Oscillatory orbits in the restricted elliptic planar three body problem

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Abstract

The restricted planar elliptic three body problem models the motion of a massless body under the Newtonian gravitational force of two other bodies, the primaries, which evolve in Keplerian ellipses.

A trajectory is called oscillatory if it leaves every bounded region but returns infinitely often to some fixed bounded region. We prove the existence of such type of trajectories for any values for the masses of the primaries provided the eccentricity of the Keplerian ellipses is small.

Contents

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1 Introduction

The restricted planar elliptic three body problem (RPE3BP from now on) models the motion of a body of zero mass under the Newtonian gravitational force of two other bodies, the primaries, which evolve in Keplerian ellipses with eccentricity e_0. Without loss of generality one can assume that the masses of the primaries are μ and $1-\mu$, their period is 2π and their positions are $-\mu q_0(t)$ and $(1-\mu)q_0(t)$, where

$$q_0(t) = (\rho(t)\cos v(t), \rho(t)\sin v(t))$$

with

$$\rho(t) = \frac{1-e_0^2}{1+e_0\cos v(t)}$$

and $v(t)$ is called the *true anomaly*, which satisfies $v(0)=0$ and

$$\frac{dv}{dt} = \frac{(1+e_0\cos v)^2}{(1-e_0^2)^{3/2}}.$$

Then, the motion of the third body is described by the following Hamiltonian

$$H(q,p,t;e_0) = \frac{\|p\|^2}{2} - V(q,t;e_0) \tag{1}$$

where

$$V(q,t;e_0) = \frac{1-\mu}{\|q+\mu q_0(t)\|} + \frac{\mu}{\|q-(1-\mu)q_0(t)\|}$$

where $q,p\in\mathbb{R}^2$.

The eccentricity of the ellipses satisfies $e_0\in[0,1)$. If $e_0=0$, the primaries describe circular orbits. This case is known as the restricted planar circular three body problem (RPC3BP from now on). In this paper we consider $e_0>0$ and small. Note that the Hamiltonian also depends on $\mu\in[0,1/2]$. We do not write this dependence explicitly since for us μ is a fixed positive parameter.

The purpose of this paper is to analyze some particular orbits of this Hamiltonian system: the *oscillatory motions*, that is, orbits which leave every bounded region but return infinitely often to some fixed bounded region. To prove the existence of such orbits, we use the framework usually considered in the study of Arnold diffusion in nearly integrable Hamiltonian systems.

If one takes $\mu = 0$, the system reduces to a central force problem which is integrable and therefore cannot have oscillatory motions. That is, they can take place for the three body problem but not in its limit $\mu \to 0$. In this paper we prove that oscillatory motions are possible for any value of $\mu \in (0, 1/2]$ and $e_0 > 0$ small enough. The existence of such motions for any μ and $e_0 = 0$ (the circular problem) has been recently proved by the authors of this paper in [GMS15].

To obtain oscillatory orbits we work in a nearly integrable setting and we use perturbative methods that provide invariant objects which can be analyzed and act as a skeleton that such orbits follow. Nevertheless, since we give results for any $\mu \in (0, 1/2]$, the nearly integrable setting cannot be in terms of $\mu \to 0$. Instead, we consider the following regime. Take the body of zero mass very far away from the two primaries. Then, at first order the third body perceives the two primaries as just one body at the center of mass plus a periodic perturbation whose smallness comes from the ratio of the distance between the two primaries over the distance between the third body and the center of mass. Then, taking the ratio of distances small enough, one has an integrable Hamiltonian plus a small periodic perturbation (see [GMS15]).

1.1 Final motions in the three body problem

One of the most important questions in the analysis of the three body problem (either restricted or non restricted, planar or spatial) is the study of the final motions. That is, what type of behaviors can happen as time $t \to \pm\infty$. Its analysis was initiated by Chazy in 1922, when he gave a complete classification of the possible final motions (see Section 2.4 of [AKN88]). In the restricted setting, the possible final motions are the following:

- $H^\pm$ (hyperbolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
- $P^\pm$ (parabolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
- $B^\pm$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
- $OS^\pm$ (oscillatory): $\limsup_{t\to\pm\infty} \|q\| = +\infty$ and $\liminf_{t\to\pm\infty} \|q\| < +\infty$.

Examples of all these behaviors, except the oscillatory motions, were already known by Chazy. The other three behaviors certainly already exist for the two body problem, where motion is confined to conics in the state space and hyperbolic motions arises on hyperbolas, parabolic motions on parabolas and bounded motion on ellipses. As already explained, oscillatory motions cannot appear in the two body problem.

The study of oscillatory motions was initiated by Sitnikov in the sixties [Sit60]. He proved their existence in a very carefully chosen symmetrical model consisting of two bodies of equal mass revolving in planar ellipses around their center of mass and a third body of mass zero moving along the perpendicular axis at the center of mass. His work also proved that $X^- \cap Y^+ \neq \emptyset$ for $X, Y = H, P, B, OS$.

Later, Moser [Mos73] gave a new proof of the Sitnikov result. The subsequent results in the area strongly rely on the ideas developed by Moser. The present paper certainly also relies on some of them but also uses other ideas recently developed for the study of Arnold diffusion. Using Moser ideas, Llibre and Simó [LS80b] obtained oscillatory motions for the collinear three body problem.

For the planar three body problem, the first result was also by Llibre and Simó [LS80a], where they proved the existence of oscillatory motions for the RPC3BP for small enough values of the mass ratio μ (Hamiltonian (1) with $e_0 = 0$). Using a real-analyticity argument, Xia [Xia92] extended their result to any $\mu \in (0, 1/2]$ except a finite (unknown) number of values (Moser [Mos73] had previously used this argument in the Sitnikov problem). The first general result was obtained by three of the authors of the present paper [GMS15], who proved the existence of oscillatory motions of the restricted planar circular three body problem for any value of the mass ratio $\mu \in (0, 1/2]$.

A completely different approach using Aubry-Mather theory and semi-infinite regions of instability was developed in [GK11, GK10b, GK10a]. The authors considered the RPC3BP with a realistic mass ratio for the Sun-Jupiter system. Using computer-assisted methods, they proved the existence of orbits with initial conditions in the range of our Solar System which become oscillatory as time tends to infinity.

All the mentioned results deal with systems which can be reduced to a two dimensional area preserving map. This makes the proof of existence of oscillatory motions considerably simpler than if one considers other three body problems (restricted elliptic, restricted spatial, non-restricted) which have more degrees of freedom. For higher dimensional systems, Moser ideas are harder to apply and the results are more scarce.

In more degrees of freedom the first result is due to Alexeev [Ale69] (actually before Moser ideas) who generalized Sitnikov result [Sit60] to small mass for the third body. Positive mass increases the number of degrees of freedom of the system to three. Moeckel [Moe84, Moe07] has proven results concerning oscillatory motions in the regime of small angular momentum (and thus close to triple collision).

Concerning the RPE3BP (1), Robinson gave a conditional result on the existence of oscillatory orbits [Rob84, Rob15]. He proved their existence provided certain homoclinic points were present. As far as the authors know, the existence of such points has not been proved yet.

A fundamental problem concerning oscillatory motions is to measure how abundant they are. As pointed out in [GK12], in the conference in honor of the 70th anniversary of Alexeev, Arnold posed the following question: *Is the Lebesgue measure of the set of oscillatory motions positive?* Arnold considered this problem the central problem of celestial mechanics. Alexeev conjectured in [Ale71] that the Lebesgue measure is zero (in the English version [Ale81] he attributes this conjecture to Kolmogorov). This conjecture remains wide open.

The only result dealing with the abundance of oscillatory motions is the recent paper [GK12]. The authors study the Hausdorff dimension of the set of oscillatory motions for the Sitnikov example and the RPC3BP. Using [Mos73] and [LS80a], they prove that the Hausdorff dimension of the set of oscillatory motins is maximal for a Baire generic subset of an open set of parameters (the eccentricity of the primaries in the Sitnikov example and the mass ratio and the Jacobi constant in the RPC3BP).

1.2 Arnold diffusion in the three body problem and growth in angular momentum

As we have said, we study a regime where the RPE3BP is close to a two body problem. In the latter, the angular momentum is a first integral. This fact is no longer true in the former. Thus, a natural question is whether the angular momentum of the zero mass body only varies by a small amount (with respect to the perturbative parameter) or can make big excursions.

This question fits into the framework of Arnold diffusion [Arn64]. Consider a nearly integrable system in action-angle coordinates

$$H(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I), \quad \varphi \in \mathbb{T}^n,\ I \in V \subset \mathbb{R}^n,\ \varepsilon \ll 1.$$

For $\varepsilon = 0$, the action variables I are constants of motion. Arnold diffusion analyzes the drastic changes that the actions can undergo for small $\varepsilon > 0$.

In the setting of nearly integrable N-body problems, the existence of Arnold diffusion can be analyzed in very different regions in the phase space and for very different ranges of the involved parameters. As far as the authors know, the first paper dealing with Arnold diffusion in Celestial Mechanics is [Moe96] (later completed in [Zhe10]) who considers the five body problem. In [DGR11], the authors analyze unstable behavior for the three body problem close to the Lagrangian point L_1.

Concerning to the growth of angular momentum, the paper [FGKR11] obtains such behavior for the RPE3BP (Hamiltonian (1)) along the mean motion resonances, which implies a change of eccentricty in the osculating ellipse of the body of mass zero. More related to our paper, is the recent [DKdlRS14], where such behavior is obtained in a neighborhood of parabolic motions, the so-called invariant manifolds of infinity. In [DKdlRS14], it is proven the existence of orbits whose angular momentum $G(t)$ satisfies $G(0) < G_1$, $G(T) > G_2$ for some $T > 0$ and for any given $G_2 > G_1$. Nevertheless, they need to assume that μ is exponentially small with respect to G_2 and e_0 is polynomially small with respect to G_2.

We expect that this result can be generalized to any value of the mass ratio. Nevertheless, in the present paper we only give a conditional result (see Remark 1.2).

1.3 Main result

We give in this section the main result that we obtain.

Theorem 1.1. *Fix any $\mu \in (0, 1/2]$. There exists $e_0^*(\mu) > 0$ such that for any $e_0 \in (0, e_0^*(\mu))$ there exists an orbit $(q(t), p(t))$ of (1) which is oscillatory. Namely, it satisfies*

$$\limsup_{t\to+\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to+\infty} \|q\| < +\infty.$$

This theorem shows that $OS^+ \neq \emptyset$. Proceeding analogously, one can show that $OS^- \neq \emptyset$. Nevertheless, our techniques do not allow us to show that $OS^- \cap OS^+ \neq \emptyset$. To do that, one needs to consider more sophisticated techniques. Indeed, the ideas developed by Moser rely on the construction of a horseshoe which has branches arbitrarily close to infinity. This allows to combine all possible past and future behavior. Here, as we explain in Section 1.4, we rely on different techniques, which are simpler to generalize to higher dimensions: shadowing certain invariant objects. These techniques can only be applied to one time direction, either future or past, but not to both of them at the same time.

We only obtain oscillatory motions provided the primaries perform nearly circular orbits. It is expected that such motions exist for any value of the eccentricity $e_0 \in [0, 1)$. The approach presented in this paper can be applied to this more general setting. The only additional difficulty is to extend the proof of transversality of the invariant manifolds of infinity to this wider range of parameters (see Section 2 for more details). In this paper, we use perturbative arguments and use that this transversality is already known for the circular problem [GMS15].

Remark 1.2. *The geometric framework we use to prove Theorem 1.1 can be also used to obtain orbits with large drift in the angular momentum, provided some non-degeneracy condition is* *satisfied. If such condition could be verified to be true, one would generalize the results in [DKdlRS14] by obtaining the following. Fix $\mu \in (0, 1/2]$. There exists $G_0(\mu) > 0$ such that for any $G_2 \geq G_1 \geq G_0(\mu)$, there exists $e_0^*(G_2) > 0$ such that for any $e_0 \in (0, e_0^*(G_2))$ there exist a time $T > 0$ and an orbit $(q(t), p(t))$ of (1) whose angular momentum G satisfies*

$$G(0) \leq G_1 \quad \textit{and} \quad G(T) \geq G_2.$$

The needed non-degeneracy condition is explained in Remark 2.6.

1.4 Common framework for oscillatory motions and Arnold diffusion

The purpose of this section is to relate the results of both Theorem 1.1 and Remark 1.2 and put them in the same framework. This is explained in more detail in Section 2.

Let us start by explaining Moser ideas to obtain oscillatory orbits for the Sitnikov example [Mos73] and applied in [LS80a] to the RPC3BP. The RPC3BP (Hamiltonian (1) with $e_0 = 0$) has a first integral, the Jacobi constant. Then, in suitable coordinates, the RPC3BP can be reduced to a two dimensional Poincaré map for which "infinity" $\{|q| = +\infty, \dot{q} = 0\}$ is a parabolic fixed point.

