Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations

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Abstract

We prove the interior $C^{2,\alpha}$ regularity of solutions for some nonconvex fully nonlinear elliptic equations $F(D^2u, x) = f(x)$, $x \in B_1 \subset \mathbb{R}^n$. Our hypothesis is that, for every $x \in B_1$, $F(\cdot, x)$ is the minimum of a concave operator and a convex operator of D^2u. This extends the Evans–Krylov theory for convex equations to some nonconvex operators of Isaacs type. For instance, our results apply to the 3–operator equation $F_3(D^2u) = \min\{L_1u, \max\{L_2u, L_3u\}\} = 0$ (here L_i are linear operators), which motivated the present work.

1 Introduction

In 1982 Evans [Ev] and Krylov [K] proved interior $C^{2,\alpha}$ estimates for fully nonlinear elliptic equations $F(D^2u, Du, u, x) = 0$, $x \in \Omega \subset \mathbb{R}^n$, under the assumption that F is either a convex or a concave function of D^2u. These works relied on the Harnack inequality for linear equations in nondivergence form established by Krylov and Safonov in 1979. The Evans–Krylov estimate, together with some extensions due to Safonov, Trudinger and the second author, led to interior $C^{2,\alpha}$ regularity results for *Bellman's equation*,

$$\sup_{\beta \in \mathcal{B}} \{L_\beta u(x) - f_\beta(x)\} = 0 \ , \tag{1.1}$$

associated to a family $L_\beta = a_{ij}^\beta(x)\partial_{ij}$ of linear uniformly elliptic operators (see [CC], [GT]). Equation (1.1), which is convex in D^2u, is the dynamic programming equation for the optimal cost in some stochastic control problems.

Since then, the validity of interior $C^{2,\alpha}$ estimates for nonconvex fully nonlinear uniformly elliptic equations $F(D^2u) = 0$, in space dimension $n \geq 3$, has been a challenging open question. Examples of such nonconvex equations appear in stochastic control theory and are called *Isaacs equations*. They are of the form

$$\inf_{\gamma \in \mathcal{G}} \sup_{\beta \in \mathcal{B}} \{L_{\beta\gamma} u(x) - f_{\beta\gamma}(x)\} = 0 \ , \tag{1.2}$$

where $L_{\beta\gamma} = a_{ij}^{\beta\gamma}(x)\partial_{ij}$ is a family of elliptic operators, all of them with same ellipticity constants. Isaacs equation (1.2) is the dynamic programming equation for the value of some two–player stochastic differential games (see [FS]). At the same time, every uniformly elliptic equation $F(D^2u,x)=0$ can be written in the form (1.2), for some family $L_{\beta\gamma}=a_{ij}^{\beta\gamma}\partial_{ij}$ of operators with constant coefficients and some functions $f_{\beta\gamma}$ (see Remark 1.5).

The best estimates known to be valid for all uniformly elliptic equations $F(D^2u)=0$ are $C^{1,\alpha}$ and $W^{3,\delta}$ estimates (in particular, also $W^{2,\delta}$), where α and δ are (small) constants that belong to $(0,1)$ and depend on the ellipticity constants of F. To our knowledge, before this work no interior $C^{2,\alpha}$ estimate was available for a nonconvex, not necessarily differentiable, Isaacs operator.

In this article we establish the interior $C^{2,\alpha}$ regularity of viscosity solutions, and in particular the existence of classical solutions, for a class of nonconvex fully nonlinear elliptic equations $F(D^2u,x)=f(x)$. Our assumption is that, for every $x\in B_1\subset\mathbb{R}^n$, $F(\cdot,x)$ is the minimum of a concave operator and a convex operator of D^2u (where these operators may depend on the point x). We therefore include the "simplest" nonconvex Isaacs equation

$$F_3(D^2u):=\min\left\{L_1u,\max\{L_2u,L_3u\}\right\}=0\ , \tag{1.3}$$

that we call the 3–operator equation and that motivated our work. Here

$$L_ku=a_{ij}^k\partial_{ij}u+c_k\ , \tag{1.4}$$

where $c_k=L_k0\in\mathbb{R}$, are three affine uniformly elliptic operators with constant coefficients a_{ij}^k. More generally, our results apply to equations of the form

$$F(D^2u):=\min\left\{\inf_{k\in\mathcal{K}}L_ku,\sup_{l\in\mathcal{L}}L_lu\right\}=0\ , \tag{1.5}$$

where $\mathcal{K}$ and $\mathcal{L}$ are arbitrary sets, and L_k,L_l are operators of the form (1.4), all of them with same ellipticity constants and with $\{c_k\},\{c_l\}$ bounded. Note that this class of equations is formed by those Isaacs equations $\inf_{\gamma\in\mathcal{G}}\sup_{\beta\in\mathcal{B}}L_{\beta\gamma}u=0$ for which the convex operators $\{\sup_{\beta\in\mathcal{B}}L_{\beta\gamma}u\}_{\gamma\in\mathcal{G}}$ are all linear except for at most one γ.

A $C^{2,\alpha}$ regularity theory for a larger class of Isaacs equations, including for instance

$$\begin{aligned}F_4(D^2u):&=\max\{L_4u,F_3(D^2u)\}\\&=\max\left\{L_4u,\min\left\{L_1u,\max\{L_2u,L_3u\}\right\}\right\}=0\ ,\end{aligned} \tag{1.6}$$

will be developed in a future paper.

The work [C] by the second author (see also [CC]) established interior $C^{2,\alpha}$ estimates and $C^{2,\alpha}$ regularity for viscosity solutions of equations of the form $F(D^2u,x)=f(x)$ assuming that the dependence of F and f on x is C^α and that, for every fixed x_0, the Dirichlet problem for $F(D^2u(x),x_0)=f(x_0)$ has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates, where $0<\alpha<\overline{\alpha}$ (see Remark 1.4). By means of this theory, we can reduce our study to operators $F(M,x) = F(M)$ with constant coefficients —such as (1.3) and (1.5) defined by operators of the form (1.4).

The Evans–Krylov theory establishes interior $C^{2,\alpha}$ estimates for $F(D^2u) = 0$ when F is either convex or concave. More generally, the same proofs of the theory apply when $\{M \in \mathcal{S} : F(M) = 0\}$ is a convex hypersurface in the space $\mathcal{S}$ of $n \times n$ symmetric matrices —that is, when $\{M \in \mathcal{S} : F(M) = 0\}$ is the boundary of a convex open set. Note that this does not hold for our simplest model, the 3–operator (1.3).

Recently, the second author and Yu Yuan [CY] have proved interior $C^{2,\alpha}$ estimates for solutions of $F(D^2u) = 0$ under the assumption that, for every t,

$$\{M : F(M) = 0\} \cap \{M : \mathrm{trace}(M) = t\} \tag{1.7}$$

is a strictly convex variety of codimension 2 in $\mathcal{S}$. In particular, one principal curvature of $\{M : F(M) = 0\}$ could be negative. This work requires $F \in C^2$, and its $C^{2,\alpha}$ estimate depends on bounds for DF and D^2F.

More recently, Yu Yuan [Y] has proved a $C^{2,\alpha}$ estimate for the special Lagrangian equation in $\mathbb{R}^3$:

$$F(D^2u) := (\arctan \lambda_1 + \arctan \lambda_2 + \arctan \lambda_3) - c = 0 \ ,$$

where $c \in \mathbb{R}$ and $\{\lambda_i\}$ are the eigenvalues of D^2u. This is the equation for those Lagrangian graphs $\{(x, \nabla u(x))\}$ which are minimal in $\mathbb{R}^6$. [Y] points out that, for this operator and for $|c| < \pi/2$, the set (1.7) fails to be convex.

On the other hand, Nadirashvili [N] has announced that in dimension $n = 12$, there exists a smooth uniformly elliptic operator F such that $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^{12}$ admits a $C^{1,1}$ viscosity solution u which is not of class C^2.

Throughout the paper, we follow the terminology and notation of [CC]. We say that an operator $G : \mathcal{S} \to \mathbb{R}$ is *uniformly elliptic* if there exist constants $0 < \lambda \leq \Lambda$ (called ellipticity constants) such that

$$\lambda \|N\| \leq G(M+N) - G(M) \leq \Lambda \|N\| \qquad \forall M \in \mathcal{S} \quad \forall N \geq 0 \ . \tag{1.8}$$

Here, $\mathcal{S}$ is the space of $n \times n$ symmetric matrices, $N \geq 0$ means that $N \in \mathcal{S}$ is nonnegative definite and, for $M \in \mathcal{S}$, $\|M\| := \sup_{|z|\leq 1} |Mz|$. We say that a constant C is *universal* when it depends only on n, λ and Λ.

We consider the class of operators F of the following form:

$$\begin{cases} F(M) = \min\{F^{\cap}(M), F^{\cup}(M)\} \ \text{ for all } M \in \mathcal{S}, \\ F(0) = 0, \ F^{\cap} \text{ and } F^{\cup} \text{ are uniformly elliptic,} \\ F^{\cap} \text{ is concave and } F^{\cup} \text{ is convex.} \end{cases} \tag{1.9}$$

Since (1.8) holds for both $F^{\cap}$ and $F^{\cup}$, it also holds for F. Hence, F is uniformly elliptic. We assume $F(0) = 0$ only for convenience. Indeed, after an appropriate translation in $\mathcal{S}$ (which amounts to subtract a quadratic polynomial to u), every operator F can be assumed to satisfy $F(0) = 0$ (see Remark 1 in Section 6.2 of [CC]). Moreover, the concavity of $F^\cap$ and the convexity of $F^\cup$ are preserved under translations in $\mathcal{S}$.

We do not require $F^\cap$ and $F^\cup$ to be of class C^1. In this way, our results apply to the equations of Isaacs type described above. Note also that the class (1.9) of operators F includes all concave operators. Indeed, if $F^\cap$ is concave then there is an affine, uniformly elliptic operator L with constant coefficients such that $F^\cap \leq L$ in $\mathcal{S}$. Take then $F^\cup = L$, so that $F = F^\cap$. Recall finally that convex elliptic equations $G(D^2u) = 0$ get transformed into concave ones by writing them as $-G(-D^2v) = 0$, where $v = -u$.

Our main result is the following interior $C^{2,\alpha}$ a priori estimate for classical solutions of $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^n$, where $0 < \alpha < 1$ is a (small) exponent depending only on n and on the ellipticity constants λ and Λ. We use the notation $B_r = B_r(0) = \{x \in \mathbb{R}^n : |x| < r\}$.

Theorem 1.1. *Let $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^n$, where F is of the form* (1.9)*. Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C\|u\|_{L^\infty(B_1)} , \tag{1.10}$$

where $0 < \alpha < 1$ and C are universal constants.

As pointed out in [BE], solutions u of $F(D^2u) = 0$ need not be of class C^3 —since we are considering operators F which need not be C^1. For instance, the function defined on $\mathbb{R}^2$ by $u(x) = 3x_1x_2^2 - x_1^3$ if $x_1 \geq 0$, and $u(x) = 3x_1x_2^2 - 2x_1^3$ if $x_1 \leq 0$, belongs to $C^{2,1} \setminus C^3$ and satisfies the 2–operator equation $\min\{L_1u, L_2u\} = 0$ in $\mathbb{R}^2$, where $L_1 = \Delta$ and $L_2 = (1/2)\partial_{11} + \partial_{22}$.

A first hint towards the validity of second derivative estimates came up when we realized that, for the 3–operator (1.3), $H^2 = W^{2,2}$ estimates followed easily from some variational tools used by Brezis and Evans in [BE]. Let us explain these ideas, even that we will not use them in the proof of Theorem 1.1. Paper [BE] (written in 1979, that is, before the development of the Evans–Krylov theory) established $C^{2,\alpha}$ estimates for the 2–operator convex equation

$$\max\{L_1u - f_1(x), L_2u - f_2(x)\} = 0 . \tag{1.11}$$

For simplicity let us take $L_k = a^k_{ij}\partial_{ij}$ to have constant coefficients. The first step in [BE] is to obtain an H^2 estimate using Sobolevsky's inequality, which states that

$$\|u\|^2_{H^2(B_1)} \leq C\left\{\int_{B_1} L_1u\, L_2u\, dx + \|u\|^2_{L^2(B_1)}\right\} \tag{1.12}$$

for all $u \in H^2(B_1) \cap H^1_0(B_1)$, where C is a universal constant. Then, for a sufficiently nice solution u of (1.11) in B_1, we have $(L_1u - f_1)(L_2u - f_2) \equiv 0$ and hence $L_1uL_2u = f_1L_2u + f_2L_1u - f_1f_2$. Then, if $u \equiv 0$ on ∂B_1, the previous equality, (1.12) and Cauchy–Schwarz lead to $\|u\|_{H^2} \leq C\{\|u\|_{L^2} + \|f_1\|_{L^2} + \|f_2\|_{L^2}\}$.

We realized that the same idea works for the 3–operator equation

$$\min\left\{L_1 u, \max\{L_2 u, L_3 u\}\right\} = f(x) \ , \tag{1.13}$$

among other equations. Indeed, we have $L_2 u - f \le \max\{L_2 u - f, L_3 u - f\}$ and, since $L_1 u - f \ge 0$, we deduce $(L_1 u - f)(L_2 u - f) \le (L_1 u - f)\max\{L_2 u - f, L_3 u - f\} \equiv 0$. Hence $L_1 u L_2 u \le f(L_1 u + L_2 u) - f^2$, that combined with Sobolevsky's inequality (1.12) leads to $\|u\|_{H^2} \le C\{\|u\|_{L^2} + \|f\|_{L^2}\}$ for every solution of (1.13) with $u \equiv 0$ on ∂B_1.

We do not use this tool in the present paper. Instead, the proof of Theorem 1.1 is based in the following fact of nonvariational nature. We will see that if $F(D^2 u) = 0$ in B_1 and F is of the form (1.9), then $F^{\cup}(D^2 u)$ belongs to the class $\underline{S}$ of subsolutions in B_1. Below, at the end of this Introduction, we define $\underline{S}$ in detail. Heuristically, $w \in \underline{S}$ if w satisfies $a_{ij}(x)\partial_{ij} w \ge 0$ in the viscosity sense, for some uniformly elliptic operator with bounded measurable coefficients $a_{ij}(x)$.

Let us illustrate the previous claim in the easiest situation, that is, when u is a classical solution of (1.3):

$$F_3(D^2 u) = \min\left\{\Delta u, \max\{L_2 u, L_3 u\}\right\} = 0 \quad \text{in } B_1 \ ,$$

where L_k are second order operators with constant coefficients and where we have taken $L_1 = \Delta$. Then, it is elementary to show that the continuous function

$$F^{\cup}(D^2 u) := \max\{L_2 u, L_3 u\}$$

is subharmonic in B_1. Indeed, note first that $F^{\cup}(D^2 u) \ge 0$ in B_1. Hence, it suffices to show that $F^{\cup}(D^2 u)$ is subharmonic in the open set $\Omega = \{F^{\cup}(D^2 u) > 0\}$. But $\Delta u = 0$ in Ω and, therefore, $L_2 u$ and $L_3 u$ are also harmonic in Ω. It follows that $F^{\cup}(D^2 u) = \max\{L_2 u, L_3 u\}$ is subharmonic in Ω.

In Section 2 we establish the previous fact in the generality of the class (1.9). It is remarkable that this leads immediately to interior $W^{2,p}$ estimates for every $p < \infty$. Indeed, since $0 \le F^{\cup}(D^2 u)$ is a subsolution in B_1, a local version of the ABP estimate gives an interior L^∞ bound for $F^{\cup}(D^2 u)$ (see Proposition 2.2). In particular, $F^{\cup}(D^2 u) \in L^p$ in the interior, for all $p < \infty$. Then, since $F^{\cup}$ is a convex operator, the fully nonlinear Calderón–Zygmund theory proved by the second author in [C] leads to $W^{2,p}$ estimates for u (for all $p < \infty$).

