressive transmission applications.

REFERENCES

[1] P. Burt and E. Adelson, "The Laplacian pyramid as a compact image code," IEEE Trans. Commun., vol. COM-31, pp. 532–540, Apr. 1983.
[2] L. Wang and M. Goldberg, "Progressive image transmission using vector quantization on images in pyramid form," IEEE Trans. Commun., vol. 37, pp. 1339–1349, Dec. 1989.
[3] A. Gersho and R. Gray, Vector Quantization and Signal Compression. Series in Communications and Information Theory. Norwood, MA: Kluwer Academic, 1992.
[4] N. Farvardin and J. W. Modestino, "Optimum quantizer performance for a class of nongaussian memoryless sources," IEEE Trans. Inform. Theory, vol. IT-30, pp. 485–497, May 1984.
[5] N. Jayant and P. Noll, Digital Coding of Waveforms, Principles and Applications to Speech and Audio. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[6] Y. Ho and A. Gersho, "Classified transform coding of images using vector quantization," in Proc. ICASSP'89, May 1989, pp. 1890–1893.
[7] G. Furlan, "An enhancement to universal modeling algorithm context for real-time applications to image compression," in Proc. ICASSP'91, May 1991, pp. 2777–2781.
[8] H. Torbey and H. Meadows, "System for lossless digital image coding/decoding," in Proc. SPIE, Visual Commun. Image Processing, 1989, vol. 1199, pp. 989–1002.

Flat Zones Filtering, Connected Operators, and Filters by Reconstruction

P. Salembier and J. Serra

Abstract— This correspondence deals with the notion of connected operators. Starting from the definition for operator acting on sets, it is shown how to extend it to operators acting on function. Typically, a connected operator acting on a function is a transformation that enlarges the partition of the space created by the flat zones of the functions. It is shown that from any connected operator acting on sets, one can construct a connected operator for functions (however, it is not the unique way of generating connected operators for functions). Moreover, the concept of pyramid is introduced in a formal way. It is shown that, if a pyramid is based on connected operators, the flat zones of the functions increase with the level of the pyramid. In other words, the flat zones are nested. Filters by reconstruction are defined and their main properties are presented. Finally, some examples of application of connected operators and use of flat zones are described.

I. INTRODUCTION

Morphological *filters by reconstruction* are becoming increasingly popular in image processing. Openings by reconstruction have appeared experimentally in morphology in [4]. Initially, they consisted

Fig. 1.   Example of connected operator for sets.

Fig. 2.   Partition resulting from a family of sets.
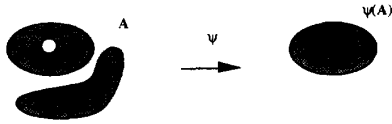
of eroding a set by a connected structuring element and in reconstructing all connected components of the set that had not been totally removed by the erosion. This original idea was fruitful because it led to geodesic operators on sets [5], [6], to markers for numerical functions [1], to multiresolution decomposition with filters by reconstruction [9], to the concept of dynamics [2], to area opening [16], and to hierarchical morphological segmentation [10]. Moreover, an intensive work has been done on the efficient implementation of these transformations [15]. These transformations by reconstruction involved not only openings but also closings, alternated filters or even alternating sequential filters. They are becoming very popular because, on experimental bases, they have been claimed to simplify the image while preserving contours. This rather surprising property makes them very attractive for a very large number of applications such as noise cancellation or segmentation.

In spite of their popularity, the fundamental reasons of their good behavior and their theoretical properties were not known until a study reported in [14]. This study revealed that filters by reconstruction belong to a larger class called *connected operators* that have the fundamental property of interacting with the signal by means of flat zones (which are the largest connected components where the signal is constant). In a few words, these operators do not remove some frequency components like linear filters or some shapes like median filters or morphological opening and closing. They remove and merge flat zones. This notion represents a new way of thinking about signal processing and allows a much better understanding of connected operators or filters by reconstruction. The results given in [14] were presented with a theoretical point of view. The objective of this corresponce is to discuss, with a more intuitive point of view, the fundamental notions behind connected operators and filters by reconstruction and their basic properties. Moreover, some examples illustrating the use of flat zones and connected operators will be described.

The organization of this correspondence is as follows: Section II is devoted to the notion of connected operators. Section III deals with a class of connected operators possessing the so called pyramidal property. Filters by reconstruction are discussed in Section IV, and Section V presents practical examples where the notions of flat zones and connected operators play a central role.

