On the Epipolar Geometry and Stereo Vision

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Abstract

We begin this paper with an introductory part of the projective, affine and euclidean geometry and the different models of projection and cameras. After this, we focus on the theory of the epipolar geometry that relates correspondences in different images, namely in two, three and four images (where we also include some new results).

1 Introduction and motivation

The geometric relation between objects (or scenes) in the world and their images, taken from different viewing positions by a pinhole camera has lately been the subject of research in computer vision. The classical approach relies on a metric interpretation of the 3D world from its projections. A metric recovery structure provides the most detailed information about the 3D structure of objects in the scene. However, this reconstruction is a problematic process characterized by the sensitivity of data and the assumption of models that do not correspond to the real case, letting aside the need of previous knowledge (or assumptions) for the internal (that leads us to the calibration process, an off-line process which is impractical and unstable in many occasions) and external parameters of the camera. Some researchers have proposed to evolve into another framework, the projective space, where these relationships become far easier to understand and to handle with. Projective geometry allows us to understand and express the geometry of the problem in a much simpler way. It is clear that the descriptions obtained this way will be much more unconstrained.

An interesting area of application in our case is computer graphics. Reprojection techniques provide a short-cut for image rendering. Given two fully rendered views of some 3D object, other views (ignoring self-occlusions) can be rendered by simply combining the reference views.

In the first section we give the basics concepts of projective, affine and euclidean geometries and we show how they form a hierarchy. In the following section, we explain the different models of projections and cameras and how they are related to the geometries. The last four sections contain the bulk of the paper which aim to explain the relationship of the correspondences of two, three and four images, (where we also include some new results) after a short introduction of the epipolar geometry.

2 Concepts of Projective Geometry

2.1 Introduction

The study of projective geometry was initiated by the painters of the Italian Renaissance, who wanted to produce a convincing illusion of 3D depth in their architectural paintings. If we look at a picture of a landscape, we realize that the lines tend to converge towards a point or line as they recede towards the horizon. These points and lines are called vanishing directions, and we can consider them as points or lines at infinity. Projective geometry can handle with these concepts in a natural way.

Projective geometry ([SK52]) will be the reference framework although the natural geometry we use is Euclidean. We will see now that it is simple to consider the Affine and Euclidean geometries as special cases of projective geometry since they form a hierarchy (see [LV96]), and how they are related.

The three-dimensional Euclidean R^3 space containing the object will be considered to be embedded in a natural way in the three-dimensional (affine and) projective space P^3 by the addition of a plane at infinity and the two-dimensional Euclidean space R^2 embedded in the two-dimensional (affine and) projective space P^2 by the addition of a line at infinity.

We will distinguish between the image itself (in 3D, a plane in space) and the view of an object which is the result of the projection onto an image. We will say that a view is the projection of an object (or a set of points) onto an image.

2.2 Projective level

The projective space of dimension n, P^n is the quotient of $R^{n+1} - (0,..,0)$ by the equivalence relation:

$$[x_1,...,x_n] = [x'_1,...,x'_n] \Leftrightarrow \exists\, \lambda \neq 0, [x_1,...,x_n] = \lambda[x'_1,...,x'_n]$$

(thus, there is no point corresponding to [0,...,0])

They are called homogeneous coordinates. Intuitively, every point in a image represents a line of sight, and any point in that line corresponds to the same point in the image. Therefore, only the ray is of importance. Let suppose the camera is at the origin (0,0,0). Then, a 3D point (x,y,t) and another point λ(x,y,t) also lies and represents the same ray, so we have the rule that rescaling homogeneous coordinates makes no difference: $(x, y, t) \cong \lambda(x, y, t)$.

In a projective framework, P^n, the location of points is measured relative to a frame of reference of n+2 points (a set of n+1 points and a unit point) such that no n+1 of them are linearly dependent (no n+1 of them lying in the same hyperplane, see [Moh92]). The set e_i = (0,..,1,..0) i=1..n+1 where 1 is in the ith position and e_{n+2} = (1,..,1,..,1) (called the unit point) is the standard projective basis of P^n. Any point x = (x_1,...,x_{n+1}) can be described as a linear combination of any n+1 points of the standard basis :

$$x = \sum_{i=1}^{n+1} x_i e_i$$

For the regular 3D projective spaces, these four first points, ie (1,0,0,0), (0,1,0,0),(0,0,1,0),(0,0,0,1), are respectively the point at infinity on the x-axis, on the y-axis, on the z-axis and the origin.

2.2.1 Homographies

A one-one correspondence between two ranges of points, two pencils of lines, or a range of points and a pencil of lines, is said to be homographic when the cross ratio of any four elements of either system is equal to the cross ratio of the four corresponding elements, taken in the corresponding order, of the other system ([SK52]). An homography is any transformation of P^n which is linear in projective coordinates (hence the terminology linear projective) and invertible. It can be represented by a (n+1)×(n+1) nonsingular matrix H defined up to scale such that the image of x is x':

$$\begin{pmatrix} x'_1 \\ \dots \\ x'_{n+1} \end{pmatrix} = H \begin{pmatrix} x_1 \\ \dots \\ x_{n+1} \end{pmatrix}$$

Homographies map any projective subspace to a subspace of the same dimension, a property which is called the conservation of incidence; homographies form a group called the projective group. If B is the scalar vector defining a projective hyperplane (the set of points X such that $B^tX = 0$) the image of this hyperplane is defined by B H^{-1}.

An homography relates one basis to another one and it is completely defined by its action on the points of a basis. There is a unique homography which transforms n given points, no n-1 of which are coplanar, into n given points, no n-1 of which are coplanar (and dually, there is a unique homography which transforms n given hyperplanes, no n-1 of which are concurrent, into n-1 given hyperplanes, no n-1 of which are concurrent).

It has to be mentioned that an homography has (n+1)×(n+1) -1 degrees of freedom. Knowing the image of each point of the basis provides us with n+1 equations up to a scaling factor, ie only n independent equations. So for n+2 points in the basis, this provides us with $n^2 + 2n$ equations, exactly the number of unknowns for the homography matrix ([Moh92]). For a proof of the uniqueness of the solution, see [SK52].

2.2.2 Cross-ratio

The cross-ratio is the ratio of ratios of distances:

1. Cross ratio of points

   Let M and N be two distinct points of a projective space. The projective line between M and N consist of all points A of the form A = α M + μ N. Then, (α,μ) are the homogeneous coordinates of the projective line P^1, expressed with respect to the linear basis M, N. Let A_1, A_2, A_3 and A_4 be four collinear points; their cross ratio is defined as:

$$[A_1, A_2, A_3, A_4] = \frac{(\alpha_1\mu_3 - \alpha_3\mu_1)(\alpha_2\mu_4 - \alpha_4\mu_2)}{(\alpha_1\mu_4 - \alpha_4\mu_1)(\alpha_2\mu_3 - \alpha_3\mu_2)} = \frac{(\frac{\alpha_1}{mu_1} - \frac{\alpha_3}{\mu_3})(\frac{\alpha_2}{\mu_2} - \frac{\alpha_4}{\mu_4})}{(\frac{\alpha_1}{\mu_1} - \frac{\alpha_4}{\mu_4})(\frac{\alpha_2}{\mu_2} - \frac{\alpha_3}{\mu_3})}$$

   and it is invariant under any linear transformation and change of basis. The cross ratio is the basic invariant in projective geometry ([Moh92]) and it does not depend on the unity vector taken on the line.

This concept has an immediate application for locating a point on a line. The position of a fourth point is defined by the cross ratio knowing three points. If the coordinates of these points are known, being A(a_1,a_2), B(b_1,b_2), C(c_1,c_2), the cross ratio of a fourth point D(x,y) is as follows.

$$[A,B,C,D] = \frac{\frac{a_1}{a_2}-\frac{c_1}{c_2}}{\frac{a_1}{a_2}-\frac{x}{y}}\frac{\frac{b_1}{b_2}-\frac{x}{y}}{\frac{b_1}{b_2}-\frac{c_1}{c_2}} = \frac{(a_1c_2-a_2c_1)ya_2}{(a_1y-a_2x)c_2a_2}\frac{(b_1y-b_2x)b_2c_2}{(b_1c_2-b_2c_1)b_2y} = \frac{(a_1c_2-a_2c_1)}{(b_1c_2-b_2c_1)}\frac{(b_1y-b_2x)}{(a_1y-a_2x)} = \frac{(a_1c_2-a_2c_1)}{(b_1c_2-b_2c_1)}\frac{(b_1y-b_2x)}{(a_1y-a_2x)}$$

2. Cross ratio of lines

   Let us now consider a pencil of four lines (see Figure 1). Let A_1, A_2, A_3, A_4 and A'_1, A'_2, A'_3, A'_4 be the intersections with two lines. Then $[A_1, A_2, A_3, A_4] = [A'_1, A'_2, A'_3, A'_4]$. Therefore the cross ratio of a pencil of lines can be defined as $[L_1, L_2, L_3, L_4] = [A_1, A_2, A_3, A_4]$.

   Let O be the origin of a pencil of lines L_1, L_2, L_3, L_4. Let A_i be points on L_i, $A_i \neq O$. Then (a theorem by Möbius)

$$[L_1,L_2,L_3,L_4] = \frac{|OA_1A_3|}{|OA_1A_4|}\frac{|OA_2A_3|}{|OA_2A_4|}$$

   where $|OA_iA_j|$ stands for the determinant of the 3×3 matrix where each column is the column of homogeneous coordinates of the points O, A_i and A_j.

3. Cross ratio of planes

   A pencil of planes in P^3 is a family of planes having a common line of intersection. The cross ratio of four planes Π_i of a pencil of planes is the same as the cross ratio of the lines l_i of intersection of the planes with a fifth, transversal plane. Different transversal planes give the same cross ratio. Let P,Q be any two distinct points on the axis of the plane pencil and A_i i=1..4 be points lying on each plane π_i (not on the axis) then (see Figure 2):

$$[\Pi_1,\Pi_2,\Pi_3,\Pi_4] = \frac{|PQA_1A_3|}{|PQA_1A_4|}\frac{|PQA_2A_4|}{|PQA_2A_3|}$$

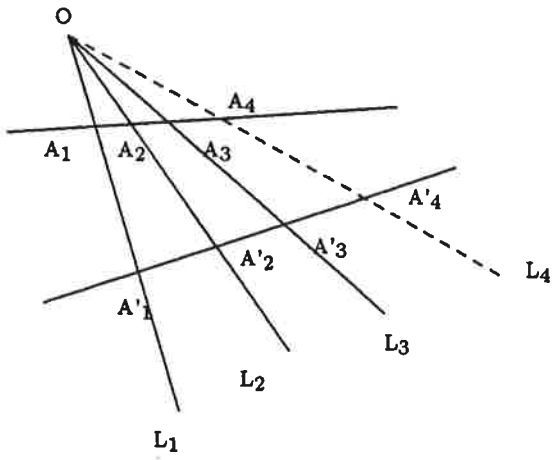

Figure 1: Cross ratio of a pencil of lines

where $|PQA_iA_j|$ stands for the determinant of the 4×4 matrix where each column is the column of homogeneous coordinates of the points P, Q, A_i and A_j.

2.2.3 Line equation in P^2

The planar line with equation $ax+by+c=0$ is represented in homogeneous coordinates by the homogeneous equation $(a,b,c).(x,y,T) = ax+by+cT = 0$.

Said in a more formal way, suppose P_1 and P_2 are fixed points, with coordinate vectors $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$ and $x^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$. If the points are distinct, they are clearly linearly independent. The set of all points linearly dependent on two given (distinct) points P_1, P_2 is called the *line* determined by P_1 and P_2, or simply the line P_1P_2. It follows that a point $P_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$ is linearly dependent on P_1 and P_2 if and only if its coordinates satisfy the following equation:

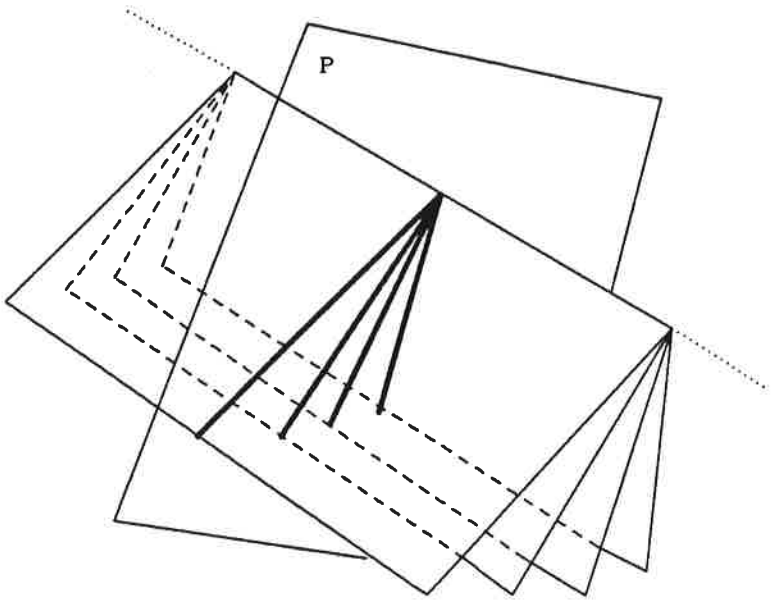

Figure 2: Models of projection

$$\begin{vmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\ x_1^{(k)} & x_2^{(k)} & x_3^{(k)} \end{vmatrix} = 0$$

As seen, the equation of a line in P^2 is the cross product of two points defining the line.

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

where

$$u_1 = x_2^{(1)} x_3^{(2)} - x_3^{(1)} x_2^{(2)} \qquad u_2 = x_3^{(1)} x_1^{(2)} - x_1^{(1)} x_3^{(2)} \qquad u_3 = x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)}$$

The line P_1P_2 admits of two kinds of algebraic representation. The first one is

$$x = \lambda x^{(1)} + \mu x^{(2)}$$

and the ratio λ:μ is uniquely determined. In this way we obtain a representation of the points of the line by the homogeneous pair of parameters (λ, μ).

The other way is by its equation $\sum_{i=1}^{3} u_i x_i = 0$. If we take the coefficients u_1, u_2, u_3 in this equation as components of a column-vector u, we may write the equation in matrix form $u^t x = 0$. The left-hand side is just the inner product (u,x) of the vectors u and x, and so the equation may also be written (u,x) = 0.

2.2.4 Plane equation in P^3

Let P_1,P_2,P_3 be three P^3 points, represented by coordinate vectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ respectively. These vectors may be taken as columns of a 3×4 matrix and the rank ρ of this matrix has a simple interpretation, for the three points are linearly independent if and only if $\rho = 3$.

If this condition is satisfied, all the points that are linearly dependent on P_1, P_2 and P_3 satisfy the linear equation $\sum_{i=0}^{3} u_i x_i = 0$ (in a similar way as seen before). That set of points defined the plane determined by P_1, P_2 and P_3. It follows at once that a plane is determined by any three of its points, said to be coplanar if they belong to a common plane; and this is the case if and only if the determinant of their coordinates (of four of them) is zero ([SK52]).

A plane may be represented in two ways :

1) If defined by P_1,P_2,P_3 (with vectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$), a general point P has a coordinate vector $x = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \lambda_3 x^{(3)}$ so we have a parametric representation of the plane by a triad of homogeneous parameters $(\lambda_1, \lambda_2, \lambda_3)$.

2) by the equation $\sum_{i=1}^{4} u_i x_i = 0$. This equation is determined by its coefficients, which may be taken as a tetrad of homogeneous plane-coordinates of the plane. If the column-vector with components (u_0, u_1, u_2, u_3) is denoted by u, the equation of the plane may be expressed in terms of the inner product of the vectors u and x: $(u, x) = u^T x = 0$.

2.2.5 Line equation in P^3

Now consider a pair of given points P_1,P_2 in P^3, with coordinate vectors $x^{(1)}$, $x^{(2)}$. The points are linearly independent if and only if the 2×4 matrix is of rank 2; and in this case their coordinates satisfy two (and no more than two) linearly independent linear equations (each one defines a projective plane u and v):

$$\sum_{i=1}^{4} u_i x_i^{(1)} \qquad \sum_{i=1}^{4} v_i x_i^{(2)} = 0$$

The points which are linearly dependent on P_1 and P_2 are the points whose coordinates satisfy these two equations simultaneously, and these points are said to make up the line determined by P_1 and P_2. Every point of the line is said to be collinear with P_1 and P_2; and, of course, the roles of P_1 and P_2 may be taken over by any two linearly independent points of the line.

A line if fixed by any two of its points P_1,P_2, and the equation $x = \lambda_1 x^{(2)} + \lambda_2 x^{(2)}$ which gives the coordinate vector of a general point of the line, leads at once to the homogeneous parametric representation of the line by the pair of parameters (λ_1, λ_2).

The homogeneous coordinates of a line form a redundant set and are connected by an identical relation. A line is defined as the set of points that are linearly dependent on two points P_1 and P_2 whose coordinate vectors in the standard projective basis are $x^{(1)}$ and $x^{(2)}$. We consider the sixteen numbers $l_{ij} = x_i^{(1)} x_j^{(2)} - x_j^{(1)} x_i^{(2)}$ i,j = 1..4. Since $l_{ij} = - l_{ji}$, there are only six of these numbers that are apparently independent. Finally, we obtain the identity $l_{41}l_{23} + l_{42}l_{31} + l_{43}l_{12} = 0$. The six numbers are referred as the Grassmann or Plücker coordinates of the line.

Two lines l and l' intersect if their Plücker coordinates satisfy the equation

$$(l_{41}l'_{23} + l'_{41}l_{23}) + (l_{42}l'_{31} + l'_{42}l_{31}) + (l_{43}l'_{31} + l'_{43}l_{31}) = 0$$

2.3 Affine level

Any point P^n $P_p = (y_1,..,y_{n+1})$ defines an hyperplane which is the set of points of P^n $P = (x_1,..,x_{n+1})$ whose coordinates satisfy

$$\sum_{i=1}^{n+1} x_i y_i = y^t x = 0$$

An hyperplane can be considered as a projective subspace of dimension P^{n-1}. Hyperplanes in P^3 are planes, hyperplanes in P^2 are lines. Affine structure of P^3 is characterized by the plane at infinity Π_∞, which is represented by the vector [0, 0, 0, 1]. The projective space can be described as the union of the affine space (points [X,λ]) and the plane at infinity Π_∞ (points [X, 0]). A primary advantage of homogeneous coordinate systems for projective spaces lies in the fact that all points may be treated alike.

Back to the example, if we suppose that the image plane of the camera is T = 1, the ray through pixel (x,y) can be represented homogeneously by the vector (x,y,1) $\cong$ (xT,yT,T) for any T $\neq$ 0 (as the homogeneous coordinates). For T = 0, these rays exist anyway but they do not correspond to any finite pixel: it is parallel to any plane T $\neq$ 0. As they can no longer be considered as finite point, they are regarded as "ideal points" or "points at infinity".

The affine transformations are the subgroup of the homographies that conserve the plane at infinity. Any two subspaces which are not contained in Π_∞ are parallel if their intersection is in Π_∞. This implies that affine transformations preserve parallelism. If H is the affine transformation in question, the last row of the matrix is of the form $[0, 0, ..., \mu]$ with $\mu \neq 0$. As defined up to scale, we can fully describe it by the matrix

$$\begin{bmatrix} M & V \\ 0_3^t & 1 \end{bmatrix} \tag{1}$$

where M is a 3×3 matrix and V is a 3×1 vector, which yields the classical description of a transformation of the affine space $x' = Mx + V$

The ratio of distances of three collinear points is invariant by an affine transformation, the center of mass, the convex hull and the ratio of volume

defined by four points in 3D space. If we add to (1) the constraint that det(M)=±1, then the volume themselves are left invariant (this is the affine unimodular group).

2.3.1 Canonical Injection of R^n to P^n

Affine space R^n can be embedded isomorphically in P^n by the standard injection $(x_1, x_2, ..., x_n) \rightarrow (x_1, x_2, ..., x_n, 1)$. Affine points can be recovered from projective ones with $x_{n+1} \neq 0$ by the mapping

$$(x_1, x_2, ..., x_{n+1}) \cong (\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}}, 1) \rightarrow (\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}})$$

2.4 Euclidean level

Any symmetric (n+1)×(n+1)matrix Q defines a hyperquadric which is the set of points P = (x_1,..,x_{n+1}) in P^n whose coordinates satisfy:

$$\sum_{1 \leq i,j \leq n+1} Q_{ij} x_i x_j = x^t Q x = 0$$

In P^3 the hyperquadratics are quadric surfaces, in P^2 they are conics and in P^1 they reduce to two points.

The Euclidean structure is characterized by the absolute conic Ω which lies in the plane at infinity Π_∞ and has a matrix identity, and therefore the equation $x_1^2 + x_2^2 + x_3^2 = 0$.

Ω can be use to define angles between planes and this implies that Euclidean transformations preserve angles. Then the Euclidean subgroup is defined by the transformations which conserve the absolute conic and this implies that they also preserve the plane at infinity and are a subgroup of the affine transformations; they are 1 with the constraint $MM^t \cong I_n$ (the matrix M is proportional to an orthogonal matrix). These transformations are called similarities (the product of a scale factor by a rigid displacement - rotation and translation).