Assume for a moment that this fixed point is hyperbolic. Then, it would have stable and unstable invariant manifolds. Assume that these invariant manifolds intersect transversally. Then, Smale Theorem would ensure that there exists a horseshoe. Suitable orbits in this horseshoe travel close to the invariant manifolds and the lim inf of the distances to the fixed point is zero. Such orbits, in the original coordinates, are oscillatory.

The infinity point of the RPC3BP is not hyperbolic but parabolic. [LS80a], as done in [Mos73] for the Sitnikov problem, shows that even if it is parabolic one can carry out the same strategy. Note that one has to face different difficult issues: prove the existence of the invariant manifolds of the parabolic point (see [McG73]), prove that they intersect transversally and prove a Lambda lemma for parabolic points which implies the existence of symbolic dynamics.

To carry out this strategy in the elliptic case is certainly more involved since the phase space has dimension five and therefore one cannot reduce the dynamics to an area preserving map. The stroboscopic Poincaré map is four dimensional. Thus, infinity cannot be reduced to a fixed point but it forms a cylinder with one angular variable and one real variable (see Section 2). This cylinder is "normally parabolic" and it has invariant manifolds. Since in this paper we consider the elliptic problem as a perturbation of the circular one, the result in [GMS15] implies that the invariant manifolds of this cylinder intersect transversally.

To prove that such transversal intersections lead to oscillatory motions we do not rely on the construction of symbolic dynamics as Moser did [Mos73]. Instead, we use the standard method of Arnold diffusion to construct a transition chain of tori [Arn64] (in the present setting of fixed points). That is, we find a sequence of fixed points belonging to the cylinder of infinity which are connected by transversal heteroclinic orbits. In Arnold diffusion problems usually it is enough to construct a finite chain. Instead, to obtain oscillatory orbits one has to construct an infinite chain and then prove the existence of an orbit which shadows the chain. This construction is much simpler than a horseshoe and leads to oscillatory motions forward (or backward) in time. Certainly a horseshoe gives much more information and imply a plethora of different types of motion. In particular, it allows to combine different types of final motions in the past and in the future (including motions which are oscillatory as both time tends to plus and minus infinity). Our simpler techniques can only show the existence of oscillatory motions in one direction of time. Nevertheless, we think that one of the main interests of this paper is to show that such a simple mechanism as the existence of transtion chains leads to oscillatory motions.

The vertical direction of the cylinder of infinity can be parameterized by the angular momentum of the orbits. To prove the existence of oscillatory orbits one needs to construct a chain which remains in a compact portion of this cylinder. The reason is that orbits shadowing chains with unbounded angular momentum cannot be oscillatory since $\liminf |q| = +\infty$. In fact, we can prove the existence of chains confined in very thin portions of the cylinder and thus with almost constant angular momentum. This is done by considering the scattering map [DdlLS08] and studing its dynamical properties. Now, one can ask the opposite question. It is possible to construct a chain which implies a large deviation in the angular momentum? Orbits shadowing such chains have a big drift in angular momentum and present the phenomenon of Arnold diffusion as explained in Remark 1.2. Summarizing, the orbits given both by Theorem 1.1 and Remark 1.2 are obtained thanks to suitable transition chains associated to the cylinder of infinity.

The structure of the paper goes as follows. First in Section 2 we analyze the invariant manifolds of infinity. We prove its existence and regularity and we analyze their transversal intersections. This allows us to construct the needed transition chain of periodic orbits. In Section 3, we state a Lambda lemma which can be applied to the invariant manifolds of the normally parabolic cylinder of infinity. This Lambda lemma allows us to prove Theorem 1.1. The prove of the Lambda Lemma is deferred to Section 4.

2 The invariant manifolds of infinity

Let (r, α) be the polar coordinates in the plane and (y, G) their symplectic conjugate momenta. Then, the Hamiltonian (1) becomes

$$\widetilde{H}(r, \alpha, y, G, t; e_0) = \frac{1}{2}\left(\frac{G^2}{r^2} + y^2\right) - U(r, \alpha, t; e_0),$$

where $U(r, \alpha, t; e_0) = V(re^{i\alpha}, t; e_0)$. The variable G is the angular momentum of the third body. As we deal with a non-autonomous system, we add the equation $\dot{s} = 1$ to the Hamiltonian equations of Hamiltonian $\widetilde{H}(r, \alpha, y, G, s; e_0)$.

Since we want to study the invariant manifolds of infinity, we consider the McGehee coordinates (x, α, y, G) where

$$r = \frac{2}{x^2}, \text{ for } x > 0.$$

We obtain the new system

$$\begin{aligned}
\dot{x} &= -\frac{1}{4}x^3 y & \dot{y} &= \frac{1}{8}G^2 x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}}{\partial x}\\
\dot{\alpha} &= \frac{1}{4}x^4 G & \dot{G} &= \frac{\partial \mathcal{U}}{\partial \alpha}\\
\dot{s} &= 1
\end{aligned} \tag{2}$$

where the potential $\mathcal{U}$ is given by

$$\mathcal{U}(x, \alpha, s; e_0) = U(2/x^2, \alpha, s; e_0) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right)$$

with

$$
\begin{aligned}
|q-q_{\mathrm{S}}|^2 &= \sigma_{\mathrm{S}}^2 = 1-\mu r_0(s)x^2\cos(\alpha-v(s))+\frac{1}{4}\mu^2 r_0^2(s)x^4,\\
|q-q_{\mathrm{J}}|^2 &= \sigma_{\mathrm{J}}^2 = 1+(1-\mu) r_0(s)x^2\cos(\alpha-v(s))+\frac{1}{4}(1-\mu)^2 r_0^2(s)x^4.
\end{aligned}
$$

We write the potential as

$$
\mathcal{U}=\frac{x^2}{2}+\Delta\mathcal{U}. \tag{3}
$$

Since

$$
U(r,\alpha,t;e_0)=\frac{1}{r}-\frac{\mu(1-\mu)}{2}(1-3\cos(\alpha-v(t)))\frac{\rho^2(t)}{r^3}+\mathcal{O}\left(\frac{\mu}{r^4}\right),
$$

the potential $\Delta\mathcal{U}$ satisfies $\Delta\mathcal{U}=\mathcal{O}(\mu x^6)$.

In view of (2), now "infinity" is foliated by the parabolic periodic orbits

$$
\widetilde{\Lambda}_{\alpha_0,G_0}=\{(x,\alpha,y,G,s)\in\mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{T}\mid x=y=0,\ \alpha=\alpha_0,\ G=G_0\}.
$$

Next theorem claims that each periodic orbit $\widetilde{\Lambda}_{\alpha_0,G_0}$ has stable and unstable 2-dimensional invariant manifolds $W^s(\widetilde{\Lambda}_{\alpha_0,G_0})$ and $W^u(\widetilde{\Lambda}_{\alpha_0,G_0})$.

Theorem 2.1. *Let $\widetilde{\phi}_t$ the flow of the system (2) and define the projections $\pi_x(x,\alpha,y,G,s)=x$ and $\pi_{(x,y)}(x,\alpha,y,G,s)=(x,y)$. Let $(\alpha_0,G_0)\in\mathbb{T}\times\mathbb{R}$. There exists ρ_0 such that for any $0<\rho<\rho_0$, the local stable set*

$$
\begin{aligned}
W^s_\rho(\widetilde{\Lambda}_{\alpha_0,G_0})=\{&(x,\alpha,y,G,s)\in\mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{T}\mid \pi_x\phi_t(x,\alpha,y,G,s)>0,\\
&|\pi_{(x,y)}\widetilde{\phi}_t(x,\alpha,y,G,s)|\le\rho,\ \lim_{t\to+\infty}\operatorname{dist}(\widetilde{\phi}_t(x,\alpha,y,G,s),\widetilde{\Lambda}_{\alpha_0,G_0})=0\},
\end{aligned}
$$

is a 2-dimensional manifold.

Moreover, there exists $u_0>0$ such that $W^s_\rho(\widetilde{\Lambda}_{\alpha_0,G_0})$ admits a $\mathcal{C}^\infty$ parametrization $\gamma^s_{\alpha_0,G_0}:[0,u_0)\times\mathbb{T}\to\mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{T}$, with $\gamma^s_{\alpha_0,G_0}(0,s)=(0,\alpha_0,0,G_0,s)$, analytic in $(0,u_0)\times\mathbb{T}$, which depends analytically on $(\alpha_0,G_0)\in\mathbb{T}\times\mathbb{R}$. The analogous result for the (local) unstable set also holds.

The proof of this theorem follows from Proposition 4.1 in Section 4 (the technical details are done in Appendix A). A related result concerning the invariant manifolds of infinity is given in [Rob84] (see also [Rob15]). Nevertheless we have included the proof of this theorem, which is based in the parameterization method [CFdlL03, BFdlLM07], for two reasons. One the one hand, we are not able to follow all the details in the proof in [Rob84, Rob15]. On the other hand, our result provides the regularity of the stable/unstable foliations of the invariant manifolds, needed in Lemma 3.1.

Theorem 2.1 shows that the points which tend asymptotically in forward (or backward) time to the periodic orbit $\widetilde{\Lambda}_{\alpha_0,G_0}$ form a manifold. Since this periodic orbit is not hyperbolic but parabolic the rate of convergence is not exponential and its invariant manifolds are not analytic at the periodic orbit but only $\mathcal{C}^\infty$. They are analytic at any other point.

As we have explained in Section 1, when $\mu=0$, the invariant manifolds $W^s(\widetilde{\Lambda}_{\alpha_0,G_0})$ and $W^u(\widetilde{\Lambda}_{\alpha_0,G_0})$ coincide and form a two-parameter family of parabolas in the configuration space (in the original cartesian coordinates). We study the splitting of these invariant manifolds for any $\mu\in(0,1/2]$ in two steps. First for the circular problem ($e_0=0$) and then for the elliptic problem ($0<e_0\ll 1$). In this study we need formulas for the homoclinic manifolds when $\mu=0$. For the derivation of such formulas, one can see [LS80a].

Lemma 2.2. *Take $\mu = 0$ and fix $\alpha_0 \in \mathbb{T}$ and $G_0 \neq 0$. Then, system (2) has a family of homoclinic orbits to the periodic orbit $\widetilde{\Lambda}_{\alpha_0,G_0}$, which is given by*

$$
\begin{aligned}
x_{\mathrm{h}}(t; G_0) =& \frac{2}{G_0(1+\tau^2)^{1/2}} \\
y_{\mathrm{h}}(t; G_0) =& \frac{2\tau}{G_0(1+\tau^2)} \\
\alpha_{\mathrm{h}}(t; \alpha_0) =& \alpha_0 + \widetilde{\alpha}_{\mathrm{h}}(t), \quad \widetilde{\alpha}_{\mathrm{h}}(t) = 2\arctan\tau \\
G_{\mathrm{h}}(t; G_0) =& G_0 \\
s_{\mathrm{h}}(t; s_0) =& s_0 + t
\end{aligned}
$$

where $s_0 \in \mathbb{T}$ is a free parameter and τ and the time t are related through

$$
t = \frac{G_0^3}{2}\left(\tau + \frac{\tau^3}{3}\right).
$$

From now on, we abuse notation and we consider the functions x_{h}, y_{h} and $\widetilde{\alpha}_{\mathrm{h}}$ both as functions of τ or t.

2.1 Scattering map for the circular problem in the extended phase space

Using the results of Theorem 2.1, for any $G_1 > 0$, the invariant set

$$
\widetilde{\Lambda}^{[G_1,+\infty)} = \bigcup_{\alpha_0\in\mathbb{T}, G_0\geq G_1} \widetilde{\Lambda}_{\alpha_0,G_0} = \{(x,\alpha,y,G,s) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{T} \mid x = y = 0,\ G \geq G_1\} \quad (4)
$$

is a “normally parabolic” 3-dimensional invariant manifold with stable and unstable 4-dimensional invariant manifolds

$$
W^{\varsigma}(\widetilde{\Lambda}^{[G_1,+\infty)}) = \bigcup_{\alpha_0\in\mathbb{T}, G_0\geq G_1} W^{\varsigma}(\widetilde{\Lambda}_{\alpha_0,G_0}), \ \ \varsigma = u, s.
$$

Theorem 2.2 of [GMS15] implies that, when $e_0 = 0$ and $\mu \in (0, 1/2]$, there exists $G^* \gg 1$ such that, for any $G_1 > G^*$, the invariant manifolds $W^s(\widetilde{\Lambda}^{[G_1,+\infty)})$ and $W^u(\widetilde{\Lambda}^{[G_1,+\infty)})$ intersect transversally in the whole space along two different 3-dimensional homoclinic manifolds.