II. CONNECTED OPERATORS

In the framework of mathematical morphology [7], [12], [13], the basic working structure is a *complete lattice*. Let us recall that a complete lattice is a set of ordered elements (partial or total order) for which each family of elements possesses a supremum (sup) and an infimum (inf). For image processing, two lattices have proved to be very useful: the lattice of sets (where the order, the sup, and the inf are respectively defined as $\subset$, $\cup$, and $\cap$), which is the classical structure for binary image processing, and the lattice of functions (where the order, the sup, and the inf are respectively defined as $\leq$, $\vee$, and $\wedge$), which is the structure for gray-level image processing. Connected operators rely on the notion of connectivity, which is rather natural for sets. However, connectivity cannot be extended simply for functions. Let us then start by the notion of connected operators for sets and then extend it for functions.
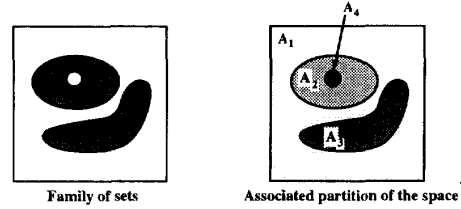
A. Connected Operators for Sets

There are several ways of defining the notion of connectivity [13]. We will restrict ourselves to the simplest one following which a connected component of a set is the set of points that may be connected by a path included in the set. In the digital case, the definition of connectivity reduces to the definition of a local neighborhood describing the connections between adjacent pixels. The classical choices involve 4, 6, or 8 connectivity. Once the connectivity has been selected, the notion of connected operators is defined as follows:

Definition 1— Connected Operators for Sets: An operator ψ working on sets is said to be *connected* when for any set A, the symmetrical difference $A \Delta \psi(A)$ is exclusively composed of connected components of A or its complement A^C.

This means that the operator acts only by preserving or removing connected components. As shown on Fig. 1, a connected operator can only remove connected components of the sets or fill connected components of the background. Although they are called connected, these operators do not preserve the connectivity.

Now, our goal is to extend the notion of connected operators to lattices of functions. This extension cannot be done directly because the connectivity has no simple equivalent in lattices of functions. Since the extension cannot be done through the connectivity itself, let us introduce an alternative definition of connected operators that is easily transposable to lattices of functions. This alternative definition relies on *partitions*.

Definition 2—Partition of a Space: A partition of a space E is a set of connected components $\{A_i\}$ which are disjoint ($A_i \cap A_j = \emptyset$, $i \neq j$) and the union of which is the entire space ($\cup A_i = E$). Each A_i is called a partition class.

A partition $\{A_i\}$ is said to be *finer* than another partition $\{B_i\}$ if any pair of points belonging to the same class A_i also belongs to a unique partition class B_j.

Consider now a family of sets and associate, to each connected component of the sets and of their complementary, a class A_i. As illustrated by Fig. 2, the sets of connected components $\{A_i\}$ constitutes a partition of the space. Let us call this partition the *associated partition* of the family of sets. The definition of connected operators can be very simply expressed with associated partitions (see Fig. 3).

Proposition 3—Connected Operators for Sets via Partitions: An operator ψ acting on sets is connected if and only if, for each family of sets A, the partition associated with $\psi(A)$ is less fine than the partition associated A.

B. Connected Operators for Functions

On the basis of Proposition 3, the extension of connected operators for lattices of functions can be easily done if we define a partition associated to a function. To this end, we propose to use the concept of *flat zones*.

Definition 4—Flat Zones: The set of *flat zones* of a gray-level function f is the set of the largest connected components of the space where f is constant.
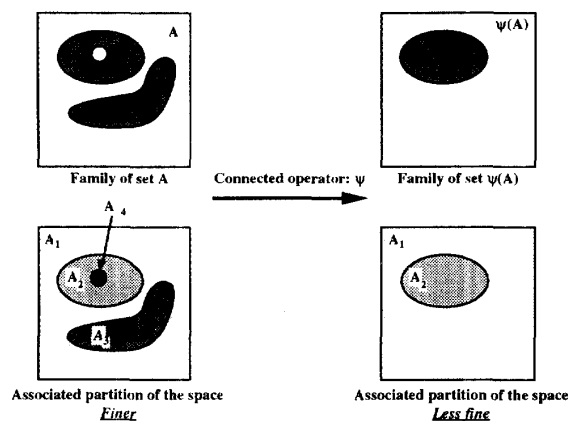
Fig. 3. Connected operators in terms of associated partition.

Fig. 4. Creation of a connected operator for functions via threshold decomposition.

Note that there is no restriction on the size of the flat zones and they can be reduced to a single point. Anyway, the important point is that it can be demonstrated [14] that the set of flat zones of a function constitutes a partition of the space. In the following, this partition will be called the *partition of flat zones of a function*. The definition of connected operators can now be extended to lattices of functions as follows:

Definition 5—Connected Operators for Functions: An operator Ψ acting on gray-level functions is said to be *connected* if, for any function f, the partition of flat zones of $\Psi(f)$ is less fine than the partition of flat zones of f.