Euclidean transformation preserve the relative distance which is the ratio of any three points. With the additional constraint that det(M) = ± 1, the absolute distances are preserved, forming the group of transformations called isometries (intersection of the affine unimodular group with the similarity group).

2.5 Hierarchy of Geometries

There is a clear hierarchy of the geometries (see [Fau95], [MBB95]):

Projective $\supset$ Affine $\supset$ Similarities $\supset$ Isometries = Euclidean

As we have seen, as we go down, the transformation groups become smaller and less general and the corresponding spatial structures become more rigid and have more invariants.

transformations	*definition*	*invariants and concepts*	*matrix*
Projective P^3			
**homography*		*cross − ratio* *incidence* *coplanarity* *duality*	4×4 *nonsingular* $\begin{bmatrix} M & V \\ w & \mu \end{bmatrix}$

Affine			
**affine transform.*	*plane at infinity* Π_∞	Π_∞ *ratio of 3 points* *center of mass* *convex hull* *ratio of volumes* *parallelism* *pointsat*∞ *sideness*	$M3 \times 3$ *nonsingular matrix* V *vector*
		betweenness	$\begin{bmatrix} M & V \\ 0_3^t & 1 \end{bmatrix}$
**unimodular group*		*absolute volume*	$det(M) \pm 1$

Euclidean			
**similarities*	*absolute conic* Ω	*angles* *ratio of 3 points* *rigid displacement* *changes of scale*	$MM^t \cong I_n$
**isometries*		*absolute distances*	$det(M) = \pm 1$

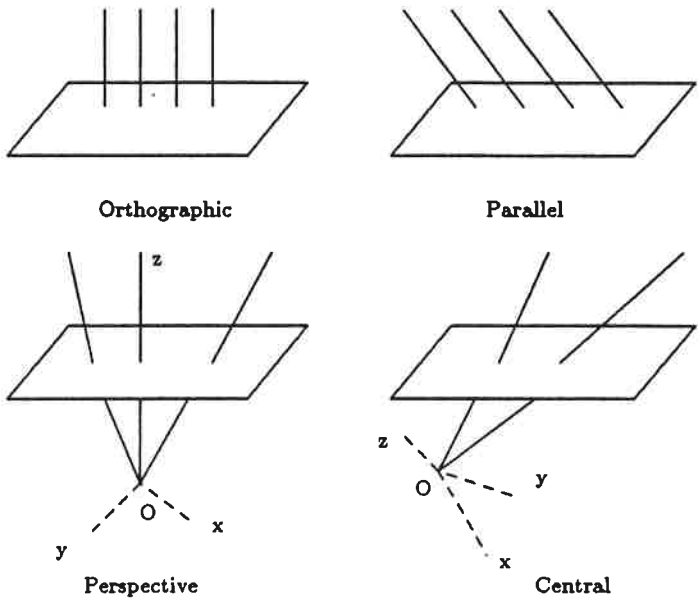

Figure 3: Models of projection

3 Models of projection and cameras

3.1 Models of projection

To find the projection of a point, a line is drawn from that point to another one, called the center of projection which can lie at any location in space. The intersection between this line and a plane in space, known as the image plane, to which the center of projection does not belong is called the projection of the point in question. The (set of) projection(s) in the image plane is called the view of that (set of) point(s) (sometimes we will use view and image plane interchangeably).

There are four models of projection: parallel, orthographic, central and perspective (see Figure 3).

There exist always a center of projection, although it may lie at infinity. In this case, the projections are called parallel: any point is projected

following one and only one determined direction. When this direction is perpendicular to the plane, the projection is called orthographic.

In the central and perspective, the center of projection does not lie at infinity. In the case of perspective projection, there exists a line passing through the center of projection, known as the principal ray which is perpendicular to the plane of the view. The point of intersection between the principal ray and the view is called the principal point. Finally, the distance between the the image plane and the center of projection along the optical ray is the focal length. In a sense, the orthographic projection is a perspective projection whose center lies at infinity.

In perspective projection (see Figure 4), the system is defined as follows: the axis z of the coordinate system is aligned with the principal ray and the axis x and y define a plane parallel to the view. The focal length is fixed and therefore, the position of the view in the coordinate system defined is perfectly known. Therefore, the change of coordinates is the result of a rigid movement: a translation of the center and a rotation of the axes, while, in the case of central projection, this rigid movement is followed by any linear projective transformation in space (see [Har80]).

3.2 Projection from 3D to 2D

3.2.1 Projective level

Given a point P in P^3, the projection matrix of image I is a 3×4 matrix $\tilde{M}$ and the projection of that point is another point p on image I:

$$\lambda \begin{pmatrix} x \\ y \\ s \end{pmatrix} = \begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{vmatrix} \begin{pmatrix} x \\ y \\ s \\ t \end{pmatrix}$$

$$\lambda p = \tilde{M} P$$

The projection matrix P must be of rank 3.

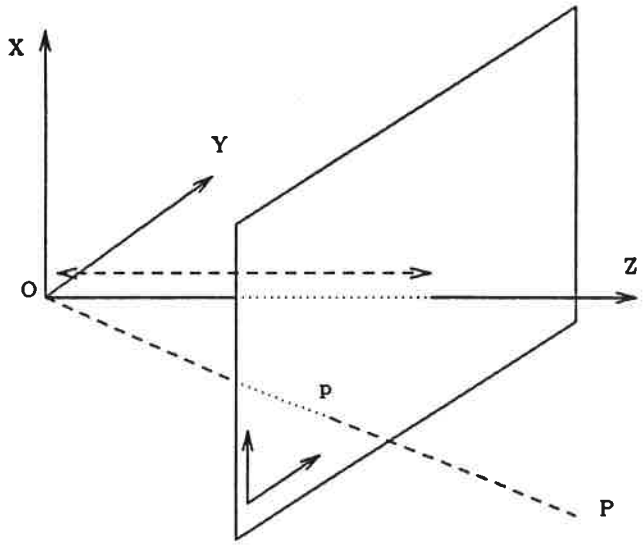

Figure 4: Perspective projection

3.2.2 Affine level

Let us decompose the projection matrix $\tilde{M}$ as the concatenation of a 3×3 submatrix M and a 3×1 vector m. The optical center is the only point whose projection is the vector (0,0,0), therefore

$$\tilde{M}\tilde{C} = [Mm][Cc] = 0$$

Thus the equation of the optical center is MC = -cm, the optical center being at infinity if det(M) = 0. If det(M)≠0 then the optical center is $\tilde{C} = [-M^{-1}m \;\; 1]$ and m = -MC. It is worth mentioning that a projection matrix from a real system will always holds this condition (the optical center must lie in the affine space).

There exists points in P^3 that are mapped to points at infinity; by multiplying M_3 (the last row of M) by any point x mapped to infinity, we get that M_3 x = 0. Obviously, this also happens to the perspective center. In fact, we can regard M_3 as a plane of all the points that are mapped to

infinity known as the principal plane (or the focal plane).

Thus, the projection can be described as a composition of an affine transformation and the projection expressed in the image system (here, the identity):

$$p = \tilde{M} P$$

$$\tilde{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M & m \\ 0_3 & 1 \end{bmatrix}$$

The projection of each point at infinity [d, 0] is the vanishing point v = Md and therefore the optical ray corresponding to a pixel p is the direction $M^{-1}p$; then, the vector m is just the projection of the origin of the world coordinate system (called the principal point).

3.2.3 Euclidean level

The nonsingular matrix M can be decomposed as two matrices which can be regarded as the matrix describing the change of the image system (5 intrinsic parameters) and the displacement in space (6 extrinsic parameters). As the matrix of the intrinsic parameters is upper triangular, it defines an affine transformation of the plane, rather that a general projective transformation.

$$p = \tilde{M} P$$

$$\tilde{M} = \lambda \begin{bmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0_3 & 1 \end{bmatrix} \tag{2}$$

$$A = \begin{bmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} R & T \\ 0_3 & 1 \end{bmatrix}$$

The matrix A does not depend on the extrinsic parameters (matrix C) and depends only on the intrinsic parameters (because the projection of the absolute conic is invariant under rigid displacements). The classic motion equation for a calibrated camera is $z'p' = zRp+T$ where R is an orthogonal matrix accounting for the rotation component of camera displacement and T is the translation component.

3.3 The camera model

The camera model used is the classical *pinhole model* (following [ZF94], [Fau93],[Har96]) and it is assumed that objects in the world are rigid.

A) Non-rigid camera configuration: the center of projection is the origin of the camera coordinate frame (central projection) and can be located anywhere in projective space. Therefore, it can be a point in Euclidean space or an ideal point (parallel projection). The image is assumed to be arbitrarily positioned with respect to the camera coordinate frame.

The motion of the camera consists of the translation of the center of projection, rotation of the coordinate frame around the new location of the center of projection and followed by tilt, pan, local length scale of the image plane with respect to the new optical axis. This model of projection is also referred to as projection with an uncalibrated camera.

B) Rigid-camera configuration: the center of projection is a point in Euclidean space and the image plane is fixed with respect to the camera coordinate frame (perspective projection).

The motion of the camera consists of translation of the center of projection followed by rotation of the coordinate frame and focal length scaling. Note that a rigid camera implicitly assumes internal calibration: the optical axis pierces a fixed point in the image and the image plane is perpendicular to the optical axis and parallel to the xy plane as in the perspective projection. This model of projection is also referred to as perspective projection with an calibrated camera.

3.3.1 Camera calibration

As the image formation process can be expressed as a projective mapping from P^3 to P^2 and assuming that the camera performs a perfect perspective projection, the projective camera calibration is the computation of the projection matrix associated with this mapping. It is done using a set of points whose location in space are known. With a number of points (at least 6) the matrix can be obtained. This matrix will contain not only the projection matrix but also the intrinsic and extrinsic parameters of the camera.

3.4 Comparison between models

The advantage of the non-rigid camera model (or the central projection model) is that images can be obtained from uncalibrated images. The price paid for this property is that the images that produce the same projective structure invariant (equivalence class of images of the object) can be produced by applying non-rigid transformation of the object, in addition to rigid transformation. With a set of arbitrary cameras with unknown possibly different calibrations it is not possible to specify the scene more precisely than up to an arbitrary projective transformation of space. This contrast with the situation for calibrated cameras in which a set of sufficiently many lines may be determined up to a scaled Euclidean transformation from three views ([Har94], [SA90], [WHA92]). In the case all of the three cameras are the same, however, or at least have the same calibration, it is possible to reconstruct the scene up to a scaled Euclidean transformation ([Har93]).

4 Concepts of Epipolar Geometry

4.1 Introduction

The epipolar geometry describes the relationship (correspondence) of a number n of projections of a point in the space on a number n of views. It contains all the information regarding the viewing transformation between n camera locations.

The subject was investigated by mathematical photogrammetrists in the 19th century but it has received much attention with the publication of [Tho68] and [LH81]. More recently, work on projective reconstruction by [Fau92], [HGC92] and [Sha95] launched a lively burst of research in this field.

Consider the case of two perspective images of a rigid scene (depicted in Figure 5) where a P^3 point P is projected onto two images I_i, I_j. It it obvious that the point P, the center of projection of two views (O_i, O_j) and the projection of P on those planes p_i, p_j must lie in a single plane, called the P's epipolar plane. This fact is known as the epipolar equation or coplanarity constraint.

The line passing through a center of projection of a view (O_i or O_j) and a point in space P is the P's epipolar line in 3D for that projection. The intersection of this line with an image plane defines the projection of the point in question (p_i or p_j). The epipolar line connecting the two centers of projection is of special interest: the projection of another image's center of projection is called the epipolar point or epipole. By construction, the intersection of the epipolar plane of a 3D point and an image gives a line, called the epipolar line in 2D, that always passes through the epipolar point and the projection of the point. All the epipolar lines in an image converge towards the corresponding epipole. An epipolar transformation defines the projective relationship between two epipolar lines in two distinct views.

Projectively, the epipolar geometry established the correspondence p_i, p_j and allows 3D reconstruction of the scene to be carried out up to a 3D projective transformation. An important practical application is to aid the search for corresponding points, reducing it from the entire second image to a single epipolar line. The epipolar geometry is sometimes obtained by

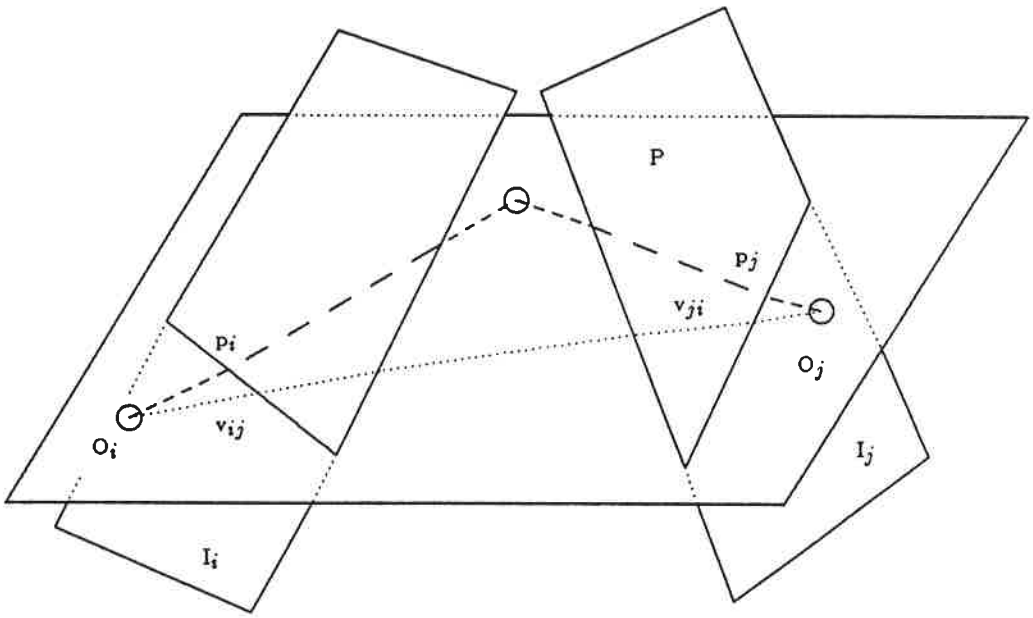

Figure 5: Epipolar coplanarity constraint

calibrating each of the cameras with respect to the same 3D frame although this is not necessary if a sufficient number of correspondences is provided.

Although invariants of a wide range of objects in the 3-dimensional projective space P^3 do exist, one is restricted to considering those that may be computed from two-dimensional projections (images) (see [MZ92]). For point sets and more structured geometrical objects lying in planes in P^3, many invariant exists, but it has been proven that no invariants of arbitrary sets of points in three dimensions can be computed from a single image.

4.2 Basic Equations

Given a point P in P^3, the projection matrix of image i is a 3×4 matrix $\tilde{M}_i$ and the projection of that point is another one p_i on image i:

$$\lambda_i p_i = \tilde{M}_i P$$

Now, given m images, we can represent the whole system as follows:

$$\begin{pmatrix} \tilde{M}_1 & p_1 & 0 & \dots & 0 \\ \tilde{M}_2 & 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \tilde{M}_m & 0 & 0 & \dots & p_m \end{pmatrix} \begin{pmatrix} P \\ -\lambda_1 \\ -\lambda_2 \\ \vdots \\ -\lambda_m \end{pmatrix} = 0 \qquad (3)$$

Say we have four images, let $\tilde{M}_1 = [e_{ij}], \tilde{M}_2 = [a_{ij}], \tilde{M}_3 = [b_{ij}], \tilde{M}_4 = [c_{ij}]$, i=1,2,3 and j=1,2,3,4 (the whole expression in Appendix A).

Let vectors 1, 2, 3 be the three rows of matrix $\tilde{M}_1$, vectors 4, 5, 6 be the three rows of matrix $\tilde{M}_2$, vectors 7, 8, 9 be the three rows of matrix $\tilde{M}_3$, vectors 10, 11, 12 be the three rows of matrix $\tilde{M}_4$; when we say [i, j, k, l] $1 \geq$i, j, k, l $\leq$12 we refer to the determinant formed by these 4 vectors.

We will see in the next sections the epipolar structures that relate the correspondences of two, three and four images.

5 Bilinear constraints: Fundamental matrix

Let us take the first two images of (3) to describe the bilinear constraints. We have a 6×6 matrix whose determinant equals 0. If we develop this determinant we can see that the elements of the fundamental matrix are 4×4 determinants as it appears in Appendix B (see also [FM95a]) :

The fundamental matrix is described as

$$p_2 \; F_{12} \; p_1 = 0$$
$$x_1x_2\gamma_{11} + y_1x_2\gamma_{12} + x_2\gamma_{13} + x_1y_2\gamma_{21} + y_1y_2\gamma_{22}$$
$$+y_2\gamma_{23} + x_1\gamma_{31} + y_1\gamma_{32} + \gamma_{33} = 0$$
$$F_{12} = [\gamma_{ij}] \qquad i,j = 1,2,3$$

Following [Sha95], if the projection matrices are recovered as $\tilde{M}_1$ = [I,0] and $\tilde{M}_2$ = [A,v_{21}] ($\lambda_1 \; p_1 = [I,0] \; P$, $\lambda_2 \; p_2 = [A, v_{21}] \; P$) then the coefficients can be described as

$$\gamma_{ij} = \begin{vmatrix} a_{i_1 j} & v_{21}^{i_1} \\ a_{i_2 j} & v_{21}^{i_2} \end{vmatrix} \qquad i,j = 1,2,3$$

i	$\Rightarrow$	i_1	i_2
1	$\Rightarrow$	2	3
2	$\Rightarrow$	3	1
3	$\Rightarrow$	1	2

The definition of the fundamental matrix is independent of the kind of projection considered.

As described in [FM95b], the epipole v_{12} is proportional to the vector ([1,4,5,6], [2,4,5,6], [3,4,5,6]) and v_{21} is proportional to ([1,2,3,4], [1,2,3,5], [1,2,3,6]).

F_{12} is of rank 2. The kernel of F_{12} is the epipole v_{12}. Given a minimum of eight correspondences in two images, it is possible to obtain the fundamental matrix and from it the epipoles by solving the systems:

$$F_{12}v_{12} = 0 \qquad F_{12}^t v_{21} = 0$$

In addition, thanks to the relationship of the different determinants, we can find the epipoles directly from the coefficients (see Appendix I, (32)).

$$\begin{array}{lll}
\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32} = v_{12}^1 v_{21}^1 & \gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{32} = v_{12}^2 v_{21}^1 & \gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22} = v_{12}^3 v_{21}^1 \\
\gamma_{21}\gamma_{33} - \gamma_{23}\gamma_{31} = v_{12}^1 v_{21}^2 & \gamma_{11}\gamma_{33} - \gamma_{13}\gamma_{31} = v_{12}^2 v_{21}^2 & \gamma_{11}\gamma_{23} - \gamma_{13}\gamma_{21} = v_{12}^3 v_{21}^2 \\
\gamma_{21}\gamma_{32} - \gamma_{22}\gamma_{31} = v_{12}^1 v_{21}^3 & \gamma_{11}\gamma_{32} - \gamma_{12}\gamma_{31} = v_{12}^2 v_{21}^3 & \gamma_{22}\gamma_{22} - \gamma_{12}\gamma_{21} = v_{12}^3 v_{21}^3
\end{array}$$

for instance, the coordinates of the epipole $v_{12} = (x_v, y_v)$ can be found with

$$\frac{\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}}{\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}} = x_v \qquad \frac{\gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{32}}{\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}} = y_v$$

Finally, they provide some nonlinear constraints of the coefficients shown in Appendix C.

The matrix F_{12} is the fundamental matrix corresponding to a pair of camera matrices P_1 and P_2 if and only of P_2^t F P_1 is skew-symmetric. Two camera matrices P_1 and P_2 with different centers of projection uniquely determine the fundamental matrix F_{12}; on the other hand, given F_{12}, the matrices P_1 and P_2 are not uniquely determined.

5.1 Estimating the fundamental matrix

Each matching pair of points between two images provide a single linear constraint of F_{ij}; this allows F to be estimated linearly from at least 8 independent correspondences (up to scale). Algebraically, however, F has only seven degrees of freedom due to the rank 2 condition that implies an additional constraint ($\det(F_{ij}) = 0$). In theory, only 7 points are strictly necessary but in practice as many points as possible are used.

Combining the equations provided by all the available correspondences gives a linear system $Af = 0$. Then, we can use a least-square process and the solution can be restricted for f to have norm 1: $min_{\|f\|=1}\|Af\|^2$, that is $min_{\|f\|=1} f^t A^t A f$, which amounts to finding the eigenvector associated with

the smallest eigenvalue of the 9×9 matrix. As A^tA is in general numerically ill-conditioned, it is a good idea to normalize the pixel coordinates to [-1,1] providing more data stability before beginning the whole process (see [LF96]).

5.2 Perspective projection in P^3

5.2.1 Epipolar geometry of two views

As we have seen in (2), the relationship between the projection matrix and the internal and external parameters of a camera i is given by the following formula:

$$\tilde{M} = A_i K C_i$$

where

$$A_i = \begin{bmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_i = \begin{bmatrix} R & T \\ 0_3 & 1 \end{bmatrix}$$

- α_u is the magnification in the x coordinate direction.

