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of Graph Layout Problems**

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Some Remarks on the Approximability of Graph layout problems *

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Abstract

In this paper we look several well known layout problems. We show that MINCUT layout and Topological bandwidth cannot be approximated unless $P = NP$, whereas the Maximum linear arrangement problem can be approximated within a constant factor. We also consider some restriction of the MINSUMCUT problem, showing that the problem is in P for, weighted trees, grids without holes and outerplanar graphs. Finally we give a strong indication that graph bisection is as hard to approximate as MINLA.

1 Introduction

Graph layout problems are a collection of graph problems, motivated as simplified mathematical models of VLSI layout. Given a set of modules, the VLSI layout problem consists in placing the modules on a board in a non-overlapping manner, and then wiring together the terminals on the different modules according to a given wiring specification and in such a way that the wires do not interfere among them [SH86, DGP⁺92]. In the graph model, given a graph $G = (V, E)$ with $|V| = n$, a *layout* of G is a one-to-one mapping φ from V to the first n integers $\{1, 2, \dots, n\}$. The term layout is also known in the literature as *linear arrangement* [Yan83, Shi79], *linear ordering* [AH73], or *numbering* [Har66]. Notice that a layout φ on V determines in a unique way a nested sequence $S_0 \subset S_1 \subset \dots \subset S_n$, such that for all $i, 1 \leq i \leq n$, $S_i = \{v \in V | \varphi(v) \leq i\}$. Clearly, $|S_i| = i$. Given a natural i , the *cut* of the layout φ at i , $\theta(i)$, is the number of edges that cross over i (i.e. $\theta(i) = \{(u, v) | \varphi(u) \leq i < \varphi(v)\}$). We can also define the “node cut” $\delta(S_i) = |\{u \in V - S_i | \exists v \in S_i : (u, v) \in E\}|$.

Som well known problems in the area of graph layout are the following. Given a graph $G(V, E)$, find the layout φ such that,

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1. *The minimum linear arrangement*, MINLA, minimizes $\sum_{\{u,v\} \in E} |\varphi(u) - \varphi(v)|$.
2. *The maximum linear arrangement*, MAXLA. It is the dual of the MINLA.
3. *The minimum cut problem*, MINCUT, minimizes the maximum over all $i, 1 \leq i \leq |V|$, of the “edge cut” $\theta(S_i)$.
4. *The graph bisection*, minimizes $\theta(S_{n/2})$.
5. *The minimal sumcut problem*, MINSUMCUT, minimizes $\sum_{i=1}^n \delta(S_i)$.
6. *The Bandwidth*, minimizes the maximum of $|\varphi(u) - \varphi(v)|$ over all $\{u, v\} \in E$.
7. *Topological Bandwidth*, minimizes the bandwidth of G' over all G' that are a homeomorphic image of G .

All the above problems are NP-Complete for general graphs [GJ76] for 1,2 and 4, [Gav77] for 3, [DGPT91] for 5, [Pap76] for 6, [MPS85]. Moreover, due to the importance of these problems, there has been some work trying to obtain polynomial time algorithms for particular types of graphs. For instance, Harper solved the MINLA for the case where the graph is a de Bruijn graph of order four [Har70]. Adolph and Hu gave a $O(n \log n)$ algorithm for the same problem for the case the graph is a rooted tree [AH73]. Even and Shiloach proved the problem was also NP-complete for bipartite graphs [ES78]. Shiloach gave an $O(n^{2.2})$ algorithm to solve the MINLA on unrooted trees with n nodes. Finally, there is a NC algorithm for MINLA on unrooted trees [DGP⁺92]. The MINCUT has a very similar trajectory. Harper solved the case where the graph is an m -dimensional hypercube ($m = 2^n$) in $O(m \log m)$ time [Har66]. Chung, Makedon, Sudborough and Turner presented a $O(n(\log n)^{d-2})$ time algorithm to solve the MINCUT problem on trees, where d is the maximum degree of any node in the tree [CMST82]. Monien and Sudborough prove that the problem is NP-Complete for weighted trees and for planar graphs with maximum vertex degree of 3 [MS86]. Yannakakis gave a $O(n \log n)$ time algorithm for general trees, [Yan83]. Also, an NC algorithm for the case of unrooted trees has been given [DGP⁺92]. Other than these results, little is known about the complexity of MINLA and MINCUT. Moreover, nothing is known about approximations for these problems. Due to its practical applications, graph bisection has been widely studied, the only positive result is a polynomial algorithm for computing it on solid grid graphs [PS90]. In fact, there are not known polynomial time approximation algorithms, although there are a number of “nice” heuristics for the problem [Bop87, BCLS87]. The MINSUMCUT is a pure graph problem, without direct applications to VLSI layout, and little work has been done on it. It is known to be in NC for the case of unrooted trees [DGPT91]. The BANDWIDTH problem is a hard