There are several ways of creating connected operators for functions. The simplest one consists in extending a connected operator acting on sets. Indeed, as shown in [3] and [13], any operator ψ acting on sets can generate an operator Ψ acting on functions by "stacking." Denote by $X_t(f)$ the planar section of a function f at level t, that is, $X_t(f) = \{x : x \in E, f(x) \geq t\}$. The set

$$A(t) = \bigcap_{s < t} \psi[X_s(f)] \tag{1}$$

is the planar section at level t of a unique gray-level function $\Psi(f)$ [3], [13]

$$X_t[\Psi(f)] = \bigcap_{s < t} \psi[X_s(f)] \tag{2}$$

Moreover, when ψ is increasing, this relation can be simplified

$$X_t[\Psi(f)] = \psi[X_t(f)]. \tag{3}$$

This last equality means that the planar section of $\Psi(f)$ at level t can be obtained by applying ψ on the planar section of f at the same level. The function $\Psi(f)$ itself can be recovered by "stacking" all its planar sections as illustrated by Fig. 4. Note, however, that in practice, it is not necessary to use the planar sections of f to build the operator $\Psi(f)$. Concerning the connectivity issue, we have the following result [14]:

Proposition 6—Extension of Connected Operators Acting on Sets: If ψ is a connected operator acting on sets, its extension Ψ to functions obtained by stacking is a connected operator acting on functions.

III. PYRAMIDS OF CONNECTED OPERATORS

In this section, we are going to define the notion of *pyramid of operators*, and we will see that when the pyramid relies on connected operators, a very interesting and strong property is obtained.
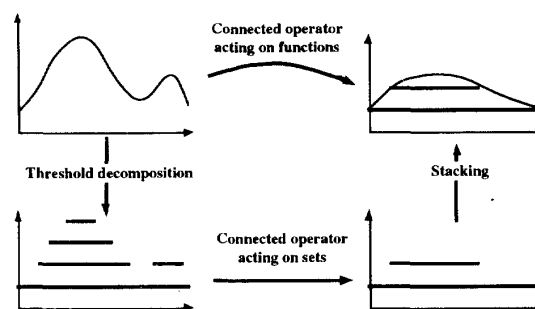
Definition 7—Pyramid of Operators: A *pyramid of operators* is a family of operators $\{\psi_\lambda\}$ depending on a positive parameter λ such that, for each $\lambda \geq \mu \geq 0$, there exists a given $\nu \geq 0$ such that $\psi_\nu \psi_\mu = \psi_\lambda$.

This definition is valid for operators acting on sets or on functions. Practically, it recovers two major points: First, the composition of two operators of the family is still an operator of the family. Second, to obtain the result of ψ_λ, one can either use ψ_λ directly or start by using ψ_μ ($\mu \leq \lambda$) and obtain an intermediate result that can be processed by an other operator ψ_ν of the family. In mathematical morphology, classical pyramids of operators are the dilation (or erosion) with a circular structuring element of radius λ or the granulometry [7] by opening with structuring element of size λ. Each case corresponds to two different composition laws. In the case of dilation, the composition rule is additive, and we have $\psi_\lambda = \psi_{\lambda - \mu} \psi_\mu$. In the case of granulometry, we have: $\psi_\lambda = \psi_\lambda \psi_\mu$, which means that in a composition, the strongest operator imposes its effects.

Suppose now that the pyramid of operators $\{\psi_\lambda\}$ relies on connected operators. It means that for any $\lambda \geq \mu \geq 0 \Rightarrow \exists \nu \geq 0$ such that $\psi_\nu \psi_\mu = \psi_\lambda$. Since the operator ψ_ν is connected, the associated partition of $\psi_\nu \psi_\mu$, and therefore of ψ_λ, is less fine than that of ψ_μ. Let us express this result for operators acting on functions:

Theorem 8—Pyramid of Connected Operators for Functions: If $\{\Psi_\lambda\}$ denotes a pyramid of connected operators acting on functions, then for any function f, the flat zones of $\Psi_\lambda(f)$ increase with λ. Since all flat zones of $\Psi_\lambda(f)$ increase with the parameter λ, they are in fact *merged*. Indeed, if two contiguous flat zones have to increase, either they remain unchanged or they are merged together. This property explains the effect of simplification while preserving the contour information that can be obtained with these filters. Pyramids of connected operators are not selective in frequency as linear filters or selective in shape/size as morphological opening or median filters, they are "selective in flat zones." This is a very strong property that allows the construction of a hierarchy where the flat zones are nested.

