Between Logic and Probability

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Abstract

Logic and Probability, as theories, have been developed quite independently and, with a few exceptions (like Boole's), have largely ignored each other. And nevertheless they share a lot of similarities, as well a considerable common ground. The exploration of the shared concepts and their mathematical treatment and unification is here attempted following the lead of illustrious researchers (Reichenbach, Carnap, Popper, Gaifman, Scott & Krauss, Fenstad, Miller, David Lewis, Stalnaker, Hintikka or Suppes, to name a few). The resulting theory, to be distinguished from the many-valued-Logics tradition, is strongly reminiscent, in its the mathematical treatment, of Probability theory, though it remains in spirit firmly inside pure Logic.

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INTRODUCTION

In the parallel history of modern Logic and Probability Theory, some intriguing similarities have baffled theoreticians: in particular, the one between a probability value of one or zero and the logical concepts of truth and falsity. Such affinities have led some to suggest using the supposedly rock-solid bases of Logic to underpin the epistemologically dubious foundations of probability. On this common ground, at least two traditions have grown. The first, dating back to Hume and Laplace, attempts to base Probability concepts on Logic and truth (as when Laplace declares probability to be the proportion of "cases" in which a proposition is *true*). In the line of Laplace, contemporary authors such as Ramsey [22], Popper [20] or Carnap [3] openly declared Probability a generalization of Logic and, accordingly, tried to build up a comprehensive theoretical construct. The contrary —or complementary— tradition mentioned is typically recent: though isolated authors like S. Watanabe [31] were insisting that Probability was 'previous' to Logic, it was paradoxically a member of the first tradition, Popper (in his 1959 appendixes [20]), who hinted that Logic might be semantically grounded on Probability valuations, thus giving way to the contemporary trend of *probabilistic semantics* of Logic of W. Harper [9], Hartry Field [7] and their recent successors. Other people chose not to proclaim either field tributary to the other but merely noticed the need to study the conceptual intersection points, especially the intriguing logical status of privileged probability values —the one and the zero, as proxies for "truth" and "falsity"— and the possibility of properly speaking of truth *values* and equating them to 1 and 0 (with intermediate values expecting their firm logical interpretation someday); such authors —Reichenbach [23] and Zawirski in the thirties or Watanabe [31] and Bodiou [1] in the sixties— were mostly Physics-oriented and insisted on the need for "compatibility" between both theoretical fields. But Logic-oriented mathematicians also have shown interest in the Logic/Probability interface, such as Gaifman [8] or Scott and Krauss [26]; the latter view Logic as "a special case of Probability" and probability valuations as "generalized theories".

The present article attempts to explore this presumed compatibility between Logic and Probability, with no claims at primacy of one over the other. Though it is independent from —but fully compatible with— the recent Popper-grounded tradition of *probabilistic semantics* (of which the author was unaware when he developed the present ideas back in 1981-82), its simple purpose is to understand Logic better through use of Probability concepts. (In this it coincides more or less with the stated aims of e.g. Gaifman [8], Scott and Krauss [26] or Fenstad [6].) By resisting classical Logic's current emphasis on compulsory two-valuedness or truth-functionality, many standard logical concepts may be seen in a new light and, hopefully, be better understood. Examples of these are: (1) the connection between classical Logic and the *possible-worlds* concept, (2) the relationship (and *difference*) between the *conditional* statement in Logic and the *conditioning* operation in Probability (that so annoyed Stalnaker [28] or David Lewis [14]), (3) the need (or redundancy) of van Fraassen's [30] *supervaluations*, (4) the standing and role of traditional 3-valued logics, (5) the truth-functional formulas for *connectives*, (6) the

standing and implications of *many-valued logics* in general, (7) some self-suggested *distances* and measures (like *imprecision*) in Logic, (8) a closer view on deduction under *uncertainty* conditions and the role of Bayesian reasoning, and (9) an analysis of ignorance and *non-rational belief*. The present treatment —that leaves points (8) and (9) to be pursued elsewhere (see [24])— may also be seen as an attempt to systematize many computations and insights casually introduced by authors such as, among others, Suppes [29], Stalnaker [28], Lewis [14] or Popper and Miller [21] in different places, often on an 'as needed' basis.