- α_v is the magnification in the y coordinate direction.

- u_0 and v_0 are the coordinates of the principal point.

- γ is a skew parameter corresponding to a skewing of the coordinate axes.

A_i is the matrix of the intrinsic parameters of the camera i (see [Fau93]) and C_i the matrix of the extrinsic parameters. Let take camera i as the one defining the coordinate system (then C_i = I).

The projection equation, relating a point out of the focal plane P = [x, y, s, t] expressed in the normalized camera frame to its projection p_i = [x_i, y_i, 1] is

$$s\, p_i = A_i K P \tag{4}$$

In the ideal case (s is just a real constant)

$$s\, p_i = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} P$$

If the relationship between two views (say image plane j) is a matrix of displacement defined by a rotation R and a translation vector T:

$$C_j = \begin{bmatrix} R & T \\ 0_3 & 1 \end{bmatrix}$$

then, assuming that the point P is not in the focal planes corresponding to the two views, we obtain the general disparity equation:

$$s' p_j = A_j K C_j P =$$
$$A_j R K P + A_j K [T\ 1] t = s A_j R A_i^{-1} p_i + t A_j T$$

then

$$s' p_j = s H_\infty p_i + t v_{ji} \tag{5}$$

where

$$v_{ji} = A_j T \qquad H_\infty = A_j R A_i^{-1} \tag{6}$$

H_∞ is the homography of the plane at infinity.

Equation (5) means that m_j lies on the line going through v_{ji} and $H_\infty p_i$, which is the epipolar line of p_i.

Then, $F_{ij} = [v_{ji}]\ H_\infty$ is the fundamental matrix, which is of rank 2 (because of $[v_{ji}]$).

If we are not using the plane at infinity but a plane defined by the vector $\Pi = [n^t\ -d]$, where n is the unitary normal, d the distance to the center and $\Pi\ P = 0$. From here, $n^t\ K\ P = d\ t$. Since $KM = s\ A_i^{-1}\ p_i$ (using (33))

$$s'p_j = sH_\infty p_i + \frac{n^t}{d}(sA_i^{-1}p_i)v_{ji} = s[H_\infty + v_{ji}\frac{n^t}{d}A_i^{-1}]p_i$$

The new form of the disparity equation (5) is

$$s'p_j = s\ H\ p_i$$

where the homography of this plane is

$$H = H_\infty + v_{ji}\frac{n^t}{d}A_i^{-1}$$

In this case, the fundamental matrix satisfy $F_{ij} = [v_{ji}]\ H$.

Either with H_∞ or H, it follows that F_{ij} provides the epipoles $v_{ji}\ F_{ij} = 0$ and $F_{ij}^t\ v_{ij} = 0$ since $H\ v_{ij} = v_{ji}$ (or H_∞) ([Fau92],[Sha94]).

Alternatively, since the epipolar line $l_{ji} = v_{ji} \times p_j$ (epipolar line of p_i on view I_j) and $p_j = H\ p_i$ we have

$$p_j^t[v_{ji}]Hp_i = 0$$

if the homography H is given because of some configuration of four coplanar points, v_{ji} follows from two or more corresponding points.

The plane at infinity, represented by [0,0,0,1], with $t = 0$, yields the following disparity equation

$$s' p_j = s H_\infty p_i$$

If A_i, A_j, R and t are known, thus H_∞ and v_{ji} (using (6)), it corresponds to a *strongly calibrated system.*

If H_∞ and F_{ij} are known, thus v_{ji} up to an unknown scalar factor, let have $\tilde{H}_\infty = \lambda H_\infty$ and $\tilde{v}_{ji} = \mu v_{ji}$, it corresponds to an *affinely calibrated system.*

If only F_{ij} is known, thus $\tilde{v}_{ji}$, it corresponds to a *weakly calibrated system.*

5.2.2 Euclidean reconstruction

If A_i, A_j, R and T are known, thus H_∞ and v_{ji} (using (6)) which corresponds to a *strongly calibrated system*, then (5) gives

$$\begin{aligned} 0 &= s(p_j \times H_\infty p_i) + t(p_j \times v_{ji}) \\ 0 &= s + t\frac{(p_j \times v_{ji})(p_j \times H_\infty p_i)}{\|p_j \times H_\infty p_i\|^2} \end{aligned}$$

$$\frac{s}{t} = -\frac{(p_j \times v_{ji})(p_j \times H_\infty p_i)}{\| p_j \times H_\infty p_i \|^2}$$

then (4) gives

$$\frac{s}{t} A_i^{-1} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x/t \\ y/t \\ s/t \end{bmatrix}$$

The projection matrices for these views I_i and I_j are

$$A_i[I_3\ 0_3] \qquad A_j[R\ T]$$

They are known up to a rigid displacement.

5.2.3 Affine reconstruction

H_∞ and F_{ij} are known, thus v_{ji} up to an unknown scalar factor, let have $\tilde{H}_\infty = \lambda H_\infty$ and $\tilde{v}_{ji} = \mu v_{ji}$, which corresponds to an *affinely calibrated system.* Since A_i and v_{ji} are unknown, equations (4) and (5) cannot be used. They are rewritten in another framework A defined by

$$C_A = \begin{bmatrix} \frac{1}{\lambda} A_i & 0_3 \\ 0_3^t & \frac{1}{\mu} \end{bmatrix}$$

If $P_A = (x_A, y_A, s_A, t_A)$ then $P_A = (\frac{1}{\lambda} A_i \; K \; P \; \frac{t_A}{\mu})$, thus $s = \lambda s_A$ and $t = \mu t_A$ and $s_A p_i = K P_A$

then (4) gives

$$\frac{s_A}{t_A} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x_A/t_A \\ y_A/t_A \\ s_A/t_A \end{bmatrix}$$

and (5) gives

$$\begin{aligned} s' p_j &= s \; H_\infty p_i + t \; v_{ji} \\ s' p_j &= s_A \lambda H_\infty p_i + t_A \mu v_{ji} \\ s' p_j &= s_A \tilde{H}_\infty p_i + t_A \tilde{v}_{ji} \\ 0 &= s_A (p_j \times \tilde{H}_\infty p_i) + t_A (p_j \times \tilde{v}_{ji}) \\ 0 &= s_A + t_A \frac{(p_j \times \tilde{v}_{ji})(p_j \times \tilde{H}_\infty p_i)}{\| p_j \times \tilde{H}_\infty p_i \|^2} \end{aligned}$$

$$\frac{s_A}{t_A} = -\frac{(p_j \times \tilde{v}_{ji})(p_j \times \tilde{H}_\infty p_i)}{\| p_j \times \tilde{H}_\infty p_i \|^2}$$

The projection matrices for these views I_i and I_j are, if

$$C_A^{-1} = \begin{bmatrix} \lambda A_i^{-1} & 0_3 \\ 0_3 & \mu \end{bmatrix}$$

then

$$A_i[I_3\ 0_3]C_A^{-1} = \lambda[I_3\ 0] \qquad A_j[R\ T]C_A^{-1} = [\tilde{H}_\infty\ \tilde{v}_{ji}]$$

5.2.4 Projective reconstruction

Only F_{ij} is known, thus $\tilde{v}_{ji}$ which corresponds to a *weakly calibrated system*. Equations (4) and (5) cannot be used. They are rewritten in another framework P where the vector representing the plane at infinity is no longer known, equal to $[0, 0, 0, 1]^t$. Let us assume that the homography of the plane is known and defined by $H = \lambda\ [H_\infty + v_{ji}\frac{n^t}{d}A_i^{-1}]$. Then

$$C_P = \begin{bmatrix} \frac{1}{\lambda}A_i & 0_3 \\ -\frac{1}{\mu}\frac{n^t}{d} & \frac{1}{\mu} \end{bmatrix}$$

If $P_P = (x_P, y_P, s_P, t_P)$ then $P_P = (\frac{1}{\lambda}\ A_i\ K\ P\ -\frac{1}{\mu}\frac{n^t}{d}\ K\ P + \frac{t}{\mu})$, thus $s = \lambda\ s_P$.

$$\begin{aligned} s'p_j &= \lambda s_P H_\infty p_i + \mu t_P v_{ji} + \lambda s_P v_{ji}\frac{n^t}{d}A_i^{-1}p_i \\ &= \lambda s_P H p_i + t_P \tilde{v}_{ji} \\ 0 &= s_P(p_j \times H p_i) + t_P(p_j \times \tilde{v}_{ji}) \\ 0 &= s_P + t_P\frac{(p_j \times \tilde{v}_{ji})(p_j \times H p_i)}{\|p_j \times H p_i\|^2} \\ \frac{s_P}{t_P} &= -\frac{(p_j \times \tilde{v}_{ji})(p_j \times H p_i)}{\|p_j \times H p_i\|^2} \end{aligned}$$

then (4), as $s_P\ m_i = K\ M_P$, gives

$$\frac{s_P}{t_P}\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x_P/t_P \\ y_P/t_P \\ s_P/t_P \end{bmatrix}$$

The projection matrices for these views I_i and I_j (back to the original framework of camera i from framework P) are

$$C_P^{-1} = \begin{bmatrix} \lambda A_i^{-1} & 0_3 \\ \frac{\lambda n^t}{d} A_i^{-1} & \mu \end{bmatrix}$$

then

$$\begin{aligned} A_i[I_3\ 0_3]C_P^{-1} &= \lambda[I_3\ 0] \\ A_j[R\ T]\ C_P^{-1} &= [H\ \tilde{v}_{ji}] \end{aligned}$$

If F_{ij} is known, we can always use the plane whose vector holds (using (34))

$$\frac{n^t}{d} = -\frac{v_{ji}^t}{\|v_{ji}\|^2} H_\infty A_i$$

then

$$H = H_\infty + v_{ji}(-\frac{v_{ji}^t}{\|v_{ji}\|^2} H_\infty A_i) A_i^{-1} =$$

$$H_\infty(1 - \frac{\|v_{ji}\|^2}{\|v_{ji}\|^2} - \frac{[v_{ji}][v_{ji}]}{\|v_{ji}\|^2}) = -H_\infty \frac{[v_{ji}][v_{ji}]}{\|v_{ji}\|^2} =$$

$$\frac{[v_{ji}]}{\|v_{ji}\|^2}([v_{ji}]H_\infty) = \frac{[v_{ji}]}{\|v_{ji}\|^2} F_{ij}$$

5.3 Recovering the epipoles and fundamental matrix of two views

The epipoles v_{ij}, v_{ji} of two views I_i, I_j are the points of intersection of the line passing through the center of projection of views I_i and I_j, ie O_i and O_j, see Figure 6.

The epipoles can be recovered from six points [Lee88], [Moh92], (four of them are assumed to be coplanar), seven points [FM90] (non-linear algorithm) or eight points, which is the most usual case [Fau92].

In the case of six points, it is a requirement that four of them must be coplanar. Let A, B, C, D be four coplanar points and F, G the two

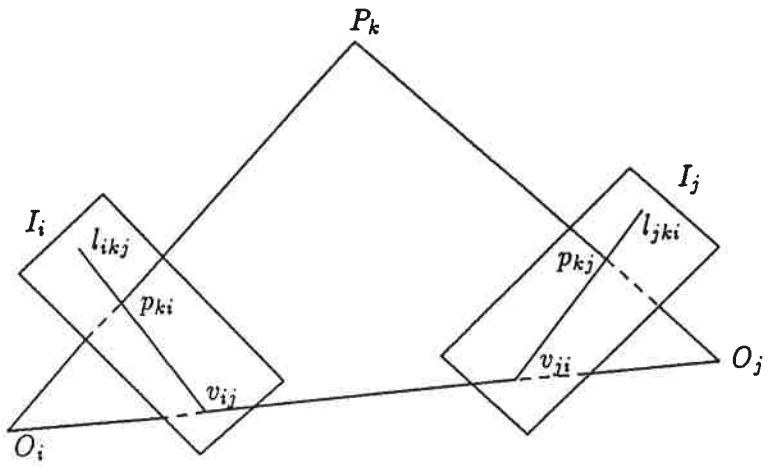

Figure 6: Epipolar points and lines of two views

remaining reference points and O_i i=1,2 the center of projection of the two views involved. Let a', a" be the projections of point A and so on for the rest of points. The intersection of the view line O_1F with the plane ABCD, F', is defined by its projective coordinates measured in view 1, taking the projections a', b', c', d' as reference frame (see Figure 7).

Now we can locate the coordinates of F' in the second view, by using the homography defined by a', b', c', d' and a", b", c", d" (let us call it f_2). Finally, we have the line determined by f' and f_2. If we proceed in the same way with G, the intersection of the lines provides the epipole on the second view. The same can be done to get the epipole on the first view.

In the case of eight points, the basic idea is based on the projective relationship between two epipolar lines. Given a point P_k and two views I_i, I_j, the epipolar line l_{ikj} is the line passing through v_{ij} and p_{ki} on view I_i and the line l_{jki} is the line passing through v_{ji} and p_{kj} on view I_j (see Figure 6). Therefore, the epipolar geometry may be represented as a 2D correlation matrix (ie a transformation of the points into lines on a plane). Let L^{ij} be an epipolar transformation, ie $L^{ij} l_{ikj} = \mu\, l_{jki}$, where $l_{ikj} = v_{ij} \times p_{ki}$ and $l_{jki} = v_{ji} \times p_{kj}$ are the corresponding epipolar lines. We can rewrite the projective relation of epipolar lines using the matrix form of cross-products:

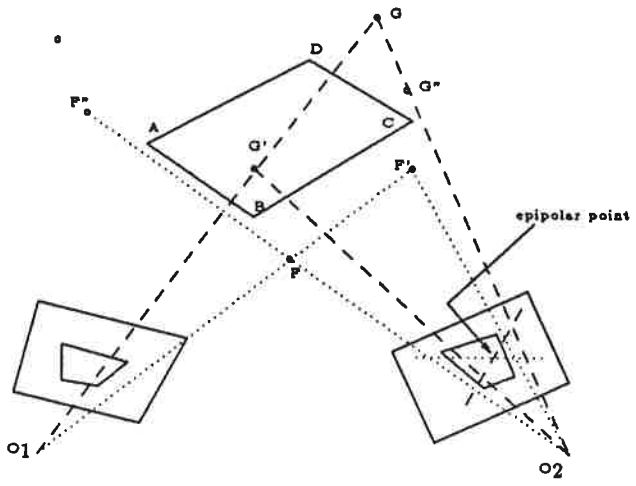

Figure 7: Epipolar points and lines of two views

$$L^{ij}(v_{ij} \times p_{ki}) = L^{ij}[v_{ij}]p_{ki} = \mu l_{jki}$$

where $[v_{ij}]$ is a skew symmetric matrix (and hence has rank 2).

From the point/line incidence property, we have that $p_{kj}^t . l_{jki} = 0$, and therefore $p_{kj}^t \; L^{ij}[v_{ij}]p_{ki}=0$, or $p_{kj}^t \; F_{ij}p_{ki}=0$ where $F_{ij}=L^{ij}[v_{ij}]$, where F_{ij} is the "fundamental" matrix.

Finally,

$$v_{ji} = F_{ij}p_{si} \times F_{ij}p_{mi}$$

where p_{si} and p_{mi} are any two points which are not on the same epipolar line (with more points, a least square method can be used) (see Figure 8).

An alternative is $F_{ij} \; v_{ij} = 0$ and $v_{ji} \; F_{ij} = 0$

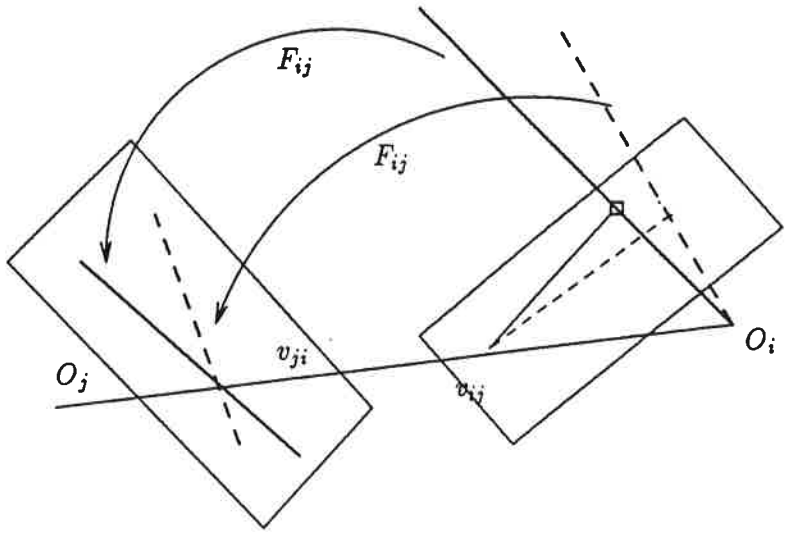

Figure 8: Epipolar lines

5.4 Finding the epipole with two points using the fundamental matrix

Given two points which belong to the same epipolar line, it is possible first to determine, without obtaining the epipole, if the epipole lies between them or which one is closer to it and second, the epipole can be obtained if necessary. In this section we will use the fundamental matrix and we will see that the method is straightforward and easy to understand; we have written it here in order to explain the same idea we will use in the next section which has more interest as we will use the coefficients of the tensor.

Say we have two images a and b and two points in image a which we believe belong to the same epipolar line. Let $v_{ab} = (x_{ab}, y_{ab})$ be the (unknown) epipole between image a and b and let sx and sy be the slopes of the epipolar line. Thus, every point in that epipolar line can be expressed as $p_a = (x_a, y_a) = (x_{ab} + n_k sx, y_{ab} + n_k sy)$, where n_k is a scalar; if $n_k = 0$ the point in question is the (unknown) epipole.

Given the fundamental matrix between these two images,

$$p_b \;\; F_{ab} \;\; p_a = 0 \qquad F_{ab} = [\gamma_{ij}] \quad i,j = 1,2,3$$

any of the rows defines one coefficient of a line in image b. If the point we are using turns out to be the the epipole, all these coefficients equal 0 (and this happens only in this case).

Assuming that p_a in image a is not the epipole

$$\begin{aligned}
\gamma_{11}x_a + \gamma_{12}y_a + \gamma_{13} &= v1_k \\
\gamma_{21}x_a + \gamma_{22}y_a + \gamma_{23} &= v2_k \\
\gamma_{31}x_a + \gamma_{32}y_a + \gamma_{33} &= v3_k
\end{aligned}$$

$$\begin{aligned}
\gamma_{11}(x_{ab} + n_k sx) + \gamma_{12}(y_{ab} + n_k sy) + \gamma_{13} &= v1_k \\
\gamma_{21}(x_{ab} + n_k sx) + \gamma_{22}(y_{ab} + n_k sy) + \gamma_{23} &= v2_k \\
\gamma_{31}(x_{ab} + n_k sx) + \gamma_{32}(y_{ab} + n_k sy) + \gamma_{33} &= v3_k
\end{aligned}$$

$$\begin{aligned}
\gamma_{11}x_{ab} + \gamma_{12}y_{ab} + \gamma_{13} + \gamma_{11}n_k sx + \gamma_{12}n_k sy &= v1_k \\
\gamma_{21}x_{ab} + \gamma_{22}y_{ab} + \gamma_{23} + \gamma_{21}n_k sx + \gamma_{22}n_k sy &= v2_k \\
\gamma_{31}x_{ab} + \gamma_{32}y_{ab} + \gamma_{33} + \gamma_{31}n_k sx + \gamma_{32}n_k sy &= v3_k
\end{aligned}$$

$$\begin{aligned}
n_k(\gamma_{11}sx + \gamma_{12}sy) &= v1_k \\
n_k(\gamma_{21}sx + \gamma_{22}sy) &= v2_k \\
n_k(\gamma_{31}sx + \gamma_{32}sy) &= v3_k
\end{aligned}$$

$$n_k = \frac{v1_k}{\gamma_{11}sx+\gamma_{12}sy} = \frac{v2_k}{\gamma_{21}sx+\gamma_{22}sy} = \frac{v3_k}{\gamma_{31}sx+\gamma_{32}sy}$$

Given two points, if the sign of n_k is different, the epipole lies between them; if the same, the epipole is closer to that one whose absolute value of n_k is smaller. Once the value of n_k is known, the epipole can be found from $x_{ab} = x_a - n_k sx$ and $y_{ab} = y_a - n_k sy$.