Following [DdlLS08], this transversality allows us to define two scattering maps $\widetilde{\mathcal{S}}_0^{\pm}$ associated to the two different transversal homoclinic intersections between $W^s(\widetilde{\Lambda}^{[G_1,+\infty)})$ and $W^u(\widetilde{\Lambda}^{[G_1,+\infty)})$ and to obtain formulas for these maps. To this end, we consider the Poincaré function for $e_0 = 0$,

$$
L(\alpha_0, G_0, s_0, \sigma; 0) = \int_{-\infty}^{\infty} \Delta\mathcal{U}(x_{\mathrm{h}}(\sigma + t; G_0), \alpha_0 + \widetilde{\alpha}_{\mathrm{h}}(\sigma + t; G_0), s_0 + t; 0)\, dt \qquad (5)
$$

where $\Delta\mathcal{U}$ is the potential defined in (3) and $(x_{\mathrm{h}}, \widetilde{\alpha}_{\mathrm{h}})$ are components of the parameterization of the unperturbed separatrix given in Lemma 2.2.

For any (α_0, G_0, s_0), the function $\sigma \mapsto L(\alpha_0, G_0, s_0, \sigma; 0)$ has two critical points $\sigma_{\pm}^*$ given by:

$$
\sigma_-^* = s_0 - \alpha_0,\ \ \sigma_+^* = \pi + s_0 - \alpha_0. \qquad (6)
$$

This fact is given by Proposition 3.1 of [GMS15] (note that this proposition is stated in certain scaled variables).

The reason to obtain such simple formulas for the critical points is twofold. On the one hand, when $e_0 = 0$, the potential only depends on the angles α_0 and s_0 through $\alpha_0 - s_0$. Thus, since

$$L(\alpha_0, G_0, s_0, \sigma; 0) = L(\alpha_0, G_0, s_0 - \sigma, 0; 0),$$

the Poincaré function only depends on one angular variable $\alpha_0 - s_0 + \sigma$. On the other hand, its Fourier expansion only contains cosines (see the Appendix of [GMS15]).

Associated to the zero σ_+^*, there is a heteroclinic connection between two periodic orbits in $\widetilde{\Lambda}^{[G_1,+\infty)}$, which are μG_0^{-4}–close to $\widetilde{\Lambda}_{\alpha_0,G_0}$ (analogously for σ_-^*). These heteroclinic connections satisfy

$$\widetilde{\Gamma}_\pm^{\alpha_0,G_0,s_0}(t) = \big(x_{\mathrm{h}}(\sigma_\pm^* + t; G_0), y_{\mathrm{h}}(\sigma_\pm^* + t; G_0), \alpha_0 + \alpha_{\mathrm{h}}(\sigma_\pm^* + t; G_0), G_0, s_0 + t\big) + \mathcal{O}\left(\mu G_0^{-4}\right)$$

(see [GMS15]). Note that these heteroclinic orbits are well defined for any $\alpha_0 \in \mathbb{T}$, $s_0 \in \mathbb{T}$ and $G_0 \geq G_1$ (see (4)). This implies that there is a homoclinic channel (see [DdlLS06]) which is defined for all points in $\widetilde{\Lambda}^{[G_1,+\infty)}$. Thus, we can define global scattering maps

$$\widetilde{\mathcal{S}}_0^\pm : \widetilde{\Lambda}^{[G_1,+\infty)} \longrightarrow \widetilde{\Lambda}^{[G_1,+\infty)}.$$

Following [DdlLS08], recall that $x_+ = \widetilde{\mathcal{S}}_0^+(x_-)$ if there exists a heteroclinic connection between these two points through the prescribed homoclinic channel (see [DdlLS08] for a more precise definition and properties of the scattering map). Usually it is not possible to define globally the scattering map since it is only defined locally in open sets. Then, globally, it can be multivaluated (see [DdlLS06]). The particular form of the circular problem allows us to define it globally. Note also that, in principle, the scattering map should map $\widetilde{\Lambda}^{[G_1,\infty)}$ to a bigger cylinder. Nevertheless, as it is shown in the next proposition, in the circular case the image is the same cylinder.

Proposition 2.3. *Let $G_1 > G^*$. The scattering maps $\widetilde{\mathcal{S}}_0^\pm : \mathbb{T}\times[G_1,+\infty)\times\mathbb{T} \longrightarrow \mathbb{T}\times[G_1,+\infty)\times\mathbb{T}$ are of the following form,*

$$\widetilde{\mathcal{S}}_0^\pm(\alpha, G, s) = (\alpha + f^\pm(G), G, s),$$

where

$$f^\pm(G) = -\mu(1-\mu)\frac{3\pi}{2G^4} + \mathcal{O}\left(G^{-8}\right). \tag{7}$$

This proposition is proven in Appendix B.

2.2 Reduction to the Poincaré map

We reduce the dimension of the system by considering the stroboscopic-Poincaré map associated to the section $\Sigma = \{s = s_0\}$,

$$\begin{array}{cccc} \mathcal{P}: & \Sigma & \longrightarrow & \Sigma \\ & (x, y, \alpha, G) & \mapsto & \mathcal{P}(x, y, \alpha, G) \end{array} \tag{8}$$

Then $\Lambda_{\alpha_0,G_0} = \widetilde{\Lambda}_{\alpha_0,G_0} \cap \Sigma$ is a two parameter family of parabolic fixed points of $\mathcal{P}$ with 1-dimensional stable and unstable manifolds

$$W^\varsigma(\Lambda_{\alpha_0,G_0}) = W^\varsigma(\widetilde{\Lambda}_{\alpha_0,G_0}) \cap \Sigma, \quad \varsigma = u, s.$$

Analogously, $\Lambda^{[G_1,+\infty)} = \widetilde{\Lambda}^{[G_1,+\infty)} \cap \Sigma$ is the 2-dimensional normally parabolic invariant cylinder of infinity with 3-dimensional invariant stable and unstable manifolds

$$W^{\varsigma}(\Lambda^{[G_1,+\infty)}) = W^{\varsigma}(\widetilde{\Lambda}^{[G_1,+\infty)}) \cap \Sigma, \quad \varsigma = u, s,$$

which, for $e_0 = 0$, intersect transversally along two 2-dimensional homoclinic channels. The two scattering maps associated to these homoclinic channels are given by

$$\mathcal{S}_0^{\pm}(\alpha, G) = (\alpha + f^{\pm}(G), G), \tag{9}$$

where $f^{\pm}$ is the function given in (7). They do not depend on the section Σ.

2.3 Scattering map of the elliptic problem

Once we have analyzed the splitting of the invariant manifolds of $\Lambda^{[G_1,+\infty)}$ for the circular problem and derived formulas for the two scattering maps, now we consider the elliptic problem for e_0 small enough. Note that e_0 is a regular parameter of the elliptic problem and, therefore, we can apply classical perturbative arguments to the stroboscopic-Poincaré map $\mathcal{P}$ (see [DdlLS08]).

Let $G_2 > G_1 > G^*$ (see (4) and Proposition 2.3) be fixed. We call $\Lambda^{[G_1,G_2]}$ to $\Lambda^{[G_1,+\infty)} \cap \{G_1 \leq G \leq G_2\}$, which is compact and invariant. Then, for e_0 small enough, the stable and unstable manifolds of $\Lambda^{[G_1,G_2]}$ intersect transversally. Note that the smallness of e_0 depends on the chosen interval. The perturbative arguments imply that there are two global homoclinic channels as in the circular problem. These two channels define two scattering maps

$$\mathcal{S}^{\pm} : \Lambda^{[G_1,G_2]} \longrightarrow \Lambda^{[G^*,+\infty)},$$

which depend regularly on e_0,

$$\mathcal{S}^{\pm} = \mathcal{S}_0^{\pm} + e_0 \mathcal{S}_1^{\pm} + \mathcal{O}\left(e_0^2\right), \tag{10}$$

where $\mathcal{S}_0^{\pm}$ are the scattering maps of the circular problem given by (9). The theory developed in [DdlLS08] does not directly apply to this case since the invariant cylinder is not hyperbolic, but parabolic. Nevertheless, the arguments in [DKdlRS14, Proposition 4] show that the theory of scattering maps do apply also to this problem. Hence, the maps $\mathcal{S}^{\pm}$ are area preserving maps on the cylinder.

To construct oscillatory orbits of the elliptic problem, we need an infinite transition chain: a sequence of fixed points belonging to $\Lambda^{[G_1,G_2]}$ with transversal heteroclinic connections between consecutive points of the chain. By the definition of the scattering map, any bounded (forward or backward) orbit of one of the scattering maps provides such a chain.

To obtain bounded orbits of the scattering maps we observe that, if e_0 is small enough, the maps $\mathcal{S}^{\pm}$ possess invariant curves. Indeed, in view of (7), $\mathcal{S}^{\pm}$ are twist maps if e_0 is small enough. Then, we can apply the following twist theorem, due to Herman, from [Her83]:

Theorem 2.4 (Twist theorem). *Let $f : [0,1] \times \mathbb{T} \to [0,1] \times \mathbb{T}$ be an exact symplectic $\mathcal{C}^l$ map with $l > 4$. Assume that $f = f_0 + \delta f_1$, where $f_0(I, \psi) = (I, \psi + A(I))$, A is $\mathcal{C}^l$, $|\partial_I A| > M$ and $\|f_1\|_{\mathcal{C}^l} \leq 1$. Then, if $\delta^{1/2} M^{-1} = \rho$ is sufficiently small, for a set of ω of Diophantine numbers of exponent $\theta = 5/4$, we can find invariant curves which are the graph of $\mathcal{C}^{l-3}$ functions u_ω, the motion on them is $\mathcal{C}^{l-3}$ conjugate to the rotation by ω, and $\|u_\omega\|_{\mathcal{C}^{l-3}} \leq C\delta^{1/2}$.*

In our setting we have that the twist condition satisfies $|\partial_G A| \gtrsim \mu G^{-5}$ whereas the $\delta \lesssim e_0$. Therefore, for any $G_2 > G_1 > G^*$ and e_0 small enough, the map $\mathcal{S}^\pm$ has KAM curves inside $\Lambda^{[G_1,G_2]}$. Any orbit of $\mathcal{S}^\pm$ in any of these KAM curves is bounded, and so are all orbits between any two of these KAM curves.

Corollary 2.5. *Consider any constants $G_2 > G_1 > G^*$. Then, for e_0 small enough (which may depend on G_2 and G_1), the scattering maps $\mathcal{S}^\pm$ considered in* (10) *have orbits which remain for all time in $\Lambda^{[G_1,G_2]}$.*

As we have explained, coming back to the original Poincaré map (15), this bounded orbit of the scattering map corresponds to a sequence of fixed points $\{\Lambda_{\alpha_k,G_k}\}_{k\in\mathbb{N}} \subset \Lambda^{[G_1,G_2]}$ such that $W^u(\Lambda_{\alpha_k,G_k})$ intersects transversally $W^s(\Lambda^{[G_1,G_2]})$ at a point P which belongs to $W^s(\Lambda_{\alpha_{k+1},G_{k+1}})$.

Remark 2.6. *To obtain oscillatory motions we have looked for an infinite transition chain with bounded angular momentum, or equivalently, for a bounded orbit of one of the scattering maps $\mathcal{S}^\pm$. If we want orbits with a drift in the angular momentum (see Remark 1.2), we have to look for a transition chain connecting fixed points of $\mathcal{P}$ in* (15) *with a large difference in angular momentum. In this case it suffices to look for a finite transition chain.*

For the oscillatory motions, we have used the fact that the scattering maps $\mathcal{S}^\pm$ are nearly integrable and possess KAM tori. KAM tori act as barriers for the orbits of the scattering maps and therefore, using only one scattering map, it is impossible to construct orbits with a large drift in G. The usual strategy to prove the existence of Arnold diffusion (see for instance [Arn64, DdlLS06]) is to combine one scattering map with the inner dynamics of the invariant cylinder induced by the Poincaré map (15). *Combining these two dynamics one obtains Arnold diffusion. Here this approach is not possible since the inner dynamics is trivial (the cylinder $\Lambda^{[G_1,G_2]}$ is filled by fixed points). Thus, we rely on an idea developed in [DKdlRS14] which is to combine the two scattering maps $\mathcal{S}^+$ and $\mathcal{S}^-$.*

In [LC07], Le Calvez showed the following: consider two area preserving twist maps on a cylinder. Assume that they do not have common invariant curves. Then, combining them one can obtain orbits with a drift in the action component. Thus, as long as the maps $\mathcal{S}^\pm$ did not have common invariant curves, we could obtain transition chains with a large drift in G. This fact has proven to be true for μ small enough in [DKdlRS14]. Nevertheless, it is not straightforward to verify it in the present setting. That is the non-degeneracy assumption mentioned in Remark 1.2.