Section 3 deals with a second important ingredient in the proof of the $C^{2,\alpha}$ estimates. It applies to more general equations than those of the form (1.9). Its statement, Theorem 3.3, assumes that u is a solution of $G(D^2 u) = 0$ in B_1, where G is uniformly elliptic and $G(0) = 0$, and that H is a uniformly elliptic operator with $C^{2,\alpha}$ estimates. The conclusion is that if G and H coincide in a ball in $\mathcal{S}$ centered at 0 of sufficiently large radius compared to $\|u\|_{L^\infty(B_1)}$, then $H(D^2 u) = 0$ in the smaller ball $B_{1/2}$.

After translations in $\mathcal{S}$, this result allows to control $F^{\cup}(D^2 P)$ for every quadratic polynomial P with $F(D^2 P) = 0$ —unless $F^{\cap}(D^2 u) = 0$ in $B_{1/2}$. This will be crucial when deriving $C^{2,\alpha}$ estimates through approximations of u by quadratic polynomials P.

In Section 4 we prove Theorem 1.1 using the two previous tools and the $C^{2,\alpha}$ iteration scheme developed in [C]. The goal is to approximate u by polynomials of degree two in $L^\infty(B_{\mu^k}(0))$–norm, where $0 < \mu < 1$, and to do it better and better as k increases. For this, we set $S_0 := \sup_{B_{1/2}} F^\cup(D^2u)$ and we distinguish two cases (see Figure 1 in Section 4). The first case is when most points x, in measure, have $F^\cup(D^2u(x))$ close to S_0. Then we can approximate u by a solution of $F^\cup(D^2v) = S_0$, which is $C^{2,\alpha}$ at the origin since $F^\cup$ is convex. In the other case, the weak Harnack inequality of Krylov–Safonov, applied to the supersolution $S_0 - F^\cup(D^2u) \geq 0$, forces the supremum of $F^\cup(D^2u)$ in a smaller ball to decrease by a factor (with respect to S_0). Heuristically, if this second case happens "often" as $k \to \infty$, then $F^\cup(D^2u)$ is concentrating near $\{F^\cup = 0\}$, and hence u can be approximated by the quadratic part of a solution of $F^\cup(D^2v) = 0$.

The proof of Theorem 1.1 requires $u \in C^2$ and does not apply to viscosity solutions. We need $u \in C^2$ to make sense of the statement that $F^\cup(D^2u)$ is a viscosity subsolution. It would be interesting to adapt the proof to viscosity solutions —for instance, by approximating $F^\cup(D^2u)$ in the spirit of the regularity theory for convex operators developed by the authors in [CaC] (see also Section 6.2 of [CC]).

Recall that the Dirichlet problem associated to every uniformly elliptic operator F always admits a unique viscosity solution. However, the $C^{2,\alpha}$ estimate of Theorem 1.1 requires the solution to be C^2. Hence, to complete our theory we need to show that $F(D^2u) = 0$ admits C^2 solutions whenever F is of the form (1.9). This is done in Section 5, where we prove the following:

Theorem 1.2. *Let F be of the form* (1.9). *Then, there exists a universal constant $\overline{\alpha} \in (0,1)$ such that for every $\alpha \in (0,\overline{\alpha})$, $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C(\partial B_1)$, the problem*

$$\begin{cases} F(D^2u) = f(x) & \text{in } B_1 \\ u = \varphi(x) & \text{on } \partial B_1 \end{cases}$$

admits a unique solution $u \in C^{2,\alpha}(B_1) \cap C(\overline{B}_1)$. Moreover, we have that

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C_\alpha \Big\{ \|f\|_{C^\alpha(\overline{B}_1)} + \|\varphi\|_{L^\infty(\partial B_1)} \Big\} ,$$

for some constant C_α depending only on n, λ, Λ and α.

Even that we only state interior $C^{2,\alpha}$ regularity, it is also true that $u \in C^{2,\alpha}(\overline{B}_1)$ if the boundary value φ is good enough. This can be proved using the boundary $C^{2,\alpha}$ estimates of Krylov (see Chapter 9 of [CC]).

Theorem 1.2 is proved in Section 5, where we establish a related result (Theorem 5.1) for a very large class of uniformly elliptic operators H. It assumes the existence of a path of operators H_t joining H and the Laplacian Δ in an appropriate Lipschitz manner and such that all equations $H_t(D^2u) =$ constant have *interior a priori* $C^{2,\overline{\alpha}}$ estimates. The conclusion is that every Dirichlet problem associated to $H(D^2u) = 0$ admits a classical solution. As a consequence (see Corollary 5.2), viscosity solutions of $H(D^2u) = 0$ are $C^{2,\alpha}$ in the interior for all $\alpha < \overline{\alpha}$.

The proof of this result is based on a continuation argument. Once we know that the operator H_t admits classical solutions, we write $H_{t+h}(D^2u) = 0$ in B_1, $u = \varphi$ on ∂B_1, as the fixed point problem

$$\begin{cases} H_t(D^2u) = (H_t - H_{t+h})(D^2u) & \text{in } B_1 \\ u = \varphi(x) & \text{on } \partial B_1 \ , \end{cases} \tag{1.14}$$

that we solve using Schauder's fixed point theorem. We cannot apply the implicit function theorem or the contraction mapping principle (as it is usually done for fully nonlinear elliptic equations) since we do not assume $H_t - H_{t+h}$ to be of class C^2 (not even C^1).

Since we only assume the validity of interior (not global) $C^{2,\alpha}$ estimates for H_t, we carry out the method of continuity in some well–known Banach spaces adapted to weighted interior $C^{2,\alpha}$ norms —a method due to Michael [M] in the linear case.

The existence of classical solutions, Theorem 1.2, and the a priori estimate of Theorem 1.1 lead immediately to the $C^{2,\alpha}$ regularity of every viscosity solution of $F(D^2u) = f(x) \in C^\alpha$. Furthermore, we also have $W^{2,p}$ regularity for $n \le p < \infty$ in case that $f \in L^p$ (this follows from [C] when $n < p < \infty$, and from the results of Escauriaza [Es] when $n - \varepsilon(n, \lambda, \Lambda) \le p \le n$). The precise statement, which is proved in Section 5, is the following:

Corollary 1.3. *Let $u \in C(B_1)$ be a viscosity solution of $F(D^2u) = f(x)$ in B_1, where f is a continuous function in B_1 and F is an operator of the form* (1.9). *Then*:
(i) If $f \in C^\alpha(B_1)$ for some $0 < \alpha < \overline{\alpha}$, where $\overline{\alpha} \in (0,1)$ is a universal constant, then $u \in C^{2,\alpha}(B_1)$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C_\alpha \{\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_{3/4})}\} \ ,$$

for some constant C_α depending only on n, λ, Λ and α.
(ii) If $f \in L^p(B_1)$ and $n \le p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \le C_p \{\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\} \ ,$$

for some constant C_p depending only on n, λ, Λ and p.

Remark 1.4. Similar interior $C^{2,\alpha}$ estimates and $C^{2,\alpha}$ regularity of viscosity solutions hold for equations with variable coefficients

$$F(D^2u, x) = f(x) \ \text{ in } B_1 \subset \mathbb{R}^n \ ,$$

under the following two assumptions:
(a) for every $x_0 \in B_1$, the operator $F(\cdot, x_0)$ is the minimum of a concave operator and a convex operator (which may depend on x_0), and

(b) $F(M,\cdot)$ and f are C^α functions of $x \in B_1$.

In particular, $F(D^2u, x)$ can be an Isaacs operator of the type (1.5) where L_k and L_l are linear operators with variable Hölder continuous coefficients.

This result is a consequence of the $C^{2,\alpha}$ perturbation theory of [C] (see Theorem 8.1 of [CC]). A similar remark also holds regarding $W^{2,p}$ regularity for $n \le p < \infty$ (see [Es] and Theorem 7.1 of [CC]).

Throughout the paper, we will use Pucci's extremal operators, as well as the class $\underline{S}$ of subsolutions. We recall that Pucci's maximal operator is defined by

$$\mathcal{M}^+(M) = \mathcal{M}^+(M,\lambda,\Lambda) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M = \max_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M \ ,$$

where $e_i = e_i(M)$ are the eigenvalues of $M \in \mathcal{S}$, $A \in \mathcal{A}_{\lambda,\Lambda}$ means that A is a symmetric matrix whose eigenvalues belong to $[\lambda,\Lambda]$, and $L_A M = a_{ij} m_{ij} = \mathrm{trace}(AM)$ (see Section 2.2 of [CC]).

The class $\underline{S} = \underline{S}(\lambda,\Lambda)$ in B_1 is formed by those continuous functions u in B_1 such that $\mathcal{M}^+(D^2u,\lambda,\Lambda) \ge 0$ in the viscosity sense in B_1 (see Section 2.1 of [CC] for the definition of the viscosity sense). Similarly, one defines the class $\overline{S}$ of supersolutions through the inequality $\mathcal{M}^-(D^2u) \le 0$, where $\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$ is Pucci's minimal operator. The class S of viscosity solutions is defined by $S = \underline{S} \cap \overline{S}$.

More generally, given a continuous function f in B_1, the class $\underline{S}(f) = \underline{S}(\lambda,\Lambda,f)$ contains those continuous functions u such that $\mathcal{M}^+(D^2u,\lambda,\Lambda) \ge f(x)$ in the viscosity sense in B_1. Similarly, one defines $\overline{S}(f)$ and $S(f)$.

Finally, we recall that Isaacs equations (1.2) cover all possible fully nonlinear elliptic equations.

Remark 1.5. Let $F(\cdot,x)$ be uniformly elliptic, with ellipticity constants $0 < \lambda \le \Lambda$ for every x. Then, for M and N in $\mathcal{S}$,

$$\begin{aligned} F(M,x) - F(N,x) &\le \Lambda \|(M-N)^+\| - \lambda \|(M-N)^-\| \\ &\le \mathcal{M}^+(M-N, \lambda/n, \Lambda) = \max_{A\in\mathcal{A}} L_A(M-N) \ , \end{aligned}$$

where $\mathcal{A} = \mathcal{A}_{\lambda/n,\Lambda}$ (see Chapter 2 of [CC]). Since there is equality when $N = M$ we deduce that, for all M and x,

$$\begin{aligned} F(M,x) &= \min_{N\in\mathcal{S}} \max_{A\in\mathcal{A}} \{L_A(M-N) + F(N,x)\} \\ &= \min_{N\in\mathcal{S}} \max_{A\in\mathcal{A}} \{L_A M + (F(N,x) - L_A N)\} \ . \end{aligned}$$

This is an operator of Isaacs type (1.2) associated to a family $\{L_A\}$ of linear operators with constant coefficients.

2 L^∞ bound for $F^\cup(D^2u)$ and $W^{2,p}$ estimates for u

In this section we establish an interior L^∞ bound for $F^\cup(D^2u)$ and, as a consequence, interior $W^{2,p}$ a priori estimates ($n < p < \infty$) for every classical solution of $F(D^2u) = 0$, where F is an operator of the form (1.9).

The L^∞ bound for $F^\cup(D^2u)$, that will be used when proving $C^{2,\alpha}$ estimates in future sections, is based on the following proposition. It is here where we use the structural assumptions on the operator F in a more crucial way.

Proposition 2.1. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form (1.9). Then*

$$0 \leq F^\cup(D^2u) \in \underline{S}(\lambda/n, \Lambda) \quad \text{in } B_1 \ .$$

The proposition states that $F^\cup(D^2u)$, a continuous and nonnegative function in B_1, is a viscosity subsolution of an homogeneous linear elliptic equation in nondivergence form.

Proof of Proposition 2.1. Since $0 = F(D^2u) \leq F^\cup(D^2u)$, we have that $F^\cup(D^2u)$ is a nonnegative continuous function in B_1. To establish that $F^\cup(D^2u) \in \underline{S}(\lambda/n, \Lambda)$ in B_1, we need to show that if $x_0 \in B_1$, $E \subset B_1$ is an open neighborhood of x_0, $\varphi \in C^2(E)$, $F^\cup(D^2u) \leq \varphi$ in E and $F^\cup(D^2u(x_0)) = \varphi(x_0)$, then $\mathcal{M}^+(D^2\varphi(x_0)) \geq 0$, where $\mathcal{M}^+$ is Pucci's maximal operator with constants λ/n and Λ.

Consider the open set

$$\Omega = \{x \in B_1 : F^\cup(D^2u(x)) > 0\} \subset B_1 \ .$$

If $x_0 \notin \Omega$ then $F^\cup(D^2u(x_0)) = 0$, and hence $\varphi(x_0) = 0 \leq \varphi$ in E. We deduce that $D^2\varphi(x_0) \geq 0$ and, in particular, $\mathcal{M}^+(D^2\varphi(x_0)) \geq 0$.

Suppose from now on that $x_0 \in \Omega$. Note that $F^\cap(D^2u) = 0$ in the open set Ω.

Since $F^\cup$ is a convex function on the space $\mathcal{S}$ of $n \times n$ symmetric matrices, it follows that $F^\cup$ has a supporting hyperplane L at $D^2u(x_0)$. That is, there exists an affine functional L on $\mathcal{S}$ such that $L(D^2u(x_0)) = F^\cup(D^2u(x_0))$ and $L(M) \leq F^\cup(M)$ for all $M \in \mathcal{S}$. By the ellipticity of $F^\cup$, this easily implies (see the proof of Theorem 6.6 of [CC]) that L is of the form $L(M) = a_{ij}m_{ij} + c = \text{trace}(AM) + c$, for some positive definite symmetric matrix $A = (a_{ij})$ with all eigenvalues in $[\lambda, \Lambda]$ and for some $c \in \mathbb{R}$.

Denoting the function $L(D^2u(x))$ by $Lu(x)$, we have that $E \cap \Omega$ is an open neighborhood of x_0,

$$Lu \leq \varphi \quad \text{in } E \cap \Omega \quad \text{and} \quad Lu(x_0) = \varphi(x_0) \ . \tag{2.1}$$

We claim that

$$Lu \in \underline{S}(\lambda/n, \Lambda) \quad \text{in } \Omega \ . \tag{2.2}$$

This claim and (2.1) imply that $\mathcal{M}^+(D^2\varphi(x_0)) \geq 0$, as desired.