Let us illustrate this result by a simple example. Consider the operator acting on sets that consists of eroding the input sets A by a structuring element of size λ and by keeping all connected components of A that have not been totally removed by the erosion. The associated connected operator acting on functions obtained via (3) is classically known as *opening by reconstruction of erosion* Γ_λ [9], [17]. This filter is illustrated on Fig. 5, where three filtering results corresponding to three structuring elements (11 * 11, 26 * 26, and 51 * 51) are shown. The filter effect is to remove bright components smaller than the structuring element. One can also observe the merging of flat zones which allows a very good preservation of the
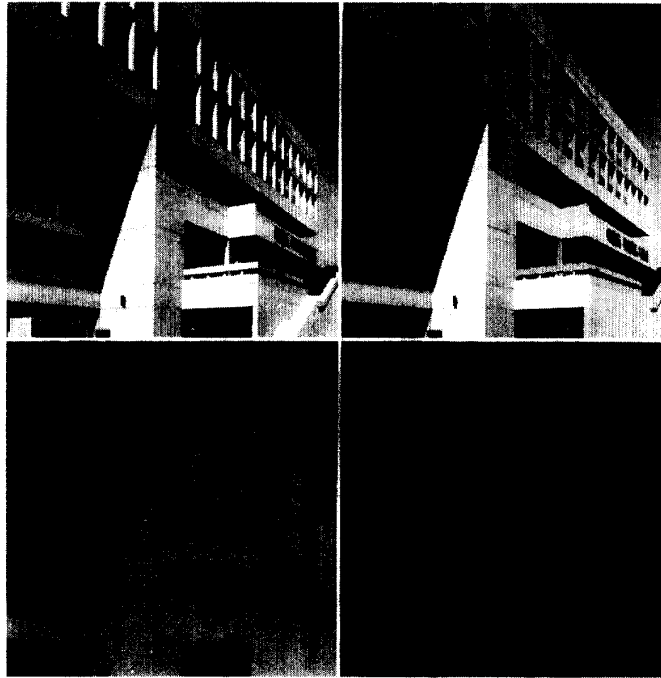
Fig. 5. Example of flat zones merging with opening by reconstruction.

contour information. Fig. 6 presents the histogram of the number of flat zones, as a function of the size of the structuring element. The original image contains more than 30 000 flat zones for 65 536 pixels. The majority of flat zones are reduced to one or two pixels. The histogram illustrates how flat zones are merged by the filtering process.

IV. FILTERS BY RECONSTRUCTION

The introduction of the notion of filters by reconstruction requires a new property called *connected invariance*. In the following, the term filter denotes an increasing and idempotent operators [12], [13].

Definition 9—Connected Invariant Filters (Ci-Filters): A filter ψ acting on sets (resp. Ψ acting on functions) is said to be a *Ci-filter* *("Connected invariant" filter)* when, for any input A (resp. f), the connected components of $\psi(A)$ (resp. the flat zones of $\Psi(f)$) are invariant under ψ (resp. Ψ).

As can be seen, this definition concerns filters that are not, *a priori*, connected. The class of Ci-filters is extremely large and, for example, all morphological opening (or closing) by connected structuring elements as well as their union (or intersection) are Ci-filters. By contrast, an opening by a bipoint is not a Ci-filter.

All the filtering techniques presented in the introduction share two fundamental properties: They are Ci-filters and connected operators. Let us use these two properties as definition of the class of filters by reconstruction:

Definition 10—Filter by Reconstruction: A filter by reconstruction is any connected Ci-Filter.

Most of the filters by reconstruction used in practice are obtained by composition of openings by reconstruction and their dual (closings by reconstruction). Moreover, the two major classes of openings by reconstruction are obtained by means of either an increasing criterion or a marker.
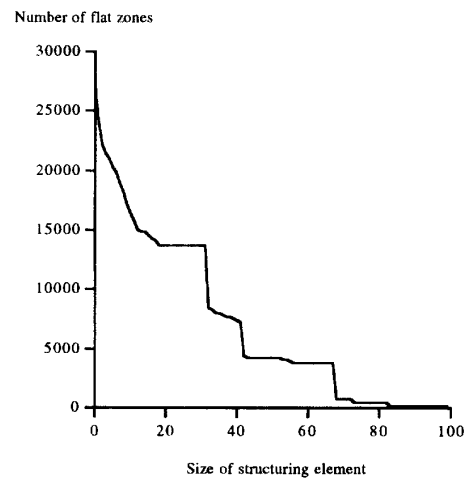
Number of flat zones

Fig. 6. Histogram of the number of flat zones.