3 Shadowing orbits

The second ingredient in the proof of Theorem 1.1 is what is usually called a Lambda or Shadowing lemma. That is, to analyze how orbits close to the stable manifold of one of the fixed points of $\Lambda^{[G_1,G_2]}$ (see Theorem 2.1) evolve and stretch along the unstable invariant manifold of the fixed point. Such results allow to shadow a concatenation of heteroclinic orbits which connect different (or the same) fixed points.

Usually in the literature of Arnol'd diffusion, one considers more general Lambda lemmas which allow to shadow invariant manifolds of more general objects (see for instance [Mar96, Cre97, FM00, Cre01, GZ04, ZG04, Bol06, DGR13, Sab13]). Nevertheless, such results deal with invariant manifolds of objects which are (normally or partially) hyperbolic and not parabolic as in the present setting.

We generalize such result to the parabolic setting. Nevertheless, we are dealing with the simplest case, that is, fixed points for the 4-dimensional Poincaré map, which correspond to periodic orbits for the flow. In the general hyperbolic setting usually one considers $\mathcal{C}^1$ Lambda lemmas even if for the shadowing results only a $\mathcal{C}^0$ version is needed. The reason is that to have $\mathcal{C}^0$ estimates including the tangential directions, one needs to compute the $\mathcal{C}^1$ estimates at the same time. In the present setting, since we are dealing with the invariant manifolds of fixed points of maps, there are not tangential directions and, therefore, one can directly compute the $\mathcal{C}^0$ estimates *à la Shilnikov* (see [Šil67]). This is a considerable simplification since in the parabolic setting the classical $\mathcal{C}^1$ estimates are no longer true, as shown in [GK12].

Lemma 3.1. *Let Γ be a curve which transversally intersects $W^s(\Lambda^{[G_1,G_2]})$ at $P \in W^s(\Lambda_{\alpha_0,G_0})$ for some $\Lambda_{\alpha_0,G_0} \in \Lambda^{[G_1,G_2]}$. Let Z be a point on $W^u(\Lambda_{\alpha_0,G_0})$. For any neighborhood $\mathcal{U}$ of Z in $\mathbb{R}^4$ and any $\varepsilon > 0$, there exists a point $a \in B_\varepsilon(P) \cap \Gamma$ and a positive integer n (which depends on Z, ε and $\mathcal{U}$) such that $\mathcal{P}^n(a) \in \mathcal{U}$.*

As a consequence $W^u(\Lambda_{\alpha_0,G_0}) \subset \overline{\cup_{j\geq 0}\mathcal{P}^j(\Gamma)}$.

The proof of Lemma 3.1 is placed in Section 4. The lemma follows from Proposition 4.3.

Remark 3.2. *By reversing time, one can consider curves Γ intersecting transversally $W^u(\Lambda^{[G_1,G_2]})$ at $P \in W^u(\Lambda_{\alpha_0,G_0})$. Then, we have the following statement. Let Z be a point on $W^s(\Lambda_{\alpha_0,G_0})$. For any neighborhood $\mathcal{U}$ of Z in $\mathbb{R}^4$ and any $\varepsilon > 0$ there exists a point $a \in B_\varepsilon(P) \cap \Gamma$ and a positive integer n (which depends on Z, ε and $\mathcal{U}$) such that $\mathcal{P}^{-n}(a) \in \mathcal{U}$.*

As a consequence $W^s(\Lambda_{\alpha_0,G_0}) \subset \overline{\cup_{j\geq 0}\mathcal{P}^{-j}(\Gamma)}$.

Now it only remains to apply Lemma 3.1 to shadow the transition chain given in Corollary 2.5. This argument is standard, one can see, for instance, [DdlLS00]. We include it here for completeness.

Proposition 3.3. *Let $\{\Lambda_{\alpha_k,G_k}\}_{k\geq 0}$ be a family of parabolic fixed points in $\Lambda^{[G_1,G_2]}$ of the Poincaré map $\mathcal{P}$ in* (15) *such that, for all k, $W^u(\Lambda_{\alpha_k,G_k})$ intersects transversally $W^s(\Lambda^{[G_1,G_2]})$ at $p_k \in W^s(\Lambda_{\alpha_{k+1},G_{k+1}})$. Consider two sequences of real numbers $\{\delta_k\}_{k\geq 0}$ and $\{\widetilde{\delta}_k\}_{k\geq 0}$, $\delta_k, \widetilde{\delta}_k > 0$. Then, there exist $a \in B_{\delta_0}(\Lambda_{\alpha_0,G_0})$ and two sequences of natural numbers $\{N_k\}_{k\geq 0}$, $\{\widetilde{N}_k\}_{k\geq 0}$, $N_k < \widetilde{N}_k < N_{k+1} < \widetilde{N}_{k+1}$ for all k, such that $\mathcal{P}^{N_k}(a) \in B_{\delta_k}(\Lambda_{\alpha_k,G_k})$ and $\mathcal{P}^{\widetilde{N}_k}(a) \in B_{\tilde{\delta}_k}(p_k)$ for all k.*

Proof. The proof follows closely the arguments in [DdlLS00].

We are going to construct a sequence of nested non-empty compact sets $\overline{U}_i \subset B_{\delta_0}(\Lambda_{\alpha_0,G_0})$ with the following property: if $p \in \overline{U}_i$, its forward orbit by $\mathcal{P}$ visits the balls $\overline{B}_{\delta_k}(\Lambda_{\alpha_k,G_k})$ and $B_{\tilde{\delta}_k}(p_k)$ for $0 \leq k \leq i$.

Let $x_0 \in B_{\delta_0}(\Lambda_{\alpha_0,G_0}) \cap W^s(\Lambda_{\alpha_0,G_0})$. We can choose an open neighborhood U_0 of x_0 such that $U_0 \subset \overline{U}_0 \subset B_{\delta_0}(\Lambda_{\alpha_0,G_0})$. By Lemma 3.1 (see Remark 3.2), there exist a point $y_0 \in B_{\tilde{\delta}_0}(p_0) \cap W^s(\Lambda_{\alpha_1,G_1})$ and an integer $n_0 > 0$ such that $\mathcal{P}^{-n_0}(y_0) \in U_0$. By continuity of the map $\mathcal{P}$, there exists an open neighborhood V_0 of y_0 such that $\overline{\mathcal{P}^{-n_0}(V_0)} \subset U_0$. Since $y_0 \in W^s(\Lambda_{\alpha_1,G_1})$, there exists $n_1 > 0$ such that $\mathcal{P}^{n_1}(y_0) = x_1 \in B_{\delta_1}(\Lambda_{\alpha_1,G_1}) \cap W^s(\Lambda_{\alpha_1,G_1})$. By continuity, there exists an open neighborhood $\tilde{U}_1$ of x_1, such that $\mathcal{P}^{-n_1}(\tilde{U}_1) \subset V_0$. We define $U_1 = \mathcal{P}^{-n_0-n_1}(\tilde{U}_1)$. By construction, $\overline{U_1} \subset U_0 \subset \overline{U_0}$ and if $x \in \overline{U_1}$, $\mathcal{P}^{n_0}(x) \in V_0 \subset B_{\tilde{\delta}_0}(p_0)$ and $\mathcal{P}^{n_0+n_1}(x) \in \overline{U_1} \subset B_{\delta_1}(\Lambda_{\alpha_1,G_1})$. We take $N_0 = 0$, $\tilde{N}_0 = n_0$ and $N_1 = n_0 + n_1$.

The proof follows by induction. Then, any point in $\cap_{i\geq 0}\overline{U}_i \neq \emptyset$, satisfies the claim.

□

Proof of Theorem 1.1 is a direct consequence of this proposition.

Proof of Theorem 1.1. Let $\{\Lambda_{\alpha_k,G_k}\}_{k\geq 1}$ be one of the bounded orbits given in Corollary 2.5. Since $\Lambda_{\alpha_k,G_k} = \mathcal{S}^+(\Lambda_{\alpha_{k-1},G_{k-1}})$, $k \geq 1$, the unstable manifold of $\Lambda_{\alpha_{k-1},G_{k-1}}$ intersects transversally the stable manifold of the cylinder in a point p_k in the stable fiber of Λ_{α_k,G_k}.

Now we apply Proposition 3.3. We take $\delta_k = 1/k$ (or any other positive sequence with limit 0) and $\tilde{\delta}_k = \tilde{\delta}$ small enough so that the balls $B_{\tilde{\delta}}(p_k)$ do not intersect the cylinder $\Lambda^{[G_1,G_2]}$. Let N_k and $\tilde{N}_k$ be the natural numbers and $a \in B_{\delta_0}(\Lambda_{\alpha_0,G_0})$ be the point given by Proposition 3.3. Then, the orbit of a is oscillatory. Indeed, $\liminf_{k\to+\infty} \mathrm{dist}(\mathcal{P}^{N_k}(a), \Lambda_{\alpha_k,G_k}) = 0$ since $\delta_k \to 0$ as $k \to +\infty$ and $\limsup_{t\to+\infty} \mathrm{dist}(\mathcal{P}^{\tilde{N}_k}(a), \Lambda_{\alpha_k,G_k}) > 0$ since the balls $B_{\tilde{\delta}}(p_k)$ do not intersect $\{x = y = 0\}$. Pulling back the McGehee change of variables $r = 2/x^2$ and considering the original coordinates, we obtain oscillatory orbits. $\square$

4 A $\mathcal{C}^0$ Lambda lemma: proof of Lemma 3.1

4.1 Local behavior close to infinity

System (2) can be written as $r = 2/x^2$,

$$\begin{aligned}
\dot{x} &= -\frac{1}{4}x^3 y \\
\dot{y} &= \frac{1}{8}G^2 x^6 + \partial_r U\left(2x^{-2}, \alpha, t\right) = -\frac{1}{4}x^4 + x^6\mathcal{O}_1 \\
\dot{\alpha} &= \frac{1}{4}Gx^4 \\
\dot{G} &= \partial_\alpha U\left(2x^{-2}, \alpha, t\right) = \beta(\alpha, t)x^6 + x^8\mathcal{O}_1,
\end{aligned}$$

where β is a function which is 2π-periodic in its variables and $\mathcal{O}_k$ stands for $\mathcal{O}(\|(x,y)\|^k)$.

To straighten the lowest order terms, we make the change

$$\begin{aligned}
q &= \frac{1}{2}(x - y) \\
p &= \frac{1}{2}(x + y)
\end{aligned}$$

and to make α a central variable, we consider the new variable $\theta = \alpha + Gy$. Then, we have the new system

$$\begin{aligned}
\dot{q} &= \frac{1}{4}(q+p)^3(q + (q+p)^3\mathcal{O}_0) \\
\dot{p} &= -\frac{1}{4}(q+p)^3(p + (q+p)^3\mathcal{O}_0) \\
\dot{\theta} &= (q+p)^6\mathcal{O}_0 \\
\dot{G} &= (q+p)^6\mathcal{O}_0 \\
\dot{t} &= 1.
\end{aligned} \tag{11}$$

This system is a particular case of a system of the form

$$
\begin{aligned}
\dot{q} &= \frac{1}{4}(q+p)^3(q+(q+p)^3\mathcal{O}_0)\\
\dot{p} &= -\frac{1}{4}(q+p)^3(p+(q+p)^3\mathcal{O}_0)\\
\dot{z} &= (q+p)^6\mathcal{O}_0\\
\dot{t} &= 1
\end{aligned}
\tag{12}
$$

where $(q,p,z)\in\mathbb{R}\times\mathbb{R}\times K$, $K\subset\mathbb{R}^n$ a compact set. Now $\mathcal{O}_k=\mathcal{O}(\|(q,p)\|^k)$ are T-periodic functions on t and the estimate is uniform for $z\in K$.

Notice that for any $z_0\in\mathbb{R}^n$, the set

$$\widetilde{\Lambda}_{z_0}=\{q=p=0,\ z=z_0,\ t\in\mathbb{T}\}$$

is a periodic orbit of system (12). In this section we study its invariant manifolds. The set $\widetilde{\Lambda}=\cup_{z_0\in K}\widetilde{\Lambda}_{z_0}$ is invariant. In the case case of system (11), when $z_0=(\alpha_0,G_0)$, the set $\widetilde{\Lambda}$ corresponds to the "parabolic infinity".

Proposition 4.1. *Consider the system* (12). *The set $\widetilde{\Lambda}_{z_0}$ possesses invariant stable and unstable manifolds, $W^u_{z_0}$ and $W^s_{z_0}$. More concretely,*

$$W^u_{z_0}=\{(q,p,z,t)\mid (p,z)=\gamma^u(q,z_0,t),\ q\in[0,q_0),t\in\mathbb{T}\}$$

where

1. *γ^u is $\mathcal{C}^\infty$ with respect to q and analytic with respect to z_0 and t,*
2. *$\pi_p\gamma^u(q,z_0,t)=\mathcal{O}(q^2)$, $\pi_z\gamma^u(q,z_0,t)=z_0+\mathcal{O}(q^3)$, where π_p and π_z are the corresponding projections.*

The analogous statement holds for W^s, as a graph over p.