5.5 Relationship of the slopes of two epipolar lines

Given a fundamental matrix F between two images

$$p_2 \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} p_1 = 0$$

and the epipoles $v_{12} = (x_{12}, y_{12})$ and $v_{21} = (x_{21}, y_{21})$, we know that

$$\begin{aligned} \gamma_{11}\, x_{12} + \gamma_{12}\, y_{12} + \gamma_{13} &= 0 \qquad & \gamma_{11}\, x_{21} + \gamma_{21}\, y_{21} + \gamma_{31} &= 0 \\ \gamma_{21}\, x_{12} + \gamma_{22}\, y_{12} + \gamma_{23} &= 0 \qquad & \gamma_{12}\, x_{21} + \gamma_{22}\, y_{21} + \gamma_{32} &= 0 \\ \gamma_{31}\, x_{12} + \gamma_{32}\, y_{12} + \gamma_{33} &= 0 \qquad & \gamma_{13}\, x_{21} + \gamma_{23}\, y_{21} + \gamma_{33} &= 0 \end{aligned}$$

Besides, any point belongs to an epipolar line and therefore it can be expressed as $p_1 = (x_{12} + k\, A_1, y_{12} + k\, B_1)$ and $p_2 = (x_{21} + m\, A_2, y_{21} + m\, B_2)$, where A_i, B_i are the slopes of the corresponding epipolar lines.

If now we develop the equation of the fundamental matrix with p_1 and p_2, we can see that all of the terms involving v_{12} or v_{21} become null and we can eliminate the scalars k and m; we are left with the following

$$\gamma_{11}\, A_1\, A_2 + \gamma_{12}\, B_1\, A_2 + \gamma_{21}\, B_2\, A_1 + \gamma_{22}\, B_1\, B_2 = 0$$

in other words,

$$(A_2 \;\; B_2) \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0$$

6 Trilinear Constraints

Let us take the first three images of (3) to describe the trilinear constraints. As said in [Tri95], the elements of the tensor of images i,j,k are 4×4 determinants α_i^{jk} as shown in Appendix D. These trilinear functions can be recovered by linear methods with a minimal configuration of seven points.

In [Sha95], the same algebraic connections across three views of a 3D scene are established with a different derivation process (see [Sha95], [Sha92] and [Sha96]).

Following [Sha95], if the projection matrices are recovered as [I,0], [A,v_{21}], [B,v_{31}] ($\lambda_1\ p_1 = [I,0]\ P,\ \lambda_2\ p_2 = [A, v_{21}]\ P,\ \lambda_3\ p_3 = [B, v_{31}]\ P$) then the trilinear tensor is an array of 27 entries (α_i^{jk} with i,j,k=1..3), each of them is the determinant of a 2×2 matrix:

$$\alpha_i^{j,k} = \begin{vmatrix} b_{jk} & v_{31}^j \\ a_{ik} & v_{21}^i \end{vmatrix} \qquad i,j,k = 1,2,3$$

As established in [Sha95], only four of the nine equations are independent, called trilinearities.

If $\alpha_i^{jk} p_t^k = \alpha_i^{j1} x_t + \alpha_i^{j2} y_t + \alpha_i^{j3}$ t=1,2,3 then they can be written ([Sha95]):

$$x_3\alpha_1^{3k}p_1^k - x_3x_2\alpha_3^{3k}p_1^k + x_2\alpha_3^{1k}p_1^k - \alpha_1^{1k}p_1^k = 0 \tag{7}$$

$$y_3\alpha_1^{3k}p_1^k - y_3x_2\alpha_3^{3k}p_1^k + x_2\alpha_3^{2k}p_1^k - \alpha_1^{2k}p_1^k = 0 \tag{8}$$

$$x_3\alpha_2^{3k}p_1^k - x_3y_2\alpha_3^{3k}p_1^k + y_2\alpha_3^{1k}p_1^k - \alpha_2^{1k}p_1^k = 0 \tag{9}$$

$$y_3\alpha_2^{3k}p_1^k - y_3y_2\alpha_3^{3k}p_1^k + y_2\alpha_3^{2k}p_1^k - \alpha_2^{2k}p_1^k = 0 \tag{10}$$

6.1 Epipolar lines

Combining (7), (8), (9) and (10) we find the equations of the epipolar lines in the second and third images:

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)y_2 + (\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)x_2 + (\alpha_2^{3k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_2^{1k}p_1^k * \alpha_1^{3k}p_1^k) = 0 \quad (11)$$

$$(\alpha_3^{2k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{2k}p_1^k)y_2 + (\alpha_2^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{2k}p_1^k)x_2 + (\alpha_2^{3k}p_1^k * \alpha_1^{2k}p_1^k - \alpha_2^{2k}p_1^k * \alpha_1^{3k}p_1^k) = 0 \quad (12)$$

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)y_3 + (\alpha_1^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_1^{3k}p_1^k * \alpha_3^{2k}p_1^k)x_3 + (\alpha_3^{2k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k * \alpha_1^{2k}p_1^k) = 0 \quad (13)$$

and also

$$(\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)y_3 + (\alpha_3^{2k}p_1^k * \alpha_2^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_2^{2k}p_1^k)x_3 + (\alpha_2^{2k}p_1^k * \alpha_3^{1k}p_1^k - \alpha_3^{2k}p_1^k * \alpha_2^{1k}p_1^k) = 0 \quad (14)$$

We need only a point in image 1 and one of the coordinates in image 2 (3) to get the other coordinate in image 2 (3) and the point in image 3 (2).

A) given (x_1, y_1) and x_2

$$x_3 = \frac{\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k x_2}{\alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k x_2} \quad (15)$$

$$y_3 = \frac{\alpha_1^{2k}p_1^k - \alpha_3^{2k}p_1^k x_2}{\alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k x_2} \quad (16)$$

B) given (x_1, y_1) and x_3

$$x_2 = \frac{\alpha_1^{1k} p_1^k - \alpha_1^{3k} p_1^k x_3}{\alpha_3^{1k} p_1^k - \alpha_3^{3k} p_1^k x_3} \tag{17}$$

$$y_2 = \frac{\alpha_2^{1k} p_1^k - \alpha_2^{3k} p_1^k x_3}{\alpha_3^{1k} p_1^k - \alpha_3^{3k} p_1^k x_3} \tag{18}$$

C) given (x_1, y_1) and y_2

$$x_3 = \frac{\alpha_2^{1k} p_1^k - \alpha_3^{1k} p_1^k y_2}{\alpha_2^{3k} p_1^k - \alpha_3^{3k} p_1^k y_2} \tag{19}$$

$$y_3 = \frac{\alpha_2^{2k} p_1^k - \alpha_3^{2k} p_1^k y_2}{\alpha_2^{3k} p_1^k - \alpha_3^{3k} p_1^k y_2} \tag{20}$$

D) given (x_1, y_1) and y_3

$$x_2 = \frac{\alpha_1^{2k} p_1^k - \alpha_1^{3k} p_1^k y_3}{\alpha_3^{2k} p_1^k - \alpha_3^{3k} p_1^k y_3} \tag{21}$$

$$y_2 = \frac{\alpha_2^{2k} p_1^k - \alpha_2^{3k} p_1^k y_3}{\alpha_3^{2k} p_1^k - \alpha_3^{3k} p_1^k y_3} \tag{22}$$

The list of constraints can be found in Appendix E.

6.2 Finding the epipolar points using the E matrices

Say we would like to to calculate the epipole in the second (of first camera) (see [SW94a]): if $Ei^tF + F^tEi = 0$ and $[v']Ei = F$ where [u]w is u×w (cross product), then $El^t[v']Ek - Ek^t[v']El = 0$. Let assume that Ek = $[a_1, a_2, a_3]$, El = $[b_1, b_2 b_3]$. Then from the diagonal: $(a_i \times b_i)^t v' = 0$ (so we have 3 equations) and from the rest: $[(a_i \times b_j) - (b_i \times a_j)]^t v' = 0$. Just replace E by W in the case of the third image.

6.3 Other previous results

We will just mention some of the properties of the tensor (see [Sha95]). Let E_j j = 1,2,3 be three matrices corresponding to $\alpha_{.j}$.

* these matrices are the homographies of three distinct, non-singular (not coplanar with any of the camera centers) planes. In the case the third camera is calibrated ($M_3 = I$), the planes are perpendicular to the main axed of the third camera coordinate frame ie, E_1 is the homography associated with the plane whose normal is the x-axis of the third camera frame and whose distance to the origin O_1 is $1/x_{v31}$ ($v'' = v_{31}$) and so forth (recall $v_{ij} = (x_{vij}, y_{vij}, s_{vij})$). [SW94b] refers to them as the intrinsic homography matrices.

* $[v']E_j \cong F$. It provides a method for recovering this epipole from the tensor (p'Fp = 0, then p'$[v']E_j$p = 0). Thus for any choice of j, given the corresponding points already used to recover the tensor, we have a linear system for v' (it can be over-determined if more than two pairs of corresponding points).

* The (over-determined) system of linear equations resulting from $F^t E_j + E_j^t F = 0$, j= 1,2,3 provides a unique solution for F (up to scale). The solution is unique when at least two of the E_j are used in the system, ie j= j_1, j_2, $j_1 \neq j_2$, j_1, j_2 = 1,2,3.

* The projective structure of the scene is captured by the equation $p' \cong E_1 p + kE_2 p$ where k is a projective invariant of p^3 (projective depth) The invariant k does not depend on the choice of the first two views (ψ_1, ψ_2). The third view ψ_3 determines the projective reference frame.

This structure is recovered without any fiducial (reference) points (four of them needed to define two planes and the fifth to establish the scale), since these planes are provided by the tensor. The fifth point is not required as well, because the homographies are already determined up to a common scale. This fifth point is at the intersection of the three planes associated with the homographies $E_1 + E_2$, $E_1 + E_3$, $E_2 + E_3$.

So the third view provides the projective reference; changing it will result in reconstruction that are projectively related to one another. In practical terms, reconstruction using the tensor treats all points equally likely, ie the tensor does not depend on the choice of seven corresponding points, which

is generally a desirable property in the presence of noise.

* the relative affine structure of the scene is captured by the equation p' $\cong E_1\ p + kv'$

* The intrinsic homography matrices are generically full rank. In the case the three camera centers are collinear, they have rank 2 (the corresponding fiducial planes are coplanar with O'_2. Finally, E_1, E_2 and E_3 has rank 1 iff $x_{v31} = 0$, $y_{v31} = 0$, $z_{v31} = 0$ respectively

* The trilinear function F reduces to a bilinear form (any perspective view ψ can be obtained by a rational linear function of two orthographic views) which implies that simpler recognition schemes are possible if the two reference views (model views) stored in memory are orthographic. Besides, only five points are necessary to obtain the tensor (see [Sha95]).

* The equivalence class of views of an object (ignoring self occlusions) undergoing 3D rigid, affine or projective transformations can be captured by storing a 3D model of the object, or simply by storing at least two arbitrary "model" views of the object, assuming that the correspondence problem between the model views can somehow be solved.

* All trilinear tensors live in a manifold of P^26. The space of all trilinear tensors with two of the views fixed, is a 12'th dimensional linear sub-space of R^27 (see [Sha96]).

6.4 Tensor T_{213} from tensor T_{123}

In this section it is shown the relationship between the coefficients of the tensor $T_{1,2,3}$ (with elements α_i^{jk}) and those of tensor $T_{2,1,3}$ (with elements ξ_i^{jk})

It is known from the trilinearities that the tensor ξ_k^{ij} must satisfy for each correspondence:

$$x_1 = \frac{\xi_1^{1k}p_2^k - \xi_1^{3k}p_2^k x_3}{\xi_3^{1k}p_2^k - \xi_3^{3k}p_2^k x_3} \qquad y_1 = \frac{\xi_2^{1k}p_2^k - \xi_2^{3k}p_2^k x_3}{\xi_3^{1k}p_2^k - \xi_3^{3k}p_2^k x_3}$$

$$x_1 = \frac{\xi_1^{2k}p_2^k - \xi_1^{3k}p_2^k y_3}{\xi_3^{2k}p_2^k - \xi_3^{3k}p_2^k y_3} \qquad y_1 = \frac{\xi_2^{2k}p_2^k - \xi_2^{3k}p_2^k y_3}{\xi_3^{2k}p_2^k - \xi_3^{3k}p_2^k y_3}$$

Say we take equation (9), isolate y_1 and substitute it into (7). What we get is an equation that depends on x_2, y_2 and x_1, x_3 with the following form:

$$x_1 = \frac{x_3^2 A + x_3 B + C}{x_3^2 A' + x_3 B' + C'}$$

where A, B, C, A', B' and C' are functions which depend only on the coefficients of α_i^{jk} and the coordinates of p_2.

As we have two equations of second degree, they can be decomposed

$$x_1 = \frac{(Ax_3 - r_1)(x_3 - r_3)}{(A'x_3 - r_2)(x_3 - r_4)}$$

where r_1, r_2, r_3 and r_4 are the roots of these equations. Now, since we know that $x_1 = (\xi_1^{1k} p_2^k - \xi_1^{3k} p_2^k x_3)/(\xi_3^{1k} p_2^k - \xi_3^{3k} p_2^k x_3)$ holds, it seems that two of the roots, say r_3 and r_4, have the same value and can be eliminated. By comparing the two formulas, it is clear that

$$\begin{aligned} r_1 &\cong \xi_1^{1k} p_2^k & r_2 &\cong \xi_3^{1k} p_2^k \\ A &\cong \xi_1^{3k} p_2^k & A' &\cong \xi_3^{3k} p_2^k \end{aligned}$$

If we do the same with the rest of the equations, we have the following list of comparisons:

$$\xi_1^{1k} p_2^k \cong (\alpha_3^{13}\alpha_2^{12} - \alpha_3^{12}\alpha_2^{13})x_2 + (\alpha_1^{13}\alpha_3^{12} - \alpha_1^{12}\alpha_3^{13})y_2 + (\alpha_1^{12}\alpha_2^{13} - \alpha_1^{13}\alpha_2^{12})$$

$$\xi_1^{2k} p_2^k \cong (\alpha_3^{23}\alpha_2^{22} - \alpha_3^{22}\alpha_2^{23})x_2 + (\alpha_1^{23}\alpha_3^{22} - \alpha_1^{22}\alpha_3^{23})y_2 + (\alpha_1^{22}\alpha_2^{23} - \alpha_1^{23}\alpha_2^{22})$$

$$\xi_1^{3k} p_2^k \cong (\alpha_3^{33}\alpha_2^{32} - \alpha_3^{32}\alpha_2^{33})x_2 + (\alpha_1^{33}\alpha_3^{32} - \alpha_1^{32}\alpha_3^{33})y_2 + (\alpha_1^{32}\alpha_2^{33} - \alpha_1^{33}\alpha_2^{32})$$

$$\xi_2^{1k} p_2^k \cong (\alpha_3^{11}\alpha_2^{13} - \alpha_3^{13}\alpha_2^{11})x_2 + (\alpha_1^{11}\alpha_3^{13} - \alpha_1^{13}\alpha_3^{11})y_2 + (\alpha_1^{13}\alpha_2^{11} - \alpha_1^{11}\alpha_2^{13})$$

$$\xi_2^{2k} p_2^k \cong (\alpha_3^{21}\alpha_2^{23} - \alpha_3^{23}\alpha_2^{21})x_2 + (\alpha_1^{21}\alpha_3^{23} - \alpha_1^{23}\alpha_3^{21})y_2 + (\alpha_1^{23}\alpha_2^{21} - \alpha_1^{21}\alpha_2^{23})$$

$$\xi_2^{3k} p_2^k \cong (\alpha_3^{31}\alpha_2^{33} - \alpha_3^{33}\alpha_2^{31})x_2 + (\alpha_1^{31}\alpha_3^{33} - \alpha_1^{33}\alpha_3^{31})y_2 + (\alpha_1^{33}\alpha_2^{31} - \alpha_1^{31}\alpha_2^{33})$$

$$\xi_3^{1k} p_2^k \cong (\alpha_3^{12}\alpha_2^{11} - \alpha_3^{11}\alpha_2^{12})x_2 + (\alpha_1^{12}\alpha_3^{11} - \alpha_1^{11}\alpha_3^{12})y_2 + (\alpha_1^{11}\alpha_2^{12} - \alpha_1^{12}\alpha_2^{11})$$

$$\xi_3^{2k} p_2^k \cong (\alpha_3^{22}\alpha_2^{21} - \alpha_3^{21}\alpha_2^{22})x_2 + (\alpha_1^{22}\alpha_3^{21} - \alpha_1^{21}\alpha_3^{22})y_2 + (\alpha_1^{21}\alpha_2^{22} - \alpha_1^{22}\alpha_2^{21})$$

$$\xi_3^{3k} p_2^k \cong (\alpha_3^{32}\alpha_2^{31} - \alpha_3^{31}\alpha_2^{32})x_2 + (\alpha_1^{32}\alpha_3^{31} - \alpha_1^{31}\alpha_3^{32})y_2 + (\alpha_1^{31}\alpha_2^{32} - \alpha_1^{32}\alpha_2^{31})$$

If we consider the value of the coefficients, we can see (done with Maple) that the scaling factors are the coordinates of the epipole between image 1 and 3, v_{31} (see also Appendix I, (32))

$$\begin{array}{lll}
v_{31}^1\xi_1^{11} = (\alpha_3^{12}\alpha_2^{13} - \alpha_3^{13}\alpha_2^{12}) & v_{31}^1\xi_1^{12} = (\alpha_1^{12}\alpha_3^{13} - \alpha_1^{13}\alpha_3^{12}) & v_{31}^1\xi_1^{13} = (\alpha_1^{12}\alpha_2^{13} - \alpha_1^{13}\alpha_2^{12}) \\
v_{31}^2\xi_1^{21} = (\alpha_3^{22}\alpha_2^{23} - \alpha_3^{23}\alpha_2^{22}) & v_{31}^2\xi_1^{22} = (\alpha_1^{22}\alpha_3^{23} - \alpha_1^{23}\alpha_3^{22}) & v_{31}^2\xi_1^{23} = (\alpha_1^{22}\alpha_2^{23} - \alpha_1^{23}\alpha_2^{22}) \\
v_{31}^3\xi_1^{31} = (\alpha_3^{32}\alpha_2^{33} - \alpha_3^{33}\alpha_2^{32}) & v_{31}^3\xi_1^{32} = (\alpha_1^{32}\alpha_3^{33} - \alpha_1^{33}\alpha_3^{32}) & v_{31}^3\xi_1^{33} = (\alpha_1^{32}\alpha_2^{33} - \alpha_1^{33}\alpha_2^{32}) \\
v_{31}^1\xi_2^{11} = (\alpha_3^{13}\alpha_2^{11} - \alpha_3^{11}\alpha_2^{13}) & v_{31}^1\xi_2^{12} = (\alpha_1^{13}\alpha_3^{11} - \alpha_1^{11}\alpha_3^{13}) & v_{31}^1\xi_2^{13} = (\alpha_1^{13}\alpha_2^{11} - \alpha_1^{11}\alpha_2^{13}) \\
v_{31}^2\xi_2^{21} = (\alpha_3^{23}\alpha_2^{21} - \alpha_3^{21}\alpha_2^{23}) & v_{31}^2\xi_2^{22} = (\alpha_1^{23}\alpha_3^{21} - \alpha_1^{21}\alpha_3^{23}) & v_{31}^2\xi_2^{23} = (\alpha_1^{23}\alpha_2^{21} - \alpha_1^{21}\alpha_2^{23}) \\
v_{31}^3\xi_2^{31} = (\alpha_3^{33}\alpha_2^{31} - \alpha_3^{31}\alpha_2^{33}) & v_{31}^3\xi_2^{32} = (\alpha_1^{33}\alpha_3^{31} - \alpha_1^{31}\alpha_3^{33}) & v_{31}^3\xi_2^{33} = (\alpha_1^{33}\alpha_2^{31} - \alpha_1^{31}\alpha_2^{33}) \\
v_{31}^1\xi_3^{11} = (\alpha_3^{12}\alpha_2^{11} - \alpha_3^{11}\alpha_2^{12}) & v_{31}^1\xi_3^{12} = (\alpha_1^{12}\alpha_3^{11} - \alpha_1^{11}\alpha_3^{12}) & v_{31}^1\xi_3^{13} = (\alpha_1^{12}\alpha_2^{11} - \alpha_1^{11}\alpha_2^{12}) \\
v_{31}^2\xi_3^{21} = (\alpha_3^{22}\alpha_2^{21} - \alpha_3^{21}\alpha_2^{22}) & v_{31}^2\xi_3^{22} = (\alpha_1^{22}\alpha_3^{21} - \alpha_1^{21}\alpha_3^{22}) & v_{31}^2\xi_3^{23} = (\alpha_1^{22}\alpha_2^{21} - \alpha_1^{21}\alpha_2^{22}) \\
v_{31}^3\xi_3^{31} = (\alpha_3^{32}\alpha_2^{31} - \alpha_3^{31}\alpha_2^{32}) & v_{31}^3\xi_3^{32} = (\alpha_1^{32}\alpha_3^{31} - \alpha_1^{31}\alpha_3^{32}) & v_{31}^3\xi_3^{33} = (\alpha_1^{32}\alpha_2^{31} - \alpha_1^{31}\alpha_2^{32})
\end{array}$$

and in a general way,

$$v_{31}^{j}\xi_{i}^{jk} = (\alpha_{k_1}^{ji_2}\alpha_{k_1}^{ji_1} - \alpha_{k_1}^{ji_1}\alpha_{k_2}^{ji_2}) \qquad \begin{array}{ccccc} s & \Rightarrow & s_1 & s_2 & \\ 1 & \Rightarrow & 3 & 2 & \\ 2 & \Rightarrow & 1 & 3 & s = i,k \\ 3 & \Rightarrow & 1 & 2 & \end{array}$$

The same can be done to find tensor $T_{3,2,1}$ from tensor $T_{1,2,3}$.