Opening by Reconstruction Following an Increasing Criterion: Consider a transformation ψ acting on sets consisting, first, of measuring an increasing criterion such as the area or the Ferret's diameter of each connected component of the input sets and, second, of keeping only the connected components for which the criterion is higher than a given limit. This transformation, as well as the filter acting on functions Ψ obtained via (3), are filters by reconstruction. Since they are moreover anti-extensive, they are opening by reconstruction. A typical example of such a filter is the area opening [16].

Opening by Reconstruction of Markers: Assume that in the case of sets, we have two different inputs: the actual input A and

a second input called *maker B*. Now consider the operator acting on sets that consists in keeping all connected components of A hitting B. This operator, as well as its associated operator acting on functions (3), are openings by reconstruction. As an example, the example described in Fig. 4 is based on this kind of opening where the marker is obtained by erosion of the original input signal.

In [14], a large number of properties are demonstrated and discussed. In the following, we simply report the most important ones. The reader interested in the details and the demonstrations is referred to [14].

- The dual closing of a filter by reconstruction is itself a filter by reconstruction.
- If γ and γ' (resp. φ and φ') are two openings (resp. closing) by reconstruction, they commute, that is, $\gamma\gamma' = \gamma'\gamma$ (resp. $\varphi\varphi' = \varphi'\varphi$).
- The class of filters obtained by composition of any family of openings and closings by reconstruction, is composed of filters by reconstruction.
- If $\{\gamma_\lambda\}$ is a granulometry of openings (that is if $\lambda \geq \mu \Rightarrow \gamma_\mu \geq \gamma_\lambda$) by reconstruction and $\{\varphi_\lambda\}$ is a antigranulometry of closings (that is if $\lambda \geq \mu \Rightarrow \varphi_\lambda \geq \varphi_\mu$) by reconstruction, then the alternating sequential filter $\psi_\lambda = \gamma_\lambda\varphi_\lambda \cdots \gamma_1\varphi_1$ is a strong filter, that is, $\psi_\lambda = \psi_\lambda(I \vee \psi_\lambda) = \psi_\lambda(I \wedge \psi_\lambda)$.

V. EXAMPLES OF FLAT ZONE PROCESSING

The goal of this section is to illustrate some possible use of the notion of flat zones and connected operators. To this end, a size-oriented and a contrast-oriented connected operator will be used.

A. Examples of Connected Operators

Size-Oriented Connected Operator—Area Open-Close: Consider the opening γ_λ acting on sets that consist of keeping all connected components of the input of area larger than a limit λ. It can be easily shown that this is an opening by reconstruction. Let us denote by Γ_λ its associated opening acting on function (defined by (3)). This opening is known as gray scale area opening [16]. Finally, the area open-close, denoted by Σ_λ, is defined as the composition of the area opening with its dual. It is a filter by reconstruction which is size-oriented in the sense that it removes image components that are smaller in area than a given limit.

Contrast-Oriented Connected Operator: Consider the operator ψ_λ acting on sets that consists of keeping all connected components of the input A hitting a second input B called a marker. Let us denote by Ψ_λ the associated operator acting on function [defined by (3)]. Assume, moreover, that the original signal is denoted by f and that the marker signal is obtained by subtracting a constant λ to f. This operator is known as λ-maxima extraction [11] (it is not a filter because it is not idempotent). Finally, the contrast operator, denoted by X_λ, is defined as the composition of the Ψ_λ with its dual. It is a connected operator that is contrast-oriented in the sense that it removes image components that have a contrast smaller than a given limit.

B. Hierarchical Decomposition

Decomposition techniques are wildly used in image processing. They consist of splitting the information contained in the original image into several images that are expected to be simpler. The most classical approach is to use the decomposition scheme of Fig. 7 with
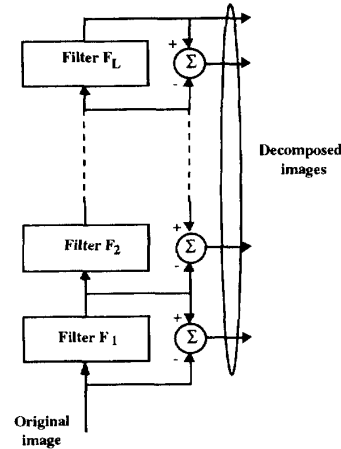

Fig. 7. Decomposition scheme.