Hence, it only remains to prove (2.2). This follows from standard results, using that $F^\cap(D^2u) = 0$ in Ω and that $F^\cap$ is concave. Indeed, since A is symmetric and positive definite, A has a symmetric square root $B = (b_{ij})$. For $k \in \{1, \dots, n\}$, let $e_k = (b_{k1}, \dots, b_{kn}) \in \mathbb{R}^n$. For $x \in \Omega$, we have

$$\begin{aligned} Lu(x) - c = a_{ij}u_{ij}(x) = b_{ik}b_{kj}u_{ij}(x) &= \sum_{k=1}^{n} \langle e_k, D^2u(x)e_k\rangle \\ &= \lim_{h\to 0} \frac{1}{h^2} \sum_{k=1}^{n} \{u(x+he_k) + u(x-he_k) - 2u(x)\} \\ &= \lim_{h\to 0} \frac{2n}{h^2} \left\{ \left[\frac{1}{2n} \sum_{k=1}^{n} (u(x+he_k) + u(x-he_k)) \right] - u(x) \right\} . \end{aligned}$$

Let $\Omega_h = \{x \in \Omega : d(x, \partial\Omega) > |h|(\|e_1\| + \cdots + \|e_n\|)\}$. Since all functions $u(\cdot + he_k)$ are classical solutions of $F^\cap(D^2w) = 0$ in Ω_h and $F^\cap$ is concave, it immediately follows that $v_h := \frac{1}{2n}\sum_{k=1}^{n}(u(\cdot + he_k) + u(\cdot - he_k))$ is a classical subsolution of $F^\cap(D^2w) = 0$ in Ω_h. Then, since u is a classical (super)solution of $F^\cap(D^2w) = 0$ in Ω, Proposition 2.13 of [CC] gives that the difference $v_h - u$ and hence also $(2n/h^2)(v_h - u)$, belong to $\underline{S}(\lambda/n, \Lambda)$ in Ω_h. Letting $h \to 0$ and using the closedness of $\underline{S}$ under uniform limits in compact sets, we finally obtain that $Lu \in \underline{S}(\lambda/n, \Lambda)$ in Ω. □

In case that $u \in C^4(B_1)$ and that both $F^\cap$ and $F^\cup$ are of class C^2, Proposition 2.1 can also be proved by direct computations. Indeed, let

$$w := F^\cup(D^2u) \geq 0 \quad \text{in } B_1 .$$

Denoting by $F^\cup_{kl} = F^\cup_{kl}(D^2u(x))$ and by $F^\cup_{kl,rs} = F^\cup_{kl,rs}(D^2u(x))$ the first and second derivatives of $F^\cup$, we have

$$w_i = F^\cup_{kl}u_{kli}$$

and

$$w_{ij} = F^\cup_{kl}u_{klij} + F^\cup_{kl,rs}u_{kli}u_{rsj} .$$

In the open set $\Omega = \{w > 0\}$ we have $F^\cap(D^2u) = 0$, and therefore

$$F^\cap_{ij}u_{ijk} = 0$$

and

$$F^\cap_{ij}u_{ijkl} + F^\cap_{ij,rs}u_{ijk}u_{rsl} = 0$$

for every pair (k, l). We infer that, in $\{w > 0\}$,

$$\begin{aligned} F^\cap_{ij}w_{ij} &= F^\cup_{kl}F^\cap_{ij}u_{klij} + F^\cup_{kl,rs}F^\cap_{ij}u_{kli}u_{rsj} \\ &= -F^\cap_{ij,rs}F^\cup_{kl}u_{ijk}u_{rsl} + F^\cup_{kl,rs}F^\cap_{ij}u_{kli}u_{rsj} \\ &\geq 0 , \end{aligned}$$

since both terms in the last expression are nonnegative. Indeed, since $(F^{\cap}_{ij})$ is positive definite, it has a symmetric square root (g_{ij}). Hence

$$F^{\cup}_{kl,rs}F^{\cap}_{ij}u_{kli}u_{rsj} = F^{\cup}_{kl,rs}g_{im}u_{kli}g_{jm}u_{rsj} \geq 0 \ ,$$

since $F^{\cup}$ is convex. The same argument gives $-F^{\cap}_{ij,rs}F^{\cup}_{kl}u_{ijk}u_{rsl} \geq 0$, since $F^{\cap}$ is concave.

We conclude that $F^{\cap}_{ij}(D^2u(x))w_{ij} \geq 0$ in $\{w > 0\}$. Since $w \geq 0$ in B_1, it follows that $w = F^{\cup}(D^2u)$ is a viscosity subsolution in B_1.

Next, we use a local maximum principle of Alexandroff–Bakelman–Pucci type for subsolutions to deduce an interior L^∞ bound for $F^{\cup}(D^2u)$.

Proposition 2.2. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form* (1.9)*. Then*

$$\sup_{B_{1/2}} F^{\cup}(D^2u) \leq C\{\|u\|_{L^\infty(B_1)} + F^{\cup}(0)\} \ ,$$

where C is a universal constant.

Proof. Since $F(D^2u) = 0$ in B_1 and $F(0) = 0$, we have that $u \in S(\lambda/n, \Lambda)$ in B_1. By a result of Fanghua Lin (see Proposition 7.4 of [CC]), it follows that

$$\|D^2u\|_{L^\delta(B_{3/4})} \leq C\|u\|_{L^\infty(B_1)} \tag{2.3}$$

for some universal constants $\delta > 0$ and $C > 0$.

By Proposition 2.1 we know that $F^{\cup}(D^2u) \in \underline{S}(\lambda/n, \Lambda)$ in B_1. Applying the local ABP estimate (Theorem 4.8(2) of [CC]) to this function, we obtain

$$\sup_{B_{1/2}} F^{\cup}(D^2u) \leq C\|F^{\cup}(D^2u)\|_{L^\delta(B_{3/4})} \tag{2.4}$$

for some universal constant C. Since $F^{\cup}(0) \geq 0$, we have $|F^{\cup}(D^2u)| \leq \Lambda\|D^2u\| + F^{\cup}(0)$ in B_1. This inequality, (2.3) and (2.4) finish the proof. □

Since $F^{\cup}(D^2u) \geq 0$, Proposition 2.2 controls $F^{\cup}(D^2u)$ in $L^\infty(B_{1/2})$ and, in particular, in $L^p(B_{1/2})$ for all $p < \infty$. The interior $W^{2,p}$ estimates for convex equations established in [C] for $n < p < \infty$ (see also Theorem 7.1 and Remark 1 in Section 7.1 of [CC]) lead immediately to the following $W^{2,p}$ estimate for u.

Corollary 2.3. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form* (1.9)*. Then, for every $1 \leq p < \infty$,*

$$\|u\|_{W^{2,p}(B_{1/4})} \leq C_p\{\|u\|_{L^\infty(B_1)} + F^{\cup}(0)\} \tag{2.5}$$

for some constant C_p depending only on n, λ, Λ and p.

The results of Section 3 will show that estimate (2.5) can be improved by removing the term $F^{\cup}(0)$ in its right hand side. Note also that in the Introduction we have already stated a more general $W^{2,p}$ estimate (Corollary 1.3) for the equation $F(D^2u) = f(x)$, that we will prove in Section 5.

3 A sufficient condition for only one operator to act

Recall that, by assumption, $F(0) = \min(F^\cap(0), F^\cup(0)) = 0$. The goal of this section is to prove that if $F^\cup(0)$ is positive and too large compared to $\|u\|_{L^\infty(B_1)}$, then we have $F^\cap(D^2u) = 0$ in $B_{1/2}$ —that is, only $F^\cap$ acts on D^2u in the smaller ball $B_{1/2}$. More precisely, we have:

Proposition 3.1. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form* (1.9)*. Then, there exists a universal constant $c_f > 0$ such that*

$$\text{if } F^\cup(0) > c_f \|u\|_{L^\infty(B_1)} \text{ then } F^\cap(D^2u) = 0 \quad \text{in } B_{1/2} \ .$$

This result allows (after a translation in $\mathcal{S}$) to control $F^\cup(D^2P)$ whenever P is a quadratic polynomial with $F(D^2P) = 0$. This will be crucial in next section to derive $C^{2,\alpha}$ estimates through approximations of u by quadratic polynomials P. Hence, we need Proposition 3.1 even if we initially assume $F^\cup(0) = 0$.

Proposition 3.1 is a particular case of the following theorem, in which the concavity and convexity of $F^\cap$ and $F^\cup$, respectively, are not needed. To state the theorem, let us introduce a terminology that we will also use in future sections.

Definition 3.2. Let H be a uniformly elliptic operator, and let $\overline{\alpha} \in (0,1)$ and $c_e > 0$ be constants. We say that *equation $H = 0$ has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates with constant c_e* if, for every $M \in \mathcal{S}$ with $H(M) = 0$ and every $w_0 \in C(\partial B_1)$, there exists $w \in C^2(B_1) \cap C(\overline{B}_1) \cap C^{2,\overline{\alpha}}(\overline{B}_{1/2})$ such that

$$\begin{cases} H(D^2w + M) = 0 & \text{in } B_1 \\ w = w_0 & \text{on } \partial B_1 \ , \end{cases}$$

and

$$\|w\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/2})} \le c_e \|w_0\|_{L^\infty(\partial B_1)} \ .$$

The main result of this section is the following. We suppose that

$$\begin{cases} u \in C^2(B_1) \text{ satisfies } G(D^2u) = 0 \text{ in } B_1,\ G \text{ and } H \text{ are} \\ \text{uniformly elliptic, } G(0) = 0, \text{ and equation } H = 0 \text{ has classical} \\ \text{solutions and interior } C^{2,\overline{\alpha}} \text{ estimates with constant } c_e. \end{cases} \tag{3.1}$$

Theorem 3.3. *Assume* (3.1)*. Then, there exists a constant $\overline{c}_f > 0$ depending only on $n, \lambda, \Lambda, \overline{\alpha}$ and c_e, such that if we suppose*

$$G(N) = H(N) \text{ for all } N \in \mathcal{S} \text{ with } \|N\| \le \overline{c}_f \|u\|_{L^\infty(B_1)}$$

then

$$H(D^2u) = 0 \quad \text{in } B_{1/2} \ .$$

Proposition 3.1 follows immediately from this theorem, applied with $G = F$ and $H = F^\cap$. Notice that, since $H = F^\cap$ is a concave operator, it satisfies the assumptions of Theorem 3.3 for some universal $\overline{\alpha}$ and c_e, by the Evans–Krylov theory (see Proposition 9.8 of [CC]). Note also that, taking $c_f := 2\Lambda\overline{c}_f$, if $F^\cup(0) > c_f\|u\|_{L^\infty(B_1)}$ and $\|N\| \le \overline{c}_f\|u\|_{L^\infty(B_1)}$ then $F^\cup(0) > 0$ (and hence $F^\cap(0) = 0$, since $F(0) = 0$) and

$$\begin{aligned} F^\cup(N) &\ge F^\cup(0) - \Lambda\|N\| \ge c_f\|u\|_{L^\infty(B_1)} - \Lambda\|N\| \\ &\ge (c_f/\overline{c}_f)\|N\| - \Lambda\|N\| = \Lambda\|N\| = \Lambda\|N\| + F^\cap(0) \ge F^\cap(N) \ . \end{aligned}$$

In particular, $(G(N) =)\ F(N) = F^\cap(N)\ (= H(N))$.

The proof of Theorem 3.3 uses the $C^{2,\alpha}$ iteration scheme developed by the second author in [C], together with the following approximation lemma based on compactness and uniqueness properties of viscosity solutions.

Lemma 3.4. *Assume* (3.1) *and* $\|u\|_{L^\infty(B_1)} \le 1$*. Then, for every* $\varepsilon > 0$*, there exists a constant* $\overline{c}(\varepsilon) > 0$ *depending only on* $\varepsilon, n, \lambda, \Lambda$ *and* c_e*, such that if we suppose*

$$G(N) = H(N) \text{ for all } N \in \mathcal{S} \text{ with } \|N\| \le \overline{c}(\varepsilon)\ , \tag{3.2}$$

we then have

$$\|u - w\|_{L^\infty(B_{1/2})} \le \varepsilon\ ,$$

where $w \in C^2(B_{1/2}) \cap C(\overline{B}_{1/2})$ *is the solution of*

$$\begin{cases} H(D^2w) = 0 & \text{in } B_{1/2} \\ \quad\ \ w = u & \text{on } \partial B_{1/2}\ . \end{cases}$$

Moreover, $G(D^2w(0)) = H(D^2w(0)) = 0$*,*

$$G(N) = H(N) \text{ for all } N \in \mathcal{S} \text{ with } \|N - D^2w(0)\| \le \frac{\overline{c}(\varepsilon)}{2}\ , \tag{3.3}$$

and $\|w\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/4})} \le Cc_e$ *for some constant* C *depending only on* n *and* $\overline{\alpha}$*.*

Proof. The last statements of the lemma follow easily from the assumptions, if we take $\overline{c}(\varepsilon)$ such that $\overline{c}(\varepsilon) \ge 8c_e$. Indeed, since $v(x) := 4w(x/2)$ solves $H(D^2v) = 0$ in B_1, $v(x) = 4u(x/2)$ on ∂B_1, and since (3.1) and (3.2) imply $H(0) = G(0) = 0$, the hypothesis on $C^{2,\overline{\alpha}}$ estimates for H (see Definition 3.2) gives $\|v\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/2})} \le 4c_e\|u\|_{L^\infty(\partial B_{1/2})} \le 4c_e \le \overline{c}(\varepsilon)/2$. Hence $\|w\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/4})} \le C(n,\overline{\alpha})c_e$, $\|D^2w(0)\| = \|D^2v(0)\| \le \overline{c}(\varepsilon)/2 \le \overline{c}(\varepsilon)$ and, by (3.2), $G(D^2w(0)) = H(D^2w(0))$. Moreover, $H(D^2w(0)) = 0$ by construction. Next, (3.3) follows from (3.2) and $\|D^2w(0)\| \le \overline{c}(\varepsilon)/2$.

Now, to prove $\|u - w\|_{L^\infty(B_{1/2})} \le \varepsilon$, we argue by contradiction. Suppose that there are constants $\varepsilon, n, \lambda, \Lambda$ and c_e such that for $k \in \mathbb{N}$ we have

$$G_k(D^2 u_k) = 0 \text{ in } B_1 \ , \quad G_k(0) = 0 \ , \quad \|u_k\|_{L^\infty(B_1)} \leq 1 \ ,$$

$$\begin{cases} H_k(D^2 w_k) = 0 & \text{in } B_{1/2} \\ w_k = u_k & \text{on } \partial B_{1/2} \ , \end{cases} \tag{3.4}$$

$$\|u_k - w_k\|_{L^\infty(B_{1/2})} \geq \varepsilon \ , \quad \text{and} \tag{3.5}$$

$$G_k(N) = H_k(N) \ \text{ if } \|N\| \leq k \ , \tag{3.6}$$

for two sequences of uniformly elliptic operators G_k and H_k with ellipticity constants λ and Λ.

By (3.6) and $G_k(0) = 0$, we also have $H_k(0) = 0$. Hence, G_k and H_k are families of equi–Lipschitz and locally bounded functions on $\mathcal{S}$. Taking subsequences (still denoted by G_k and H_k), we may assume that, for some uniformly elliptic operators G_∞ and H_∞,

$$G_k \to G_\infty \ \text{ and } \ H_k \to H_\infty \equiv G_\infty$$

uniformly in compact sets of $\mathcal{S}$, as $k \to \infty$. The last identity, $H_\infty \equiv G_\infty$, follows from (3.6).

By the Krylov–Safonov theory (see Proposition 4.10 of [CC]), $\|u_k\|_{C^\alpha(\overline{B}_{1/2})} \leq C$ for some universal $\alpha \in (0,1)$ and $C > 0$. This bound, (3.4), and Hölder estimates up to the boundary (see Proposition 4.13 of [CC]) lead to $\|w_k\|_{C^{\alpha/2}(\overline{B}_{1/2})} \leq C$. Therefore, we may assume that (again up to subsequences) $u_k \to u_\infty$ and $w_k \to w_\infty$ uniformly in $\overline{B}_{1/2}$.

It follows that u_∞ and w_∞ are viscosity solutions of the same equation $G_\infty(D^2 v) = 0$ (see Proposition 2.9 of [CC]). Moreover, $u_\infty \equiv w_\infty$ on $\partial B_{1/2}$, by (3.4). Jensen's uniqueness theorem for viscosity solutions (see Corollary 5.4 of [CC]) implies that $u_\infty \equiv w_\infty$ in $B_{1/2}$. But this contradicts $\|u_\infty - w_\infty\|_{L^\infty(B_{1/2})} \geq \varepsilon$, a consequence of (3.5). □

Next, we prove Theorem 3.3 using Lemma 3.4. We approximate the solution u of $G(D^2 u) = 0$ by the quadratic part P of w at the origin, which solves $H(D^2 P) = 0$.

Lemma 3.5. *Assume* (3.1). *Then, there exist constants* $0 < \mu \leq 1/4$ *and* $\overline{c}_f > 0$ *depending only on* $n, \lambda, \Lambda, \overline{\alpha}$ *and* c_e*, such that if we suppose*

$$G(N) = H(N) \ \textit{for all } N \in \mathcal{S} \textit{ with } \|N\| \leq \overline{c}_f \|u\|_{L^\infty(B_1)} \ , \tag{3.7}$$

then

$$\mu^{-2} \|u - P\|_{L^\infty(B_\mu)} \leq \frac{1}{2} \|u\|_{L^\infty(B_1)} \tag{3.8}$$

for some polynomial P *of degree two. Moreover,* P *satisfies* $G(D^2 P) = H(D^2 P) = 0$ *and*

$$G(N) = H(N) \ \textit{for all } N \in \mathcal{S} \textit{ with } \|N - D^2 P\| \leq \frac{\overline{c}_f}{2} \|u\|_{L^\infty(B_1)} \ . \tag{3.9}$$

We recall that a polynomial P of degree two (to be precise we should say of degree at most two) is of the form

$$P(x) = a + \langle b, x\rangle + \frac{1}{2}x^t M x$$

for some $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and some symmetric matrix $M = D^2 P$.