The case of tensor $T_{1,3,2}$ from $T_{1,2,3}$ is very simple as it is well known that if the coefficients of $T_{1,2,3}$ are α_i^{jk}, then the coefficients of $T_{1,3,2}$ are α_j^{ik})

6.5 Tensor 124 from tensor 123 and tensor 234

Say we have tensor $T_{i,j,k}$ and $T_{j,k,l}$ and we want to obtain $T_{i,j,l}$. To make it clear, let put it with numbers: $T_{1,2,3}$ (tensor α_i^{jk}) and $T_{2,3,4}$ (tensor β_i^{jk}) and we want to obtain $T_{1,2,4}$ (tensor γ_i^{jk})

Say point p in image m is noted p_m with coordinates (x_m,y_m,1) and $p_m^1 = x_m$, $p_m^2 = y_m$,$p_m^3 = 1$.

Say we have the trilinearities of $T_{2,3,4}$

$$\begin{aligned} x_3(\beta_3^{1k}p_2^k - x_4\beta_3^{3k}p_2^k) &= (\beta_1^{1k}p_2^k - x_4\beta_1^{3k}p_2^k) \\ x_3(\beta_3^{2k}p_2^k - y_4\beta_3^{3k}p_2^k) &= (\beta_1^{2k}p_2^k - y_4\beta_1^{3k}p_2^k) \\ y_3(\beta_3^{1k}p_2^k - x_4\beta_3^{3k}p_2^k) &= (\beta_2^{1k}p_2^k - x_4\beta_2^{3k}p_2^k) \\ y_3(\beta_3^{2k}p_2^k - y_4\beta_3^{3k}p_2^k) &= (\beta_2^{2k}p_2^k - y_4\beta_2^{3k}p_2^k) \end{aligned} \tag{23}$$

Now, what we want is to find the epipolar line in image 4 for every point in image 1.

What we have in images 2 and 3 is the projection of the line defined by a point in image 1. Our goal is to find this projection in image 4.

Say we have the following line in image 2 and image 3:

$$y_2 = a_2x_2 + b_2 \qquad y_3 = a_3x_3 + b_3$$

which (according to (12) and (13) are

$$a_2 = \frac{(\alpha_2^{2k}p_1^k\alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k\alpha_3^{2k}p_1^k)}{(\alpha_3^{3k}p_1^k\alpha_1^{2k}p_1^k - \alpha_3^{2k}p_1^k\alpha_1^{3k}p_1^k)} \qquad b_2 = \frac{(\alpha_2^{3k}p_1^k\alpha_1^{2k}p_1^k - \alpha_2^{2k}p_1^k\alpha_1^{3k}p_1^k)}{(\alpha_3^{3k}p_1^k\alpha_1^{2k}p_1^k - \alpha_3^{2k}p_1^k\alpha_1^{3k}p_1^k)}$$

$$a_3 = \frac{(\alpha_1^{2k}p_1^k\alpha_3^{3k}p_1^k - \alpha_1^{3k}p_1^k\alpha_3^{2k}p_1^k)}{(\alpha_3^{3k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{3k}p_1^k)} \qquad b_3 = \frac{(\alpha_3^{2k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{2k}p_1^k)}{(\alpha_3^{3k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{3k}p_1^k)}$$

Then equations (23) become

$$x_3(\beta_3^{1k}p_2^k - x_4\beta_3^{3k}p_2^k) = (\beta_1^{1k}p_2^k - x_4\beta_1^{3k}p_2^k) \tag{24}$$

$$x_3(\beta_3^{2k}p_2^k - y_4\beta_3^{3k}p_2^k) = (\beta_1^{2k}p_2^k - y_4\beta_1^{3k}p_2^k) \tag{25}$$

$$(a_3x_3 + b_3)(\beta_3^{1k}p_2^k - x_4\beta_3^{3k}p_2^k) = (\beta_2^{1k}p_2^k - x_4\beta_2^{3k}p_2^k) \tag{26}$$

$$(a_3x_3 + b_3)(\beta_3^{2k}p_2^k - y_4\beta_3^{3k}p_2^k) = (\beta_2^{2k}p_2^k - y_4\beta_2^{3k}p_2^k) \tag{27}$$

Say we take (24), (26)

$$a_3(\beta_1^{1k}p_2^k - x_4\beta_1^{3k}p_2^k) + b_3(\beta_3^{1k}p_2^k - x_4\beta_3^{3k}p_2^k) = (\beta_2^{1k}p_2^k - x_4\beta_2^{3k}p_2^k)$$

which is an equation with variables x_2 and x_4 only. We can rearrange it to get (15) (but with tensor γ). Then, we do the same with (25), (27), to get an equation with variables x_2 and y_4, to get (16).

In short, by comparing the coefficients we get, there are the following relations:

$$\gamma_1^{1k}p_1^k \cong \beta_2^{12}b_2 + \beta_2^{13} - \beta_1^{12}b_2a_3 - \beta_1^{13}a_3 - \beta_3^{12}b_2b_3 - \beta_3^{13}b_3$$
$$\gamma_3^{1k}p_1^k \cong \beta_1^{11}a_3 + \beta_1^{12}a_2a_3 + \beta_3^{11}b_3 + \beta_3^{12}a_2b_3 - \beta_2^{11} - \beta_2^{12}a_2$$

$$\gamma_1^{3k}p_1^k \cong -\beta_1^{32}b_2a_3 - \beta_1^{33}a_3 - \beta_3^{32}b_2b_3 - \beta_3^{33}b_3 + \beta_2^{32}b_2 + \beta_2^{33}$$
$$\gamma_3^{3k}p_1^k \cong \beta_1^{31}a_3 + \beta_1^{32}a_2a_3 + \beta_3^{31}b_3 + \beta_3^{32}a_2b_3 - \beta_2^{31} - \beta_2^{32}a_2$$
$$\gamma_1^{2k}p_1^k \cong \beta_2^{22}b_2 + \beta_2^{23} - \beta_1^{22}b_2a_3 - \beta_1^{23}a_3 - \beta_3^{22}b_2b_3 - \beta_3^{23}b_3$$
$$\gamma_3^{2k}p_1^k \cong \beta_1^{21}a_3 + \beta_1^{22}a_3a_2 - \beta_3^{21}b_3 + \beta_3^{22}a_2b_3 - \beta_2^{21} - \beta_2^{22}a_2$$
$$\gamma_2^{1k}p_1^k \cong \gamma_1^{1k}p_1^ka_2 + \gamma_3^{1k}p_1^kb_2$$
$$\gamma_2^{3k}p_1^k \cong \gamma_1^{3k}p_1^ka_2 + \gamma_3^{3k}p_1^kb_2$$

($\gamma_2^{2k}p_1^k$ depends on the rest of the coefficients)

for any point p_1^k in image 1 (in fact, tensor α just defines the line coefficients). The scale factor in this case is the same for any point p_1.

Now, by using points (1, 0, 0), (0, 1, 0), (0, 0, 1) the coefficients of γ can directly be estimated. There are, though, three groups of coefficients which depend on the value of k.

For instance γ_3^{33} :

$$a_2^{(0,0,1)} = \frac{(\alpha_2^{23}\alpha_3^{33} - \alpha_2^{33}\alpha_3^{23})}{(\alpha_3^{33}\alpha_1^{23} - \alpha_3^{23}\alpha_1^{33})} \qquad b_2^{(0,0,1)} = \frac{(\alpha_2^{33}\alpha_1^{23} - \alpha_2^{23}\alpha_1^{33})}{(\alpha_3^{33}\alpha_1^{23} - \alpha_3^{23}\alpha_1^{33})}$$
$$a_3^{(0,0,1)} = \frac{(\alpha_1^{23}\alpha_3^{33} - \alpha_1^{33}\alpha_3^{23})}{(\alpha_3^{33}\alpha_1^{13} - \alpha_3^{13}\alpha_1^{33})} \qquad b_3^{(0,0,1)} = \frac{(\alpha_3^{23}\alpha_1^{13} - \alpha_3^{13}\alpha_1^{23})}{(\alpha_3^{33}\alpha_1^{13} - \alpha_3^{13}\alpha_1^{33})}$$

$$\gamma_3^{33} \cong \beta_1^{31}a_3^{(0,0,1)} + \beta_1^{32}a_2^{(0,0,1)}a_3^{(0,0,1)}$$
$$+\beta_3^{31}b_3^{(0,0,1)} + \beta_3^{32}a_2^{(0,0,1)}b_3^{(0,0,1)} - \beta_2^{31} - \beta_2^{32}a_2^{(0,0,1)}$$

Now, since the values of $\gamma_2^{3k}p_1^k$, we can use any point to find the right scale factors as follows. First, in case we do not have any point on the third image (image number 4), it is easy to get one by using (15) and (16), for instance with point p_1 = (0,1,1) and x_2 = 0. The values are put in any of the trilinearities and the scale factor corresponding to k=2 is estimated. The same should be done with point p_1 = (1,0,1).

6.6 Finding the epipole with two points using the tensor

In this section I show two results: given two points which belong to the same epipolar line, it is possible first to determine, without obtaining the epipole, if the epipole lies between them or which one is closer to it and second, the epipole can be obtained if necessary.

Say we have two images a and b and two points in image a which we believe belong to the same epipolar line. Let $v_{ab} = (x_{ab}, y_{ab})$ be the (unknown) epipole between image a and b and let sx and sy be the slopes of the epipolar line. Thus, every point in that epipolar line can be expressed as $p_a = (x_a, y_a) = (x_{ab} + n_k sx, y_{ab} + n_k sy)$, where n_k is a scalar; if $n_k = 0$ the point in question is the (unknown) epipole.

Say the tensor $T_{a,b,c} = \alpha_i^{jk}$ is at hand and we obtain the epipolar lines from the trilinearities (as in (11), (12), (13), (14)):

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)y_b + (\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)x_b + (\alpha_2^{3k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_2^{1k}p_1^k * \alpha_1^{3k}p_1^k) = 0$$

$$(\alpha_3^{2k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{2k}p_1^k)y_b + (\alpha_2^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{2k}p_1^k)x_b + (\alpha_2^{3k}p_1^k * \alpha_1^{2k}p_1^k - \alpha_2^{2k}p_1^k * \alpha_1^{3k}p_1^k) = 0$$

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)y_c + (\alpha_1^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_1^{3k}p_1^k * \alpha_3^{2k}p_1^k)x_c + (\alpha_3^{2k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k * \alpha_1^{2k}p_1^k) = 0$$

$$(\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)y_c + (\alpha_3^{2k}p_1^k * \alpha_2^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_2^{2k}p_1^k)x_c (\alpha_2^{2k}p_1^k * \alpha_3^{1k}p_1^k - \alpha_3^{2k}p_1^k * \alpha_2^{1k}p_1^k) = 0$$

Let us look for instance at the first epipolar line in image b from a point in image a :

$$(\alpha_3^{1k}p_a^k * \alpha_1^{3k}p_a^k - \alpha_3^{3k}p_a^k * \alpha_1^{1k}p_a^k)y_b + (\alpha_2^{1k}p_a^k * \alpha_3^{3k}p_a^k - \alpha_2^{3k}p_a^k * \alpha_3^{1k}p_a^k)x_b (\alpha_2^{3k}p_a^k * \alpha_1^{1k}p_a^k - \alpha_2^{1k}p_a^k * \alpha_1^{3k}p_a^k)$$

If p_a were the epipole between images a and b, then, instead of getting a line in b we would have obtained a point and therefore every coefficient of the line would have been zero. In general, they will not be null:

$$\begin{aligned}
\alpha_3^{1k} p_a^k * \alpha_1^{3k} p_a^k - \alpha_3^{3k} p_a^k * \alpha_1^{1k} p_a^k &= v1_k \\
\alpha_2^{1k} p_a^k * \alpha_3^{3k} p_a^k - \alpha_2^{3k} p_a^k * \alpha_3^{1k} p_a^k &= v2_k \\
\alpha_2^{3k} p_a^k * \alpha_1^{1k} p_a^k - \alpha_2^{1k} p_a^k * \alpha_1^{3k} p_a^k &= v3_k
\end{aligned}$$

Each of these coefficients can be written as a product of two vectors (for instance the first one).

$$\begin{gathered}
(\alpha_3^{1k} p_a^k * \alpha_1^{3k} p_a^k - \alpha_3^{3k} p_a^k * \alpha_1^{1k} p_a^k) = p_a^k (\alpha_3^{1k} * \alpha_1^{3k} - \alpha_3^{3k} * \alpha_1^{1k}) p_a^k = \\
M_{31,13,33,11}^t (x_a^2, y_a^2, x_a y_a, x_a, y_a, 1) = v1_k
\end{gathered}$$

where $M_{33,11,31,13}$ is a 6×1 vector of $(\alpha_3^{1k})^t * \alpha_1^{3k} - (\alpha_3^{3k})^t * \alpha_1^{1k}$ conveniently arranged (see Appendix I, (??)):

$$\begin{aligned}
M_{31,13,33,11}^t &= M1 \qquad & M1(x_a^2, y_a^2, x_a y_a, x_a, y_a, 1) &= v1_k \\
M_{21,33,23,31}^t &= M2 \qquad & M2(x_a^2, y_a^2, x_a y_a, x_a, y_a, 1) &= v2_k \\
M_{23,11,21,13}^t &= M3 \qquad & M3(x_a^2, y_a^2, x_a y_a, x_a, y_a, 1) &= v3_k
\end{aligned}$$

Let take any of the Mj j=1,2,3 matrices. Then, if the point were the epipole

$$Mj(x_{ab}^2, y_{ab}^2, x_{ab} y_{ab}, x_{ab}, y_{ab}, 1) = 0$$

therefore, for any point

$$Mj(x_a^2, y_a^2, x_a y_a, x_a, y_a, 1) =$$
$$Mj((x_{ab} + n_k sx)^2, (y_{ab} + n_k sy)^2, (x_{ab} + n_k sx)(y_{ab} + n_k sy),$$
$$x_{ab} + n_k sx, y_{ab} + n_k sy, 1) =$$
$$Mj(x_{ab}^2, y_{ab}^2, x_{ab} y_{ab}, x_{ab}, y_{ab}, 1) +$$
$$Mj((n_k sx)^2 + 2x_{ab} n_k sx, (n_k sy)^2 + 2y_{ab} n_k sy,$$
$$n_k sx y_{ab} + n_k sy x_{ab} + n_k^2 sxsy, n_k sx, n_k sy, 0) =$$
$$Mj((n_k sx)^2 + 2x_{ab} n_k sx, (n_k sy)^2 + 2y_{ab} n_k sy,$$
$$n_k sx y_{ab} + n_k sy x_{ab} + n_k^2 sxsy, n_k sx, n_k sy, 0) = vj_k$$

For any point we have three equations (j=1,2,3) whose unknowns are x_{ab}, y_{ab} and n_k. The question is whether or not we can obtain these values (since the equations are not linear). If Mj = $(Mj_1, Mj_2, Mj_3, Mj_4, Mj_5, Mj_6)$ then

$$Mj_1(n_k sx)^2 + 2Mj_1 x_{ab} n_k sx + Mj_2(n_k sy)^2 + 2Mj_2 y_{ab} n_k sy + Mj_3 n_k sx y_{ab} +$$
$$Mj_3 n_k sy x_{ab} + Mj_3 n_k^2 sxsy + Mj_4 n_k sx + Mj_5 n_k sy - vj_k$$

$$n_k^2(Mj_1 sx^2 + Mj_2 sy^2 + Mj_3 sxsy) + n_k x_{ab}(2Mj_1 sx + Mj_3 sy)$$
$$+n_k y_{ab}(2Mj_2 sy + Mj_3 sx) + n_k(Mj_4 sx + Mj_5 sy) - vj_k = 0$$
$$n_k^2 aj + n_k x_{ab} bj + n_k y_{ab} cj + n_k dj - vj_k = 0$$

where

$$aj = Mj_1 sx^2 + Mj_2 sy^2 + Mj_3 sxsy \qquad bj = Mj_1 2sx + Mj_3 sy$$
$$cj = Mj_2 2sy + Mj_3 sx \qquad dj = Mj_4 sx + Mj_5 sy \quad j = 1, 2, 3$$

As the solution must be unique, the quadratic coefficient must equal zero and we can get a unique value for the unknowns. Using Maple, we can find the value of n_k without computing the epipole.

$$n_k = \frac{\begin{vmatrix} b1 & b2 & b3 \\ c1 & c2 & c3 \\ v1_k & v2_k & v3_k \end{vmatrix}}{\begin{vmatrix} b1 & b2 & b3 \\ c1 & c2 & c3 \\ d1 & d2 & d3 \end{vmatrix}}$$

The value of n_k expresses the distance of the point to the epipole, according to the slope of the line. If the signs of two n_k are different, the epipole lies in between; if the same sign, then the absolute value of n_k gives idea of the distance: the closest point has the smaller number.

The values of the coordinates of the epipole are as follows:

$$x_{ab} = \frac{(a2c1 - c2a1)n_k^2 + (d2c1 - c2d1)n_k + (c2v1 - v2c1)}{(b1c2 - c1b2)n_k}$$

$$y_{ab} = \frac{(a1b2 - b1a2)n_k^2 + (d1b2 - b1d2)n_k + (b1v2 - v1b2)}{(b1c2 - c1b2)n_k};$$

If just one point, the slopes of the line and the tensor α are known, the epipole itself can be estimated if required.

6.7 The epipoles from the coefficients of the tensor

In this section I show how the coordinates of the epipoles in images 2 and 3 can be easily obtained from the corresponding tensor $\mathrm{T}_{1,2,3} = \alpha_i^{jk}$.

We know that every epipolar line passes through the epipole. Therefore, this point on the second image, for example, does not depend on the corresponding projection value of the point in the third image. Let assume the epipole does not lie at infinity and let us look at the formulas we have from the tensor (see (17), (18), (21), (22)):

$$x_2 = \frac{\alpha_1^{1k}p_1^k - \alpha_1^{3k}p_1^k x_3}{\alpha_3^{1k}p_1^k - \alpha_3^{3k}p_1^k x_3} \qquad y_2 = \frac{\alpha_2^{1k}p_1^k - \alpha_2^{3k}p_1^k x_3}{\alpha_3^{1k}p_1^k - \alpha_3^{3k}p_1^k x_3}$$

$$x_2 = \frac{\alpha_1^{2k} p_1^k - \alpha_1^{3k} p_1^k y_3}{\alpha_3^{2k} p_1^k - \alpha_3^{3k} p_1^k y_3} \qquad y_2 = \frac{\alpha_2^{2k} p_1^k - \alpha_2^{3k} p_1^k y_3}{\alpha_3^{2k} p_1^k - \alpha_3^{3k} p_1^k y_3}$$

Now, say we are looking for the x coordinate of the epipole in the second image, we can get rid of points in the third image if we find p_1^k such that:

a) $\alpha_1^{3k} p_1^k$ and $\alpha_3^{3k} p_1^k$ equal 0 (intersection of two lines) or

b) $\alpha_1^{1k} p_1^k$ and $\alpha_3^{1k} p_1^k$ equal 0 or

c) $\alpha_1^{2k} p_1^k$ and $\alpha_3^{2k} p_1^k$ equal 0.

which give us a vertical[1] line in the second image because the value of x_2 is determined and the value of y_2 depends on x_3 or y_3. This gives us three points in the first image and four ways (2+1+1) of finding the x coordinate of this epipole. This vertical line in the second image is the intersection of the image plane with a plane which contains the center of projection of views 1 and 2 (see Figure 9).

Now, we can do the same with the formulas where y_2 appears, in order to find the horizontal epipolar line in the second image.