linear lowpass filters as simplification operators. As illustrated by the first row of Fig. 8, the resulting decomposition is frequency-oriented in the sense that each image is a passband version of the original image. However, the scheme of Fig. 7 is a general decomposition structure since the sum of the decomposed images restores the original image whatever simplification operators are used. Fig. 8 presents the decomposition results with linear lowpass filters (first row), morphological open-close filter with square structuring element (second row), the area open-close Σ_λ (third row), and the contrast operator X_λ (fourth row). The major drawback of using a linear lowpass filter in the decomposition can be seen around contours: Any transition contains a large range of frequencies and therefore appears on all levels of the decomposition. The use of open-close filters partially solves this problem; however, the decomposition is strongly influenced by the shape of the structuring element (square). The area open-close filters successfully classify the image components as a function of their size. Moreover, the information concerning one object is concentrated mainly on one level of the decomposition and is not corrupted by an arbitrary shape such as a square structuring element. The decomposition with the contrast operator has similar characteristics except that the classification criterion is the relative contrast. In conclusion, the main advantage of using connected operators within a decomposition scheme is the better classification of the information and representation of the image components on each level.

C. Segmentation

Connected operators being able to simplify while preserving the contour information are very attractive for segmentation purpose. Let us illustrate two very simple examples.

A fairly general approach to morphological segmentation [8] involves three steps: image simplification, marker extraction, and contour definition. The goal of the image simplification is to remove the useless information. The marker extraction identifies the presence of homogeneous regions. It results in a set of connected components indicating the *interior* of the regions. Finally, the contour definition precisely locates the transitions between the previously extracted regions. This third step is classically performed by the so called *watershed* algorithm [8].

Fig. 9 illustrates a size-oriented segmentation. The first row presents the original image, the result of a simplification with an

Fig. 8. Decomposition with linear lowpass filter (first row), morphological open/close (second row), area open/close (third row), and contrast operator (last row).

area open–close (area limit of 60), and the sets of flat zones after simplification. There is a very large number of flat zones; however, only 91 are larger than the size limit of 60. The presence of small flat zones after the area filter corresponds to transition areas like ramps between large flat zones. Let us take as markers all flat zones of size larger than 60. The resulting set of markers is shown in the second row of Fig. 9, together with the segmentation result obtained with the watershed algorithm. The segmentation result is in two forms: a segmentation map and a synthetic image where each region has been filled by the mean of the original image. The segmentation contains 91 regions and represents all "large" regions.

Fig. 10 presents a similar result for a contrast-oriented segmentation. The first row presents the original image: the simplified images for positive and negative contrast (contrast parameter = 40). The marker extraction is simply achieved by taking all the flat zones of the image after contrast simplification where the difference between the original image and the simplified one is equal to λ at least in one point. As shown in [11], these flat zones correspond to the components that have a contrast higher or equal to λ.

The last row of Fig. 10 shows the set of markers as well as the corresponding segmentation result. The segmentation involves 104 regions. A comparison with the result of Fig. 9 reveals that the segmented regions are of very different size and that the main segmentation criterion is actually the contrast.