Proof. It is an immediate consequence of Theorem A.7 in the Appendix. Indeed, after changing the sign of time and introducing the new variables

$$\tilde{z}=\frac{1}{q+p}(z-z_0),$$

system (11) becomes

$$
\begin{aligned}
\dot{q} &= -\frac{1}{4}(q+p)^3(q+(q+p)^3\mathcal{O}_0)\\
\dot{p} &= \frac{1}{4}(q+p)^3(p+(q+p)^3\mathcal{O}_0)\\
\dot{\tilde{z}} &= \frac{1}{4}(q+p)^2(q-p)\tilde{z}+(q+p)^5\mathcal{O}_0\\
\dot{t} &= 1
\end{aligned}
\tag{13}
$$

and depends analytically on z_0 and t. It satisfies the hypotheses of Theorem A.7 with $N=4$ and $c=1/4$. Then, the origin of (13) has an invariant stable manifold parameterized by

$$\tilde{\gamma}(u,z_0,t)=(u,0,0,0)+\mathcal{O}(u^2).$$

From this parametrization, we have that the invariant manifold is the graph of a function over q satisfying $(p, \tilde{z}) = \mathcal{O}(q^2)$. The claim follows simply restoring the original variables.

The proof for the stable manifold is analogous. $\square$

Once we have the existence and regularity of the invariant manifolds, their straightening can be easily accomplished.

Proposition 4.2. *There exists a C^∞ change of variables which transforms system* (12) *into*

$$\begin{aligned} \dot{q} &= f(q,p)^3 q(1+\mathcal{O}_1) \\ \dot{p} &= -f(q,p)^3 p(1+\mathcal{O}_1) \\ \dot{z} &= f(q,p)^2 qp\mathcal{O}_1 \\ \dot{t} &= 1 \end{aligned} \tag{14}$$

defined for $q, p \geq 0$, where $f(q,p) = q + p + \mathcal{O}_2$ is a $\mathcal{C}^\infty$ function in $\{q, p \geq 0\}$ with bounded derivatives.

Proof. The change is the composition of two consecutive changes of variables. The first one is defined as follows. By Proposition 4.1, the map

$$\Gamma : (q, z_0, t) \mapsto (q, \pi_z \gamma^u(q, z_0, t), t)$$

is a $\mathcal{C}^\infty$ diffeomorphism from $\{(q, z_0, t) \mid 0 < q < \delta, t \in \mathbb{T}\}$ onto its image, for some $\delta > 0$. By item 2 of Proposition 4.1, there exists $\delta' > 0$ such that $\{(q, z, t) \mid 0 < q < \delta'\} \subset \Gamma(\{(q, z_0, t) \mid 0 < q < \delta\})$. Hence,

$$(q, z_0(q, z, t), t) = \Gamma^{-1}(q, z, t) = (q, z, t) + \mathcal{O}(q^3)$$

is a diffeomorphism.

The first change that we perform is given by

$$\tilde{q} = q, \quad \tilde{p} = p - \pi_p \gamma^u(q, z_0(q, z, t)), \quad \tilde{z} = z_0(q, z, t).$$

It is clearly a change of variables if q is small, and straightens the unstable leaves. Analogously, one straightens the stable leaves. The second change is the identity on the unstable manifold. $\square$

4.2 The Lambda lemma

To state the Lambda Lemma 3.1 it is more convenient to work with the Poincaré map

$$\mathcal{P} : \{t = t_0\} \longrightarrow \{t = t_0 + 2\pi\} \tag{15}$$

associated to the flow of the system (14).

Lemma 3.1 follows from the next statement.

Proposition 4.3. *Consider a point $Q = (q_f, 0, z_0) \in W^u(0, 0, z_0)$ and a $\mathcal{C}^1$ curve C parameterized by $\alpha(\delta) = (\delta, \alpha_p(\delta), \alpha_z(\delta))$, $\delta \in [0, \delta_0]$ for some $\delta_0 > 0$, which is transversal to $\{q = 0\}$ in a point $P = \alpha(0) = (0, p_0, z_0)$. For any $\rho \ll 1$, there exists a sequence $\{\delta_k\}_{k\geq 1} \searrow 0$ as $k \to \infty$ with $0 < \delta_k < \delta_0$, and an increasing sequence of $\{n_k\}_{k\geq 1} \subset \mathbb{N}$, $n_k \to +\infty$ as $k \to +\infty$, such that*

$$\mathcal{P}^{n_k}(\alpha(\delta_k)) \in B_\rho(Q).$$

Proof. We first study the associated equation (14) and later we show how to deduce the statement for the Poincaré map.

As $\{q=0\}$ and $\{p=0\}$ are invariant manifolds for system (14), the region $\{q>0, p>0\}$ is invariant. In this region, by Proposition 4.2, f is strictly positive for (q,p) small enough and satisfies $|f(q,p)| \geq K\|(q,p)\|$, for some $K>0$ depending only on the domain.

We rescale time by setting $\frac{ds}{dt} = f(q,p)^3$. In the new time s, in the region $\{q>0, p>0\}$, the system becomes

$$\begin{aligned} \dot{q} &= q(1+\mathcal{O}_1) \\ \dot{p} &= -p(1+\mathcal{O}_1) \\ \dot{z} &= qp\mathcal{O}_0 \\ \dot{t} &= (f(q,p))^{-3}. \end{aligned} \tag{16}$$

Call Ψ^s the flow associated with this equation.

Fix $\varepsilon_0>0$ and $\zeta_0>4$ such that the equation (16) is well defined for

$$(q,p,z,t) \in D = (0,\varepsilon_0]^2 \times B_{\zeta_0}(z_0) \times \mathbb{T} \subset \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{T}.$$

There exists $K>0$ such that the terms $\mathcal{O}_0$ and $\mathcal{O}_1$ appearing in (16) satisfy $|\mathcal{O}_0| \leq K$ and $|\mathcal{O}_1| \leq K\|(q,p)\| \ll 1$. Choose $\varepsilon \in (0,\varepsilon_0)$ such that $\widetilde{\varepsilon} = K\varepsilon \in (0, 1/10)$. Then, for any point in $(0,\varepsilon]^2 \times [G_1, G_2] \times \mathbb{T}^2$,

$$\begin{array}{rcll} (1-\tilde{\varepsilon})q & \leq \dot{q} \leq & (1+\tilde{\varepsilon})q & \\ -(1+\tilde{\varepsilon})p & \leq \dot{p} \leq & -(1-\tilde{\varepsilon})p & \\ -Kqp & \leq \dot{z}_i \leq & Kqp, & i=1,\ldots,n \end{array} \tag{17}$$

where $z=(z_1,\ldots,z_n)$.

The points P and Q introduced in the statement of the proposition can be chosen such that $0<q_f, p_0<\varepsilon/10$. Fix $\widetilde{\delta}_0>0$ with $\widetilde{\delta}_0 < \min\{\delta_0, \varepsilon/10\}$. Fix $\rho>0$ and define $\mathcal{U} = B_\rho(Q) \subset \mathbb{R}^4$. For any $0<\delta<\widetilde{\delta}_0$, there exists $T^* = T^*(\varepsilon,\delta)>0$ such for any time in $[0,T^*]$, the orbit of the point $\alpha(\delta)$ under the flow Ψ^s of system (16) does not leave the domain D.

Applying Gronwall estimates to the first two equations of (17) with initial condition $(q(0),p(0),z(0)) = \alpha(\delta)$ and $s\in[0,T^*]$, one obtains the following inequalities

$$\begin{array}{rcl} q(0)e^{(1-\tilde{\varepsilon})s} & \leq q(s) \leq & q(0)e^{(1+\tilde{\varepsilon})s} \\ p(0)e^{-(1+\tilde{\varepsilon})s} & \leq p(s) \leq & p(0)e^{-(1-\tilde{\varepsilon})s}. \end{array} \tag{18}$$

Moreover,

$$\|z(s)-z(0)\| \leq \left\| \int_0^s qp\mathcal{O}_0 \right\| \leq 5Ke^{s/5}q(0)p(0). \tag{19}$$

It is clear that the variable that leaves first the domain D is q and therefore T^* satisfies

$$\frac{1}{1+\widetilde{\varepsilon}} \ln\frac{\varepsilon}{\delta} \leq T^* \leq \frac{1}{1-\widetilde{\varepsilon}} \ln\frac{\varepsilon}{\delta}.$$

Since $q_f \leq \varepsilon/10$, by continuity, for $\delta \leq \widetilde{\delta}_0$ there exists $S=S(\delta) \leq T^*$ such that $\Psi^S(\alpha(\delta),0) \in \{q=q_f\}$. Moreover, by the Gronwall estimates in (18),

$$\frac{1}{1+\widetilde{\varepsilon}} \ln\frac{q_f}{\delta} \leq S \leq \frac{1}{1-\widetilde{\varepsilon}} \ln\frac{q_f}{\delta}. \tag{20}$$

For such S, using the second inequality in (18) we get that

$$p(S) \leq p(0) e^{\frac{1-\tilde{\varepsilon}}{1+\tilde{\varepsilon}} \log \frac{\delta}{q_f}} \leq p(0) \left(\frac{\delta}{q_f} \right)^{\frac{1-\tilde{\varepsilon}}{1+\tilde{\varepsilon}}}.$$

Note that, since $\widetilde{\varepsilon} \ll 1$, the right hand side converges to zero with δ. Now we use (19), to bound $\|z(S) - z(0)\|$. We have that

$$\begin{aligned}
\|z(S) - z_0\| &\leq \|z(S) - z(0)\| + \|z(0) - z_0\| \\
&\leq 5K e^{S/5} q(0) p(0) + \|\alpha\|_{\mathcal{C}^1} \delta \\
&\leq 5K p(0) q_f^{\frac{1}{5(1-\tilde{\varepsilon})}} \delta^{1 - \frac{1}{5(1-\tilde{\varepsilon})}} + \|\alpha\|_{\mathcal{C}^1} \delta
\end{aligned}$$

where we have used that $q(0) = \delta$.

We want the final point in $\{q = q_f\}$ to belong to $B_\rho(Q)$. We choose δ such that this is true. We need

$$|p(S) - p_0| \leq \frac{\rho}{4}, \quad \text{and} \quad \|z(S) - z_0\| \leq \frac{\rho}{4}.$$

Then, we need to choose δ such that

$$\|\gamma\|_{\mathcal{C}^0} \left(\frac{\delta}{q_f} \right)^{\frac{1-\tilde{\varepsilon}}{1+\tilde{\varepsilon}}} \leq \frac{\rho}{4} \quad \text{and} \quad 5K \|\gamma\|_{\mathcal{C}^0} q_f^{\frac{1}{5(1-\tilde{\varepsilon})}} \delta^{1 - \frac{1}{5(1-\tilde{\varepsilon})}} + \|\gamma\|_{\mathcal{C}^1} \delta \leq \frac{\rho}{4}.$$

All is left to prove now is that, in the original time variable (before rescaling), that is for the flow associated to the equation (14), the time T needed for the transition from $\gamma(\delta)$ to $B_\rho(Q)$ can be chosen as a multiple of 2π. This is achieved by taking δ small enough and allows us to say that what we obtained makes sense for the Poincaré map. The argument goes as follows. Fix the starting section $p = p(0)$ and the final final section $q = q_f$. The transition time S in the rescaled time depends continuously on δ and, by (20), satisfies that $S(\delta) \to \infty$ as $\delta \to 0$. Since the orbit never leaves the domain D, we have that $dt/ds \geq 1/\varepsilon^3$. Therefore, the transition time $T(\delta) \to \infty$ as $\delta \to 0$. By continuity, it must pass through a multiple of 2π. In fact, it must pass through a multiple of 2π infinitely many times $T(\delta_k)$ with $\delta_k \to 0$ as stated in the proposition. $\square$

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A Invariant manifolds of parabolic points and dependence with respect to parameters

Here we present a version of Theorem 2.1 in [BFdlLM07] on the invariant manifolds of parabolic fixed points where we also include their dependence with respect to parameters. We need this statement for Proposition 4.1, which provides the proper set of coordinates in which we derive de Lambda Lemma. Here we deal with the analytic case, which is the one relevant in the present work, unlike the setting of [BFdlLM07], where $\mathcal{C}^k$ maps were considered. The proof of the theorem we present here follows the same lines of the one in [BFdlLM07], but is simpler and shorter. The main difference is that, to deal with analyticity, we work with complex domains.

We first present the statement for maps and then we deduce the analogous result for time periodic flows.

