Problems and conjectures on parameterized H-coloring*

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Recall that given two directed or undirected graphs $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ an *homomorphism* of G into G' is a map ha $\theta : V(G) \rightarrow V(G')$ with the property that $\{v, w\} \in E(G) \Rightarrow \{\theta(v), \theta(w)\} \in E(G')$. The *$H$-coloring problem* is the problem of checking whether, for fixed H, there exists an homomorphism (H-coloring) from an input graph G to H.

It is known that if H is bipartite or it has a loop the H-coloring problem can be trivially solved in polynomial time, but in the case that H is loop less and not-bipart ite the problem is known to be NP-complete [HN90]. An interesting generalization of the H-coloring problem is the *list* H-coloring problem where each vertex of G carries a list of the vertices of H where it is allowed to be mapped [FH98, FHH99]. In [DST01] we consider the following parameterized version of the H-coloring problem:
Set up a weighting $K = \{k_j | j \in C\}$ of $C \subseteq V(G)$ with non negative integers , we say that an input graph G has a (H, C, K)-coloring if there exists an H-homomorphism $\chi : V(G) \rightarrow V(H)$ such that $\forall v \in C, |\chi^{-1}(v)| = k_v$. Denote (H, C, K) a *partial weighted assignment.* If we additionally assign to each vertex of G a list permissible images when we have a more general version of the (H, C, K)-coloring problem that we call *list* (H, C, K)-coloring. We can consider the integers in K to be small fixed constants, which constitutes a parameterization of the above problems problems. A weighted assignment (H, C, K) is a *weighted extension* of a graph F if $H - C = F$.

In [DST01] we prove that if F is a graph where the F-coloring is NP-complete then for any weighted extension (H, C, K) of F the (H, C, K)-coloring is also NP-complete. On the other hand, if F-coloring is in P then *there exist* a weighted extension (H, C, K) of F so that the (H, C, K)-coloring is in P.

In the same reference we also prove that if F is a graph where the list F-coloring is in P then for any weighted extension (H, C, K) of F, the

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(H, C, K)-coloring is also in P and from the other hand, we provided some examples where the F-list coloring is NP-complete and F has a weighted extension (H, C, K) such that (H, C, K)-coloring is also NP-complete.

The above results seem to imply that there are graphs that for different weighted extensions produce parameterized coloring problems with different complexities. It seems to be a hard problem to achieve a dichotomy distinguishing those parameterized assignments (H, C, K) of a given graph F for which the (H, C, K)-coloring problem is NP-complete or in P. We conjecture the following.

Conjecture 1. *For any graph F such that the list F-coloring is NP-complete, there is a weighted extension (H, C, K) of F such that the (H, C, K)-coloring is also NP-complete.*

All of our results on NP-completeness indicate that this frontier depends not only on $H-C$ but also on the structure imposed by the weighted vertices. However, we observe that in all our complexity results – positive or negative – this dichotomy does not depend on the choice of the numbers in K when K is positive.

Another conjecture, let (H, C, K) be a weighted assignment. We call it *compact* when each connected component H_i of H satisfies one of the following:

(1) $E(H_i - C) = \emptyset$,

(2) $H_i[C]$ is a non-empty reflexive clique with all its vertices adj acent with one looped vertex of $H_i - C$, or

(3) $V(H_i) \cap C = \emptyset$ and H_i contains at least one looped vertex.

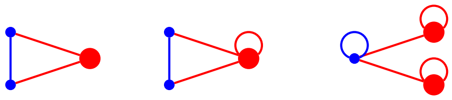
Conjecture 2. *For any partial weighted assignment (H, C, K), if (H, C, K) is compact then the (H, C, K)-coloring problem, with the numbers in K as parameters, is in FPT, otherwise it is W[1]-hard.*

In support of our dichotomy conjecture we recall that the parameterized independent set problem, that is known to be W[1]-complete [DF99] falls in this area.

We mention that our second conjecture is quite general to express several open problems in parameterized complexity such as

1. (Parameter: k) Does G contain an independent set S, where $|S| = k$, and such that $G[V(G) - S]$ is bipartite? (This problem can be seen as a parameterization of 3-coloring where some color should be used exactly k times.)

2. (Parameter: k) Is there a set $S \subseteq V(G)$, where $|S| = k$ and $G[V(G)-S]$ is bipartite?



(H_1, C_1, K_1) (H_2, C_2, K_2) (H_3, C_3, K_3)

Figura 1: Three weighted extensions conjectured to be W[1]-hard when K is parameterized

3. (Parameters: k, l) Does G contain $K_{k,l}$ as subgraph?

This problems correspond to the the parameterized colorings given in figure 1. The (H_3, C_2, K_3)-problem is equivalent with asking if the complement of G contains K_{k_1,k_2} as a subgraph.

Referencias

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