Throughout the paper, *probability* is to be understood in a strictly technical sense, i.e. as a *normalized measure* on *sentences* from a language (or, alternatively, on *sets* from a universe). Nothing popularly considered relevant to Probability beyond this should be considered essential here: neither should there be anything like "events" ("happening" in time somewhere) nor any kind of frequential or objectivist interpretation going beyond mere *rational belief*. (Actually, we will speak, simply, of *valuations*.) Thus, we will insist now and then to take 'probability' everywhere as a mere technical synonym for 'normalized measure', thus avoiding any connotation associated with 'chance' so that the concept is being amplified to embrace not just ordinary probability but also poorly characterized notions such as pure 'rational belief' or 'truth value' (and even 'degree of error', 'truth content', 'informativeness' or 'verisimilitude', as the case will later arise). But, this being a *logic* context and the valued objects being *sentences*, we will often indulge in preferring *truth value* to mere *value* or *measure* or the rather more misleading "probability" as the linguistic label for a numerical value.

(One previous general remark: results presented here can be easily worked out by any careful reader, so formal or detailed proofs —most being trivial computations— are generally omitted, and left to the reader's initiative.)

1. PRELIMINARIES

A standard presentation of sentential logic uses to be like this: first, assume sentences are constructed by recursive application of the $\wedge$, $\vee$ and $\neg$ connectives to the (possibly infinite) set of propositional letters P, Q, ... Second, assume sentences form a Boolean algebra (with respect to the three connectives and two special sentences $\bot$ and $\top$). We will have then a complete Proof Theory by identifying the " $\vdash$ " order defined by the Boolean algebra with the deductive consequence relation. So the algebra of sentences we started with automatically becomes the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation " $\dashv\vdash$ " given by the $\vdash$ order (i.e. $A \dashv\vdash B$ iff $A = B$). Third, assume that all sentences are valued in $\{0, 1\}$. This can be done in the standard way of a normalized measure, by just requiring that the valuation is additive and that $\top$ gets a value of 1; we will then have also the whole Model Theory of Sentential Logic.

In that orthodox presentation we now vary *one* single point: the valuation we propose is on [0,1], not just on $\{0,1\}$. What we get is a new way of looking at Sentential Logic that in the sequel this paper will attempt to explore. Only two remarks before we begin our analysis. First: as it is obvious, the situation is relatively novel to Logic but traditional in Probability, for the proposed valuation is no more nor less than a *probability* in all technical senses. However, only a few mathematicians have defined probability as a valuation on *sentences* and many, since Kolmogorov [12], have said they did but immediately substituted the more docile *sets* for the sentences. (Only Carnap persisted in treating sentences, and only Popper stood by the need to keep a double interpretation permanently open.) Also, the similarities —or differences— between Logic and Probability (or truth/falsity and a probability value of one/zero) have never been clear, despite efforts by Reichenbach [23], Carnap [4] and others. (Again, Popper opened

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the way to considering both two flips of the same coin.) We will explore how far we can go on making valuations (normalized measures) in [0,1] on *sentences*. (Note also that we do not require the valuations —even when interpreted as "truth" valuations— to be "extensional" or "truth-functional" as done in many-valued logics.)

The second remark concerns the Booleanity of the sentences. Either this is assumed (imposed) or it just arises naturally from a "minimal algebra" of sentences with only two connectives (say, ¬ and ∧): Popper [20] showed that by giving values in [0,1] to each pair of sentences (or sets, for he deliberately chose to leave the objects uninterpreted) we could define very naturally a binary function (that we can —predictably— call "conditional probability") from which we can define the usual (unary) probability; the interesting point here is that, by requesting that the binary Popper functions satisfy a few very undemanding and reasonable conditions, the algebra of sentence classes modulo equiprobability becomes automatically a Boolean algebra. The Boolean structure of sentences we postulated above can thus be either imposed by fiat or supposed to be the product of such valuations (and our *sentences* are then mere representatives of all member-sentences of Popper's minimal algebra assigned the same conditional-probability values). Though we could justify the Booleanity of the sentences by recurring to Popper valuations —and then our values would simply be Popper's induced unary probabilities— in this paper we choose to leave the origins of both (Booleanity and valuation) unexplained and *given*. So we begin.

Suppose, first, we have a propositional language $\mathcal{L}$, whose members (henceforth denoted $A, B, \ldots$) are sentences. We shall assume that $\mathcal{L}$ is closed by the propositional operators $\wedge$, $\vee$ and $\neg$ and, moreover, that it has the structure of a Boolean algebra. Thus, the structure $\langle \mathcal{L}, \wedge, \vee, \neg, \perp, \top \rangle$ (with signature $\langle 2, 2, 1, 0, 0 \rangle$) has the following properties (for arbitrary A, B and C):

1. $A \wedge A = A$ and $A \vee A = A$ *(Idempotency)*   (1)