If now we transform again the equations for the coordinates x_3 and y_3 (see (15), (16), (19) and (20)):

$$x_3 = \frac{\alpha_1^{1k} p_1^k - \alpha_3^{1k} p_1^k x_2}{\alpha_1^{3k} p_1^k - \alpha_3^{3k} p_1^k x_2} \qquad y_3 = \frac{\alpha_1^{2k} p_1^k - \alpha_3^{2k} p_1^k x_2}{\alpha_1^{3k} p_1^k - \alpha_3^{3k} p_1^k x_2}$$

$$x_3 = \frac{\alpha_2^{1k} p_1^k - \alpha_3^{1k} p_1^k y_2}{\alpha_2^{3k} p_1^k - \alpha_3^{3k} p_1^k y_2} \qquad y_3 = \frac{\alpha_2^{2k} p_1^k - \alpha_3^{2k} p_1^k y_2}{\alpha_2^{3k} p_1^k - \alpha_3^{3k} p_1^k y_2}$$

Say we are looking for the epipole in the third image, we can get rid of points in the second image if we find p_1^k such that:

a) $\alpha_3^{1k} p_1^k$ and $\alpha_3^{3k} p_1^k$ equal 0 or

b) $\alpha_1^{1k} p_1^k$ and $\alpha_1^{3k} p_1^k$ equal 0 or

c) $\alpha_2^{1k} p_1^k$ and $\alpha_2^{3k} p_1^k$ equal 0.

which give us a vertical line in the third image because the value of x_3 is

[1] when we say vertical(horizontal), we mean the line parallel to the abcisa(ordinate) of the image, even if they are not perpendicular

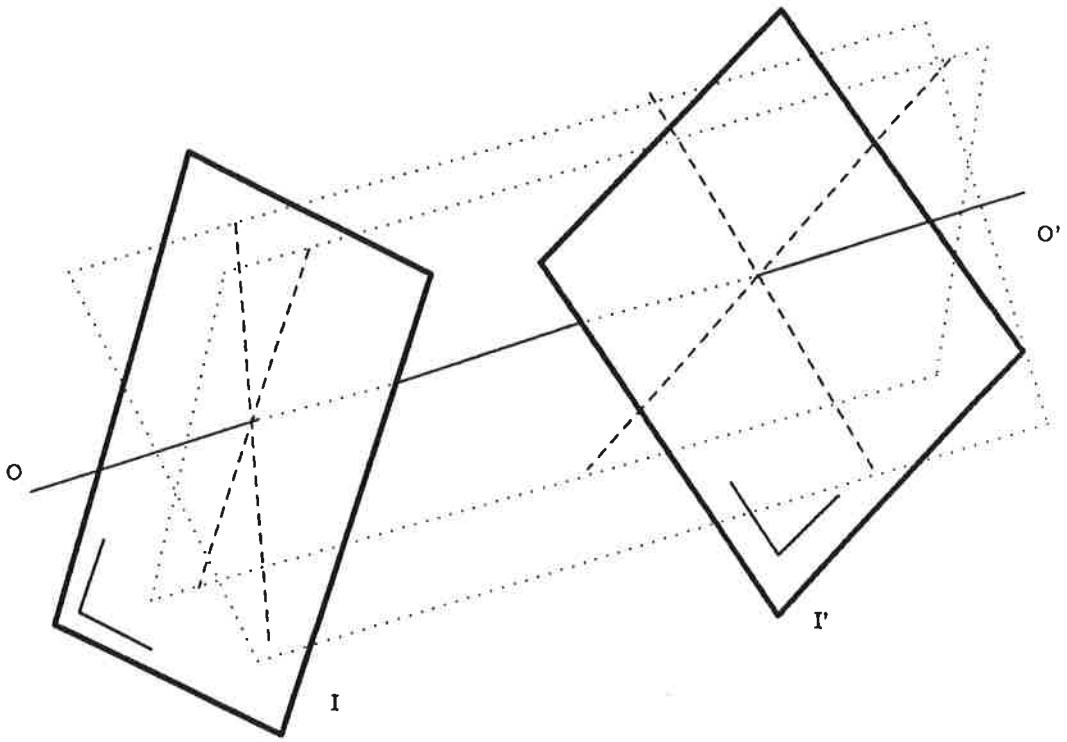

Figure 9: Vertical and horizontal lines in the second image

determined and the value of y_3 depends on x_2 or y_2. This give us again three points and four ways (2+1+1) of finding the x coordinate of this epipole.

We can use the same idea to find points whose epipolar line in the third image is horizontal and therefore the y value of the epipole.

The expression of points and epipoles are written in Appendix F.

In any case we have (two sets of) three points in the first image that produce the same vertical (horizontal) epipolar line in the second (third) image. In consequence, all three points lie on the same epipolar line in the first image; this line is the intersection of a plane which passes through the

line connecting the center of views of the first image and the second (or the third) and which is perpendicular (vertical or horizontal) to the coordinate system of the second (third) image. Therefore, the epipole in the first image can be found by intersecting these two lines (of three points each, although only two of them are necessary). This is numerically very simple. Since we have three points in each line, we have in fact three lines for the vertical epipolar line and another three for the horizontal epipolar line and therefore, nine possible ways to find the epipole of the second image onto the first. In case one these points were the actual epipole, we have a way to check it with the other two.

6.8 Lines defined by the tensor

Given a fundamental matrix between two images $F_{1,2}$,

$$(x_2 \quad y_2 \quad 1) \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = 0$$

We can see that this structure naturally defines three epipolar lines in each of the images:

a) the horizontal epipolar line in image 2 is the corresponding epipolar line $\gamma_{11}\, x_1 + \gamma_{12}\, y_1 + \gamma_{13}$ in image 1.

b) the vertical epipolar line in image 2 is the corresponding epipolar line $\gamma_{21}\, x_1 + \gamma_{22}\, y_1 + \gamma_{23}$ in image 1 and

c) the epipolar line in image 2 that passes through (0,0) is the corresponding epipolar line $\gamma_{31}\, x_1 + \gamma_{32}\, y_1 + \gamma_{33}$ in image 1.

If we exchange the indices of every γ_{ij}, we get the corresponding epipolar lines in image 2. Thus, each line (column) of the fundamental matrix defines an epipolar line in image 1 (2) and therefore these lines must intersect in the epipole of that image.

In Section 6.7, we have shown that it is possible to find some specific points in the first image that belong to the same epipolar lines.

Now, say we interpret that the coefficients of a tensor actually define lines in the first image (α_a^{bk} defines the line α_a^{b1} x + α_a^{b2} y + α_a^{b3} = 0, a,b = 1,2,3) and the epipolar lines in the second and third images:

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)yk_b + (\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)xk_b$$
$$(\alpha_2^{3k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_2^{1k}p_1^k * \alpha_1^{3k}p_1^k) = 0$$

$$(\alpha_3^{2k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{2k}p_1^k)yk_b + (\alpha_2^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{2k}p_1^k)xk_b +$$
$$(\alpha_2^{3k}p_1^k * \alpha_1^{2k}p_1^k - \alpha_2^{2k}p_1^k * \alpha_1^{3k}p_1^k) = 0$$

$$(\alpha_3^{1k}p_1^k * \alpha_1^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_1^{1k}p_1^k)yk_c + (\alpha_1^{2k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_1^{3k}p_1^k * \alpha_3^{2k}p_1^k)xk_c +$$
$$(\alpha_3^{2k}p_1^k * \alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k * \alpha_1^{2k}p_1^k) = 0$$

$$(\alpha_2^{1k}p_1^k * \alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k * \alpha_3^{1k}p_1^k)yk_c + (\alpha_3^{2k}p_1^k * \alpha_2^{3k}p_1^k - \alpha_3^{3k}p_1^k * \alpha_2^{2k}p_1^k)xk_c +$$
$$(\alpha_2^{2k}p_1^k * \alpha_3^{1k}p_1^k - \alpha_3^{2k}p_1^k * \alpha_2^{1k}p_1^k) = 0$$

As seen before, we can take some of the intersections of these lines (see Figure 11) :

* the vertical epipolar line in the second image (x coordinate): its corresponding epipolar line in the first image is a line passing through the following points:

1) intersection of lines α_1^{3k} and α_3^{3k}

2) intersection of lines α_1^{1k} and α_3^{1k}

3) intersection of lines α_1^{2k} and α_3^{2k}

* the horizontal epipolar line in the second image (y coordinate): its corresponding epipolar line in the first image is a line passing through the following points:

4) intersection of lines α_2^{3k} and α_3^{3k}

5) intersection of lines α_2^{1k} and α_3^{1k}

6) intersection of lines α_2^{2k} and α_3^{2k}

* the vertical epipolar line in the third image (x coordinate): its corresponding epipolar line in the first image is a line passing through the following points:

7) intersection of lines α_3^{1k} and α_3^{3k}

8) intersection of lines α_1^{1k} and α_1^{3k}

9) intersection of lines α_2^{1k} and α_2^{3k}

* the horizontal epipolar line in the second image (y coordinate): its corresponding epipolar line in the first image is a line passing through the following points:

10) intersection of lines α_3^{2k} and α_3^{3k}

11) intersection of lines α_1^{2k} and α_1^{3k}

12) intersection of lines α_2^{2k} and α_2^{3k}

Now we can see by inspection of the equations of the epipolar lines that there are two more lines defined by any tensor. They are the epipolar lines whose corresponding epipolar line (in 2 and 3) passes through the center of the coordinate system (in other words, point (0,0)) (see Figure 10).

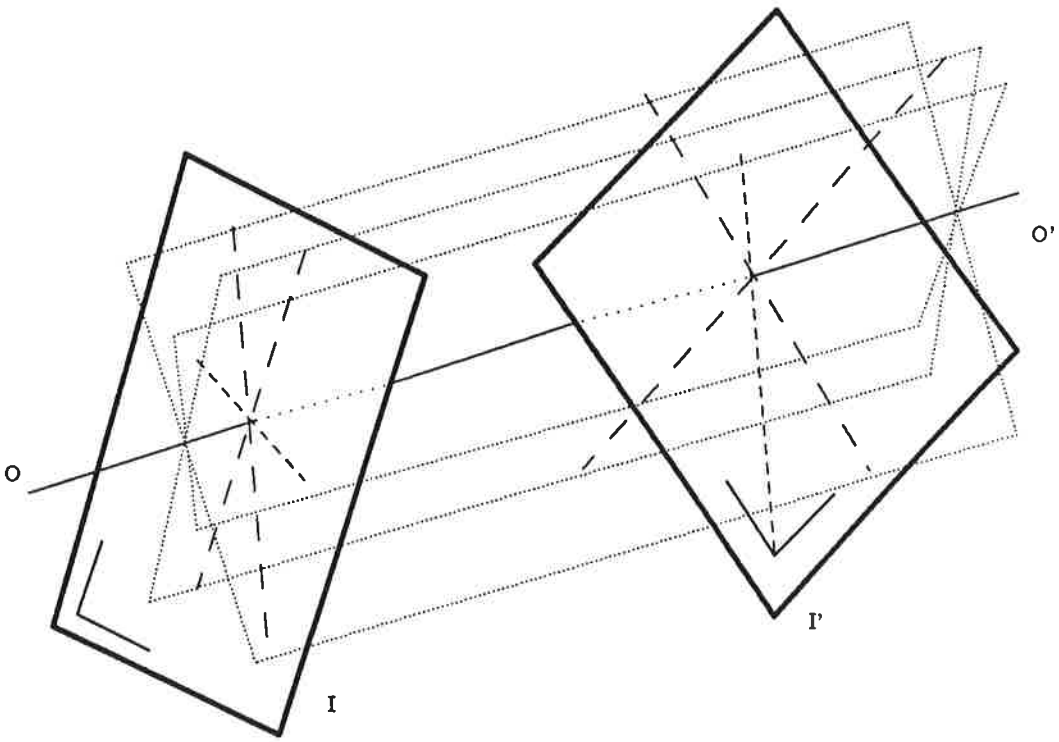

Figure 10: Vertical and horizontal lines and line passing through (0,0) in the second image

* the epipolar line in the second image that passes through (0,0): its corresponding epipolar line in the first image is a line passing through the following points:

13) intersection of lines α_1^{3k} and α_2^{3k}

14) intersection of lines α_1^{1k} and α_2^{1k}

15) intersection of lines α_1^{2k} and α_2^{2k}

* the epipolar line in the third image that passes through (0,0): its corresponding epipolar line in the first image is a line passing through the following points:

16) intersection of lines α_1^{1k} and α_1^{2k}

17) intersection of lines α_2^{1k} and α_2^{2k}

18) intersection of lines α_3^{1k} and α_3^{2k}

Say we take the dual (ie lines become points and vice versa) and represent this structure (see Figure 13). As we can see, each point (representing the epipolar lines) follows this invariant in P^2: Given three lines l_1, l_2, l_3 that intersect in one point E, if we take two points in each line and intersect the lines defined by them, the three new points A, B and C lie in a line (see Figure 12).

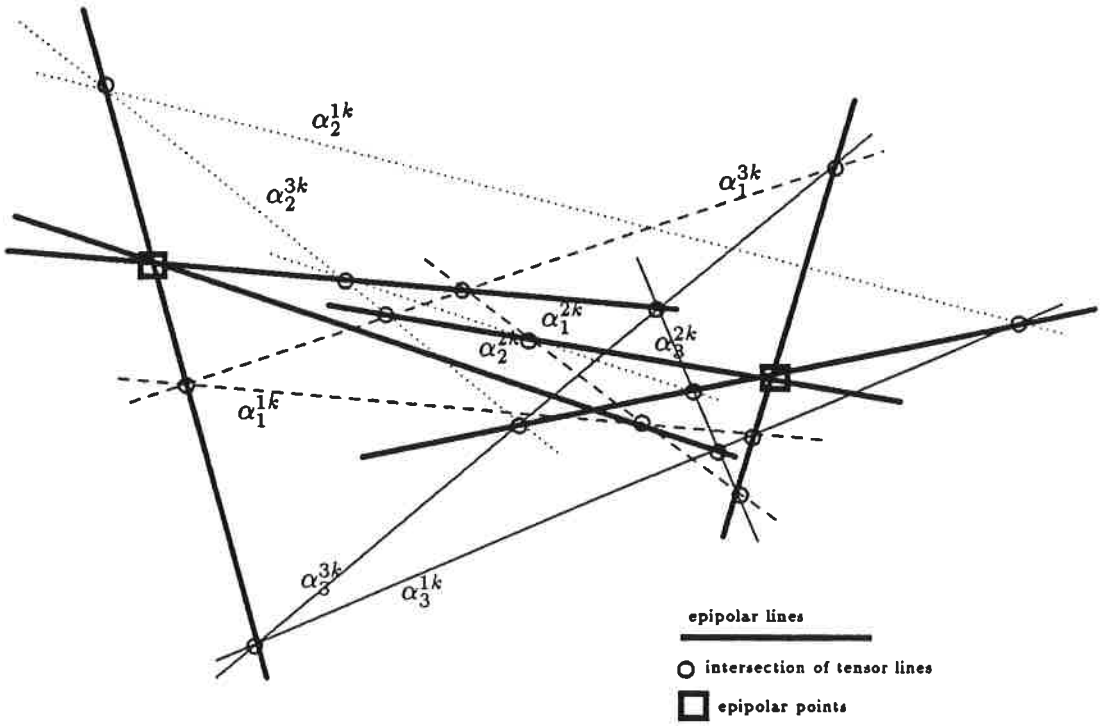

Figure 11: Lines defined by the tensor in the first image

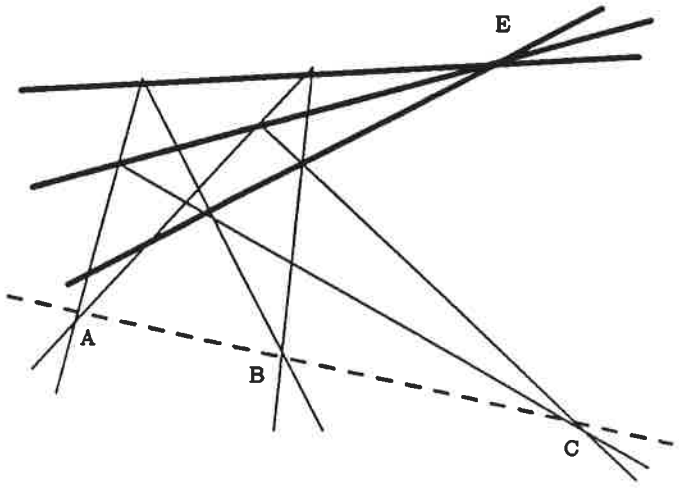

Figure 12: Invariant in P^2

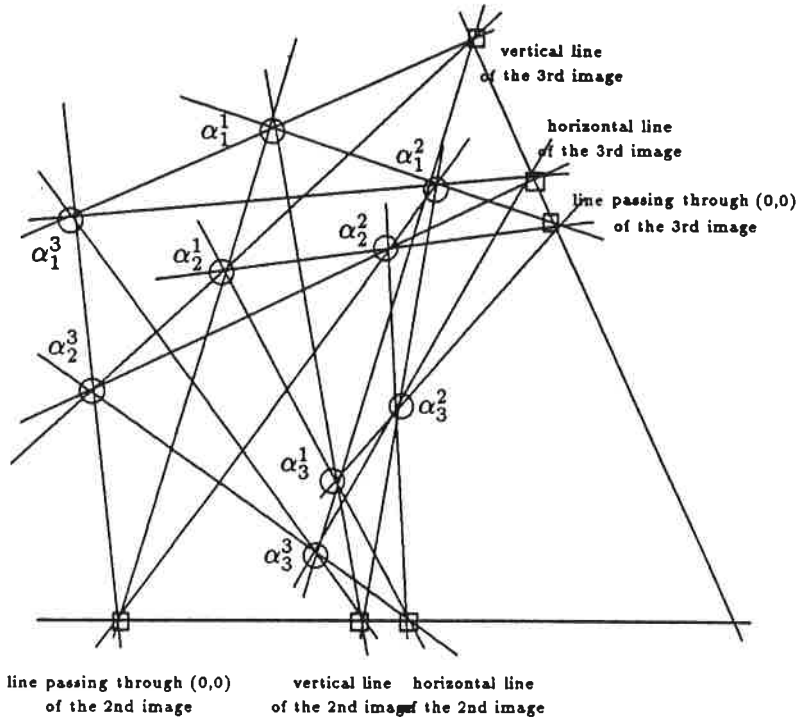

Figure 13: Lines defined by a tensor (dual)

As seen, these intersections define 18 lines in the first image; every line (α_i^j) intersects exactly four times with other lines, and, for their part, the lines defined by these points give as a result six epipolar lines that define the epipoles of image 2 and 3 in image 1; as we can see, both images are treated alike.

As seen, these intersections define 18 lines in the first image; every line (α_i^j) intersects exactly four times with other lines, and, for their part, the lines defined by these points give as a result six epipolar lines that define the epipolar points of image 2 and 3 in image 1; as we can see, both images are treated alike.

7 Quadrilinear constraints

Finally, let us take the first four images to describe the quadrilinear constraints. As said in [Tri95], the elements of the fundamental matrix are 4×4 determinants as shown in Appendix G.

Following [Sha95], if the projection matrices are [I,0], [A,v_{21}], [B,v_{31}], [C,v_{41}]:

$$\lambda_1\ p_1 = [I,0]\ P \quad \lambda_2\ p_2 = [A,v_{21}]\ P \quad \lambda_3\ p_3 = [B,v_{31}]\ P \quad \lambda_4\ p_4 = [C,v_{41}]\ P$$

then the quadrilinear tensor is an array of 81 entries, each of them is the determinant of a 3×3 matrix:

$$\omega_{i,j,k,l} = \begin{vmatrix} a_{il_1} & a_{il_2} & v_{21}^i \\ b_{jl_1} & b_{jl_2} & v_{31}^j \\ c_{kl_1} & c_{kl_2} & v_{41}^k \end{vmatrix} \qquad i,j,k,l = 1,2,3$$

$$\begin{array}{cccc} l & \Rightarrow & l_1 & l_2 \\ 1 & \Rightarrow & 1 & 2 \\ 2 & \Rightarrow & 1 & 3 \\ 3 & \Rightarrow & 2 & 3 \end{array}$$

Finally, using Maple, we have found the linear expression of the 16 independent equations of the quadrilinear constraints (shown in Appendix ?? where only the coordinates of the correspondences are needed. As each correspondence provides 16 equations and there are 81 unknowns, we need 5 or more correspondences to find these coefficients.