Proof of Lemma 3.5. Let $K = \|u\|_{L^\infty(B_1)}$ and consider (we may assume $K > 0$)

$$\tilde{u} := \frac{u}{K} = \frac{u}{\|u\|_{L^\infty(B_1)}} \ .$$

We apply Lemma 3.4 to the function $\tilde{u}$. We have $\|\tilde{u}\|_{L^\infty(B_1)} = 1$ and $\tilde{G}(D^2\tilde{u}) = 0$ in B_1, where $\tilde{G}$ is the uniformly elliptic operator

$$\tilde{G}(M) := \frac{1}{K} G(KM) \ ,$$

which has same ellipticity constants as G. We consider also the operator $\tilde{H}(M) := K^{-1}H(KM)$. It is easy to verify that equation $\tilde{H} = 0$ still has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates with constant c_e.

Now, take $0 < \mu \leq 1/4$ such that $C^* c_e \mu^{\overline{\alpha}} \leq 1/4$, where $C^* = C^*(n, \overline{\alpha}) > 0$ is a constant to be specified below and depending only on n and $\overline{\alpha}$. With this choice of μ, take $\varepsilon := \mu^2/4$ and let $\overline{c}_f := \overline{c}(\varepsilon)$ be the constant given by Lemma 3.4.

Let us verify assumption (3.2). If $\|N\| \leq \overline{c}(\varepsilon) = \overline{c}_f$ then $\|KN\| \leq \overline{c}_f \|u\|_{L^\infty(B_1)}$ and, by (3.7), $G(KN) = H(KN)$, i.e., $\tilde{G}(N) = \tilde{H}(N)$. Therefore, we can apply Lemma 3.4 and obtain a function $\tilde{w} \in C^2(B_{1/2})$ such that

$$\|\tilde{u} - \tilde{w}\|_{L^\infty(B_{1/2})} \leq \varepsilon \quad \text{and} \quad \|\tilde{w}\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/4})} \leq C(n, \overline{\alpha}) c_e \ , \tag{3.10}$$

$\tilde{G}(D^2\tilde{w}(0)) = \tilde{H}(D^2\tilde{w}(0)) = 0$, and

$$\tilde{G}(M) = \tilde{H}(M) \quad \text{if } \|M - D^2\tilde{w}(0)\| \leq \frac{\overline{c}(\varepsilon)}{2} = \frac{\overline{c}_f}{2} \ . \tag{3.11}$$

Let P be the quadratic part of $K\tilde{w}$ at 0, i.e.,

$$P(x) = K\tilde{w}(0) + \langle K\nabla\tilde{w}(0), x\rangle + \frac{1}{2}x^t K D^2\tilde{w}(0) x \ .$$

We have that $D^2P = KD^2\tilde{w}(0)$, and hence $G(D^2P) = H(D^2P) = 0$. To verify (3.9), note that if $\|N - D^2P\| \leq (\overline{c}_f/2)\|u\|_{L^\infty(B_1)}$ then $\|N/K - D^2\tilde{w}(0)\| \leq \overline{c}_f/2$. Now, (3.11) gives $\tilde{G}(N/K) = \tilde{H}(N/K)$, i.e., $G(N) = H(N)$.

It remains to check (3.8). By (3.10) and $\mu \leq 1/4$,

$$\|u - K\tilde{w}\|_{L^\infty(B_{1/2})} \leq \varepsilon \|u\|_{L^\infty(B_1)}$$

and

$$\|K\tilde{w}-P\|_{L^\infty(B_\mu)} \le C^*(n,\overline{\alpha})c_e\mu^{2+\overline{\alpha}}\|u\|_{L^\infty(B_1)} \ .$$

Adding these two inequalities and dividing by μ^2, we find

$$\begin{aligned}\mu^{-2}\|u-P\|_{L^\infty(B_\mu)} &\le \left(\mu^{-2}\varepsilon + C^* c_e\mu^{\overline{\alpha}}\right)\|u\|_{L^\infty(B_1)}\\ &\le \left(\frac{1}{4}+\frac{1}{4}\right)\|u\|_{L^\infty(B_1)} = \frac{1}{2}\|u\|_{L^\infty(B_1)}\end{aligned}$$

by our choice of μ and ε. This concludes the proof. □

Lemma 3.5 is ready to be iterated in the sequence of balls B_{μ^k}, to obtain:

Lemma 3.6. *Assume* (3.1) *and* (3.7). *Then, there is a sequence of polynomials* P_k *of degree two, with* $P_0 = 0$*, such that for* $k \ge 1$ *we have*

$$\mu^{-2k}\|u-P_k\|_{L^\infty(B_{\mu^k})} \le \frac{1}{2}\mu^{-2(k-1)}\|u-P_{k-1}\|_{L^\infty(B_{\mu^{k-1}})} \ , \tag{3.12}$$

$G(D^2P_k) = H(D^2P_k) = 0$, *and*

$$G(N) = H(N) \ \textit{if} \ \|N - D^2P_k\| \le \frac{\overline{c}_f}{2}\mu^{-2(k-1)}\|u-P_{k-1}\|_{L^\infty(B_{\mu^{k-1}})} \ , \tag{3.13}$$

where $\overline{c}_f$ *and* μ *are the constants of Lemma* 3.5.

Proof. For $k=1$, all statements follow from Lemma 3.5, taking $P_1 = P$ and $P_0 = 0$. Suppose now that $P_0, P_1, \dots, P_k$ have been found satisfying (3.12) and (3.13), and let us construct P_{k+1}. Consider

$$\tilde{u}(x) := \mu^{-2k}(u-P_k)(\mu^k x) \ \text{ for } x \in B_1 \ .$$

We have $\tilde{G}(D^2\tilde{u}) = 0$ in B_1, where $\tilde{G}(M) := G(M + D^2P_k)$. Note that $\tilde{G}(0) = 0$. Define the operator $\tilde{H}(M) := H(M + D^2P_k)$, and note that $\tilde{H} = 0$ also satisfies the assumptions on existence and $C^{2,\overline{\alpha}}$ estimates of Lemma 3.5, with same constant c_e as H.

Let us verify assumption (3.7) of Lemma 3.5 for $\tilde{G}$, $\tilde{H}$ and $\tilde{u}$. If $M \in \mathcal{S}$ and

$$\|M\| \le \overline{c}_f\|\tilde{u}\|_{L^\infty(B_1)} = \overline{c}_f\mu^{-2k}\|u-P_k\|_{L^\infty(\mu^k)}$$

then, by (3.12),

$$\|M\| \le \frac{\overline{c}_f}{2}\mu^{-2(k-1)}\|u-P_{k-1}\|_{L^\infty(B_{\mu^{k-1}})}$$

and hence $G(M + D^2P_k) = H(M + D^2P_k)$, by (3.13). We conclude $\tilde{G}(M) = \tilde{H}(M)$.

Lemma 3.5 gives the existence of a polynomial P of degree two such that

$$\mu^{-2}\|\tilde{u}-P\|_{L^\infty(B_\mu)} \le \frac{1}{2}\|\tilde{u}\|_{L^\infty(B_1)} \ , \tag{3.14}$$

$\tilde{G}(D^2P) = \tilde{H}(D^2P) = 0$, and

$$\tilde{G}(M) = \tilde{H}(M) \quad \text{if} \quad \|M - D^2P\| \le \frac{\overline{c}_f}{2}\|\tilde{u}\|_{L^\infty(B_1)} \ . \tag{3.15}$$

Rescaling back (3.14), we have

$$\begin{aligned}
\frac{1}{2}\mu^{-2k}\|u - P_k\|_{L^\infty(B_{\mu^k})} &= \frac{1}{2}\|\tilde{u}\|_{L^\infty(B_1)} \\
&\ge \mu^{-2}\|\tilde{u} - P\|_{L^\infty(B_\mu)} = \mu^{-2(k+1)}\|u(\mu^k x) - P_k(\mu^k x) - \mu^{2k}P(x)\|_{L^\infty(B_\mu)} \\
&= \mu^{-2(k+1)}\|u - P_{k+1}\|_{L^\infty(B_{\mu^{k+1}})}
\end{aligned}$$

if we define $P_{k+1}(y) := P_k(y) + \mu^{2k}P(\mu^{-k}y)$. We have proved (3.12) with k replaced by $k+1$. Moreover, since $D^2P_{k+1} = D^2P_k + D^2P$ and $\tilde{G}(D^2P) = \tilde{H}(D^2P) = 0$, we deduce that $G(D^2P_{k+1}) = H(D^2P_{k+1}) = 0$. Finally, to verify (3.13) with k replaced by $k+1$, suppose that

$$\|N - D^2P_{k+1}\| \le \frac{\overline{c}_f}{2}\mu^{-2k}\|u - P_k\|_{L^\infty(B_{\mu^k})} = \frac{\overline{c}_f}{2}\|\tilde{u}\|_{L^\infty(B_1)} \ .$$

Then $\|(N - D^2P_k) - D^2P\| \le (\overline{c}_f/2)\|\tilde{u}\|_{L^\infty(B_1)}$, and (3.15) gives $\tilde{G}(N - D^2P_k) = \tilde{H}(N - D^2P_k)$, that is, $G(N) = H(N)$. $\square$

Theorem 3.3 follows from Lemma 3.6, since the sequence of polynomials P_k guarantees that u is $C^{2,\alpha}$ at the origin for some $\alpha \in (0,1)$, and that $H(D^2u(0)) = 0$. Let us state this general and simple fact as a lemma that we will also use in next section.

Lemma 3.7. *Let $u \in L^\infty(B_1)$, H be a uniformly elliptic operator with $H(0) = 0$, and let $0 < \mu < 1$ be a constant. For $k \in \mathbb{N} \setminus \{0\}$, define*

$$A_0 := \|u\|_{L^\infty(B_1)} \quad \text{and}$$

$$A_k := \mu^{-2k} \inf\left\{\|u - P\|_{L^\infty(B_{\mu^k})} \ : \ H(D^2P) = 0, P \in \mathcal{P}_2\right\} \ , \tag{3.16}$$

where $\mathcal{P}_2$ is the space of polynomials of degree two. Assume

$$A_k < d_0\tau^k \qquad \forall k \ge 0 \ ,$$

for some constants $d_0 > 0$ and $0 < \tau < 1$. Then u is $C^{2,\alpha}$ at the origin; more precisely, there exist constants $0 < \alpha < 1$ and $C > 0$ depending only on μ and τ, such that

$$\|u - P\|_{L^\infty(B_r(0))} \le Cd_0 r^{2+\alpha} \qquad \forall r \le 1 \ , \tag{3.17}$$

for some polynomial P of degree two which satisfies

$$H(D^2P) = 0 \quad \text{and} \quad |P(0)| + |DP(0)| + \|D^2P\| \le Cd_0 \ .$$

Proof. Take $P_k(x) = a_k + \langle b_k, x\rangle + x^t M_k x/2$, with $P_0 \equiv 0$, such that $H(D^2 P_k) = 0$ and

$$\|u - P_k\|_{L^\infty(B_{\mu^k})} \le d_0 \mu^{2k} \tau^k \le d_0 \mu^{(2+\alpha)k} , \qquad k \ge 0 .$$

Here, we have chosen $\alpha \in (0,1)$ such that $\tau \le \mu^\alpha$. It follows that

$$\|P_{k+1} - P_k\|_{L^\infty(B_{\mu^{k+1}})} \le 2d_0 \mu^{(2+\alpha)k} , \qquad k \ge 0 . \tag{3.18}$$

In particular, $|a_{k+1} - a_k| = |(P_{k+1} - P_k)(0)| \le 2d_0\mu^{(2+\alpha)k}$. Next, since

$$\begin{aligned} &|(P_{k+1} - P_k)\left(\mu^{k+1}(b_{k+1} - b_k)/|b_{k+1} - b_k|\right) \\ &\qquad - (P_{k+1} - P_k)\left(-\mu^{k+1}(b_{k+1} - b_k)/|b_{k+1} - b_k|\right)| = 2\mu^{k+1}|b_{k+1} - b_k| , \end{aligned}$$

(3.18) implies $|b_{k+1} - b_k| \le 2d_0\mu^{-1}\mu^{(1+\alpha)k}$. This bound, the one for $|a_{k+1} - a_k|$, and (3.18) give

$$\|x^t(M_{k+1} - M_k)x/2\|_{L^\infty(B_{\mu^{k+1}})} \le 6d_0\mu^{(2+\alpha)k} .$$

Therefore $\|M_{k+1} - M_k\| \le 12d_0\mu^{-2}\mu^{\alpha k}$.

It follows that $a_k \to a$, $b_k \to b$, $M_k \to M$, and if $P(x) := a + \langle b, x\rangle + x^t M x/2$ then $H(D^2 P) = 0$ and

$$|P(0)| + |DP(0)| + \|D^2 P\| \le \sum_{k=0}^{\infty} 16 d_0 \mu^{-2}\mu^{\alpha k} = C d_0 .$$

Finally, using the previous bounds on $|a_{i+1} - a_i|$, $\mu^k|b_{i+1} - b_i|$ and $\mu^{2k}\|M_{i+1} - M_i\|$, we find

$$\|u - P\|_{L^\infty(B_{\mu^k})} = \|u - P_k - \sum_{i=k}^{\infty}(P_{i+1} - P_i)\|_{L^\infty(B_{\mu^k})} \le C d_0 \mu^{(2+\alpha)k} ,$$

which implies (3.17) and the lemma. □

We can now give the

Proof of Theorem 3.3. By iterating (3.12) in Lemma 3.6, we know that

$$\mu^{-2k}\|u - P_k\|_{L^\infty(B_{\mu^k})} \le \frac{1}{2^k}\|u\|_{L^\infty(B_1)} ,$$

for a sequence of polynomials of degree two with $H(D^2 P_k) = 0$. Hence, the quantity A_k defined by (3.16) (with μ given by Lemma 3.6) satisfies

$$A_k \le \frac{1}{2^k}\|u\|_{L^\infty(B_1)} < \frac{1}{2^k} 2\|u\|_{L^\infty(B_1)} ,$$

unless $u \equiv 0$. Lemma 3.7, applied with $\tau = 1/2$, gives the existence of a polynomial P such that $H(D^2 P) = 0$ and, by (3.17), $D^2 u(0) = D^2 P$. Hence, $H(D^2 u(0)) = H(D^2 P) = 0$.

The assertion of Theorem 3.3,

$$H(D^2u) = 0 \ \text{ in } B_{1/2} \ ,$$

follows from the previous argument, taking as origin any point $x_0 \in B_{1/2}$. Note that $B_{1/2}(x_0) \subset B_1$, and hence $G(D^2v) = 0$ in $B_1 = B_1(0)$, where $v(x) = 4u(x_0 + x/2)$, $x \in B_1$. Since $\|v\|_{L^\infty(B_1)} \le 4\|u\|_{L^\infty(B_1)}$, the conclusion $H(D^2u(x_0)) = H(D^2v(0)) = 0$ follows from the previous argument by simply replacing the constant $\overline{c}_f$ by $4\overline{c}_f$. □

We finish this section with a simple lemma that will be useful in next section. This lemma is also used in [CY] and [Y].

Lemma 3.8. *Let G be a uniformly elliptic operator, and $u \in C(\overline{B}_\mu)$ a viscosity solution of $G(D^2u) = 0$ in B_μ, for some $\mu > 0$. Let Q be a polynomial of degree two.*

Then, there exists a polynomial P of degree two such that $G(D^2P) = 0$ and

$$\|u - P\|_{L^\infty(B_\mu)} \le C\|u - Q\|_{L^\infty(B_\mu)} \ ,$$

where C is a universal constant.