VI. CONCLUSION

In this correspondence, the notion of connected operators has been defined and discussed. Starting from the definition for set operator, it has been extended to function operators. Typically, a connected operator acting on functions is a transformation that enlarges the partition of the space created by the flat zones of the functions. It has been shown that from any connected operator acting on sets, one can construct a connected operator for functions (however, it is not the unique way of generating connected operators for functions). Then, the concept of pyramid has been introduced in a formal way. One of the most important results of this study is that if a pyramid is based on connected operators, the flat zones of the functions increase with the
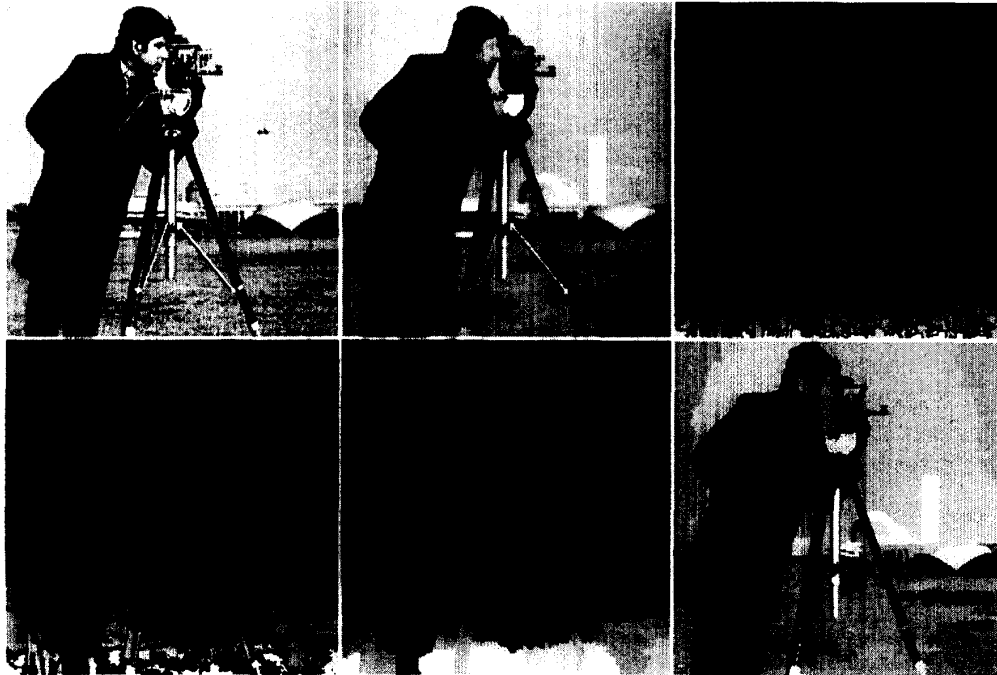
Fig. 9. Size-oriented segmentation. First row: original image, simplified image, and flat zones. Second row: markers, segmented regions, and segmentation results.
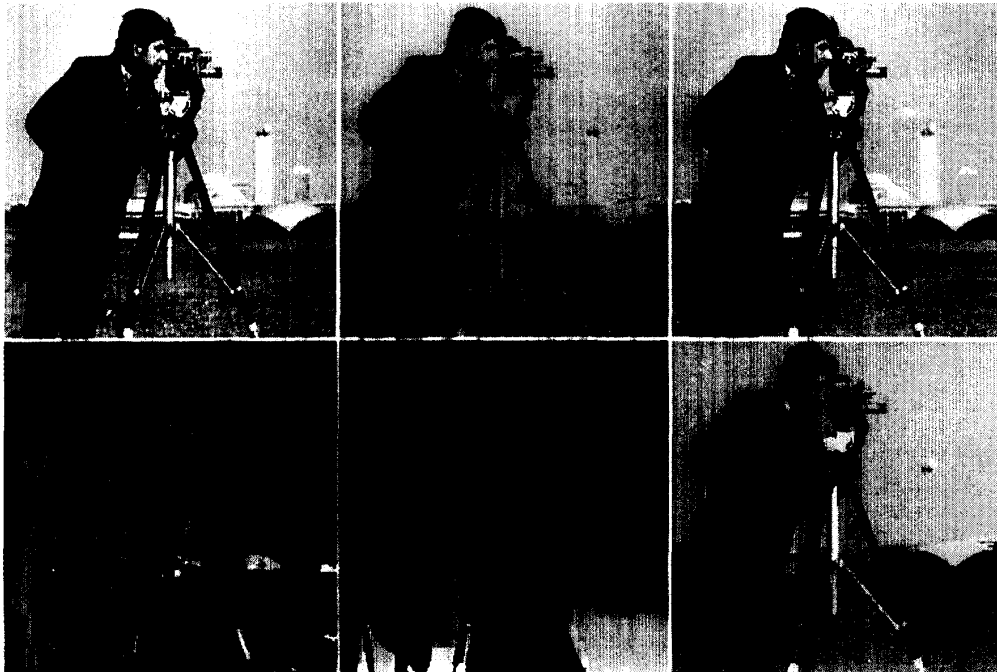

Fig. 10. Contrast-oriented segmentation. First row: original image, simplified image (positive contrast), and simplified image (negative contrast). Second row: markers, and segmented regions, segmentation results.

level of the pyramid. In other words, the flat zones are nested. Filters by reconstruction have then been defined, and their main properties have been stated. Finally, some examples of application of connected operators and use of flat zones have been discussed. Their main advantage is to be able to simplify the image while preserving the contour information.