A Basic equations

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & x_1 & 0 & 0 & 0 \\ e_{21} & e_{22} & e_{23} & e_{24} & y_1 & 0 & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} & 1 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & 0 & x_2 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & y_2 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 1 & 0 & 0 \\ b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0 & x_3 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & 0 & 0 & y_3 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 & 1 & 0 \\ c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 & 0 & x_4 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 & 0 & y_4 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \\ -\lambda_1 \\ -\lambda_2 \\ -\lambda_3 \\ -\lambda_4 \end{pmatrix} = 0$$

B Fundamental matrix

$$F_{1,2} = \begin{bmatrix} [2,3,5,6] & -[1,3,5,6] & [1,2,5,6] \\ -[2,3,4,6] & [1,3,4,6] & -[1,2,4,6] \\ [2,3,4,5] & -[1,3,4,5] & [1,2,4,5] \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

C Nonlinear constraints

of the coefficients of the fundamental matrix:

$$\frac{\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}}{\gamma_{21}\gamma_{33} - \gamma_{23}\gamma_{31}} = \frac{\gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{32}}{\gamma_{11}\gamma_{33} - \gamma_{13}\gamma_{31}} = \frac{\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}}{\gamma_{11}\gamma_{23} - \gamma_{13}\gamma_{21}} = \frac{v_{21}^1}{v_{21}^2}$$

$$\frac{\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}}{\gamma_{21}\gamma_{32} - \gamma_{22}\gamma_{31}} = \frac{\gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{32}}{\gamma_{11}\gamma_{32} - \gamma_{12}\gamma_{31}} = \frac{\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}}{\gamma_{22}\gamma_{22} - \gamma_{12}\gamma_{21}} = \frac{v_{21}^1}{v_{21}^3}$$

$$\frac{\gamma_{21}\gamma_{32} - \gamma_{22}\gamma_{31}}{\gamma_{21}\gamma_{33} - \gamma_{23}\gamma_{31}} = \frac{\gamma_{11}\gamma_{32} - \gamma_{12}\gamma_{31}}{\gamma_{11}\gamma_{33} - \gamma_{13}\gamma_{31}} = \frac{\gamma_{22}\gamma_{22} - \gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{23} - \gamma_{13}\gamma_{21}} = \frac{v_{21}^2}{v_{21}^3}$$

$$\frac{\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{32}}{\gamma_{12}\gamma_{33}-\gamma_{13}\gamma_{32}}=\frac{\gamma_{21}\gamma_{33}-\gamma_{23}\gamma_{31}}{\gamma_{11}\gamma_{33}-\gamma_{13}\gamma_{31}}=\frac{\gamma_{21}\gamma_{32}-\gamma_{22}\gamma_{31}}{\gamma_{11}\gamma_{32}-\gamma_{12}\gamma_{31}}=\frac{v_{12}^1}{v_{12}^2}$$
$$\frac{\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{32}}{\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}}=\frac{\gamma_{21}\gamma_{33}-\gamma_{23}\gamma_{31}}{\gamma_{11}\gamma_{23}-\gamma_{13}\gamma_{21}}=\frac{\gamma_{21}\gamma_{32}-\gamma_{22}\gamma_{31}}{\gamma_{22}\gamma_{22}-\gamma_{12}\gamma_{21}}=\frac{v_{12}^1}{v_{12}^3}$$
$$\frac{\gamma_{12}\gamma_{33}-\gamma_{13}\gamma_{32}}{\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}}=\frac{\gamma_{11}\gamma_{33}-\gamma_{13}\gamma_{31}}{\gamma_{11}\gamma_{23}-\gamma_{13}\gamma_{21}}=\frac{\gamma_{11}\gamma_{32}-\gamma_{12}\gamma_{31}}{\gamma_{22}\gamma_{22}-\gamma_{12}\gamma_{21}}=\frac{v_{12}^2}{v_{12}^3}$$

D Trilinear constraints

$$T_{i,j,k}=\begin{bmatrix} [2,3,4,7] & [1,3,4,7] & [1,2,4,7] \\ [2,3,4,8] & [1,3,4,8] & [1,2,4,8] \\ -[2,3,4,9] & -[1,3,4,9] & -[1,2,4,9] \\ [2,3,5,7] & [1,3,5,7] & [1,2,5,7] \\ [2,3,5,8] & [1,3,5,8] & [1,2,5,8] \\ -[2,3,5,9] & -[1,3,5,9] & -[1,2,5,9] \\ -[2,3,6,7] & -[1,3,6,7] & -[1,2,6,7] \\ -[2,3,6,8] & -[1,3,6,8] & -[1,2,6,8] \\ [2,3,6,9] & [1,3,6,9] & [1,2,6,9] \end{bmatrix} = \begin{bmatrix} \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \\ \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{bmatrix}$$

E Constrains of the tensor

E.1 From Homographies E_j, W_i T_k

Let call E the homographies from the first to the second view, and W those from the first to the third and T matrices:

$$E_j=\begin{bmatrix} \alpha_1^{j1} & \alpha_1^{j2} & \alpha_1^{j3} \\ \alpha_2^{j1} & \alpha_2^{j2} & \alpha_2^{j3} \\ \alpha_3^{j1} & \alpha_3^{j2} & \alpha_3^{j3} \end{bmatrix} \qquad W_i=\begin{bmatrix} \alpha_i^{11} & \alpha_i^{12} & \alpha_i^{13} \\ \alpha_i^{21} & \alpha_i^{22} & \alpha_i^{23} \\ \alpha_i^{31} & \alpha_i^{32} & \alpha_i^{33} \end{bmatrix} \qquad T_k=\begin{bmatrix} \alpha_1^{1k} & \alpha_1^{2k} & \alpha_1^{3k} \\ \alpha_2^{1k} & \alpha_2^{2k} & \alpha_2^{3k} \\ \alpha_3^{1k} & \alpha_3^{2k} & \alpha_3^{3k} \end{bmatrix}$$

The constrains are as follows

$$\text{with } E_j: \quad \begin{vmatrix} \alpha_1^{11} & \alpha_1^{21} & \alpha_1^{31} \\ \alpha_2^{11} & \alpha_2^{21} & \alpha_2^{31} \\ \alpha_3^{11} & \alpha_3^{21} & \alpha_3^{31} \end{vmatrix} = 0 \quad \begin{vmatrix} \alpha_1^{12} & \alpha_1^{22} & \alpha_1^{32} \\ \alpha_2^{12} & \alpha_2^{22} & \alpha_2^{32} \\ \alpha_3^{12} & \alpha_3^{22} & \alpha_3^{32} \end{vmatrix} = 0, \quad \begin{vmatrix} \alpha_1^{13} & \alpha_1^{23} & \alpha_1^{33} \\ \alpha_2^{13} & \alpha_2^{23} & \alpha_2^{33} \\ \alpha_3^{13} & \alpha_3^{23} & \alpha_3^{33} \end{vmatrix} = 0$$

$$\text{with } W_i: \quad \begin{vmatrix} \alpha_1^{11} & \alpha_2^{11} & \alpha_3^{11} \\ \alpha_1^{21} & \alpha_2^{21} & \alpha_3^{21} \\ \alpha_1^{31} & \alpha_2^{31} & \alpha_3^{31} \end{vmatrix} = 0 \quad \begin{vmatrix} \alpha_1^{12} & \alpha_2^{12} & \alpha_3^{12} \\ \alpha_1^{22} & \alpha_2^{22} & \alpha_3^{22} \\ \alpha_1^{32} & \alpha_2^{32} & \alpha_3^{32} \end{vmatrix} = 0 \quad \begin{vmatrix} \alpha_3^{13} & \alpha_3^{13} & \alpha_3^{13} \\ \alpha_3^{23} & \alpha_3^{23} & \alpha_3^{23} \\ \alpha_3^{33} & \alpha_3^{33} & \alpha_3^{33} \end{vmatrix} = 0$$

$$\text{with } T_i: \quad \begin{vmatrix} \alpha_1^{11} & \alpha_1^{21} & \alpha_1^{31} \\ \alpha_2^{11} & \alpha_2^{21} & \alpha_2^{31} \\ \alpha_3^{11} & \alpha_3^{21} & \alpha_3^{31} \end{vmatrix} = 0 \quad \begin{vmatrix} \alpha_1^{12} & \alpha_1^{22} & \alpha_1^{32} \\ \alpha_2^{12} & \alpha_2^{22} & \alpha_2^{32} \\ \alpha_3^{12} & \alpha_3^{22} & \alpha_3^{32} \end{vmatrix} = 0 \quad \begin{vmatrix} \alpha_1^{13} & \alpha_1^{23} & \alpha_1^{33} \\ \alpha_2^{13} & \alpha_2^{23} & \alpha_2^{33} \\ \alpha_3^{13} & \alpha_3^{23} & \alpha_3^{33} \end{vmatrix} = 0$$

but there are only three of them

$$\begin{vmatrix} \alpha_1^{1k} & \alpha_1^{2k} & \alpha_1^{3k} \\ \alpha_2^{1k} & \alpha_2^{2k} & \alpha_2^{3k} \\ \alpha_3^{1k} & \alpha_3^{2k} & \alpha_3^{3k} \end{vmatrix} = 0$$

$$\begin{aligned} &\alpha_1^{1k}\alpha_2^{2k}\alpha_3^{3k} + \alpha_2^{1k}\alpha_3^{2k}\alpha_1^{3k} + \alpha_3^{1k}\alpha_1^{2k}\alpha_2^{3k} \\ &-\alpha_3^{1k}\alpha_2^{2k}\alpha_1^{3k} - \alpha_1^{1k}\alpha_3^{2k}\alpha_2^{3k} - \alpha_2^{1k}\alpha_1^{2k}\alpha_3^{3k} = 0 \end{aligned} \tag{28}$$

Equation (28) means that a row or a column in T_k matrices is a linear combination of the other two.

The derivation is as follows:

From equations (29) and (30) we get the following constrain:

$$\begin{aligned} &\alpha_1^{1k}p_1^k\alpha_2^{2k}p_1^k\alpha_3^{3k}p_1^k + \alpha_2^{1k}p_1^k\alpha_3^{2k}p_1^k\alpha_1^{3k}p_1^k + \alpha_3^{1k}p_1^k\alpha_1^{2k}p_1^k\alpha_2^{3k}p_1^k - \\ &\alpha_3^{1k}p_1^k\alpha_2^{2k}p_1^k\alpha_1^{3k}p_1^k - \alpha_1^{1k}p_1^k\alpha_3^{2k}p_1^k\alpha_2^{3k}p_1^k - \alpha_2^{1k}p_1^k\alpha_1^{2k}p_1^k\alpha_3^{3k}p_1^k = 0 \end{aligned}$$

Now, if we expand this product and group the terms (x_1^3, y_1^3, $x_1^2y_1$, $x_1y_1^2$, x_1^2, y_1^2, x_1y_1, x_1, y_1 and constant, we get a vector in which each of its 10 elements must be zero (because the product is null to any point p_1).

Let $(E_j)_m$ denote the mth column of matrix E_j.

These elements are as follows:

A) (coefficient x_1^3)

$$0 = \begin{vmatrix} \alpha_1^{11} & \alpha_1^{21} & \alpha_1^{31} \\ \alpha_2^{11} & \alpha_2^{21} & \alpha_2^{31} \\ \alpha_3^{11} & \alpha_3^{21} & \alpha_3^{31} \end{vmatrix}$$

$$det((E_1)_1(E_2)_1(E_3)_1) = 0$$

The first row of matrices E_1,E_2,E_3 or the first columns of matrices W_1,W_2,W_3 are a 3x3 matrix of rank 2.

This constrain is the same as (28) for k=1.

B) (coefficient y_1^3)

$$0 = \begin{vmatrix} \alpha_1^{12} & \alpha_1^{22} & \alpha_1^{32} \\ \alpha_2^{12} & \alpha_2^{22} & \alpha_2^{32} \\ \alpha_3^{12} & \alpha_3^{22} & \alpha_3^{32} \end{vmatrix}$$

$$det((E_1)_2(E_2)_2(E_3)_2) = 0$$

The second row of matrices E_1,E_2,E_3 or the second columns of matrices W_1,W_2,W_3 are a 3x3 matrix of rank 2.

This constrain is the same as (28) for k=2.

C) (constant coefficient)

$$0 = \begin{vmatrix} \alpha_1^{13} & \alpha_1^{23} & \alpha_1^{33} \\ \alpha_2^{13} & \alpha_2^{23} & \alpha_2^{33} \\ \alpha_3^{13} & \alpha_3^{23} & \alpha_3^{33} \end{vmatrix}$$

$$det((E_1)_3(E_2)_3(E_3)_3) = 0$$

The third row of matrices E_1,E_2,E_3 or the third columns of matrices W_1,W_2,W_3 are a 3x3 matrix of rank 2.

This constrain is the same as (28) for k=3.

D) (coefficient $x_1^2 y_1$)

$$det((E_1)_1(E_2)_1(E_3)_2) + det((E_1)_1(E_2)_2(E_3)_1) + det((E_1)_2(E_2)_1(E_3)_1) = 0$$

The addition of the determinants composed by any combination of first, first and second rows of matrices E_1,E_2,E_3 or the first, first and second columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 112,121,211).

E) (coefficient $x_1 y_1^2$)

$$det((E_1)_1(E_2)_2(E_3)_2) + det((E_1)_2(E_2)_1(E_3)_2) + det((E_1)_2(E_2)_2(E_3)_1) = 0$$

The addition of the determinants composed by any combination of first, second and second rows of matrices E_1,E_2,E_3 or the first, second and second columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 122,221,212).

F) (coefficient x_1^2)

$$det((E_1)_1(E_2)_1(E_3)_3) + det((E_1)_1(E_2)_3(E_3)_1) + det((E_1)_3(E_2)_1(E_3)_1) = 0$$

The addition of the determinants composed by any combination of first, first and third rows of matrices E_1,E_2,E_3 or the first, first and third columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 113,131,311).

G) (coefficient y_1^2)

$$det((E_1)_2(E_2)_2(E_3)_3) + det((E_1)_2(E_2)_3(E_3)_2) + det((E_1)_3(E_2)_2(E_3)_2) = 0$$

The addition of the determinants composed by any combination of second, second and third rows of matrices E_1,E_2,E_3 or the second, second, and third columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 223,232,322).

H) (coefficient x_1y_1)

$$det((E_1)_1(E_2)_2(E_3)_3) + det((E_1)_1(E_2)_3(E_3)_2) + det((E_1)_2(E_2)_1(E_3)_3)$$
$$+det((E_1)_2(E_2)_3(E_3)_1) + det((E_1)_3(E_2)_1(E_3)_2) + det((E_1)_3(E_2)_2(E_3)_1) = 0$$

The addition of the determinants composed by any combination of first, second and third rows of matrices E_1,E_2,E_3 or the first, second, and third columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be six determinants (rows/columns 123,132,213,231,312,321).

I) (coefficient x_1)

$$det((E_1)_1(E_2)_3(E_3)_3) + det((E_1)_3(E_2)_1(E_3)_3) + det((E_1)_3(E_2)_3(E_3)_1) = 0$$

The addition of the determinants composed by any combination of first, third and third rows of matrices E_1,E_2,E_3 or the first, third, and third columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 133,313,331).

J) (coefficient y_1)

$$det((E_1)_2(E_2)_3(E_3)_3) + det((E_1)_3(E_2)_2(E_3)_3) + det((E_1)_3(E_2)_3(E_3)_2) = 0$$

The addition of the determinants composed by any combination of second, third and third rows of matrices E_1,E_2,E_3 or the second, third, and third columns of matrices W_1,W_2,W_3 equals 0. Therefore, there will be three determinants (rows/columns 233,323,332).

E.2 From the epipolar lines

A) from equations (11) and (12)

$$\frac{(\alpha_2^{3k}p_1^k\alpha_1^{1k}p_1^k - \alpha_2^{1k}p_1^k\alpha_1^{3k}p_1^k)}{(\alpha_2^{3k}p_1^k\alpha_1^{2k}p_1^k - \alpha_2^{2k}p_1^k\alpha_1^{3k}p_1^k)} = \frac{(\alpha_2^{1k}p_1^k\alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k\alpha_3^{1k}p_1^k)}{(\alpha_2^{2k}p_1^k\alpha_3^{3k}p_1^k - \alpha_2^{3k}p_1^k\alpha_3^{2k}p_1^k)} = \frac{(\alpha_3^{3k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{3k}p_1^k)}{(\alpha_3^{3k}p_1^k\alpha_1^{2k}p_1^k - \alpha_3^{2k}p_1^k\alpha_1^{3k}p_1^k)} \quad (29)$$

B) from equations (13) and (14)

$$\frac{(\alpha_3^{2k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{2k}p_1^k)}{(\alpha_2^{2k}p_1^k\alpha_3^{1k}p_1^k - \alpha_3^{2k}p_1^k\alpha_2^{1k}p_1^k)} = \frac{(\alpha_1^{2k}p_1^k\alpha_3^{3k}p_1^k - \alpha_1^{3k}p_1^k\alpha_3^{2k}p_1^k)}{(\alpha_3^{2k}p_1^k\alpha_2^{3k}p_1^k - \alpha_3^{3k}p_1^k\alpha_2^{2k}p_1^k)} = \frac{(\alpha_3^{3k}p_1^k\alpha_1^{1k}p_1^k - \alpha_3^{1k}p_1^k\alpha_1^{3k}p_1^k)}{(\alpha_2^{3k}p_1^k\alpha_3^{1k}p_1^k - \alpha_2^{1k}p_1^k\alpha_3^{3k}p_1^k)} \quad (30)$$

Now

$$\alpha_a^{bk}p_1^k * \alpha_c^{dk}p_1^k = (p_1^k)^t(\alpha_a^{bk})^t * \alpha_c^{dk}p_1^k =$$
$$\alpha_c^{dk}p_1^k * \alpha_a^{bk}p_1^k = (p_1^k)^t(\alpha_c^{dk})^t * \alpha_a^{bk}p_1^k =$$
$$(p_1^k)^t \begin{bmatrix} \alpha_a^{b1}\alpha_c^{d1} & \alpha_a^{b1}\alpha_c^{d2} & \alpha_a^{b1}\alpha_c^{d3} \\ \alpha_a^{b2}\alpha_c^{d1} & \alpha_a^{b2}\alpha_c^{d2} & \alpha_a^{b2}\alpha_c^{d3} \\ \alpha_a^{b3}\alpha_c^{d1} & \alpha_a^{b3}\alpha_c^{d2} & \alpha_a^{b3}\alpha_c^{d3} \end{bmatrix} p_1^k = (p_1^k)^t \begin{bmatrix} \alpha_c^{d1}\alpha_a^{b1} & \alpha_c^{d1}\alpha_a^{b2} & \alpha_c^{d1}\alpha_a^{b3} \\ \alpha_c^{d2}\alpha_a^{b1} & \alpha_c^{d2}\alpha_a^{b2} & \alpha_c^{d2}\alpha_a^{b3} \\ \alpha_c^{d3}\alpha_a^{b1} & \alpha_c^{d3}\alpha_a^{b2} & \alpha_c^{d3}\alpha_a^{b3} \end{bmatrix} p_1^k$$

The diagonal elements of every matrix hold a similar condition (plugging in points (1,0,0), (0,1,0), (0,0,1) as follows):

with k=1,2,3

$$\frac{(\alpha_2^{3k}\alpha_1^{1k} - \alpha_2^{1k}\alpha_1^{3k})}{(\alpha_2^{3k}\alpha_1^{2k} - \alpha_2^{2k}\alpha_1^{3k})} = \frac{(\alpha_2^{1k}\alpha_3^{3k} - \alpha_2^{3k}\alpha_3^{1k})}{(\alpha_2^{2k}\alpha_3^{3k} - \alpha_2^{3k}\alpha_3^{2k})} = \frac{(\alpha_3^{3k}\alpha_1^{1k} - \alpha_3^{1k}\alpha_1^{3k})}{(\alpha_3^{3k}\alpha_1^{2k} - \alpha_3^{2k}\alpha_1^{3k})}$$

and

$$\frac{(\alpha_3^{2k}\alpha_1^{1k} - \alpha_3^{1k}\alpha_1^{2k})}{(\alpha_2^{2k}\alpha_3^{1k} - \alpha_3^{2k}\alpha_2^{1k})} = \frac{(\alpha_1^{2k}\alpha_3^{3k} - \alpha_1^{3k}\alpha_3^{2k})}{(\alpha_3^{2k}\alpha_2^{3k} - \alpha_3^{3k}\alpha_2^{2k})} = \frac{(\alpha_3^{3k}\alpha_1^{1k} - \alpha_3^{1k}\alpha_1^{3k})}{(\alpha_2^{3k}\alpha_3^{1k} - \alpha_2^{1k}\alpha_3^{3k})}$$

C) from equations (29) and (30) (similar to (28))

$$\alpha_1^{1k}p_1^k\alpha_2^{2k}p_1^k\alpha_3^{3k}p_1^k + \alpha_2^{1k}p_1^k\alpha_3^{2k}p_1^k\alpha_1^{3k}p_1^k + \alpha_3^{1k}p_1^k\alpha_1^{2k}p_1^k\alpha_2^{3k}p_1^k - \alpha_3^{1k}p_1^k\alpha_2^{2k}p_1^k\alpha_1^{3k}p_1^k$$
$$-\alpha_1^{1k}p_1^k\alpha_3^{2k}p_1^k\alpha_2^{3k}p_1^k - \alpha_2^{1k}p_1^k\alpha_1^{2k}p_1^k\alpha_3^{3k}p_1^k = 0$$

F Epipoles from the tensor coefficients

Points in the first image corresponding to the vertical epipolar line in the second image

$$p = (\alpha_3^{33}\alpha_1^{32} - \alpha_3^{32}\alpha_1^{33}, \alpha_1^{31}\alpha_3^{33} - \alpha_3^{31}\alpha_1^{33}, \alpha_1^{31}\alpha_3^{32} - \alpha_3^{31}\alpha_1^{32})$$

$$p = (\alpha_3^{12}\alpha_1^{13} - \alpha_3^{13}\alpha_1^{12}, \alpha_3^{11}\alpha_1^{13} - \alpha_1^{11}\alpha_3^{13}, \alpha_3^{11}\alpha_1^{12} - \alpha_1^{11}\alpha_3^{12})$$

$$p = (\alpha_1^{22}\alpha_3^{23} - \alpha_1^{23}\alpha_3^{22}, \alpha_1^{21}\alpha_3^{23} - \alpha_3^{21}\alpha_1^{23}, \alpha_1^{21}\alpha_3^{22} - \alpha_3^{21}\alpha_1^{22})$$

X coordinate of the epipole in the second view

$$x = \frac{\begin{vmatrix} \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \\ \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \end{vmatrix}}$$