Proof. Let

$$a := G(D^2Q) \ \text{ and } \ b := \|u - Q\|_{L^\infty(B_\mu)} \ .$$

By Proposition 2.13 of [CC], we have $u - Q \in S(\lambda/n, \Lambda, -G(D^2Q)) = S(-a)$. Let $R(x) := 2\mu^{-2}b|x|^2$, and note that

$$\begin{aligned}
\{R - (u - Q)\}_{|\partial B_\mu} &= 2b - (u - Q)_{|\partial B_\mu} \\
&\ge 2b - (u - Q)(0) - 2\|u - Q\|_{L^\infty(B_\mu)} \\
&= -(u - Q)(0) = (R - (u - Q))(0) \ .
\end{aligned}$$

Hence, $\min_{\overline{B}_\mu}(R - (u - Q))$ is achieved in the interior B_μ. As a consequence, we have that $\mathcal{M}^+(D^2R, \lambda/n, \Lambda) \ge -a$, and therefore $-a \le C\mu^{-2}b$. Similarly, using that $u - Q$ is also viscosity supersolution, we find $a \le C\mu^{-2}b$. That is,

$$|G(D^2Q)| = |a| \le C\mu^{-2}b = C\mu^{-2}\|u - Q\|_{L^\infty(B_\mu)} \ .$$

For $s \in \mathbb{R}$, let $P(x) = Q(x) + s|x|^2/2$. By the ellipticity of G, there exists $s \in \mathbb{R}$ such that $G(D^2P) = 0$ and $\|D^2P - D^2Q\| = |s| \le C|G(D^2Q)|$. Thus,

$$|s| \le C|G(D^2Q)| \le C\mu^{-2}\|u - Q\|_{L^\infty(B_\mu)} \ .$$

Finally, we have

$$\begin{aligned}
\|u - P\|_{L^\infty(B_\mu)} &\le \|u - Q\|_{L^\infty(B_\mu)} + \|Q - P\|_{L^\infty(B_\mu)} \\
&= \|u - Q\|_{L^\infty(B_\mu)} + \frac{|s|}{2}\mu^2 \le C\|u - Q\|_{L^\infty(B_\mu)} \ .
\end{aligned}$$

□

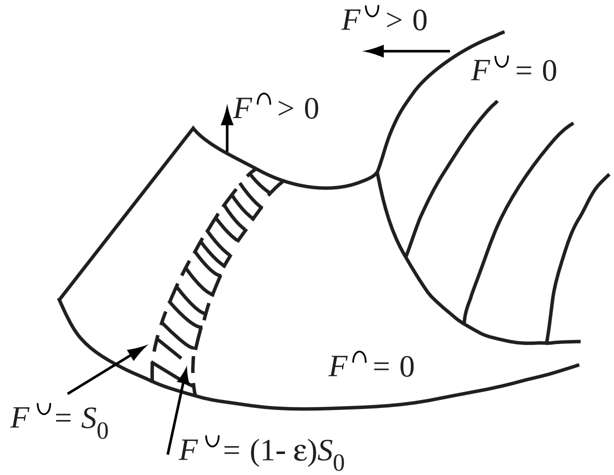

Figure 1: The hypersurface $\{F = 0\}$ in $\mathcal{S}$

4 Interior $C^{2,\alpha}$ a priori estimates

In this section we prove Theorem 1.1, that is, an interior $C^{2,\alpha}$ a priori estimate for classical solutions of $F(D^2u) = 0$, where $F = \min(F^\cap, F^\cup)$ is of the form (1.9) and $\alpha \in (0, 1)$ is universal.

Let us describe the main ideas in the proof. By Proposition 3.1 we know that if $F^\cup(0)$ is positive and too large then $F^\cap(D^2u) = 0$ in a smaller ball. In this case there is nothing to prove, since u will be $C^{2,\overline{\alpha}}$ by interior regularity for the concave operator $F^\cap$. Hence, we can assume that $F^\cup(0)$ is under control. Then, the L^∞ bound of Proposition 2.2 gives that the quantity

$$S_0 := \sup_{B_{1/2}} F^\cup(D^2u)$$

is also under control (see Figure 1). By Proposition 2.1, we know that $F^\cup(D^2u)$ is a subsolution.

Now we distinguish two possibilities. First suppose that, except for a set of very small measure, D^2u lives in $\{(1-\varepsilon)S_0 \leq F^\cup \leq S_0\}$. In this case, taking $\varepsilon > 0$ small and using the ABP estimate, we will see that u is very close in L^∞ norm to the polynomial given by the quadratic part of a solution of $F^\cup(D^2v) = S_0$, since this convex equation has $C^{2,\overline{\alpha}}$ estimates.

In the second case we have that the set $\{F^\cup < (1-\varepsilon)S_0\} = \{S_0 - F^\cup > \varepsilon S_0\}$ has positive and not too small measure. Since $S_0 - F^\cup$ is a nonnegative supersolution, this will imply that $S_0 - F^\cup \geq \tilde{\varepsilon} S_0$, that is $F^\cup \leq (1-\tilde{\varepsilon})S_0$, *everywhere* in a smaller ball. Hence, the supremum of $F^\cup$ has decreased by a factor.

We iterate this process in smaller balls. Heuristically, if the second case occurs "often" then D^2u is concentrating near the set $\{F^\cup = 0\}$, and hence u can be approximated by the quadratic part of a solution of $F^\cup(D^2v) = 0$.

The iteration process in the actual proof is more delicate than the previous outline, and it is described in detail in the rest of this section. We start with two lemmas.

Lemma 4.1. *Let $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in B_1, where F is of the form (1.9). Assume that*

$$F^{\cup}(0) \leq c_f \|u\|_{L^\infty(B_1)} \ , \tag{4.1}$$

where c_f is the universal constant of Proposition 3.1. For $0 < \mu < 1/8$ define

$$\delta_\mu(u) := \frac{1}{|B_{1/4}|}\left| B_{1/4} \cap \left\{ F^{\cup}(D^2u) \leq (1-\mu^3) \sup_{B_{1/2}} F^{\cup}(D^2u) \right\} \right| \in [0,1] \ . \tag{4.2}$$

Then,
(a) there exists a polynomial P of degree two such that $F(D^2P) = 0$ and

$$\mu^{-2}\|u - P\|_{L^\infty(B_\mu)} \leq C\left\{ \mu^{\overline{\beta}}\|u\|_{L^\infty(B_1)} + \mu^{-2}\delta_\mu(u)^{1/n} \sup_{B_{1/2}} F^{\cup}(D^2u) \right\} \ , \tag{4.3}$$

and
(b) we have that

$$\sup_{B_{\mu/2}} F^{\cup}(D^2u) \leq \{1 - c\mu^3 \delta_\mu(u)^{1/p_0}\} \sup_{B_{1/2}} F^{\cup}(D^2u) \ , \tag{4.4}$$

where $0 < \overline{\beta} < 1$, $p_0 > 0$, $c > 0$ and $C > 0$ are universal constants.

A consequence of this lemma is the following result, that we will iterate later in the sequence of balls B_{μ^k}. Here, we do not assume the upper bound (4.1) on $F^{\cup}(0)$.

Lemma 4.2. *Let $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in B_1, where F is of the form (1.9). Then, there exist universal constants $0 < \mu < 1/8$, $0 < \nu < 1$ and $C > 0$ such that, for*

$$A_0 := \|u\|_{L^\infty(B_1)} \ , \quad A_1 := \mu^{-2} \inf\left\{ \|u - P\|_{L^\infty(B_\mu)} \ : \ F(D^2P) = 0, \ P \in \mathcal{P}_2 \right\} \ ,$$

(here $\mathcal{P}_2$ denotes the polynomials of degree two) and

$$S_0 := \sup_{B_{1/2}} F^{\cup}(D^2u) \ , \quad S_1 := \sup_{B_{\mu/2}} F^{\cup}(D^2u) \ ,$$

we have either
(i) $A_1 \leq \frac{1}{2}A_0$
or
(ii) $S_1 \leq \nu S_0$ and $A_1 \leq CS_0$.

Proof of Lemma 4.1. Let $A_0 := \|u\|_{L^\infty(B_1)}$ and $S_0 := \sup_{B_{1/2}} F^{\cup}(D^2u)$. Note that Proposition 2.2 and hypothesis (4.1) lead to

$$S_0 \leq CA_0 \ . \tag{4.5}$$

We start proving (b). By Proposition 2.1, we know that $0 \leq S_0 - F^{\cup}(D^2u) \in \overline{S}(\lambda/n, \Lambda)$ in $B_{1/2}$. The Krylov–Safonov weak Harnack inequality (see Theorem 4.8(1) of [CC]) and (4.2) lead to

$$\begin{aligned}\inf_{B_{1/3}} (S_0 - F^{\cup}(D^2u)) &\geq c\|S_0 - F^{\cup}(D^2u)\|_{L^{p_0}(B_{1/4})} \\ &\geq c\mu^3 \delta_\mu(u)^{1/p_0} S_0 \ .\end{aligned}$$

This establishes (4.4).

We now prove (a). Since $F^{\cup}$ is convex, the problem

$$\begin{cases} F^{\cup}(D^2v) = S_0 & \text{in } B_{1/4} \\ v = u & \text{on } \partial B_{1/4} \end{cases}$$

has a unique classical solution v (see Proposition 9.8 of [CC]). Moreover, v satisfies $\|v\|_{C^{2,\overline{\beta}}(\overline{B}_{1/8})} \leq C(A_0 + |F^{\cup}(0) - S_0|) \leq CA_0$ for some universal constants $\overline{\beta} \in (0,1)$ and C. Here we have used (4.1) and (4.5). Hence, we have

$$\|v - Q\|_{L^\infty(B_\mu)} \leq C\mu^{2+\overline{\beta}} A_0 \ , \tag{4.6}$$

where Q is the polynomial of degree two given by the (affine +) quadratic part of v at the origin.

Next, we apply the Alexandroff–Bakelman–Pucci estimate to $u - v$. Since $u - v = 0$ on $\partial B_{1/4}$,

$$|F^{\cup}(D^2u) - F^{\cup}(D^2v)| \leq \begin{cases} S_0 \ \text{ in } \ \omega := B_{1/4} \cap \{F^{\cup}(D^2u) \leq (1 - \mu^3)S_0\} \\ \mu^3 S_0 \ \text{ in } \ B_{1/4} \setminus \omega \ , \end{cases}$$

and $|\omega| = |B_{1/4}|\delta_\mu(u)$, we obtain

$$\|u - v\|_{L^\infty(B_{1/4})} \leq C\big(\delta_\mu(u)^{1/n} + \mu^3\big)S_0 \ .$$

This estimate and (4.6) lead to

$$\begin{aligned}\mu^{-2}\|u - Q\|_{L^\infty(B_\mu)} &\leq C\{\mu^{\overline{\beta}} A_0 + (\mu^{-2}\delta_\mu(u)^{1/n} + \mu)S_0\} \\ &\leq C\{\mu^{\overline{\beta}} A_0 + \mu^{-2}\delta_\mu(u)^{1/n} S_0\} \ ;\end{aligned}$$

we have used (4.5) and $\mu \leq \mu^{\overline{\beta}}$. Now, Lemma 3.8 gives the existence of a polynomial P of degree two such that $F(D^2P) = 0$ and $\|u - P\|_{L^\infty(B_\mu)} \leq C\|u - Q\|_{L^\infty(B_\mu)}$. Together with the previous inequality, this proves (4.3) and the lemma. □

Proof of Lemma 4.2. We distinguish two cases.

Case 1. Assume that $F^{\cup}(0) > c_f A_0$. Then, by Proposition 3.1, we have $F^{\cap}(D^2u) = 0$ in $B_{1/2}$. Since $F^{\cup}(0) > 0$, we also have $F^{\cap}(0) = 0$. Interior $C^{2,\overline{\beta}}$ estimates for the concave operator $F^{\cap}$ (here we can take the same $\overline{\beta}$ of Lemma 4.1 for convex operators) give $\|u\|_{C^{2,\overline{\beta}}(\overline{B}_{1/4})} \leq CA_0$.

Taking Q to be the (affine +) quadratic part of u at the origin, then $\|u-Q\|_{L^\infty(B_\mu)} \leq C\mu^{2+\overline{\beta}}A_0$ for all $\mu < 1/8$. Now we apply Lemma 3.8 with $G = F$ and obtain a new polynomial P of degree two such that $F(D^2P) = 0$ and

$$\|u-P\|_{L^\infty(B_\mu)} \leq C\|u-Q\|_{L^\infty(B_\mu)} \leq C\mu^{2+\overline{\beta}}A_0 \ .$$

It follows that

$$A_1 \leq \mu^{-2}\|u-P\|_{L^\infty(B_\mu)} \leq C\mu^{\overline{\beta}}A_0 \leq \frac{1}{2}A_0 \tag{4.7}$$

if we take μ (universally) small enough. We have seen that in this case 1, (i) in Lemma 4.2 holds.

Case 2. Assume now that $F^{\cup}(0) \leq c_f A_0$. Proposition 2.2 gives that

$$S_0 \leq CA_0 \ , \tag{4.8}$$

while Lemma 4.1 reads

$$A_1 \leq C^*\{\mu^{\overline{\beta}}A_0 + \mu^{-2}\delta_\mu(u)^{1/n}S_0\} \tag{4.9}$$

and

$$S_1 \leq \{1 - c\mu^3\delta_\mu(u)^{1/p_0}\}S_0 \ , \tag{4.10}$$

for positive universal constants $0 < \overline{\beta} < 1$, p_0, c, C and C^*. We now make the universal choice of $\mu \in (0, 1/8)$ to ensure $2C^*\mu^{\overline{\beta}} \leq 1/2$ —where C^* is the constant in (4.9)— and, at the same time, to ensure the smallness condition on μ needed in the last inequality of (4.7) in the previous case 1.

With this choice of μ, if we further assume $\mu^{-2}\delta_\mu(u)^{1/n}S_0 \leq \mu^{\overline{\beta}}A_0$, then (4.9) leads to $A_1 \leq 2C^*\mu^{\overline{\beta}}A_0 \leq A_0/2$. That is, option (i) in the lemma holds.

Hence, from now on we may assume

$$\mu^{\overline{\beta}}A_0 < \mu^{-2}\delta_\mu(u)^{1/n}S_0 \ . \tag{4.11}$$

We shall show that in this case, option (ii) of the lemma holds. Using (4.8) and (4.11), we find

$$\mu^{\overline{\beta}}A_0 < \mu^{-2}\delta_\mu(u)^{1/n}S_0 \leq C\mu^{-2}\delta_\mu(u)^{1/n}A_0 \ ,$$

and therefore $\delta_\mu(u) \geq (C^{-1}\mu^{2+\overline{\beta}})^n$. This and (4.10) give

$$S_1 \leq \nu S_0 \ ,$$

with $0 < \nu < 1$ universal. Finally, since $\delta_\mu(u) \leq 1$, (4.9) and (4.11) lead to $A_1 \leq C^*\{\mu^{\overline{\beta}}A_0 + \mu^{-2}S_0\}$ and $\mu^{\overline{\beta}}A_0 \leq \mu^{-2}S_0$, respectively. It follows that $A_1 \leq 2C^*\mu^{-2}S_0 = CS_0$, and hence option (ii) in Lemma 4.2 holds. □

Lemma 4.2 can be iterated in balls B_{μ^k}, since our class (1.9) of operators is invariant under translations in $\mathcal{S}$. For this purpose let us define, for $k \geq 1$,

$$\begin{cases} A_k := \mu^{-2k} \inf \left\{ \|u - P\|_{L^\infty(B_{\mu^k})} \ : \ F(D^2 P) = 0, \ P \in \mathcal{P}_2 \right\} \\ S_k := \sup\limits_{B_{\mu^k/2}} F^\cup(D^2 u) \ , \end{cases} \tag{4.12}$$

where $0 < \mu < 1$ is a given constant, and $\mathcal{P}_2$ is the set of polynomials of degree two. Note that A_k and S_k rescale in the same manner. Recall also that

$$A_0 := \|u\|_{L^\infty(B_1)} \quad \text{and} \quad S_0 := \sup_{B_{1/2}} F^\cup(D^2 u) \ .$$

The goal is to prove power decay for the sequence A_k, and then conclude $C^{2,\alpha}$ regularity by Lemma 3.7. First, we have:

Lemma 4.3. *Let $u \in C^2(B_1)$ satisfy $F(D^2 u) = 0$ in B_1, where F is of the form* (1.9). *Let A_k and S_k be defined as above with the universal constant μ of Lemma* 4.2, *and let $0 < \nu < 1$ and $C > 0$ be the universal constants of that lemma.*

Then, for every $k \geq 0$, we have either
(i) $A_{k+1} \leq \frac{1}{2} A_k$
or
(ii) $S_{k+1} \leq \nu S_k$ and $A_{k+1} \leq C S_k$.