2. $A \wedge B = B \wedge A$ and $A \vee B = B \vee A$ *(Commutativity)*   (2)

3. $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ and $A \vee (B \vee C) = (A \vee B) \vee C$ *(Associativity)*   (3)

4. $A \wedge (B \vee A) = A$ and $A \vee (B \wedge A) = A$ *(Absorption)*   (4)

5. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ and its dual *(Distributivity)*   (5)

6. $A \wedge \perp = \perp$ and $A \vee \perp = A$ and $A \wedge \top = A$ and $A \vee \top = \top$ *(Bounds)*   (6)

7. $A \wedge \neg A = \perp$ and $A \vee \neg A = \top$ *(Complementation)*   (7)

The resulting structure is, of course, the well-known Lindenbaum-Tarski algebra of propositions. It is, if *finite*, of cardinality 2^N (for some N) and *atomic*, with exactly N atoms. If there are n propositional letters in the language —acting as generators of the resulting *free* Boolean algebra— the structure has 2^{2^n} elements, with 2^n atoms. In case there are infinitely many letters (generators) the algebra has the same cardinality as the set of generators, and is positively *not* atomic. The Lindenbaum-Tarski character of this algebra may be taken on word, as we do, or may be induced by Popper valuations (in case they are assumed); or we can simply leave the question open until we conclude —as we do (see (31) below)— that the equality of $\mathcal{L}$ simply coincides with the interderivability relation, and so $\mathcal{L}$ is a Lindenbaum-Tarski algebra. (Note that more standard presentations, as in Gaifman [8] or Scott and Krauss [26], define theoremhood or derivability first and then define their valuations on the subsequent quotient algebra.)

As in all Boolean algebras, a partial order (that we denote by $\vdash$) is definable in the structure —which is then characterized as a Boolean lattice, with bounds $\perp$ and $\top$— through the equivalence:

8. $A \vdash B \iff A \wedge B = A \iff A \vee B = B$ *(Partial order)* (8)

Suppose, secondly, that we impose a valuation on this structure, that takes any sentence A in the language and assigns it a real number $[\![A]\!]$, that we could either call "probability of A" or simply "value of A" but that we will rather biasedly call "truth value of A" (whatever sense we may make of this). This valuation is required to be a non-trivial real-valued measure of the sentences in the Lindenbaum-Tarski algebra $\mathcal{L}$ satisfying the following three standard conditions (that give it all the technical characteristics of a *probability*):

i) For any A in $\mathcal{L}$ $[\![A]\!] \geq 0$

ii) For any A and B in $\mathcal{L}$ such that $A \wedge B = \perp$ $[\![A \vee B]\!] = [\![A]\!] + [\![B]\!]$ *(Finite additivity)*

iii) $[\![\top]\!] = 1$.

though we prefer —for compatibility with a weaker presentation elsewhere [24]— this equivalent formulation:

9. There is a valuation $v : \mathcal{L} \longrightarrow [0,1] : A \mapsto [\![A]\!]$ such that:

    9a. $[\![\perp]\!] = 0$, and $[\![\top]\!] = 1$. (9)

    9b. If $A \vdash B$ then $[\![A]\!] \leq [\![B]\!]$ *(Monotonicity)* (10)

    9c. For any A and B, $[\![A \wedge B]\!] + [\![A \vee B]\!] = [\![A]\!] + [\![B]\!]$ *(Finite additivity)* (11)

Once we are given the nine conditions 1-9 above —admittedly over-redundant (actually, 2 and 5-7 plus 9a and 9c would suffice)— we have all we need to proceed.

Observe that if the valuation described in 9 had been defined in $\{0, 1\}$ instead of $[0, 1]$, conditions 1-9 would have completely characterized the standard (two-valued) Propositional Logic. In particular, from 1-8 arises all of its *proof theory*, while conditions 9a-c —together with the immediately derivable facts (12-13) (see below)— describe completely its *model theory* (its truth tables).

It is to be specially emphasized that condition 9 as stated above —that the v valuation has $[0,1]$ as its range of values rather than merely $\{0,1\}$— is the *single* differing point between our presentation of Propositional Logic and the ordinary two-valued view of it. The first naturally subsumes the second as a special case.

Notice that we have used *finite additivity* here. In this we follow the trend of questioning whether countable additivity, usually required in Probability Theory, is really necessary. In our case it is not. (Moreover, this condition can be obtained through compactness, if need be.)

(A remark, on notation: $[\![A]\!] = 1$ will be sometimes abbreviated as "$\models_v A$", which has the advantage of showing explicitly that a particular valuation v is involved.)