Given a local diffeomorphism F from a neighborhood of $(0,0) \in \mathbb{R}\times\mathbb{R}^n$ to $\mathbb{R}\times\mathbb{R}^n$ such that $F(0,0) = (0,0)$, an open convex set $\widetilde{V}$ such that $(0,0) \in \partial\widetilde{V}$ and $r > 0$, we define the *local stable set of* $(0,0)$ *with respect to* $\widetilde{V}$ as

$$W^s_{\widetilde{V},r} = \{(x,y) \in \widetilde{V} \cap B_r \mid F^k(x,y) \in \widetilde{V} \cap B_r, \text{ for all } k \geq 0, \lim_{k\to\infty} F^k(x,y) = (0,0)\},$$

where B_r denotes the ball of radius r in $\mathbb{R}\times\mathbb{R}^n$.

Theorem A.1. *Let $U \subset \mathbb{R}\times\mathbb{R}^n$ be a neighborhood of the origin $(x,y) = (0,0)$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$, $\mathcal{A} \subset \mathbb{R}^p$ an open set and $F = (F^1, F^2) : U \times \mathcal{A} \to \mathbb{R}\times\mathbb{R}^n$ a real analytic map such that its derivatives up to order $N \geq 2$ at the origin are independent of the parameters $\lambda \in \mathcal{A}$, $F(0,0,\lambda) = 0$, $DF(0,0,\lambda) = \mathrm{Id}$*

$$D^j F(0,0,\lambda) = 0, \qquad \text{for} \quad 2 \leq j \leq N-1$$

and

$$\frac{\partial^N F^1}{\partial x^N}(0,0,\lambda) =: -c < 0, \qquad \frac{\partial^N F^2}{\partial x^N}(0,0,\lambda) = 0$$

and

$$\operatorname{spec} \frac{\partial^N F^2}{\partial x^{N-1}\partial y}(0,0,\lambda) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

Then there exists $t_0 > 0$ and a $\mathcal{C}^\infty$ map $K : [0,t_0) \times \mathcal{A} \subset \mathbb{R}\times\mathbb{R}^p \to \mathbb{R}\times\mathbb{R}^n$ of the form $K(t) = (t,0) + O(t^2)$, analytic in $(0,t_0)\times\mathcal{A}$, and a polynomial $R(t) = t - ct^N + \tilde{c}(\lambda)t^{2N-1}$, with $\tilde{c}$ analytic in $\mathcal{A}$, such that

$$F \circ K = K \circ R. \tag{21}$$

Furthermore, there exists an open convex set $\widetilde{V}$, with $(0,0) \in \partial\widetilde{V}$, containing the line $\{x > 0, y = 0\}$, $r > 0$ and t_0 such that the range of K is $W^s_{\widetilde{V},r}$.

Proof. Along the proof we skip the dependence of the functions on λ.

First of all, we remark that the hypotheses of Theorem A.1 imply those of Theorem 3.1 in [BF04]. Hence, there exists an open convex set $\widetilde{V}$, with $(0,0) \in \partial\widetilde{V}$, containing the line $\{x > 0, y = 0\}$ and $r > 0$ such that the local stable set $W^s_{\widetilde{V},r}$ is the graph of a Lipschitz function $\psi : [0,s_0) \to \mathbb{R}^n$. The fact that the set $\widetilde{V}$ can be chosen independently of λ follows from the estimates of Theorem 3.1 in [BF04], where it is proven that it contains a cone whose size only depends on the derivatives of F up to order N. Then, if K is the solution of (21) given by the theorem, since $K(t,0) = (t,0) + O(t^2)$, its range is the graph of a Lipschitz function, which must coincide with $W^s_{\widetilde{V},r}$. This proves the last statement of the theorem, assuming that the previous ones are true.

Now we prove the existence and properties of such a K.

Under the current hypotheses, we can apply Lemma 3.1 of [BFdlLM07] and we obtain that for any $k \geq N$ there exist polynomials $K^{\leq k} : \mathbb{R} \to \mathbb{R}^{1+n}$, of degree k, and $R(t) = t - ct^N + \tilde{c}(\lambda)t^{2N-1}$ such that

$$F \circ K^{\leq k}(t) - K^{\leq k} \circ R(t) = \mathcal{O}(t^{k+N}). \tag{22}$$

It is straightforward from the proof of Lemma 3.1 that the coefficients of $K^{\leq k}$ and R are analytic functions of the coefficients of the Taylor expansion of F at $(0,0)$ up to order k. Hence, they depend analytically on λ.

We claim

Lemma A.2. *Let $\alpha = 1/(N-1)$. Let $\widehat{\mathcal{A}}$ be a neighborhood of $\mathcal{A}$ in $\mathbb{C}^p$ in which the map F is analytic. There exists $a_0 > 0$ such that for any $0 < a < a_0$, there exists $\rho' > 0$ such that the set $V = \{t \in \mathbb{C} \mid |\arg t| \leq a,\ 0 < |t| \leq \rho\}$ satisfies $R(V) \subset V$, for any $0 < \rho < \rho'$ and $\lambda \in \widehat{\mathcal{A}}$.*

Proof of Lemma A.2. We remark that, for $t \in V$,

$$\begin{aligned}
\arg(1 - ct^{N-1} + \tilde{c}t^{2N-2}) &= -i \log \frac{(1 - ct^{N-1} + \tilde{c}t^{2N-2})}{|(1 - ct^{N-1} + \tilde{c}t^{2N-2})|} \\
&= -c|t|^{N-1} \sin((N-1)\arg t) + \mathcal{O}(|t|^{2N-2}) \\
&= -(N-1)c|t|^{N-1} \arg t + |t|^{N-1}\mathcal{O}(a^2) + \mathcal{O}(|t|^{2N-2}).
\end{aligned}$$

If ρ is such that $1 - (N-1)c\rho^{N-1} > 0$, for some $M > 0$ we have

$$\begin{aligned}
|\arg R(t)| &= |\arg(t(1 - ct^{N-1} + \tilde{c}t^{2N-2}))| \\
&= |\arg t + \arg(1 - ct^{N-1} + \tilde{c}t^{2N-2})| \\
&= |(1 - (N-1)c|t|^{N-1}) \arg t + |t|^{N-1}\mathcal{O}(a^2) + \mathcal{O}(|t|^{2N-2})| \\
&\leq (1 - (N-1)c|t|^{N-1})|\arg t| + M|t|^{N-1}a^2 + M|t|^{2N-2}.
\end{aligned}$$

From the last inequality we obtain that, if ρ satisfies,

$$\rho^{N-1} < (N-1)ca\left(1 - \frac{Ma}{(N-1)c}\right),$$

then $|\arg R(t)| < a$, which proves the claim. $\square$

We choose the constants a and ρ accordingly to Lemma A.2. Taking a and ρ smaller if necessary, it is clear that there exists $0 < b < c < d$ such that for all $t \in V$ and any $\lambda \in \widehat{\mathcal{A}}$,

$$R_d(|t|) := |t| - d|t|^N \leq |R(t)| \leq |t| - b|t|^N =: R_b(|t|). \tag{23}$$

Notice that b and d can be chosen arbitrarily close to c taking a and ρ small enough.

Next lemma describes the contraction provided by the nonlinear terms.

Lemma A.3. *Let b_0, d_0 and $s > 0$ such that $b_0^{N-1} = \alpha b^{-1}$, $d_0^{N-1} = \alpha d^{-1}$ and $b_0 s^{-\alpha} = \rho$. Then there exist two sequences, $(b_i)_{i\geq 0}$ and $(d_i)_{i\geq 0}$ such that for any $0 < \beta < 1$,*

$$b_i = b_0(1 + \mathcal{O}(i^{-\beta})), \quad d_i = d_0(1 + \mathcal{O}(i^{-\beta})), \quad i \geq 0 \tag{24}$$

for $i \geq 0$,

$$\frac{d_{i+1}}{(s+i+1)^\alpha} < \frac{b_i}{(s+i)^\alpha}, \tag{25}$$

and

$$R_b\left(\frac{b_i}{(s+i)^\alpha}\right) = \frac{b_{i+1}}{(s+i+1)^\alpha}, \quad R_d\left(\frac{d_i}{(s+i)^\alpha}\right) = \frac{d_{i+1}}{(s+i+1)^\alpha}. \tag{26}$$

Furthermore, the sets

$$V_i = \left\{ t \in V \mid \frac{d_{i+1}}{(s+i+1)^\alpha} \le |t| \le \frac{b_i}{(s+i)^\alpha} \right\} \tag{27}$$

satisfy $V = \cup_{i\ge 0} V_i$ *and* $R(V_i) \subset V_{i+1}$. *Consequently, if* $t \in V_i$, *for any* $j \ge 0$

$$\frac{d_0(1+\mathcal{O}(i^{-\beta}))}{(s+i+j+1)^\alpha} \le |R^j(t)| \le \frac{b_0(1+\mathcal{O}(i^{-\beta}))}{(s+i+j)^\alpha}. \tag{28}$$

Proof of Lemma A.3. The two relations in (26) define the numbers b_k and d_k for $k \ge 1$. To prove (25) we proceed by induction. For the first step, since $b < d$, we have that $d_0 < b_0$. Then, using (23), the fact that $R_d(r), R_b(r) < r$, for $0 < r < r_0$ and are strictly increasing,

$$\frac{d_1}{(s+1)^\alpha} = R_d\left(\frac{d_0}{s^\alpha}\right) \le R_b\left(\frac{d_0}{s^\alpha}\right) < R_b\left(\frac{b_0}{s^\alpha}\right) < \frac{b_0}{s^\alpha}.$$

Since

$$R_b\left(\frac{b_0}{s^\alpha}\right) = \frac{b_1}{(s+1)^\alpha},$$

we have that $d_1 < b_1$ and we can perform the induction procedure.

Relations (24) follow from Lemma 4.4 in [BFdlLM07].

From (24) and (25) follows that $V = \cup_{k\ge 0} V_k$. Inclusion $R(V_k) \subset V_{k+1}$ follows from (26) and inequality (23). □

We choose $k > 2N-1$. Let $K^\le = K^{\le k} : \mathbb{R} \to \mathbb{R}^{1+n}$, polynomial of degree k given in (22), and $R(t) = t - ct^N + \tilde{c}(\lambda)t^{2N-1}$ be such that

$$E(t) := F \circ K^\le(t) - K^\le \circ R(t) = \mathcal{O}(t^{k+N}). \tag{29}$$

Since from now on k is fixed, we skip the dependence on k of $K^\le$ and E. It is worth to remark that if one chooses $k' > k$ and finds the corresponding $K^\le$ of degree k', its terms up to degree k coincide with the former (since we keep R fixed in the construction).

We want to find a solution of

$$F \circ (K^\le + \varphi) - (K^\le + \varphi) \circ R = 0.$$

We rewrite this equation as

$$\mathcal{L}(\varphi) = \mathcal{F}(\varphi), \tag{30}$$

where

$$\mathcal{L}(\varphi) = (DF \circ K^\le)\varphi - \varphi \circ R \tag{31}$$

and

$$\mathcal{F}(\varphi) = -E - F \circ (K^\le + \varphi) + F \circ K^\le + (DF \circ K^\le)\varphi. \tag{32}$$

The operator defined by (31) is linear. A formal inverse is given by the formula

$$\mathcal{G}(\psi) = \sum_{j\ge 0} \left(\prod_{i=0}^{j} (DF)^{-1} \circ K^\le \circ R^i \right) \psi \circ R^j. \tag{33}$$

(see [BFdlLM07]).