Proof. For $k = 0$ the statements are contained in Lemma 4.2. Assume now $k \geq 1$, and let $\varepsilon > 0$. By definition of A_k, there exists $P_k \in \mathcal{P}_2$ with $F(D^2 P_k) = 0$ and $\mu^{-2k} \|u - P_k\|_{L^\infty(B_{\mu^k})} \leq A_k + \varepsilon$. Consider

$$\tilde{u}(x) := \mu^{-2k} (u - P_k)(\mu^k x) \quad \text{for } x \in B_1 \ .$$

We have $\tilde{F}(D^2 \tilde{u}) = 0$ in B_1, where $\tilde{F}$ is defined by

$$\tilde{F}(M) := F(M + D^2 P_k) \ .$$

Note the $\tilde{F}(0) = 0$ and that $\tilde{F}$ is of the form (1.9), with $\tilde{F}^\cap(M) = F^\cap(M + D^2 P_k)$ and $\tilde{F}^\cup(M) = F^\cup(M + D^2 P_k)$. We have that

$$\tilde{A}_0 := \|\tilde{u}\|_{L^\infty(B_1)} \leq A_k + \varepsilon$$

and

$$\tilde{S}_0 := \sup_{B_{1/2}} \tilde{F}^\cup(D^2 \tilde{u}) = S_k \ .$$

We apply Lemma 4.2 to $\tilde{u}$ and $\tilde{F}$. In case that (i) in the lemma holds, we have the existence of a polynomial $\tilde{P}$ with $\tilde{F}(D^2 \tilde{P}) = 0$ and

$$\mu^{-2} \|\tilde{u} - \tilde{P}\|_{L^\infty(B_\mu)} \leq \frac{1}{2} \tilde{A}_0 + \frac{\varepsilon}{2} \leq \frac{1}{2} A_k + \varepsilon \ .$$

Using $\mu^{-2}(\tilde{u}-\tilde{P})(x) = \mu^{-2(k+1)}\{u(\mu^k x) - P_k(\mu^k x) - \mu^{2k}\tilde{P}(x)\}$ and taking $P_{k+1}(y) = P_k(y) + \mu^{2k}\tilde{P}(\mu^{-k}y)$, we have that $\mu^{-2(k+1)}\|u - P_{k+1}\|_{L^\infty(B_{\mu^{k+1}})} \leq A_k/2 + \varepsilon$. Since $F(D^2P_{k+1}) = 0$, we conclude that $A_{k+1} \leq A_k/2 + \varepsilon$.

If (ii) in Lemma 4.2 holds, then

$$\mu^{-2}\|\tilde{u} - \tilde{Q}\|_{L^\infty(B_\mu)} \leq C\tilde{S}_0 + \varepsilon = CS_k + \varepsilon$$

for some polynomial $\tilde{Q}$ with $\tilde{F}(D^2\tilde{Q}) = 0$, and also

$$\sup_{B_{\mu/2}} \tilde{F}^{\cup}(D^2\tilde{u}) \leq \nu\tilde{S}_0 = \nu S_k \ .$$

Proceeding as before, we find $A_{k+1} \leq CS_k + \varepsilon$ and $S_{k+1} \leq \nu S_k$.

We now let $\varepsilon \to 0$. Note that for every fixed k, there is a sequence $\varepsilon_j \to 0$ such that either (i) holds for every ε_j or (ii) holds for every ε_j. This proves the alternative of Lemma 4.3. □

We can now establish the power decay of A_k. For this, we will only use Lemma 4.3 together with the inequality

$$A_{k+1} \leq \mu^{-2}A_k \qquad \forall k \geq 0 \ , \tag{4.13}$$

and the fact that $\{S_k\}$ is a nonincreasing sequence:

$$S_{k+1} \leq S_k \qquad \forall k \geq 0 \ . \tag{4.14}$$

Both (4.13) and (4.14) follow from the definition (4.12) of these quantities.

Lemma 4.4. *Under the assumptions of Lemma* 4.3*, we have*

$$A_i \leq C\tau^i(A_0 + S_0) \qquad \forall i \geq 1 \ , \tag{4.15}$$

where $0 < \mu < 1$*,* $0 < \tau < 1$*, and* C *are universal constants.*

Proof. Let $i \geq 1$ and consider

$$\begin{aligned} I &= \{k \in \{0,1,\dots,i-1\} \ : \ S_{k+1} \leq \nu S_k \text{ and } A_{k+1} \leq CS_k\} \\ &= \{k \in \{0,1,\dots,i-1\} \ : \ \text{(ii) in Lemma 4.3 holds}\} \ , \end{aligned}$$

and let

$$n_i = \#I \in [0,i]$$

be the cardinal of I. Then, by Lemma 4.3, option (i) in the lemma happens at least $i - n_i$ times. This and (4.13) lead to

$$A_i \leq \left(\frac{1}{2}\right)^{i-n_i} \mu^{-2n_i} A_0 = \frac{(2\mu^{-2})^{n_i}}{2^i} A_0 \ .$$

In case that $n_i \le \beta i$, where $\beta > 0$ is a universally small constant such that $(2\mu^{-2})^\beta/2 \le 3/4$, then

$$A_i \le \left(\frac{(2\mu^{-2})^\beta}{2}\right)^i A_0 \le \left(\frac{3}{4}\right)^i A_0 \le \tau^i(A_0+S_0)$$

if we take $3/4 \le \tau < 1$. In this case, (4.15) is proved.

Hence, we may assume that $n_i > \beta i$. Let

$$\overline{k} = \max I \in \{0,1,\dots,i-1\} \ .$$

Since $\overline{k} \in I$, we have $A_{\overline{k}+1} \le CS_{\overline{k}}$. Since (i) in Lemma 4.3 holds for every $k \in \{\overline{k}+1,\dots,i-1\}$ then, in particular, A_k is decreasing for these indexes. Hence $A_i \le A_{\overline{k}+1}$, which combined with the previous inequality gives

$$A_i \le CS_{\overline{k}} \ . \tag{4.16}$$

Using $n_i > \beta i$ and that $\{S_k\}$ is a decreasing sequence, we find $S_{\overline{k}} \le \nu^{n_i-1}S_0 \le \nu^{-1}(\nu^\beta)^i S_0$. Combined with (4.16), this leads to $A_i \le C\nu^{-1}(\nu^\beta)^i S_0 \le C\tau^i(A_0+S_0)$, if we take $\tau \in [\nu^\beta, 1)$. The proof is now finished. □

Finally, Theorem 1.1 follows from the previous lemma.

Proof of Theorem 1.1. Assume first that $F^\cup(0) \le c_f\|u\|_{L^\infty(B_1)}$, where c_f is the constant of Proposition 3.1. Then, Proposition 2.2 gives that

$$S_0 = \sup_{B_{1/2}} F^\cup(D^2u) \le C\|u\|_{L^\infty(B_1)} = CA_0 \ .$$

Hence, by (4.15), $A_k < C\tau^k\|u\|_{L^\infty(B_1)}$ for $k \ge 1$ (unless $u \equiv 0$). Using Lemma 3.7, we have

$$\|u-P_0\|_{L^\infty(B_r(0))} \le Cr^{2+\alpha}\|u\|_{L^\infty(B_1)} \qquad \forall r \le 1 \tag{4.17}$$

for some polynomial P_0 of degree two with

$$|P_0(0)| + |DP_0(0)| + \|D^2P_0\| \le C\|u\|_{L^\infty(B_1)} \ , \tag{4.18}$$

where $\alpha \in (0,1)$ is universal.

In case that $F^\cup(0) > c_f\|u\|_{L^\infty(B_1)}$, Proposition 3.1 states that $F^\cap(D^2u) = 0$ in $B_{1/2}$. Interior $C^{2,\alpha}$ regularity for this concave equation (which in this second case satisfies $F^\cap(0) = 0$) gives that (4.17) and (4.18) also hold in this case, for some polynomial P_0 of degree two.

Next, given $x_0 \in \overline{B}_{1/2}$, we apply the previous argument to $F(D^2v) = 0$ in B_1, where $v(x) = 4u(x_0 + x/2)$. We find that (4.17) and (4.18), with P_0 replaced by a new polynomial P_{x_0} and with $B_r(0)$ replaced by $B_r(x_0)$, $r \le 1/2$, also hold.

It follows that $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and that $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C\|u\|_{L^\infty(B_1)}$. □

5 Existence of classical solutions

In this section we prove that, for a large class of uniformly elliptic operators H, the validity of interior $C^{2,\overline{\alpha}}$ a priori estimates for classical solutions of the equations $H =$ constant implies the existence of classical solutions to the Dirichlet problems for the operator H, and consequently, it also implies the $C^{2,\alpha}$ regularity of viscosity solutions of $H(D^2u) = 0$ —and more generally, of $H(D^2u) = f(x) \in C^\alpha$— for every $\alpha \in (0, \overline{\alpha})$.

To carry this out, we use the method of continuity, the Schauder fixed point theorem in Banach spaces adapted to weighted interior $C^{2,\alpha}$ norms, and the interior $C^{2,\alpha}$ perturbation theory of [C]. We need to assume interior $C^{2,\overline{\alpha}}$ estimates not only for the equations $H =$ constant, but for a path of equations connecting H with the Laplacian. This hypothesis will be satisfied by every operator F of the form (1.9). Hence, Theorem 1.2 and Corollary 1.3 will be particular cases of the more general result that we present in this section.

To state it, let us introduce the following terminology. Given a uniformly elliptic operator H and constants $\overline{\alpha} \in (0,1)$, $c_e > 0$ and $c \in \mathbb{R}$, we say that *equation $H = c$ has interior $C^{2,\overline{\alpha}}$ estimates with constant c_e* if, for every $M \in \mathcal{S}$ with $H(M) = c$ and for every classical solution $w \in C^2(B_1)$ of

$$H(D^2w(x) + M) = c \quad \text{in } B_1 \ ,$$

we have $w \in C^{2,\overline{\alpha}}(\overline{B}_{1/2})$ and

$$\|w\|_{C^{2,\overline{\alpha}}(\overline{B}_{1/2})} \le c_e \|w\|_{L^\infty(B_1)} \ . \tag{5.1}$$

Note that this is a weaker hypothesis than that of Definition 3.2, in the sense that here we do not assume the existence of classical solutions for the equation $H - c = 0$ —such existence result is precisely our goal here.

The main result of this section is the following:

Theorem 5.1. *Let H be a uniformly elliptic operator with $H(0) = 0$. Assume:*
(i) for every $t \in [0,1]$, there exists a uniformly elliptic operator H_t with $H_t(0) = 0$, such that $H_0 = \Delta$, $H_1 = H$ and

$$\|H_t - H_s\|_{\mathrm{Lip}} \le c_l |t - s| \qquad \forall t, s \in [0,1] \tag{5.2}$$

for some positive constant c_l. Here $\|\cdot\|_{\mathrm{Lip}}$ denotes the Lipschitz norm, i.e., we are assuming

$$|(H_t - H_s)(M) - (H_t - H_s)(N)| \le c_l |t - s| \, \|M - N\| \tag{5.3}$$

with c_l independent of $M, N \in \mathcal{S}$ and $t, s \in [0,1]$, and
(ii) there exist constants $0 < \overline{\alpha} < 1$ and $c_e > 0$ such that, for every $t \in [0,1]$ and $c \in \mathbb{R}$, equation $H_t = c$ has interior $C^{2,\overline{\alpha}}$ estimates with constant c_e.

Let $0 < \alpha < \overline{\alpha}$. Then, for every $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C(\partial B_1)$, there exists a unique solution $u \in C^{2,\alpha}(B_1) \cap C(\overline{B}_1)$ of

$$\begin{cases} H(D^2u) = f(x) & \text{in } B_1 \\ \qquad u = \varphi(x) & \text{on } \partial B_1 \ . \end{cases}$$

Moreover, we have

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C\left\{\|f\|_{C^\alpha(\overline{B}_1)} + \|\varphi\|_{L^\infty(\partial B_1)}\right\} \tag{5.4}$$

for some constant C depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

The constant C in (5.4) does not depend on the constant c_l in (5.2). Note also that we need to assume that $f \in C^\alpha(\overline{B}_1)$ is C^α up to the boundary to have existence of solution u. Instead, the next corollary on regularity of viscosity solutions requires f to be C^α only in the interior.

Corollary 5.2. *Let H satisfy the assumptions of Theorem 5.1 and let $f \in C^\alpha(B_1)$, where $0 < \alpha < \overline{\alpha}$ and $\overline{\alpha}$ is the exponent in assumption (ii) of Theorem 5.1. Then, every viscosity solution u of $H(D^2u) = f(x)$ in B_1 is a $C^{2,\alpha}$ function in B_1, and satisfies*

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C\left\{\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_{3/4})}\right\} ,$$

where C depends only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

Proof. The corollary follows easily from Theorem 5.1. Indeed, for $r < 1$, there exists a solution $v_r \in C^{2,\alpha}(B_r) \cap C(\overline{B}_r)$ of

$$\begin{cases} H(D^2v_r) = f(x) & \text{in } B_r \\ \qquad v_r = u(x) & \text{on } \partial B_r \end{cases}$$

—we simply consider the Dirichlet problem in B_1 for $w_r(x) = r^{-2}v_r(rx)$ and apply Theorem 5.1. Since u solves in the viscosity sense the same Dirichlet problem as v_r and $v_r \in C^2(B_r)$, the definition of viscosity solution easily implies that $u \equiv v_r$ in B_r. Hence, u is $C^{2,\alpha}$ in B_1. Finally, estimate (5.4) for $v_{3/4}$ and a covering argument give $\|u\|_{C^\alpha(\overline{B}_{1/2})} \le C\{\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_{3/4})}\}$, as stated in the corollary. □

Next, we show that Theorem 1.2 and Corollary 1.3 follow from Theorem 5.1 and Corollary 5.2.

Proof of Theorem 1.2 and Corollary 1.3. First, we need to verify that every operator F of the form (1.9) satisfies the hypotheses of Theorem 5.1. We simply define, for $t \in [0,1]$,

$$\begin{aligned} F_t(D^2u) :&= (1-t)\Delta u + tF(D^2u) \\ &= (1-t)\Delta u + t\min\left\{F^\cap(D^2u), F^\cup(D^2u)\right\} \\ &= \min\left\{(1-t)\Delta u + tF^\cap(D^2u), (1-t)\Delta u + tF^\cup(D^2u)\right\} \end{aligned}$$

which is again an operator of the form (1.9) with ellipticity constants $\min(1,\lambda)$ and $\max(n,\Lambda)$. Note that $\|F_t - F_s\|_{\mathrm{Lip}} = \|(t-s)(F-\Delta)\|_{\mathrm{Lip}} \le (\Lambda+n)|t-s|$, and hence (5.2) holds.