From conditions (1-11) the five formulas below follow immediately:

$$[\![\neg A]\!] = 1 - [\![A]\!] \tag{12}$$

$$[\![A \wedge B]\!] \leq [\![A]\!] \leq [\![A \vee B]\!] \tag{13}$$

$$[\![A \wedge B]\!] \leq \min([\![A]\!], [\![B]\!]) \tag{14}$$

$$[\![A \vee B]\!] \geq \max([\![A]\!], [\![B]\!]) \tag{15}$$

$$\text{If } A_i \wedge A_j = \perp \ (i \neq j) \text{ then } \left[\!\left[\bigvee_{i=1}^{i=k} A_i\right]\!\right] = \sum_{i=1}^{i=k} [\![A_i]\!] \tag{16}$$

If we now define the *conditional* or *if then* connective in the usual —classical— manner:

$$A \to B =_{def} \neg A \vee B \tag{17}$$

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then the following formula immediately obtains:

$$[\![A \to B]\!] = 1 - [\![A]\!] + [\![A \land B]\!] \tag{18}$$

and, from these, also:

$$[\![A \to B]\!] - [\![B \to A]\!] = [\![B]\!] - [\![A]\!] \tag{19}$$

If we then define the *biconditional* or *equivalence* connective in the usual manner:

$$A \leftrightarrow B =_{def} (A \to B) \land (B \to A) \tag{20}$$

then the following formulas are immediately obtainable:

$$[\![A \leftrightarrow B]\!] = [\![A \to B]\!] + [\![B \to A]\!] - 1 \tag{21}$$

$$[\![A \leftrightarrow B]\!] = 1 - [\![A \lor B]\!] + [\![A \land B]\!]$$

As is to be expected in a Boolean algebra,

$$A \vdash B \iff \neg A \lor B = \top.$$

If we denote $A = \top$ by "$\vdash A$", this can be written as:

$$A \vdash B \iff \vdash A \to B, \tag{22}$$

more in line with the usual formulation of the Deduction Theorem of elementary logic.

Now, we define the relation between A and B given by $[\![A \to B]\!] = 1$ —that we note by "$A \models_v B$" (notice it depends on the particular valuation v chosen)—:

[*Definition:*] "$A \models_v B$" if and only if $[\![A \to B]\!] = 1$.

This relation is a *quasiordering* that contains the lattice order $A \vdash B$ (i.e. $A \vdash B \Rightarrow A \models_v B$) and satisfies the following conditions:

$$A \models_v B \iff [\![A]\!] = [\![A \land B]\!] \iff [\![B]\!] = [\![A \lor B]\!]$$

If $A \models_v B$ then $[\![A]\!] \le [\![B]\!]$ (but note the converse is *not* true in general)

We now define the relation between A and B given by $[\![A \leftrightarrow B]\!] = 1$ and we note it by "$A =_v B$" (notice the dependence on the particular valuation v chosen):

[*Definition:*] "$A =_v B$" if and only if $[\![A \leftrightarrow B]\!] = 1$. $\tag{23}$

Note that "$=_v$" (which we can call "equality under a valuation") is an equivalence relation and contains the ordinary propositional (Lindenbaum-Tarski) identity (i.e. $A = B \Rightarrow A =_v B$). It yields the following two properties:

$A =_v B$ if and only if $A \models_v B$ and $B \models_v A$.

If $A =_v B$ then $[\![A]\!] = [\![B]\!]$ (*Truth-value equality*). (The converse is *not* true in general)

Now, the definition below follows the usual line:

[*Definition:*] "$\models A$" if and only if $[\![A]\!] = 1$ *for all valuations* $\tag{24}$

(*Remark:* Here "$[\![A]\!] = 1$ *for all valuations*" means "$v(A) = 1$ for all $[0,1]$-valuations v of A". From now on, "*for all valuations*" will be sometimes informally shortened to "$(\forall v)$".)

As is obvious, $\models_v$ contains $\models$ (i.e. $\models A \Rightarrow \models_v A$), and the given definition can be informally written in this way: "$\models A$ iff $(\forall v) \models_v A$".

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Naturally,

$$\text{If } A = B \text{ then } [\![A]\!] = [\![B]\!] \text{ for all valuations} \qquad (25)$$

(because the valuation is meant to be a function in the mathematical sense).

This has a corollary:

$$\text{If } \vdash A \text{ then } \models A \quad (\textit{Soundness}) \qquad (26)$$

(because $A = \top$ yields $[\![A]\!] = [\![\top]\!] = 1$ for any valuation).