As explained at the beginning of Section 4.4 in [BFdlLM07], we make the rescaling $\overline{y} = \delta y$ for some parameter δ and from now on we work in this rescaled variable without changing the notation for $K^{\leq}$ and F. For any $\sigma > 0$, choosing appropriately the norm in $\mathbb{R} \times \mathbb{R}^n$ and taking a, ρ and δ small enough, we can assume that, for some $C > 0$,

$$\|DF^{-1}(K^{\leq}(t))\| \leq 1 + (N+\sigma)d|t|^{N-1} + C|t|^N, \quad \text{for all } t \in V. \tag{34}$$

Lemma A.4. *Let $\{V_i\}_{i\geq 0}$ be the family of sets defined in (27) and $s > 0$ the number defined in Lemma A.3. There exists $C > 0$ such that for any $l \geq 0$, $t \in V_l$, the following inequalities hold*

$$\left\| \prod_{i=0}^{j} (DF)^{-1} \circ K^{\leq} \circ R^i(t) \right\| \leq C \left(\frac{s+l+j}{s+l} \right)^{(N+\sigma)\alpha d b^{-1}}$$

Proof of Lemma A.4. By Lemma A.3, $R^i(V_l) \subset V_{l+i}$. By (34), and using the definition and properties of the sets V_l in Lemma A.3, if $t \in V_l$,

$$\begin{aligned}
\left\|(DF)^{-1} \circ K^{\leq} \circ R^i(t)\right\| &\leq 1 + (N+\sigma)d \left(\frac{b_i}{(s+i+l)^\alpha} \right)^{N-1} + C \left(\frac{b_i}{(s+i+l)^\alpha} \right)^N \\
&\leq 1 + \frac{(N+\sigma)\alpha d b^{-1}}{s+i+l} + \frac{C}{(s+i+l)^{1+\gamma}},
\end{aligned}$$

where $\gamma = \min\{\alpha, \beta\}$ (see (24)). Then, redefining C,

$$\begin{aligned}
\left\| \prod_{i=0}^{j} (DF)^{-1}(K^{\leq}(R^i(t))) \right\| &\leq \exp \left(\sum_{i=0}^{j} \log \left(1 + \frac{(N+\sigma)\alpha d b^{-1}}{s+i+l} + \frac{C}{(s+i+l)^{1+\gamma}} \right) \right) \\
&\leq \exp \left(\sum_{i=0}^{j} \left(\frac{(N+\sigma)\alpha d b^{-1}}{s+i+l} + \frac{C}{(s+i+l)^{1+\gamma}} \right) \right) \\
&\leq \exp \left(\log \left(\frac{s+j+l}{s+l-1} \right)^{(N+\sigma)\alpha d b^{-1}} + \frac{C}{\gamma} \left(\frac{1}{(s+l-1)^\gamma} - \frac{1}{(s+l+j)^\gamma} \right) \right),
\end{aligned}$$

which implies the claim with a suitable C. □

In order to solve equation (30), we introduce the space

$$\mathcal{X}_m = \{\varphi : V \times \widehat{\mathcal{A}} \to \mathbb{C}^{1+n} \mid \varphi \text{ analytic}, \sup_{t\in V, \lambda \in \widehat{\mathcal{A}}} |t^{-m}\varphi(t,\lambda)| < \infty\},$$

which is a Banach space with the norm

$$\|\varphi\|_m = \sup_{t\in V, \lambda\in\widehat{\mathcal{A}}} |t^{-m}\varphi(t,\lambda)|.$$

By (29), the function E in (29) belongs to $\mathcal{X}_{k+N}$.

Lemma A.5. *Assume $m \in \mathbb{N}$ satisfies $m > N + (N+\sigma)\alpha d b^{-1} + 2$. Let $\mathcal{F}$ and $\mathcal{G}$ be the operators defined in (31) and (33). Then $\mathcal{G} : \mathcal{X}_{m+N-1} \to \mathcal{X}_m$ is linear, bounded and $\mathcal{L} \circ \mathcal{G} = \mathrm{Id}$ on $\mathcal{X}_{m+M-1}$.*

Proof of Lemma A.5. Let $\psi \in \mathcal{X}_{m+N-1}$. For any $l \geq 0$, $t \in V_l$, by Lemma A.4, and inequalities (28), we have that for some constant C independent of ψ and l,

$$\begin{aligned}
|\mathcal{G}(\psi)(t)| &\leq \sum_{j\geq 0} \left\| \prod_{i=0}^{j} (DF)^{-1} \circ K^{\leq} \circ R^i(t) \right\| \left| \psi \circ R^j(t) \right| \\
&\leq C \sum_{j\geq 0} \left(\frac{s+l+j}{s+l} \right)^{(N+\sigma)\alpha d b^{-1}} \left(\frac{b_0(1+\mathcal{O}(l^{-\beta}))}{(s+l+j)^{\alpha}} \right)^{m+N-1} \|\psi\|_{m+N-1} \\
&\leq \frac{C}{(s+l)^{\alpha(m+N-1)-1}} \|\psi\|_{m+N-1}
\end{aligned}$$

Hence, since $V = \cup_{l\geq 0} V_l$,

$$\|\mathcal{G}(\psi)\|_m = \sup_{l\geq 0} \sup_{t\in V_l, \lambda\in\widehat{\mathcal{A}}} |t^{-m}\mathcal{G}(\psi)(t)| \leq C \sup_{l\geq 0} \frac{(s+l+1)^{\alpha m}}{(s+l)^{\alpha(m+N-1)-1}} \|\psi\|_{m+N-1} \leq C\|\psi\|_{m+N-1}.$$

The above calculations show that the sums that define $\mathcal{G}$ are absolutely convergent and can be reordered if $\psi \in \mathcal{X}_{m+N-1}$. Then a simple computation shows that $\mathcal{L} \circ \mathcal{G} = \mathrm{Id}$ $\square$

To complete the proof it is enough to check that the operator $\mathcal{F}$ defined in (32) is well defined and Lipschitz between appropriate Banach spaces with Lipschitz constant small enough. Recall that $\mathcal{F}(0) = -E \in \mathcal{X}_{k+N}$.

Lemma A.6. *Let $r > 0$ be such that the map F is analytic and bounded at $B_r(0,0) \times \mathcal{A} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$, where $B_r(0,0)$ is the ball of radius r centered at the origin in $\mathbb{R} \times \mathbb{R}^n$. Let $\mathcal{B} \subset \mathcal{X}_{k+1}$ the ball of radius $R > 0$. Assume that $\rho < (r/R)^{1/(k+1)}$. Then $\mathcal{F} : \mathcal{B} \to \mathcal{X}_{k+N}$ is well defined and Lipschitz with*

$$\mathrm{lip}\, \mathcal{F} \leq C R \rho^{k-N+1}.$$

for some $C > 0$ independent of R and ρ.

Proof of Lemma A.6. Let $\varphi \in \mathcal{B}$. It satisfies $|\varphi(t)| \leq \|\varphi\|_{k+1} |t|^{k+1} \leq R\rho^{k+1} < r$. Hence, the function $\mathcal{F}(\varphi)$ is well defined and

$$\|\mathcal{F}(\varphi)\|_{k+N} \leq \|E\|_{k+N} + \sup_{t\in V, \lambda\in\mathcal{A}} C|t|^{-k-N} |\varphi(t)|^2 \leq \|E\|_{k+N} + \sup_{t\in V, \lambda\in\mathcal{A}} C|t|^{k-N+1} \|\varphi\|_{k+1}^2 < \infty.$$

As for the Lipschitz constant of $\mathcal{F}$, for any $\varphi, \varphi' \in \mathcal{B}$, since $k > 2N-1$,

$$\begin{aligned}
\|\mathcal{F}(\varphi) - \mathcal{F}(\varphi')\|_{k+N} &\leq \sup_{t\in V, \lambda\in\mathcal{A}} C|t|^{-k-N} \max\{|\varphi(t)|, |\varphi'(t)|\} |(\varphi - \varphi')(t)| \\
&\leq \sup_{t\in V, \lambda\in\mathcal{A}} C|t|^{k-N+1} R \|\varphi - \varphi'\|_{k+1} \\
&\leq C R \rho^{k-N+1} \|\varphi - \varphi'\|_{k+1}.
\end{aligned}$$

$\square$

We consider the fixed point equation

$$\varphi = \mathcal{G} \circ \mathcal{F}(\varphi). \tag{35}$$

Let $k > \min\{2N-1, N+(N+\sigma)\alpha d b^{-1}+1\}$ and choose $m = k+1$ in Lemma A.5. In Lemma A.6, take $R = 2\|\mathcal{G}\|\|E\|_{k+N}$ and $\rho < \min\{(r/R)^{1/(k+1)}, (R\|\mathcal{G}\|)^{-1/(k-N+1)}\}$. Then, the map $\mathcal{G} \circ \mathcal{F} : \mathcal{B} \to \mathcal{B}$ is well defined and Lipschitz with $\operatorname{lip}(\mathcal{G} \circ \mathcal{F}) < 1$. Hence, equation (35) has a unique solution $\varphi^* \in \mathcal{B} \subset \mathcal{X}_{k+1}$. By Lemma A.5, φ^* is a solution of equation (30).

Up to this point we have found a solution of equation (21) of the form $K^{\leq} + \varphi^*$. Since $K^{\leq}$ is a polynomial of degree k, it is $\mathcal{C}^k$ at the origin. Also, since $\varphi^* \in \mathcal{X}_{k+1}$, $\lim_{t \to 0} D^j \varphi^*(t) = 0$, for $0 \leq j \leq k$. Hence, φ^* is also $\mathcal{C}^k$ at the origin. Since k is arbitrary and a solution of the equation (35) for $k' > k$ also provides (conveniently rewritten) a solution of the equation for k, taking ρ smaller, if necessary, we have that K is $\mathcal{C}^\infty$, for real t, at the origin.

This completes the proof of Theorem A.1.

□

In what follows, given $T > 0$, $\mathbb{T}$ stands for the torus $\mathbb{R}/T$.

Given a local T-periodic vector field X from a neighborhood of $\{(0,0,t),\ t \in \mathbb{T}\} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{T}$ to $\mathbb{R} \times \mathbb{R}^n$ such that $X(0,0,t) = (0,0)$ for all $t \in \mathbb{T}$, an open convex set $\widetilde{V} \subset \mathbb{R} \times \mathbb{R}^n$ such that $(0,0) \in \partial \widetilde{V}$ and $r > 0$, we define de *local stable set of* $(0,0)$ *with respect to* $\widetilde{V}$ as

$$W^s_{\widetilde{V},r} = \{(x,y) \in \widetilde{V} \cap B_r \mid \phi_{t,\tau}(x,y) \in \widetilde{V} \cap B_r, \text{ for all } t \geq 0, \lim_{t \to \infty} \phi_{t,\tau}(x,y) = (0,0),\ \tau \in \mathbb{T}\},$$

where B_r denotes the ball of radius r in $\mathbb{R} \times \mathbb{R}^n$ and $\phi_{t,\tau}$ is the flow of the vector field X, that is,

$$\frac{d}{dt}\phi_{t,\tau}(x,y) = X(\phi_{t,\tau}(x,y), t), \quad \phi_{\tau,\tau}(x,y) = (x,y).$$

Theorem A.7. *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be a neighborhood of the origin $(x,y) = (0,0)$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$, $\mathcal{A} \subset \mathbb{R}^p$ an open set and $X = (X^1, X^2) : U \times \mathbb{T} \times \mathcal{A} \to \mathbb{R} \times \mathbb{R}^n$ a real analytic T-periodic vector field, such that its derivatives up to order $N \geq 2$ are independent of the parameters $\lambda \in \mathcal{A}$ and t at $(x,y) = (0,0)$, $X(0,0,t,\lambda) = 0$*

$$D^j X(0,0,t,\lambda) = 0, \qquad \textit{for} \quad 1 \leq j \leq N-1$$

and

$$\frac{\partial^N X^1}{\partial x^N}(0,0,t,\lambda) =: -c < 0, \qquad \frac{\partial^N X^2}{\partial x^N}(0,0,t,\lambda) = 0,$$

and

$$\operatorname{spec} \frac{\partial^N X^2}{\partial x^{N-1} \partial y}(0,0,t,\lambda) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

Then there exists an open convex set $\widetilde{V}$, with $(0,0) \in \partial \widetilde{V}$, containing the line $\{x > 0, y = 0\}$, $r > 0$ and a $\mathcal{C}^\infty$ map $K : [0, s_0) \times \mathbb{T} \times \mathcal{A} \subset \mathbb{R} \times \mathbb{T} \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^n$ of the form $K(s,\tau,\lambda) = (s,0) + O(s^2)$, analytic in $(0,s_0) \times \mathcal{A}$ such that the range of K is $W^s_{\widetilde{V},r}$.

Proof. Let F_τ be the Poincaré map of the vector field X associated to the section $\{t = \tau\}$. It depends analytically on τ and λ. A simple computation shows that it satisfies the hypotheses of Theorem A.1. Let $\widetilde{K}(s,\lambda,\tau)$ be the solution of the invariance equation

$$F_\tau \circ \widetilde{K}(s,\lambda,\tau) = \widetilde{K}(R(s,\lambda,\tau),\lambda,\tau)$$

provided by Theorem A.1. Its range is precisely $W^s_{\widetilde{V},r}$. □

B Formulas for the scattering maps of the circular problem

In this section we obtain formulas for the scattering map of the circular problem. As the scattering maps $\widetilde{\mathcal{S}}_0^{\pm}$ are defined in the extended phase space and send s to s [DdlLS08], they can be restricted to the section Σ (see (8)) giving rise to symplectic maps $\mathcal{S}_0^{\pm}$ on the cylinder $\Lambda^{[G_1,+\infty]}$. Recall that the result in [DdlLS08] deals with a normally hyperbolic cylinder, whereas in our case the manifold Λ is a "normally parabolic invariant cylinder" with stable and unstable manifolds. The arguments in [DKdlRS14] extend the proof to this case. Moreover, as it is explained in [DKdlRS14], the conservation of the Jacobi constant implies that G is a first integral of the two scattering maps.