one, it is NP-complete not only for trees [GGJK78], but also for cartepillars with hairs of length at most 3 [Mon83].

Monien and Sudborough proved the NP-Completeness of MINCUT for weighted trees and planar graphs [MS86]. Their reduction is from a restricted version of 0-1 Linear Equations, we call it *Super Restricted 0-1 Linear Equations* (0-1 SRLE);

Moreover the following extension can be proven,

Theorem 1 *If for some $\epsilon > 0$ we can approximate MINCUT within an n^ϵ factor, (even for planar graphs) then $P=NP$.*

Proof. (Sketch) Monien and Sudborough proved the NP-Completeness of MINCUT for weighted trees nad planar graphs [MS86]. Their reduction is from a particular case of 0-1 Linear Equations, call it *Super Restricted 0-1 Linear Equations* (0-1 SRLE);

Input: A non-negative integer matrix A of dimension $m \times 2n$, every row i of A of the form $(a_i, a_i, a_{i3}, a_{i4}, ..)$ with $a_i \geq \sum_{k \geq 3} a_{ik}$ and also with the condition that for every i , $\max\{a_{ij}\}_{j=1}^{2n} < \min\{a_{(i+1)j}\}_{j=1}^{2n}$; a non-negative integer vector $b = (b_1, \dots, b_m)$ such that for every $i, 1 \leq i \leq m$, we have that $b_i > b_{i+1}$ and also that the sum of the values of row i in A is equal to $2b_i$.

Output: Is there a vector $x \in \{0,1\}^{2n}$ with equal number of 0's than of 1's, and such that $Ax \leq b$?

It is easy to see that 0-1 SRLE is NP-complete. Moreover folowing techniques developed in [OM87], it can be seeing that if 0-1 SRLE can be approximated within any n^ϵ for $0 < \epsilon < 1$ then $P=NP$.

Given an instance of 0-1 SRLE, A, b , construct the following weighted tree $T(A, w)$, let $T(A, w)$ be a tree with B_j branches (one branch per column), $1 \leq j \leq 2n$ each branch a linear chain of length m , let c be the center of the tree to where all branches are attached. Let $w_{j,i}$ denote the i^{th} segment in the j^{th} branch B_j . For each $j, 1 \leq j \leq 2n$ construct its weights from the input A as follows, let $(a_{1,j}, a_{2,j}, \dots, a_{m,j})$ be the j^{th} column of A , then

1. $w_{j,1} = a_{1,j}$,
2. $w_{j,2} = a_{2,j}$,
3. $w_{j,3} = b_1 - b_2 + a_{2,1}$,
4. for all $i, 2 \leq i \leq m - 1$, $w_{j,2i} = a_{i+1,j}$, and
5. for all $i, 2 \leq i \leq m - 1$, $w_{j,2i+1} = b_1 - b_i + a_{i,1}$.

Notice that the previous reduction is approximation preserving. Thus the result follows. \square

For graphs with maximum vertex degree 3, MINCUT and topological bandwidth are known to be linearly related [MPS85], therefore Theorem 1 also applies to topological bandwidth.

Unfortunately, is not known a similar general result for approximation to MINLA . We have a partial result, about dense graphs. We adopt here the following definition, a graph G is k -dense where its complement \overline{G} is a partial k -tree [AP89].