Moreover, for every $c \in \mathbb{R}$ and $M \in \mathcal{S}$ with $F_t(M) = c$, equation

$$F_t(D^2u + M) = c$$

has $C^{2,\overline{\alpha}}$ estimates with constant c_e for some universal constants $\overline{\alpha} \in (0,1)$ and c_e. This follows from Theorem 1.1 applied to the operator $\tilde{F}_t(D^2u) = F_t(D^2u + M) - c$, which is again of the form (1.9) (note that $\tilde{F}_t(0) = 0$).

Therefore, Theorem 1.2 follows from Theorem 5.1, and part (i) of Corollary 1.3 from Corollary 5.2.

Finally, part (ii) of Corollary 1.3 is a consequence of the $W^{2,p}$ regularity theories of the second author [C] (for $n < p < \infty$) and of Escauriaza [Es] (for $p = n$). We can apply these results since, by the existence result of Theorem 1.2, we already know that the Dirichlet problem for equation $F = 0$ has classical solutions and interior $C^{1,1}$ estimates. □

To prove the existence of classical solutions, Theorem 5.1, we use the method of continuity. We cannot carry it out using the implicit function theorem, as it is usually done for fully nonlinear equations, since this would require to assume the operators H_t to be of class C^2 (we do not even assume H_t to be C^1). For operators $H = F$ of the form (1.9), one could think of regularizing F (this can be done for Bellman's equations). It is not clear, however, how to prove $C^{2,\alpha}$ estimates for regularized versions of F.

Instead, we use the continuity method combined with Schauder's fixed point theorem. For this, once we know that H_t has classical solutions, we write $H_{t+h}(D^2u) = f(x)$ in B_1, $u = \varphi(x)$ on ∂B_1, as $\tilde{u} \equiv u$ where

$$\begin{cases} H_t(D^2\tilde{u}) = (H_t - H_{t+h})(D^2u) + f(x) & \text{in } B_1 \\ \tilde{u} = \varphi(x) & \text{on } \partial B_1\ . \end{cases}$$

We then regularize the right hand side $(H_t - H_{t+h})(D^2u) + f$ with a mollifier to gain compactness, and we apply Schauder's fixed point theorem (see the proof of Theorem 5.1 for the precise map $u \mapsto \tilde{u}$ that we use).

We cannot apply the Banach contraction mapping theorem in Hölder spaces to the map $u \mapsto \tilde{u}$ since, as in the previous remark concerning the implicit function theorem, H_t is not necessarily C^2. The difficulty is to control

$$\|(H_t - H_{t+h})(D^2u_1) - (H_t - H_{t+h})(D^2u_2)\|_{C^\alpha}\ ,$$

that is

$$\begin{aligned} |x-y|^{-\alpha} \big\{ & (H_t - H_{t+h})(D^2u_1(x)) - (H_t - H_{t+h})(D^2u_2(x)) \\ & -(H_t - H_{t+h})(D^2u_1(y)) + (H_t - H_{t+h})(D^2u_2(y)) \big\}\ , \end{aligned}$$

by $\|u_1 - u_2\|_{C^{2,\alpha}}$ times a small constant.

We will carry out the method of continuity in the following well–known Banach spaces, which are adapted to interior $C^{2,\alpha}$ estimates (see Chapter 6 of [GT]). For $0 < \alpha < 1$, define

$$X_\alpha = \{u \in C^\alpha(\overline{B}_1) \cap C^{2,\alpha}(B_1) : \|u\|_{X_\alpha} < \infty\} ,$$

where

$$\|u\|_{X_\alpha} := \|u\|_{C^\alpha(\overline{B}_1)} + \sup_{x\in B_1}\{d_x^{1-\alpha}|Du(x)| + d_x^{2-\alpha}\|D^2u(x)\|\} + \sup_{x,y\in B_1} d_{x,y}^2 \frac{\|D^2u(x) - D^2u(y)\|}{|x-y|^\alpha}$$

and

$$d_x = \operatorname{dist}(x, \partial B_1) = 1 - |x| , \quad d_{x,y} = \min(d_x, d_y) \quad \text{for } x, y \in B_1 .$$

We will also use the space

$$Y_\alpha = \{f \in C^\alpha(B_1) : \|f\|_{Y_\alpha} < \infty\} ,$$

where

$$\|f\|_{Y_\alpha} := \sup_{x\in B_1} d_x^{2-\alpha}|f(x)| + \sup_{x,y\in B_1} d_{x,y}^2 \frac{|f(x) - f(y)|}{|x-y|^\alpha} .$$

We have that X_α and Y_α are Banach spaces. Note also that if $u \in X_\alpha$ then $H_t(D^2u) \in Y_\alpha$ for every uniformly elliptic operator with $H_t(0) = 0$.

Let H be an operator satisfying the assumptions (i) and (ii) of Theorem 5.1. Let $0 < \overline{\alpha} < 1$ be the constant in assumption (ii). Consider the Dirichlet problem

$$\begin{cases} H_t(D^2u) = f(x) & \text{in } B_1 \\ u = \varphi(x) & \text{on } \partial B_1 \end{cases} \tag{5.5$_t$}$$

and, for $0 < \alpha < \overline{\alpha}$, the set

$$\begin{aligned} A_\alpha = \{t \in [0,1] \ : \ &\forall f \in C^\alpha(\overline{B}_1)\ \forall \varphi \in C^{2,\alpha}(\overline{B}_1) \\ &\exists u \in C^2(B_1) \cap C(\overline{B}_1) \text{ solution of } (5.5_t)\} . \end{aligned} \tag{5.6}$$

Since $H_0 = \Delta$, we have that $0 \in A_\alpha$. Our goal is to show that $A_\alpha = [0,1]$. In particular, we will have $1 \in A_\alpha$.

Note that by the classical maximum principle, the solution $u \in C^2(B_1) \cap C(\overline{B}_1)$ in (5.6) is unique. Note also that in the definition of A_α we do not require any estimate for the classical solution, but only its existence. The key estimate to carry out the method of continuity is the following:

Proposition 5.3. *Assume that H satisfies assumptions (i) and (ii) of Theorem 5.1. Suppose that $\alpha \in (0,\overline{\alpha})$, $t \in A_\alpha$, $f \in Y_\alpha$, $\varphi \in C^2(\overline{B}_1)$, and that $u \in C^2(B_1) \cap C(\overline{B}_1)$ solves (5.5_t). Then $u \in X_\alpha$ and*

$$\|u\|_{X_\alpha} \leq C \left\{ \|f\|_{Y_\alpha} + \|\varphi\|_{C^2(\overline{B}_1)} \right\} , \tag{5.7}$$

where C is a constant depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

The proof of this estimate will use two ingredients. The first one is the following bound on $\|u\|_{C^\alpha(\overline{B}_1)}$ (note that $f \in Y_\alpha$ may blow–up on the boundary), which follows from a standard barrier argument. This is Lemma 6.21 of [GT]. For the convenience of the reader, we present its proof below.

Lemma 5.4. *Let $v \in S(\lambda, \Lambda, f)$ in B_1, where f is a continuous function in B_1. Assume also that $v \in C(\overline{B}_1)$ and $v \equiv 0$ on ∂B_1. Then, for every $\alpha \in (0,1)$,*

$$\sup_{x \in B_1} d_x^{-\alpha} |v(x)| \leq C_\alpha \sup_{x \in B_1} d_x^{2-\alpha} |f(x)| ,$$

where C_α is a constant depending only on n, λ, Λ and α.

The second tool that we use to prove (5.7) is the $C^{2,\alpha}$ interior estimate of [C] for equations of the form $H_t(D^2u) = f(x)$. Its statement for operators with constant coefficients is the following:

Theorem 5.5. ([C]) *Assume that H is a uniformly elliptic operator with $H(0) = 0$ such that, for every $c \in \mathbb{R}$, equation $H - c = 0$ has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates with constant c_e (in the sense of Definition 3.2).*

Let $0 < \alpha < \overline{\alpha}$, $f \in C^\alpha(\overline{B}_1)$ and u be a viscosity solution of $H(D^2u) = f(x)$ in B_1. Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C \left\{ \|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)} \right\} ,$$

where C is a constant depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

This estimate follows from Theorem 8.1 of [CC]. Indeed, there exists $t \in \mathbb{R}$ with $|t| \leq \lambda^{-1}|f(0)|$ and $H(tI) = f(0)$, where I is the identity matrix. We write $H(D^2u) = f(x)$ as

$$\tilde{H}(D^2\tilde{u}) = \tilde{f}(x) \quad \text{in } B_1 ,$$

where

$$\begin{aligned} \tilde{u}(x) &:= \frac{u(x) - (t/2)|x|^2}{\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)}} = \frac{u(x) - (t/2)|x|^2}{K} , \\ \tilde{H}(M) &:= K^{-1}\{H(KM + tI) - f(0)\} \qquad \text{and} \\ \tilde{f}(x) &:= K^{-1}\{f(x) - f(0)\} . \end{aligned}$$

Note that $\tilde{H}(0) = \tilde{f}(0) = 0$. If $\tilde{M} \in \mathcal{S}$ is such that $\tilde{H}(\tilde{M}) = 0$ then equation $\tilde{H}(D^2 w + \tilde{M}) = 0$ is equivalent to $H(D^2(Kw) + (K\tilde{M} + tI)) - f(0) = 0$, which has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates by the assumption of Theorem 5.5 applied with $M = K\tilde{M} + tI$ and $c = f(0)$. Hence, we can apply Theorem 8.1 of [CC] with $r_0 = 1$, $C_1 = 0$ and $C_2 = 1$. We obtain a universal approximation of $\tilde{u}$ by a polynomial of degree two. Going back to the function u, we obtain a polynomial P of degree two such that $\|u - P\|_{L^\infty(B_r(0))} \leq CKr^{2+\alpha}$ for all $r \leq 1$, and with $|P(0)| + |DP(0)| + \|D^2P\| \leq CK$. Recall that $K = \|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)}$.

Applying this result (rescaled) with the origin replaced by any $x_0 \in B_{1/2}$, we obtain a polynomial P_{x_0} approximating u in a $C^{2,\alpha}$ manner in $B_r(x_0)$. It follows that $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq CK$. This establishes Theorem 5.5.

We prove now Lemma 5.4 and Proposition 5.3.

Proof of Lemma 5.4. Let $K := \sup_{B_1} d_x^{2-\alpha}|f(x)| < \infty$. We consider the barrier function

$$\psi(x) := (1 - |x|^2)^\alpha \qquad \text{for } \ x \in \overline{B}_1 \ .$$

It is easy to check that, at the point $x = (r, 0, \dots, 0)$, we have

$$\begin{cases} \partial_{ij}\psi(x) & = 0 \quad \text{if } \ i \neq j \ , \\ \partial_{ii}\psi(x) & = -2\alpha(1 - r^2)^{\alpha-1} \leq 0 \quad \text{if } \ i > 1 \ , \\ \partial_{11}\psi(x) & = -2\alpha(1 - r^2)^{\alpha-2}(1 + r^2 - 2\alpha r^2) \\ & \leq -2\alpha(1 - \alpha)(1 - r^2)^{\alpha-2} \ . \end{cases}$$

By rotational symmetry, it follows that $\mathcal{M}^+(D^2\psi(x)) \leq -\lambda 2\alpha(1 - \alpha)(1 - |x|^2)^{\alpha-2} \leq -C_\alpha d_x^{\alpha-2}$ for every $x \in B_1$, where throughout the proof C_α denotes positive constants depending only on n, λ, Λ and α.

Since $v \in \underline{S}(f)$, we have $\mathcal{M}^+(D^2v) \geq f(x) \geq -|f(x)| \geq -Kd_x^{\alpha-2}$ in the viscosity sense in B_1. Using that $v \equiv \psi \equiv 0$ on ∂B_1, the definition of viscosity solution leads to $v \leq C_\alpha K\psi$ in B_1. The same argument applied to $-v$ gives

$$|v(x)| \leq C_\alpha K\psi(x) \leq C_\alpha K d_x^\alpha \qquad \text{for } x \in B_1 \ ,$$

that proves the lemma. □

In what follows, we will need the existence of classical solutions for continuous boundary values. This follows from existence for smooth boundary values, that is, from assumption $t \in A_\alpha$. More precisely, we have:

Remark 5.6. Assume the hypotheses of Theorem 5.1, $\alpha \in (0, \overline{\alpha})$ and that $t \in A_\alpha$. Then, for every $c \in \mathbb{R}$, equation $H_t - c = 0$ has classical solutions and interior $C^{2,\overline{\alpha}}$ estimates with constant c_e. Furthermore, if $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C(\partial B_1)$ then (5.5_t) admits a classical solution satisfying estimate (5.4).

To establish the first statement, we need to show that $H_t(D^2w+M)=c$ in B_1, $w=w_0$ on ∂B_1, admits a $C^2(B_1)$ solution w whenever $w_0\in C(\partial B_1)$. For this, we approximate w_0 uniformly on ∂B_1 by a sequence $\{w_{0,k}\}\subset C^{2,\alpha}(\overline{B}_1)$. Note that $H_t(D^2w+M)=c$ can be written as $H_t(D^2\tilde{w})=c\in C^\alpha(\overline{B}_1)$, where $\tilde{w}=w+P$ and P is a quadratic polynomial with $D^2P=M$. Hence, since we are assuming $t\in A_\alpha$, the approximate problem admits a solution $w_k\in C^2(B_1)\cap C(\overline{B}_1)$.

The maximum principle and Proposition 4.14 of [CC] imply that $\{w_k\}$ is equicontinuous in $C(\overline{B}_1)$. Hence, a subsequence of $\{w_k\}$ converges to a viscosity solution w of $H_t(D^2w+M)=c$, $w=w_0$ on ∂B_1. The interior $C^{2,\overline{\alpha}}$ estimate (5.1) (applied to H_t and w_k) and the compactness in $C^2(\overline{B}_{1/2})$ of bounded sets in $C^{2,\overline{\alpha}}(\overline{B}_{1/2})$ imply that $w\in C^{2,\overline{\alpha}}(\overline{B}_{1/2})$. The same argument (rescaled and done in every ball $B_r(x_0)\subset B_1$) leads to $w\in C^{2,\overline{\alpha}}(B_1)$.

The second statement of the remark is proved in the same manner. Now, the boundedness in $C^{2,\alpha}(\overline{B}_{1/2})$ of the approximate solutions is given by Theorem 5.5 applied to H_t. Note that we can apply this theorem since we already know that $H_t-c=0$ admits classical solutions (this has been proved in this same remark).

Proof of Proposition 5.3. We assume that H satisfies (i) and (ii) of Theorem 5.1, that $0<\alpha<\overline{\alpha}$, $t\in A_\alpha$, $f\in Y_\alpha$ and $\varphi\in C^2(\overline{B}_1)$. Let $u\in C^2(B_1)\cap C(\overline{B}_1)$ solve (5.5$_t$), and let $K:=\|f\|_{Y_\alpha}+\|\varphi\|_{C^2(\overline{B}_1)}$. We have that $u-\varphi\in S(\lambda/n,\Lambda,f(x)-H_t(D^2\varphi(x)))\cap C(\overline{B}_1)$ and $u-\varphi\equiv 0$ on ∂B_1. Lemma 5.4, applied to $v:=u-\varphi$, gives

$$|u(x)-\varphi(x)|\le CKd_x^\alpha \quad \text{for } x\in\overline{B}_1\ , \tag{5.8}$$

where C (here and throughout the proof) denotes a positive constant depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e. In particular, we deduce that $\|u\|_{L^\infty(\overline{B}_1)}\le CK$.

Take $x\in B_1$ and let $\overline{x}\in\partial B_1$ satisfy $d_x=|x-\overline{x}|$. We apply Theorem 5.5 to the operator H_t and the function $u-u(\overline{x})$, that solves $H_t(D^2(u-u(\overline{x})))=f(x)$ in $B_{d_x/2}(x)$. Note that, by Remark 5.6, the hypothesis "$H_t-c=0$ has classical solutions" is satisfied, since we assume that $t\in A_\alpha$.