REFERENCES

[1] S. Beucher, "Cours sur la morphologie numérique," in *Summer School of Paris School of Mines, Paris, France*, 1980.
[2] M. Grimaud, "La géodésie numérique en morphologie mathématique," *Application á la détection automatique des microclassifications*," Ph.D. thesis, Paris School of Mines, Paris, France, 1991.
[3] H. Heijmans, "Theoretical aspects of gray level morphology," *IEEE Trans. Patt. Anal. Machine Intell.*, vol. 13, no. 6, pp. 568–592, 1991.
[4] J. C. Klein, "Conception et réalisation d'une unité logique pour l'analyse quantitative d'images," Ph.D. dissertation, Nancy Univ., France, 1976.
[5] C. Lantuéjoul and S. Beucher, "On the use of geodesic metric in image analysis," *J. Microscopy*, no. 121, pp. 39–49, 1981.
[6] C. Lantuéjoul and F. Maisonneuve, "Geodesic methods in image analysis," *Pattern Recogn.*, vol. 17, pp. 117–187, 1984.
[7] G. Matheron, *Random Sets and Integral Geometry*. New York: Wiley, 1975.
[8] F. Meyer and S. Beucher, "Morphological segmentation," *J. Visual Commun. Image Representation*, vol. 1, no. 1, pp. 21–46, 1990.
[9] P. Salembier and M. Kunt, "Size-sensitive multiresolution decomposition of images with rank order based filters," *Signal Processing*, vol. 27, no. 2, pp. 205–241, May 1992.
[10] P. Salembier, "Morphological multiscale segmentation for image coding," *Signal Processing (Special Issue on Nonlinear Signal Processing)*, vol. 38, no. 3, pp. 359–386, 1994.
[11] ____, "Multi-criterion segmentation for image coding," in *Proc. Workshop Mathematical Morphology Applications Signal Processing*, Barcelona, Spain, May 1993, pp. 40–45.
[12] J. Serra, *Image Analysis and Mathematical Morphology*. Norwood, MA: Academic, 1982.
[13] ____, *Image Analysis and Mathematical Morphology, Th. Advances.* Vol. II. Norwood, MA: Academic, 1988.
[14] J. Serra and P. Salembier, "Connected operators and pyramids," in *Proc. SPIE Image Algebra Math. Morphology*, San Diego, CA, SPIE vol. 2030, 1993, pp. 65–76.
[15] L. Vincent, "Algorithmes morphologiques a base de files d'attente et de lacets, Extension aux graphes," Ph.D. thesis, Paris School of Mines, Paris, France, 1990.
[16] ____, "Gray scale area openings and closings, their efficient implementation and applications," in *Proc. Workshop Mathematical Morphology Applications Signal Processing*, Barcelona, Spain, May 1993, pp. 22–27.
[17] ____, "Morphological gray scale reconstruction in image analysis: Applications and efficient algorithms," *IEEE Trans. Image Processing*, vol. 2, no. 2, pp. 176–201, Apr. 1993.

Two-Dimensional Filter Bank Design for Optimal Reconstruction Using Limited Subband Information

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Abstract— In this correspondence, we propose design techniques for analysis and synthesis filters of 2-D perfect reconstruction filter banks (PRFB's) that perform optimal reconstruction when a reduced number of subband signals is used. Based on the minimization of the squared error between the original signal and some low-resolution representation of it, the 2-D filters are optimally adjusted to the statistics of the input images so that most of the signal's energy is concentrated in the first few subband components. This property makes the optimal PRFB's efficient for image compression and pattern representations at lower resolutions for classification purposes. By extending recently introduced ideas from frequency domain principal component analysis to two dimensions, we present results for general 2-D discrete nonstationary and stationary second-order processes, showing that the optimal filters are nonseparable. Particular attention is paid to separable random fields, proving that only the first and last filters of the optimal PRFB are separable in this case. Simulation results that illustrate the theoretical achievements are presented.

I. INTRODUCTION

Multiresolution signal representations can be effective in analyzing the information content of signals and especially images [7]. Representation of signals at many resolutions gained a further popularity with the use of discrete wavelet transform [3], which is implemented in a straightforward manner by filter banks using quadrature mirror filters (QMF's) [10]. In image processing, all the above ideas converge to subband processing [12], [13]. Image decomposition is performed with an appropriate filter bank of decomposition (or decimating) filters. Appropriate reconstruction (interpolating) filter banks guarantee perfect reconstruction of the original image from its subband components [2]. The design of perfect reconstruction filter banks is based on the assumption that all subband signals are available to the interpolation bank with infinite precision. In practical applications, this is not generally true. On the one hand, during coding/transmission/decoding, it is possible that some of the subband components are lost or discarded so that only a reduced number of them is used for reconstruction. On the other hand, quantization or other types of additive noise can disturb the subband signals. In both cases, perfect reconstruction filters lose their optimality.

In this correspondence, we assume noise-free subband sequences and consider the case of discarding some of the subband channels. For optimal synthesis filterbank design in the presence of subband noise, refer to [4]. This means that if we decompose the original 2-D signal in $P \times P$ subband components, we consider transmission of only the first Q of them with $Q < M \doteq P \times P$. The objective then is to minimize the reconstruction mean squared error (MSE). To achieve the minimization of the MSE, we design analysis and synthesis filters that are able to concentrate most of the signal's information into the first few, say Q, channels. The designed filters are optimally adjusted to the statistics of the input signals.