Lemma 1 *For constant k , the MINLA on k -dense graph can be $1/2k$ -approximated.*

Proof. (Sketch) We first recover the partial k -tree structure of \overline{G} , this structure always starts by a clique on k vertices. Thus we start enumerating such a clique, then enumerate a vertex connected to the clique (in \overline{G}), and compare the layout enumerating this vertex with 1 or with $k + 1$, choose the minimum enumeration (and if necessary shift the whole enumeration). Proceedy in this “Greedy” manner, following the structure of the partial k -tree. That algorithm give us a $1/2k$ -approximation. \square

On the other hand the “dual” problem of MINLA, MAXLA can be approximated within a constant for general graphs,

Lemma 2 *The problem of MAXLA on general graphs can be $1/2$ -approximated.*

Proof. Given a graph $G(V, E)$ with $|V| = n$ as input to the MAXLA problem, consider the following Greedy algorithm: At step 1, choose any $v \in V$ and make $\varphi(v) = 1$, at step i , we already have a set $S_i = \{v | \varphi(v) \leq i\}$, choose any $v' \in V - S_i$ and compute the linear arrangement with both $\varphi(v') = 1$ and $\varphi(v') = i + 1$ compare and choose the one that maximizes the layout. If $Ap(S_m)$ denotes the result of applying the above algorithm up to stem m^{th} and $Opt(S_m)$ denotes the optimum value, it follows that $(Ap(S_n)/Opt(S_n)) \geq 1 - 1/2$. \square

The lack of a non-approximability result of MINLA for planar graphs is particularly bad, because it will imply the non-approximability of graph bisection. The following reduction is approximation preserving,

Theorem 2 *There exist a polynomial time approximation preserving reduction between the MINLA on graphs and graph bisection on general graphs.*

Proof. (sketch)

Given a graph $G(V, E)$ with $|V| = n$ and a constant k as input to the MINLA problem, let us construct a reduction to the input of graph bisection $G'(V', E')$ and constant k' . $V' = V_1 \cup V_2 \cup \dots \cup V_{n+1}$ and for $1 \leq j \leq n+1$, $V_j = V$. Let $v_{1j}, v_{2j}, \dots, v_{nj}$ denote the vertices of the j^{th} copy of V in V' . The edges E' and weights w are the union of the following,

Each one of V_1 and V_{n+1} form by themselves a clique of size n , and each edge in each of the cliques has weight k^2 ,

For every i, j , there is an edge between v_{ij} and $v_{i(j+1)}$ and it has weight k ,

If $i, l \in V$ with $\{i, l\} \in E$ then for each j , $1 \leq j \leq n+1$, $\{v_{ij}, v_{lj}\} \in E'$ and it has weight 1.

Finally, the constant to test the min. bisection will be $k' = (1+n)k$.

It can be proved that $G(V, E)$ has a linear arrangement of cost less or equal to k iff $G'(V', E')$ has a bisection of cost less or equal to k' . Moreover the above reduction preserves approximations. □

This last theorem says that if as it seems MINLA is not ϵ -approximable for planar graphs (unless $P=NP$) then the same will happen for graph bisection on any general graph (weighted or unweighted).

For other graph layout problem, as the MINSUMCUT, the situation looks a bit better, as the following results indicate

Lemma 3 *The MINSUMCUT can be solved in polynomial time for the following classes of graphs,*

1. *Weighted trees,*
2. *Grids without holes,*
3. *Outerplanar graphs*

Proof. (sketch)

For weighted trees, the algorithm is an extension of the one for unweighted trees [DGPT91], taking into account the node weights when computing the cost of subtree layouts.

In the case of grids, we consider first a rectangular grid it is easy to show that the optimal enumeration is starting by a corner and go on enumerating neighbors scanning in diagonal the grid. When the grid has no holes it can be subdivided into smaller rectangular grids that are connected in a tree like manner. We proceed by computing the weight of each piece and compare, following the tree structure, to get the order in which the different rectangles must be enumerated.

For outerplanar graphs we follow a greedy approach. Start enumerating a vertex, then consider the two next vertices following the exterior face (one in each direction) and enumerate the one that gives smallest cut. Proceed in this greedy way until the whole graph has been enumerated. Computing the $|V|$ layouts (one for each vertex) and taking the one with minimum sum cost we get the optimal layout for the whole outerplanar graph. \square

There are several open problems arising from the above presentation. The most important one to prove the non-approximability for MINLA. Although we firmly believe that this is the case, so far we have been unable to prove it, even to give a proof of the NP-Completeness of the MINLA for weighted trees or planar graphs. Other related problem is the approximability of the MINSUMCUT for general graphs. Again the intuition seems to indicate that the MINSUMCUT is similar to the MAXLA, and it is approximable.

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