Points in the first image corresponding to the horizontal epipolar line in the second image

$$p = (\alpha_2^{33}\alpha_3^{32} - \alpha_2^{32}\alpha_3^{33}, \alpha_3^{31}\alpha_2^{33} - \alpha_2^{31}\alpha_3^{33}, \alpha_2^{32}\alpha_3^{31} - \alpha_3^{32}\alpha_2^{31})$$

$$p = (\alpha_2^{13}\alpha_3^{12} - \alpha_2^{12}\alpha_3^{13}, \alpha_3^{11}\alpha_2^{13} - \alpha_2^{11}\alpha_3^{13}, \alpha_3^{11}\alpha_2^{12} - \alpha_2^{11}\alpha_3^{12})$$

$$p = (\alpha_3^{22}\alpha_2^{23} - \alpha_3^{23}\alpha_2^{22}, \alpha_3^{21}\alpha_2^{23} - \alpha_2^{21}\alpha_3^{23}, \alpha_3^{21}\alpha_2^{22} - \alpha_2^{21}\alpha_3^{22})$$

Y coordinate of the epipole in the second view

$$y = \frac{\begin{vmatrix} \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \end{vmatrix}}$$

Points in the first image corresponding to the vertical epipolar line in the third image

$$p = (\alpha_3^{13}\alpha_3^{32} - \alpha_3^{12}\alpha_3^{33}, \alpha_3^{31}\alpha_3^{13} - \alpha_3^{11}\alpha_3^{33}, \alpha_3^{31}\alpha_3^{12} - \alpha_3^{11}\alpha_3^{32})$$

$$p = (\alpha_1^{13}\alpha_1^{32} - \alpha_1^{12}\alpha_1^{33}, \alpha_1^{31}\alpha_1^{13} - \alpha_1^{11}\alpha_1^{33}, \alpha_1^{31}\alpha_1^{12} - \alpha_1^{11}\alpha_1^{32})$$

$$p = (\alpha_2^{13}\alpha_2^{32} - \alpha_2^{12}\alpha_2^{33}, \alpha_2^{31}\alpha_2^{13} - \alpha_2^{11}\alpha_2^{33}, \alpha_2^{31}\alpha_2^{12} - \alpha_2^{11}\alpha_2^{32})$$

X coordinate of the epipole in the third view

$$x = \frac{\begin{vmatrix} \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{11} & \alpha_1^{12} & \alpha_1^{13} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_3^{11} & \alpha_3^{12} & \alpha_3^{13} \\ \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_2^{11} & \alpha_2^{12} & \alpha_2^{13} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \end{vmatrix}}$$

Points in the first image corresponding to the horizontal epipolar line in the third image

$$p = (\alpha_3^{32}\alpha_3^{23} - \alpha_3^{22}\alpha_3^{33}, \alpha_3^{31}\alpha_3^{23} - \alpha_3^{21}\alpha_3^{33}, \alpha_3^{31}\alpha_3^{22} - \alpha_3^{21}\alpha_3^{32})$$

$$p = (\alpha_1^{32}\alpha_1^{23} - \alpha_1^{33}\alpha_1^{22}, \alpha_1^{31}\alpha_1^{23} - \alpha_1^{21}\alpha_1^{33}, \alpha_1^{31}\alpha_1^{22} - \alpha_1^{21}\alpha_1^{32})$$

$$p = (\alpha_2^{32}\alpha_2^{23} - \alpha_2^{33}\alpha_2^{22}, \alpha_2^{31}\alpha_2^{23} - \alpha_2^{21}\alpha_2^{33}, \alpha_2^{31}\alpha_2^{22} - \alpha_2^{21}\alpha_2^{32})$$

Y coordinate of the epipole in the third view

$$y = \frac{\begin{vmatrix} \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{23} \end{vmatrix}}{\begin{vmatrix} \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{23} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{23} \end{vmatrix}}{\begin{vmatrix} \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \\ \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_1^{23} \\ \alpha_1^{31} & \alpha_1^{32} & \alpha_1^{33} \end{vmatrix}} = \frac{\begin{vmatrix} \alpha_3^{21} & \alpha_3^{22} & \alpha_3^{23} \\ \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \end{vmatrix}}{\begin{vmatrix} \alpha_3^{31} & \alpha_3^{32} & \alpha_3^{33} \\ \alpha_2^{21} & \alpha_2^{22} & \alpha_2^{23} \\ \alpha_2^{31} & \alpha_2^{32} & \alpha_2^{33} \end{vmatrix}}$$

G Quadrilinear constraints

$$Q_{1,2,3,4} = \begin{bmatrix} +[3,4,7,10] & +[2,4,7,10] & -[1,4,7,10] \\ +[3,4,7,11] & +[2,4,7,11] & -[1,4,7,11] \\ -[3,4,7,12] & -[2,4,7,12] & +[1,4,7,12] \\ +[3,4,8,10] & +[2,4,8,10] & -[1,4,8,10] \\ +[3,4,8,11] & +[2,4,8,11] & -[1,4,8,11] \\ -[3,4,8,12] & -[2,4,8,12] & +[1,4,8,12] \\ -[3,4,9,10] & -[2,4,9,10] & +[1,4,9,10] \\ -[3,4,9,11] & -[2,4,9,11] & +[1,4,9,11] \\ +[3,4,9,12] & +[2,4,9,12] & -[1,4,9,12] \\ \\ +[3,5,7,10] & +[2,5,7,10] & -[1,5,7,10] \\ +[3,5,7,11] & +[2,5,7,11] & -[1,5,7,11] \\ -[3,5,7,12] & -[2,5,7,12] & +[1,5,7,12] \\ +[3,5,8,10] & +[2,5,8,10] & -[1,5,8,10] \\ +[3,5,8,11] & +[2,5,8,11] & -[1,5,8,11] \\ +[3,5,8,12] & -[2,5,8,12] & +[1,5,8,12] \\ -[3,5,9,10] & -[2,5,9,10] & +[1,5,9,10] \\ -[3,5,9,11] & -[2,5,9,11] & +[1,5,9,11] \\ +[3,5,9,12] & +[2,5,9,12] & -[1,5,9,12] \\ \\ -[3,6,7,10] & -[2,6,7,10] & +[1,6,7,10] \\ -[3,6,7,11] & -[2,6,7,11] & +[1,6,7,11] \\ +[3,6,7,12] & +[2,6,7,12] & -[1,6,7,12] \\ -[3,6,8,10] & -[2,6,8,10] & +[1,6,8,10] \\ -[3,6,8,11] & -[2,6,8,11] & +[1,6,8,11] \\ +[3,6,8,12] & +[2,6,8,12] & -[1,6,8,12] \\ +[3,6,9,10] & +[2,6,9,10] & -[1,6,9,10] \\ +[3,6,9,11] & +[2,6,9,11] & -[1,6,9,11] \\ -[3,6,9,12] & -[2,6,9,12] & +[1,6,9,12] \end{bmatrix} = \begin{bmatrix} \omega_{1111} & \omega_{1112} & \omega_{1113} \\ \omega_{1121} & \omega_{1122} & \omega_{1123} \\ \omega_{1131} & \omega_{1132} & \omega_{1133} \\ \omega_{1211} & \omega_{1212} & \omega_{1213} \\ \omega_{1221} & \omega_{1222} & \omega_{1223} \\ \omega_{1231} & \omega_{1232} & \omega_{1233} \\ \omega_{1311} & \omega_{1312} & \omega_{1313} \\ \omega_{1321} & \omega_{1322} & \omega_{1323} \\ \omega_{1331} & \omega_{1332} & \omega_{1333} \\ \\ \omega_{2111} & \omega_{2112} & \omega_{2113} \\ \omega_{2121} & \omega_{2122} & \omega_{2123} \\ \omega_{2131} & \omega_{2132} & \omega_{2133} \\ \omega_{2211} & \omega_{2212} & \omega_{2213} \\ \omega_{2221} & \omega_{2222} & \omega_{2223} \\ \omega_{2231} & \omega_{2232} & \omega_{2233} \\ \omega_{2311} & \omega_{2312} & \omega_{2313} \\ \omega_{2321} & \omega_{2322} & \omega_{2323} \\ \omega_{2331} & \omega_{2332} & \omega_{2333} \\ \\ \omega_{3111} & \omega_{3112} & \omega_{3113} \\ \omega_{3121} & \omega_{3122} & \omega_{3123} \\ \omega_{3131} & \omega_{3132} & \omega_{3133} \\ \omega_{3211} & \omega_{3212} & \omega_{3213} \\ \omega_{3221} & \omega_{3222} & \omega_{3223} \\ \omega_{3231} & \omega_{3232} & \omega_{3233} \\ \omega_{3311} & \omega_{3312} & \omega_{3313} \\ \omega_{3321} & \omega_{3322} & \omega_{3323} \\ \omega_{3331} & \omega_{3332} & \omega_{3333} \end{bmatrix}$$

H Independent equations of the quadrilinear constraints

The 16 independent equations are as it follows :

1) $$+x_1x_2x_3x_4\ \omega_{3331} - x_1x_2x_3\ \omega_{3311} - x_1x_2x_4\ \omega_{3131} - x_1x_3x_4\ \omega_{1331} - x_2x_3x_4\ \omega_{3333}$$
$$+x_1x_2\ \omega_{3111} + x_1x_3\ \omega_{1311} + x_1x_4\ \omega_{1131} + x_2x_3\ \omega_{3313} + x_2x_4\ \omega_{3133} + x_3x_4\ \omega_{1333}$$
$$-x_1\ \omega_{1111} - x_2\ \omega_{3113} - x_3\ \omega_{1313} - x_4\ \omega_{1133} + \omega_{1113} = 0$$

2) $$+x_1x_2x_3y_4\ \omega_{3331} - x_1x_2x_3\ \omega_{3321} - x_1x_2y_4\ \omega_{3131} - x_1x_3y_4\ \omega_{1331} - x_2x_3y_4\ \omega_{3333}$$
$$+x_1x_2\ \omega_{3121} + x_1x_3\ \omega_{1321} + x_1y_4\ \omega_{1131} + x_2x_3\ \omega_{3323} + x_2y_4\ \omega_{3133} + x_3y_4\ \omega_{1333}$$
$$-x_1\ \omega_{1121} - x_2\ \omega_{3123} - x_3\ \omega_{1323} - y_4\ \omega_{1133} + \omega_{1123} = 0$$

3) $$+x_1x_2y_3x_4\ \omega_{3331} - x_1x_2y_3\ \omega_{3311} - x_1x_2x_4\ \omega_{3231} - x_1y_3x_4\ \omega_{1331} - x_2y_3x_4\ \omega_{3333}$$
$$+x_1x_2\ \omega_{3211} + x_1y_3\ \omega_{1311} + x_1x_4\ \omega_{1231} + x_2y_3\ \omega_{3313} + x_2x_4\ \omega_{3233} + y_3x_4\ \omega_{1333}$$
$$-x_1\ \omega_{1211} - x_2\ \omega_{3213} - y_3\ \omega_{1313} - x_4\ \omega_{1233} + \omega_{1213} = 0$$

4) $$+x_1y_2x_3x_4\ \omega_{3331} - x_1y_2x_3\ \omega_{3311} - x_1y_2x_4\ \omega_{3131} - x_1x_3x_4\ \omega_{2331} - y_2x_3x_4\ \omega_{3333}$$
$$+x_1y_2\ \omega_{3111} + x_1x_3\ \omega_{2311} + x_1x_4\ \omega_{2131} + y_2x_3\ \omega_{3313} + y_2x_4\ \omega_{3133} + x_3x_4\ \omega_{2333}$$
$$-x_1\ \omega_{2111} - y_2\ \omega_{3113} - x_3\ \omega_{2313} - x_4\ \omega_{2133} + \omega_{2113} = 0$$

5) $$+y_1x_2x_3x_4\ \omega_{3331} - y_1x_2x_3\ \omega_{3311} - y_1x_2x_4\ \omega_{3131} - y_1x_3x_4\ \omega_{1331} + x_2x_3x_4\ \omega_{3332}$$
$$+y_1x_2\ \omega_{3111} + y_1x_3\ \omega_{1311} + y_1x_4\ \omega_{1131} - x_2x_3\ \omega_{3312} - x_2x_4\ \omega_{3132} - x_3x_4\ \omega_{1332}$$
$$-y_1\ \omega_{1111} + x_2\ \omega_{3112} + x_3\ \omega_{1312} + x_4\ \omega_{1132} - \omega_{1112} = 0$$

6) $$+x_1x_2y_3y_4\ \omega_{3331} - x_1x_2y_3\ \omega_{3321} - x_1x_2y_4\ \omega_{3231} - x_1y_3y_4\ \omega_{1331} - x_2y_3y_4\ \omega_{3333}$$
$$+x_1x_2\ \omega_{3221} + x_1y_3\ \omega_{1321} + x_1y_4\ \omega_{1231} + x_2y_3\ \omega_{3323} + x_2y_4\ \omega_{3233} + y_3y_4\ \omega_{1333}$$
$$-x_1\ \omega_{1221} - x_2\ \omega_{3223} - y_3\ \omega_{1323} - y_4\ \omega_{1233} + \omega_{1223} = 0$$

7) $$+x_1y_2x_3y_4\ \omega_{3331} - x_1y_2x_3\ \omega_{3321} - x_1y_2y_4\ \omega_{3131} - x_1x_3y_4\ \omega_{2331} - y_2x_3y_4\ \omega_{3333}$$
$$+x_1y_2\ \omega_{3121} + x_1x_3\ \omega_{2321} + x_1y_4\ \omega_{2131} + y_2x_3\ \omega_{3323} + y_2y_4\ \omega_{3133} + x_3y_4\ \omega_{2333}$$
$$-x_1\ \omega_{2121} - y_2\ \omega_{3123} - x_3\ \omega_{2323} - y_4\ \omega_{2133} + \omega_{2123} = 0$$

8) $$+y_1x_2x_3y_4\ \omega_{3331} - y_1x_2x_3\ \omega_{3321} - y_1x_2y_4\ \omega_{3131} - y_1x_3y_4\ \omega_{1331} + x_2x_3y_4\ \omega_{3332}$$
$$+y_1x_2\ \omega_{3121} + y_1x_3\ \omega_{1321} + y_1y_4\ \omega_{1131} - x_2x_3\ \omega_{3322} - x_2y_4\ \omega_{3132} - x_3y_4\ \omega_{1332}$$
$$-y_1\ \omega_{1121} + x_2\ \omega_{3122} + x_3\ \omega_{1322} + y_4\ \omega_{1132} - \omega_{1122} = 0$$

9) $$+x_1y_2y_3x_4\ \omega_{3331} - x_1y_2y_3\ \omega_{3311} - x_1y_2x_4\ \omega_{3231} - x_1y_3x_4\ \omega_{2331} - y_2y_3x_4\ \omega_{3333}$$
$$+x_1y_2\ \omega_{3211} + x_1y_3\ \omega_{2311} - x_1x_4\ \omega_{2231} + y_2y_3\ \omega_{3313} + y_2x_4\ \omega_{3233} + y_3x_4\ \omega_{2333}$$
$$-x_1\ \omega_{2211} - y_2\ \omega_{3213} - y_3\ \omega_{2313} - x_4\ \omega_{2233} + \omega_{2213} = 0$$

10) $$+y_1x_2y_3x_4\ \omega_{3331} - y_1x_2y_3\ \omega_{3311} - y_1x_2x_4\ \omega_{3231} - y_1y_3x_4\ \omega_{1331} + x_2y_3x_4\ \omega_{3332}$$
$$+y_1x_2\ \omega_{3211} + y_1y_3\ \omega_{1311} + y_1x_4\ \omega_{1231} - x_2y_3\ \omega_{3312} + x_2x_4\ \omega_{3232} - y_3x_4\ \omega_{1332}$$
$$-y_1\ \omega_{1211} + x_2\ \omega_{3212} + y_3\ \omega_{1312} + x_4\ \omega_{1232} - \omega_{1212} = 0$$

11) $$+y_1y_2x_3x_4\ \omega_{3331} - y_1y_2x_3\ \omega_{3311} - y_1y_2x_4\ \omega_{3131} - y_1x_3x_4\ \omega_{2331} + y_2x_3x_4\ \omega_{3332}$$
$$+y_1y_2\ \omega_{3111} + y_1x_3\ \omega_{2311} + y_1x_4\ \omega_{2131} - y_2x_3\ \omega_{3312} - y_2x_4\ \omega_{3132} - x_3x_4\ \omega_{2332}$$
$$-y_1\ \omega_{2111} + y_2\ \omega_{3112} + x_3\ \omega_{2312} + x_4\ \omega_{2132} - \omega_{2112} = 0$$

12) $$+x_1y_2y_3y_4\ \omega_{3331} - x_1y_2y_3\ \omega_{3321} - x_1y_2y_4\ \omega_{3231} - x_1y_3y_4\ \omega_{2331} - y_2y_3y_4\ \omega_{3333}$$
$$+x_1y_2\ \omega_{3221} + x_1y_3\ \omega_{2321} - x_1y_4\ \omega_{2231} + y_2y_3\ \omega_{3323} + y_2y_4\ \omega_{3233} + y_3y_4\ \omega_{2333}$$
$$-x_1\ \omega_{2221} - y_2\ \omega_{3223} - y_3\ \omega_{2323} - y_4\ \omega_{2233} + \omega_{2223} = 0$$

13) $$+y_1x_2y_3y_4\ \omega_{3331} - y_1x_2y_3\ \omega_{3321} - y_1x_2y_4\ \omega_{3231} - y_1y_3y_4\ \omega_{1331} + x_2y_3y_4\ \omega_{3332}$$
$$+y_1x_2\ \omega_{3221} + y_1y_3\ \omega_{1321} + y_1y_4\ \omega_{1231} - x_2y_3\ \omega_{3322} - x_2y_4\ \omega_{3232} - y_3y_4\ \omega_{1332}$$
$$-y_1\ \omega_{1221} + x_2\ \omega_{3222} + y_3\ \omega_{1322} + y_4\ \omega_{1232} - \omega_{1222} = 0$$

14) $$+y_1y_2x_3y_4\ \omega_{3331} - y_1y_2x_3\ \omega_{3321} - y_1y_2y_4\ \omega_{3131} - y_1x_3y_4\ \omega_{2331} + y_2x_3y_4\ \omega_{3332}$$
$$+y_1y_2\ \omega_{3121} + y_1x_3\ \omega_{2321} + y_1y_4\ \omega_{2131} - y_2x_3\ \omega_{3322} - y_2y_4\ \omega_{3132} - x_3y_4\ \omega_{2332}$$
$$-y_1\ \omega_{2121} + y_2\ \omega_{3122} + x_3\ \omega_{2322} + y_4\ \omega_{2132} - \omega_{2122} = 0$$

15) $$+y_1y_2y_3x_4\ \omega_{3331} - y_1y_2y_3\ \omega_{3311} - y_1y_2x_4\ \omega_{3231} - y_1y_3x_4\ \omega_{2331} + y_2y_3x_4\ \omega_{3332}$$
$$+y_1y_2\ \omega_{3211} + y_1y_3\ \omega_{2311} - y_1x_4\ \omega_{2231} - y_2y_3\ \omega_{3312} - y_2x_4\ \omega_{3232} - y_3x_4\ \omega_{2332}$$
$$-y_1\ \omega_{2211} + y_2\ \omega_{3212} + y_3\ \omega_{2312} + x_4\ \omega_{2232} - \omega_{2212} = 0$$

16) $$+y_1y_2y_3y_4\ \omega_{3331} - y_1y_2y_3\ \omega_{3321} - y_1y_2y_4\ \omega_{3231} - y_1y_3y_4\ \omega_{2331} + y_2y_3y_4\ \omega_{3332}$$
$$+y_1y_2\ \omega_{3221} + y_1y_3\ \omega_{2321} - y_1y_4\ \omega_{2231} - y_2y_3\ \omega_{3322} - y_2y_4\ \omega_{3232} - y_3y_4\ \omega_{2332}$$
$$-y_1\ \omega_{2221} + y_2\ \omega_{3222} + y_3\ \omega_{2322} + y_4\ \omega_{2232} - \omega_{2222} = 0$$

As ech correspondence provides 16 equations and there are 81 unknowns, we need 5 or more correspondences to find these coefficients.

I Some algebraic equations

Referenced in the text:

$$(x,\ y,\ 1)\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} =$$

$$(m_{11}, m_{22}, m_{12}+m_{21}, m_{13}+m_{31}, m_{23}+m_{32}, m_{33})\begin{pmatrix} x^2 \\ y^2 \\ xy \\ x \\ y \\ 1 \end{pmatrix} \quad (31)$$

Given 4 vectors of dimension 2 C,D,E,F, it holds the following formula, where [] represente the determinant of the vectors:

$$[C,E][D,F]-[C,F][D,E]=[C,D][E,F]$$

Given 6 vectors of dimension 4 A,B,C,D,E,F, it holds the following formula:

$$[A,B,C,E][A,B,D,F]-[A,B,C,F][A,B,D,E]-[A,B,C,D][A,B,E,F]=0 \quad (32)$$

$$(u^t M v)w = (w u^t M)v \tag{33}$$

$$uu^t = \| u \|^2 I_3 + [u]^2 \tag{34}$$

Inverse of a diagonal matrix

$$\begin{bmatrix} A & 0 \\ - & - \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ - & - \\ 0 & B^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & 0 \\ - & - \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ - & - \\ \frac{-A^{-1}C}{B} & B^{-1} \end{bmatrix}$$

Skew-symmetric matrix of a vector v = [a,b,c]

$$[v] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

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