Theorem 5.5 (rescaled) gives that, for every $y\in B_1$ with $|x-y|\le d_x/4$,

$$\begin{aligned}
&\frac{|u(x)-u(y)|}{|x-y|^\alpha}+d_x^{1-\alpha}|Du(x)|+d_x^{2-\alpha}\|D^2u(x)\|+d_x^2\frac{\|D^2u(x)-D^2u(y)\|}{|x-y|^\alpha}\\
&\quad\le C\Big\{d_x^{-\alpha}\|u-u(\overline{x})\|_{L^\infty(B_{d_x/2}(x))}+d_x^{2-\alpha}\|f\|_{L^\infty(B_{d_x/2}(x))}\\
&\qquad+d_x^2\sup_{x_1,x_2\in B_{d_x/2}(x)}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\alpha}\Big\}\\
&\quad\le CK=C\{\|f\|_{Y_\alpha}+\|\varphi\|_{C^2(\overline{B}_1)}\}\ .
\end{aligned} \tag{5.9}$$

In the last inequality we have used (5.8) to obtain, for $z\in B_{d_x/2}(x)$,

$$\begin{aligned}
|u(z)-u(\overline{x})|&=|u(z)-\varphi(\overline{x})|\le|u(z)-\varphi(z)|+|\varphi(z)-\varphi(\overline{x})|\\
&\le CKd_z^\alpha+\|\varphi\|_{C^1(\overline{B}_1)}|z-\overline{x}|\le CKd_x^\alpha\ .
\end{aligned} \tag{5.10}$$

Hence, to complete the proof we only need to show that

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} + d_{x,y}^2 \frac{\|D^2u(x)-D^2u(y)\|}{|x-y|^\alpha} \le CK \tag{5.11}$$

for every $x, y \in B_1$ with $|x-y| > d_x/4$. By symmetry, we may assume that $|x-y| \ge d_x/4 \ge d_y/4$. Let $\overline{x}, \overline{y} \in \partial B_1$ be such that $d_x = |x-\overline{x}|$ and $d_y = |y-\overline{y}|$. Note that $d_y \le d_x \le 4|x-y|$ and $|\overline{x}-\overline{y}| \le d_x + |x-y| + d_y \le 9|x-y|$. Moreover, (5.10) with $z = x$ gives $|u(x) - \varphi(\overline{x})| \le CKd_x^\alpha$. Similarly, $|u(y)-\varphi(\overline{y})| \le CKd_y^\alpha$. We conclude

$$\begin{aligned} |u(x)-u(y)| &\le |u(x)-\varphi(\overline{x})| + |\varphi(\overline{x})-\varphi(\overline{y})| + |u(y)-\varphi(\overline{y})| \\ &\le CK\{d_x^\alpha + |\overline{x}-\overline{y}| + d_y^\alpha\} \le CK|x-y|^\alpha \ , \end{aligned}$$

which controls the first term in (5.11).

Finally, by (5.9) we know that $d_z^{2-\alpha}\|D^2u(z)\| \le CK$ for every $z \in B_1$. Hence

$$d_{x,y}^2 \frac{\|D^2u(x)-D^2u(y)\|}{|x-y|^\alpha} \le C\{d_x^{2-\alpha}\|D^2u(x)\| + d_y^{2-\alpha}\|D^2u(y)\|\} \le CK \ ,$$

which concludes the proof of (5.11). □

We finally give the

Proof of Theorem 5.1. It suffices to establish that $A_\alpha = [0,1]$, where A_α is the set defined by (5.6). Then the theorem follows from Remark 5.6 and the fact that $1 \in A_\alpha$.

Clearly $0 \in A_\alpha$, since $H_0 = \Delta$ and therefore (5.5_0) admits a $C^{2,\alpha}(\overline{B}_1)$ solution for every $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C^{2,\alpha}(\overline{B}_1)$. Hence, $A_\alpha = [0,1]$ will be a consequence of the following claim:

$$\text{if } t \in A_\alpha,\ 0 \le h \le (Cc_l)^{-1} \text{ and } t+h \le 1 \text{ then } t+h \in A_\alpha \ ,$$

where c_l is the Lipschitz constant in (5.2), and C (here and throughout the proof) denotes a positive constant depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

To prove the claim, assume $t \in A_\alpha$, $0 \le h \le (Cc_l)^{-1}$, $t+h \le 1$, and let $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C^{2,\alpha}(\overline{B}_1)$. We write (5.5_{t+h}) as

$$\begin{cases} H_t(D^2u) = (H_t - H_{t+h})(D^2u) + f(x) & \text{in } B_1 \\ u = \varphi(x) & \text{on } \partial B_1 \ . \end{cases} \tag{5.12}$$

We need to show the existence of a $C^2(B_1) \cap C(\overline{B}_1)$ solution u of (5.12). To do this, we look at (5.12) as a fixed point problem. To have compactness and apply Schauder's fixed point theorem, we approximate (5.12) by the following regularized problems.

Given $0 < \varepsilon < 1/4$ and $u \in X_\alpha$, consider the Dirichlet problem for u_ε:

$$\begin{cases} H_t(D^2u_\varepsilon(x)) = \Big\{\big[(H_t - H_{t+h})(D^2u) + f\big] * \rho_\varepsilon\Big\}((1-2\varepsilon)x) & \text{in } B_1 \\ u_\varepsilon(x) = \varphi(x) & \text{on } \partial B_1 \ , \end{cases} \tag{5.13}$$

where $\{\rho_\varepsilon\}$ is a family of mollifiers with support$(\rho_\varepsilon) \subset \overline{B}_\varepsilon(0)$. Note that $f \in C^\alpha(\overline{B}_1) \subset Y_\alpha$, and let

$$K := \|f\|_{Y_\alpha} + \|\varphi\|_{C^2(\overline{B}_1)} \ .$$

Given $u \in X_\alpha$, we use the notation

$$h := (H_t - H_{t+h})(D^2 u) + f \ , \quad \text{and}$$
$$g(x) := (h * \rho_\varepsilon)((1-2\varepsilon)x) \quad \text{ for } x \in \overline{B}_1 \ .$$

Since $u \in X_\alpha$, we have that $h \in Y_\alpha$. It follows that $h_{|\overline{B}_{1-\varepsilon}} \in C^\alpha(\overline{B}_{1-\varepsilon})$, and hence the right hand side $g(x) = (h * \rho_\varepsilon)((1-2\varepsilon)x)$ of (5.13) belongs to $C^\alpha(\overline{B}_1)$. Since $t \in A_\alpha$, this implies that (5.13) has a solution $u_\varepsilon \in C^2(B_1) \cap C(\overline{B}_1)$. Moreover, we have the following estimates.

Since $(H_t - H_{t+h})(0) = 0$ and $\|H_t - H_{t+h}\|_{\mathrm{Lip}} \le c_l h$ by (5.2), we have

$$\|h\|_{Y_\alpha} \le c_l h \|u\|_{X_\alpha} + \|f\|_{Y_\alpha} \ .$$

Note also that, for $x \in \overline{B}_1$,

$$g(x) = \int_{\{|z| \le \varepsilon\}} h((1-2\varepsilon)x - z)\rho_\varepsilon(z)\,dz \tag{5.14}$$

$$= \int_{\{|\xi| \le 1-\varepsilon\}} h(\xi)\rho_\varepsilon((1-2\varepsilon)x - \xi)\,d\xi \ . \tag{5.15}$$

Since $|(1-2\varepsilon)x - z| \le (1-2\varepsilon)|x| + \varepsilon \le (1-2\varepsilon)|x| + 2\varepsilon$ for $|x| \le 1$ and $|z| \le \varepsilon$, we have $d_{(1-2\varepsilon)x-z} \ge (1-2\varepsilon)\,d_x \ge d_x/2$. This and (5.14) lead to

$$\|g\|_{Y_\alpha} \le C\|h\|_{Y_\alpha} \le Cc_l h\|u\|_{X_\alpha} + C\|f\|_{Y_\alpha} \ ,$$

where throughout the proof C denotes positive constants depending only on n, λ, Λ, α, $\overline{\alpha}$ and c_e.

Next, Proposition 5.3 gives that $u_\varepsilon \in X_\alpha$ and

$$\|u_\varepsilon\|_{X_\alpha} \le Cc_l h\|u\|_{X_\alpha} + C\{\|f\|_{Y_\alpha} + \|\varphi\|_{C^2(\overline{B}_1)}\} = Cc_l h\|u\|_{X_\alpha} + CK \ . \tag{5.16}$$

Consider the map

$$T_\varepsilon : X_\alpha \to X_\alpha \, , \qquad T_\varepsilon(u)(x) = u_\varepsilon((1-\varepsilon)x) \quad \text{for } x \in \overline{B}_1 \ ,$$

where u_ε is the solution of (5.13). Since $\|T_\varepsilon(u)\|_{X_\alpha} \le \|u_\varepsilon\|_{X_\alpha}$, (5.16) and the smallness assumption $Cc_l h \le 1/2$ lead to $\|T_\varepsilon(u)\|_{X_\alpha} \le \|u\|_{X_\alpha}/2 + CK$. In particular,

$$T_\varepsilon(\overline{B}_R) \subset \overline{B}_R \ ,$$

where $\overline{B}_R = \{\|u\|_{X_\alpha} \le R\}$ and $R := 2CK$.

Next, we prove that $T_\varepsilon : X_\alpha \to X_\alpha$ is a continuous and compact map. Let us fix an exponent $\beta \in (\alpha, \overline{\alpha})$. Using (5.15), we see that $\|g\|_{C^\beta(\overline{B}_1)} \le C_\varepsilon \|h\|_{L^\infty(B_{1-\varepsilon})}$ and hence

$\|g\|_{C^\beta(\overline{B}_1)} \le C_\varepsilon \|h\|_{L^\infty(B_{1-\varepsilon})} \le C_\varepsilon \|h\|_{Y_\alpha} \le C_\varepsilon\{\|u\|_{X_\alpha} + \|f\|_{Y_\alpha}\}$, for some constants C_ε depending on ε. This estimate together with the $C^{2,\alpha}$ estimate of Theorem 5.5 (applied to problem (5.13) with α replaced by β) and a covering argument, lead to

$$\|T_\varepsilon u\|_{C^{2,\beta}(\overline{B}_1)} \le C\|u_\varepsilon\|_{C^{2,\beta}(\overline{B}_{1-\varepsilon})} \le C_\varepsilon\{\|g\|_{C^\beta(\overline{B}_1)} + \|\varphi\|_{C^2(\overline{B}_1)}\} \le C_\varepsilon\{\|u\|_{X_\alpha} + K\} \ .$$

Since the inclusion $C^{2,\beta}(\overline{B}_1) \subset\subset C^{2,\alpha}(\overline{B}_1)$ is compact (simply note that, for every function w, $[w]_\alpha \le [w]_\beta^{\alpha/\beta}\|2w\|_{L^\infty}^{1-\alpha/\beta}$) and $C^{2,\alpha}(\overline{B}_1) \subset X_\alpha$, we conclude that T_ε sends bounded sets in X_α into precompact sets in X_α.

By this compactness property, in order to prove that T_ε is a continuous map it suffices to establish the following. If $u_k \to u$ in X_α and $T_\varepsilon(u_k) \to v$ in X_α, then necessarily $T_\varepsilon(u) = v$. To show this, let $u_{k,\varepsilon}$ be the solution of (5.13) with the right hand side g replaced by

$$g_k(x) = \left\{\left[(H_t - H_{t+h})(D^2u_k) + f\right] * \rho_\varepsilon\right\}((1-2\varepsilon)x) \ .$$

Since D^2u_k converges uniformly in $\overline{B}_{1-\varepsilon}$ to D^2u, we have that g_k converges uniformly in $\overline{B}_1$ to g. The ABP estimate applied to the difference $u_{k,\varepsilon} - u_\varepsilon$ gives that $u_{k,\varepsilon}$ converges uniformly in $\overline{B}_1$ to u_ε. In particular, $T_\varepsilon(u_k) \to T_\varepsilon(u)$ in $L^\infty(B_1)$ (by definition of the operator T_ε). Finally, since $T_\varepsilon(u_k) \to v$ in X_α and hence in $L^\infty(B_1)$, we conclude that $T_\varepsilon(u) = v$.

We can therefore apply Schauder's fixed point theorem to obtain a function $u^\varepsilon \in X_\alpha$ such that

$$\|u^\varepsilon\|_{X_\alpha} \le R \quad \text{and} \quad T_\varepsilon(u^\varepsilon) = u^\varepsilon \ .$$

Recall that $R = 2CK$ is independent of ε, and that the inclusion $X_\alpha \subset C^\alpha(\overline{B}_1)$ is continuous. Hence, there exists $u \in C(\overline{B}_1)$ and a sequence $\varepsilon_k \to 0$ such that

$$u^{\varepsilon_k} \to u \quad \text{uniformly in } \overline{B}_1 \ .$$

Denote by $u_\varepsilon^\varepsilon \in C^2(B_1) \cap C(\overline{B}_1)$ the solution of (5.13) with u replaced by u^ε in the right hand side, i.e.,

$$\begin{cases} H_t(D^2u_\varepsilon^\varepsilon(x)) = \left\{\left[(H_t - H_{t+h})(D^2u^\varepsilon) + f\right] * \rho_\varepsilon\right\}((1-2\varepsilon)x) & \text{in } B_1 \\ u_\varepsilon^\varepsilon(x) = \varphi(x) & \text{on } \partial B_1 \ . \end{cases} \tag{5.17}$$

The fixed point equation $T_\varepsilon(u^\varepsilon) = u^\varepsilon$ means that

$$u^\varepsilon(x) = u_\varepsilon^\varepsilon((1-\varepsilon)x) \quad \text{for } x \in \overline{B}_1 \ , \tag{5.18}$$

and therefore

$$D^2u_\varepsilon^\varepsilon(x) = \frac{1}{(1-\varepsilon)^2} D^2u^\varepsilon\left(\frac{x}{1-\varepsilon}\right) \quad \text{for } \ x \in \overline{B}_{1-\varepsilon} \ . \tag{5.19}$$

Estimate (5.16), applied to u^ε instead of u, gives that $\|u^\varepsilon_\varepsilon\|_{X_\alpha} \le CK$ independently of ε. Hence $\|u^\varepsilon_\varepsilon\|_{C^\alpha(\overline{B}_1)} \le CK$, and using (5.17) and (5.18) we deduce, for $x \in \partial B_1$,

$$\begin{aligned} |u(x) - \varphi(x)| &= |u(x) - u^\varepsilon_\varepsilon(x)| \\ &\le |u(x) - u^\varepsilon(x)| + |u^\varepsilon(x) - u^\varepsilon_\varepsilon(x)| \\ &= |u(x) - u^\varepsilon(x)| + |u^\varepsilon_\varepsilon((1-\varepsilon)x) - u^\varepsilon_\varepsilon(x)| \\ &\le |u(x) - u^\varepsilon(x)| + CK\varepsilon^\alpha \ , \qquad x \in \partial B_1 \ . \end{aligned}$$

Taking $\varepsilon = \varepsilon_k$ and letting $k \to \infty$, we find that $u = \varphi$ on ∂B_1.

Next, since $\{u^\varepsilon\}$ is bounded in X_α, $\{u^\varepsilon\}$ is bounded in $C^{2,\alpha}(\overline{B}_r)$ for every $r < 1$. This implies that, up to a subsequence of $\{\varepsilon_k\}$, $u \in C^2(\overline{B}_r)$ and

$$u^{\varepsilon_k} \to u \quad \text{in } C^2(\overline{B}_r) \ ,$$

for every $r < 1$.

It follows that $u \in C^2(B_1)$ and that, for every fixed $x \in B_1$, the right hand side of (5.17) converges to $(H_t - H_{t+h})(D^2u(x)) + f(x)$ as $\varepsilon_k \to 0$. Using (5.19), we also see that the left hand side of (5.17) converges to $H_t(D^2u(x))$ as $\varepsilon_k \to 0$, for every $x \in B_1$. Hence, $u \in C^2(B_1) \cap C(\overline{B}_1)$ is a classical solution of (5.12). This finishes the proof. □

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