Since the scattering maps $\mathcal{S}_0^{\pm}$ are symplectic and G is preserved, they must be of the form (9). Thus, it only remains to obtain formulas for the functions $f^{\pm}$. To compute these functions, we use [DdlLS08]. Note that this paper provides formulas for the scattering map in terms of the Poincaré potential in a regular perturbation regime. The regular perturbative approach used in [DKdlRS14] is to consider system (2) as an order μ perturbation of the two body problem. As has been explained in Section 1, here we use a different nearly integrable regime, also considered in [GMS15] for the circular problem, which allow us to deal with arbitrary values of μ. We proceed as follows.

Recall that we already know that the scattering map is defined for $\alpha \in \mathbb{T}$ and $G \geq G_1$. To provide formulas for $f^{\pm}$ we need to rescale the variables to transform the problem into a perturbative setting. Consider the scaling

$$x = G_1^{-1}\widetilde{x},\ y = G_1^{-1}\widetilde{y},\ \alpha = \widetilde{\alpha},\ G = G_1\widetilde{G}$$

and the change of time $s = G_1^3 t$. It transforms system (2) for $e_0 = 0$ into a system of the same form with a new potential

$$\widetilde{\mathcal{U}}(\widetilde{x}, \widetilde{\alpha}, t; G_1) = G_1^2\mathcal{U}(G_1^{-1}\widetilde{x}, \widetilde{\alpha}, G_1^3 t; 0) = \frac{\widetilde{x}^2}{2} + \Delta\widetilde{\mathcal{U}}(\widetilde{x}, \widetilde{\alpha}, t; G_1)$$

where $\Delta\widetilde{\mathcal{U}}(\widetilde{x}, \widetilde{\alpha}, t; G_1) = G_1^2\Delta\mathcal{U}(G_1^{-1}\widetilde{x}, \widetilde{\alpha}, t; G_1)$ and is given by

$$\Delta\widetilde{\mathcal{U}}(\widetilde{x}, \widetilde{\alpha}, t; G_1) = \frac{\widetilde{x}^2}{2}\left(\frac{1-\mu}{\left(1 - \frac{\mu}{G_1^2}\widetilde{x}^2\cos\phi + (\frac{\mu}{2G_1^2}\widetilde{x}^2)^2\right)^{1/2}} + \frac{\mu}{\left(1 + \frac{1-\mu}{G_1^2}\widetilde{x}^2\cos\phi + (\frac{1-\mu}{2G_1^2}\widetilde{x}^2)^2\right)^{1/2}} - 1\right)$$

with $\phi = \widetilde{\alpha} - G_1^3 t$. Expanding $\Delta\widetilde{\mathcal{U}}$ in G_1^{-1} one can see that it can be written as $\Delta\widetilde{\mathcal{U}} = G_1^{-4}\Delta\mathcal{V}$ where $\Delta\mathcal{V}$ is bounded uniformly on $G_1 \to +\infty$. Therefore, system (2) with $e_0 = 0$ in these rescaled variables can be seen as a $\mathcal{O}(G_1^{-4})$-perturbation of the two body problem. Nevertheless, this limit is singular since $\Delta\widetilde{\mathcal{U}}$, and therefore $\Delta\mathcal{V}$, is $2\pi/G_1^3$-periodic in time. When $G_1 \to +\infty$, the frequency of the perturbation blows up. Hence, we need to adapt the theory developed in [DdlLS08] to our setting, along the lines in [DdlLS00]

To transform this problem into a regular perturbation one, we introduce $\delta = G_1^{-4}$, which we consider as an independent parameter. Later we will recover the true value of δ. We write the potential $\widetilde{\mathcal{U}}$ as

$$\frac{\widetilde{x}^2}{2} + \delta\Delta\mathcal{V}$$

and, then, system (2) is a $O(\delta)$-perturbation, $2\pi/G_1^3$-periodic in time, of an integrable system with a "normally parabolic" invariant manifold $\widetilde{\Lambda}^{[G_1,+\infty)}$ with a 3-dimensional homoclinic manifold that can be parameterized by the homoclinic orbits

$$(x_{\rm h}(t;\widetilde{G}_0), y_{\rm h}(t;\widetilde{G}_0), \alpha_{\rm h}(t;\widetilde{\alpha}_0,\widetilde{G}_0), G_{\rm h}(t;\widetilde{G}_0), \widetilde{s}_0 + G_1^3 t),$$

for any $\widetilde{G}_0$, $\widetilde{\alpha}_0$, $\widetilde{s}_0$. Here $(x_{\rm h}, y_{\rm h}, \alpha_{\rm h}, G_{\rm h}, s_{\rm h})$ is the homoclinic orbit given in Lemma 2.2 and we have changed the last component to take into account the period of the perturbation.

Since now system (2) is a δ-regular perturbation of an integrable system with a homoclinic manifold, we can use the perturbative arguments (using deformation theory) in Theorem 32 of [DdlLS08]. We consider the Poincaré function associated to the homoclinic manifold

$$\begin{aligned}
\widetilde{L}\left(\widetilde{\alpha}_0,\widetilde{G}_0,\widetilde{s}_0,\sigma\right) &= \int_{-\infty}^{\infty} \Delta\mathcal{V}\left(x_{\rm h}(t+\sigma;\widetilde{G}_0), \alpha_{\rm h}(t+\sigma;\widetilde{\alpha}_0,\widetilde{G}_0), \widetilde{s}_0+G_1^3t\right)\,dt\\
&= G_1^4\int_{-\infty}^{\infty} \Delta\widetilde{\mathcal{U}}\left(x_{\rm h}(t+\sigma;\widetilde{G}_0), \alpha_{\rm h}(t+\sigma;\widetilde{\alpha}_0,\widetilde{G}_0), \widetilde{s}_0+G_1^3t\right)\,dt\\
&= G_1^6\int_{-\infty}^{\infty} \Delta\mathcal{U}\left(G_1^{-1}x_{\rm h}(t+\sigma;G_1\widetilde{G}_0), \alpha_{\rm h}(t+\sigma;\widetilde{\alpha}_0,\widetilde{G}_0), \widetilde{s}_0+G_1^3t\right)\,dt\\
&= G_1^3\int_{-\infty}^{\infty} \Delta\mathcal{U}\left(G_1^{-1}x_{\rm h}\left(\frac{r+G_1^3\sigma}{G_1^3};\widetilde{G}_0\right), \alpha_{\rm h}\left(\frac{r+G_1^3\sigma}{G_1^3};\widetilde{\alpha}_0,\widetilde{G}_0\right), \widetilde{s}_0+r\right)\,dr\\
&= G_1^3\int_{-\infty}^{\infty} \Delta\mathcal{U}\left(x_{\rm h}\left(r+G_1^3\sigma;G_1\widetilde{G}_0\right), \alpha_{\rm h}\left(r+G_1^3\sigma;\widetilde{\alpha}_0,G_1\widetilde{G}_0\right), \widetilde{s}_0+r\right)\,dr\\
&= G_1^3 L\left(\widetilde{\alpha}_0, G_1\widetilde{G}_0, \widetilde{s}_0, G_1^3\sigma;0\right) = G_1^3 L\left(\widetilde{\alpha}_0, G_1\widetilde{G}_0, \widetilde{s}_0-G_1^3\sigma, 0;0\right),
\end{aligned}$$

where we have used that the formulas in Lemma 2.2 imply

$$G_1^{-1}x_{\rm h}\left(\frac{u}{G_1^3};\widetilde{G}_0\right) = x_{\rm h}\left(u;G_1\widetilde{G}_0\right),\ \ \alpha_{\rm h}\left(\frac{u}{G_1^3};\alpha_0,\widetilde{G}_0\right) = \alpha_{\rm h}\left(u;\alpha_0,G_1\widetilde{G}_0\right),$$

and L is the function introduced in (5). The corresponding reduced Poincaré functions are obtained by evaluating the function $\sigma\mapsto\widetilde{L}(\widetilde{\alpha}_0,\widetilde{G},\widetilde{s}_0,\sigma)$ at its nondegenerate critical points $\sigma_\pm^*$, given by $\widetilde{s}_0-\widetilde{\alpha}_0-G_1^3\sigma_-^*=0$ and $\widetilde{s}_0-\widetilde{\alpha}_0+\pi-G_1^3\sigma_+^*=0$ (see (6)). They are defined as

$$\begin{aligned}
\widetilde{\mathcal{L}}_-^*(\widetilde{\alpha}_0,\widetilde{G}_0) &= \widetilde{L}(\widetilde{\alpha}_0,\widetilde{G}_0,\widetilde{s}_0,G_1^{-3}(\widetilde{s}_0-\widetilde{\alpha}_0)) = G_1^3L(\widetilde{\alpha}_0,G_1\widetilde{G}_0,\widetilde{\alpha}_0,0;0)\\
\mathcal{L}_+^*(\widetilde{\alpha}_0,\widetilde{G}_0) &= \widetilde{L}(\widetilde{\alpha}_0,\widetilde{G}_0,\widetilde{s}_0,G_1^{-3}(\pi+\widetilde{s}_0-\widetilde{\alpha}_0)) = G_1^3L(\widetilde{\alpha}_0,G_1\widetilde{G}_0,\widetilde{\alpha}_0-\pi,0;0).
\end{aligned}$$

Since L, and, consequently, $\widetilde{L}$, only depend on the angles through $\alpha_0-s_0+\sigma$, $\widetilde{\mathcal{L}}_\pm^*$ only depend on $G_1\widetilde{G}_0$.

$$\begin{aligned}
\widetilde{\mathcal{L}}_-^*(G_1\widetilde{G}_0) &= G_1^3L(0,G_1\widetilde{G}_0,0,0;0)\\
\mathcal{L}_+^*(G_1\widetilde{G}_0) &= G_1^3L(\pi,G_1\widetilde{G}_0,0,0;0).
\end{aligned}$$

Then, the generating function of the scattering map for the rescaled system is given by

$$S_{\rm resc}^\pm(\widetilde{\alpha}_0,\widetilde{G}_0) = \widetilde{G}_0\widetilde{\alpha}_0 + \delta\widetilde{\mathcal{L}}_\pm^*(G_1\widetilde{G}_0) + \mathcal{O}(\delta^2). \tag{36}$$

Using the results in [GMS15], we have that

$$\mathcal{L}_\pm^*(G_1\widetilde{G}_0) = G_1^3L_0(G_1\widetilde{G}_0) \pm G_1^3L_{1,-1}(G_1\widetilde{G}_0) + \mathcal{O}\left(G_1^4(G_1\widetilde{G}_0)^{-\frac{3}{2}}e^{-2\frac{(G_1\widetilde{G}_0)^3}{3}}\right)$$

where

$$L_0(G_1\widetilde{G}_0) = \mu(1-\mu)\frac{\pi}{2(G_1\widetilde{G}_0)^3}\left(1+\mathcal{O}((G_1\widetilde{G}_0)^{-4})\right)$$

$$L_{1,-1}(G_1\widetilde{G}_0) = \mu(1-\mu)(1-2\mu)\sqrt{\frac{\pi}{8(G_1\widetilde{G}_0)}}e^{-\frac{(G_1\widetilde{G}_0)^3}{3}}\left(1+\mathcal{O}((G_1\widetilde{G}_0)^{-1})\right).$$

Using (36), we have that the rescaled scattering map of the circular problem is of the form

$$\mathcal{S}^{\pm}_{\mathrm{resc}}\begin{pmatrix}\widetilde{\alpha}_0\\ \widetilde{G}_0\end{pmatrix} = \begin{pmatrix}\widetilde{\alpha}_0+\widetilde{f}^{\pm}(\widetilde{G}_0)\\ \widetilde{G}_0\end{pmatrix}.$$

with

$$\widetilde{f}^{\pm}\left(\widetilde{G}_0\right) = \delta\partial_{\widetilde{G}_0}\mathcal{L}^*_{\pm}(G_1\widetilde{G}_0)+\mathcal{O}\left(\delta^2\right) = -\delta\mu(1-\mu)\frac{3\pi}{2\widetilde{G}_0^4}+\mathcal{O}\left(\delta G_1^{-4}\widetilde{G}^{-8}\right)+\mathcal{O}\left(\delta^2\right).$$

where $\mathcal{O}$ refers to uniform bounds $\widetilde{G}_0 \in [1,+\infty)$.

We recover the value of $\delta = G_1^{-4}$ and we scale back the variables. Then, we obtain formula (9) with

$$f^{\pm}(G) = -\mu(1-\mu)\frac{3\pi}{2G^4}+\mathcal{O}\left(G_1^{-8}\right).$$

To recover the claim of Proposition 2.3, we need to replace the error term $\mathcal{O}\left(G_1^{-8}\right)$ by $\mathcal{O}\left(G^{-8}\right)$. To do so, we consider the above procedure restricted to compact sets. That is, for any $k \in \mathbb{N}$, we choose $G \in [kG^*,(k+1)G^*]$ and thus we take $\delta_k = G_1^{-4} = (kG^*)^{-4}$. Then, for any $G \in [kG^*,(k+1)G^*]$, $\delta_k < CG^{-4}$, for some $C > 0$ independent of k.

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