Conversely, we are forced to admit —by convention— that:

$$\text{If } [\![A]\!] = [\![B]\!] \text{ for all valuations, then } A = B \qquad (27)$$

because we have no other way to distinguish any two propositions through the semantic means available (i.e. the $[0,1]$-valuations). Actually, we know that the ordinary $\{0,1\}$-valuations suffice to distinguish and identify all the elements of a Boolean algebra (this is a consequence of the Prime Ideal Theorem). So we have a considerably softened version of (27):

$$\text{If } [\![A]\!] = [\![B]\!] \text{ for all binary valuations, then } A = B \qquad (28)$$

As a corollary of (27) we get:

$$\text{If } \models A \text{ then } \vdash A \quad (\textit{Weak completeness}) \qquad (29)$$

The two relationships (25) and (27) shown above between propositions and values can be combined to yield this (informally stated) *semantical characterization* of propositional identity:

$$A = B \text{ if and only if } (\forall v) \, A =_v B \qquad (30)$$

where "$\forall v$" can be here —and subsequently— taken to mean "for all *binary* valuations v" (see (28)).

Now, combining (30), (23), (24), (29) and (26) we get

$$A = B \text{ if and only if } \vdash A \leftrightarrow B \qquad (31)$$

confirming that the Boolean algebra we presented in (1-8) was just the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation $\vdash A \leftrightarrow B$.

Also, the *soundness* and *weak completeness* conditions, taken together, yield this equivalence:

[*Weak completeness theorem:*] $\quad \vdash A \text{ if and only if } \models A$

Now we put forward this (that we state informally):

[*Definition:*] "$A \models B$" if and only if $(\forall v) \, \{[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1\}$.

It is easy to show that

$$A \models B \text{ if and only if } \models A \rightarrow B$$

Thus, $\models_v$ contains $\models$ (i.e. $A \models B \Rightarrow A \models_v B$), and the last definition can also be written: "$A \models B$ iff $(\forall v) \, A \models_v B$".

Recalling the above semantical characterization of propositional identity ("$A = B$ iff $(\forall v) \, [\![A]\!] = [\![B]\!]$"), we would like to have a reasonable characterization of "$A \models B$" along the same line: something as "$A \models B$ iff $(\forall v) \, [\![A]\!] \leq [\![B]\!]$". Unfortunately, this does *not* hold. The most we can have is just the rightward half:

$$\text{If } A \models B \text{ then } [\![A]\!] \leq [\![B]\!] \text{ for all valuations.}$$

Now, again from the given semantical characterization of propositional identity (30) we have:

$$A = A \wedge B \quad \text{if and only if} \quad [\![A]\!] = [\![A \wedge B]\!] \text{ for all valuations.}$$

Note the the left-hand side is equivalent to writing "$A \vdash B$", while the right-hand part amounts to saying "$[\![A \to B]\!] = 1$ for all valuations" (or else, by definition, "$\models A \to B$", that we have shown to be equivalent to "$A \models B$"). So we are led to this new characterization of the soundness and completeness condition:

[*Strong completeness theorem:*] $A \vdash B$ if and only if $A \models B$.

A could represent a list (or, better, a conjunction) of propositions $A_1, \ldots, A_n$ —the *premises*—. In that case, it would read thus, in its most general form:

[*Completeness theorem:*] $A_1, \ldots, A_n \vdash B$ if and only if $A_1, \ldots, A_n \models B$

where the left-hand A's are the conjuncts of $A = A_1 \wedge \ldots \wedge A_n$, while the right-hand term is (demonstrably) equivalent to stating "$\{ [\![A_1]\!] = \ldots = [\![A_n]\!] = 1 \Rightarrow [\![B]\!] = 1 \}$ *for all valuations*".

2. SENTENCES AS SET EXTENSIONS

Two widely-known and yet under-exploited results in Boolean Algebra Theory (see [13]) have to do with *representations* of Boolean algebras on set structures. They can be stated thus:

A. Every *finite* Boolean algebra has a representation in the algebra of *all* subsets of a (finite) set.

B. Every Boolean algebra is *representable* on —isomorphic to— a field of sets. (*Stone's Representation Theorem.*)

In particular, the free sentential algebra $\mathcal{L}$ generated by countably many propositional letters has a representation in —is isomorphic to— :

(a) The lattice $\hat{\mathcal{L}}$ of principal filters of $\mathcal{L}$. Principal filters are theories deductively derived from a single sentence. If $\mathcal{L}$ is finite, all filters are principal. (In Tarski's terminology theories are "closed deductive systems".)

(b) A field $\mathcal{B}$ of sets. More specifically: $\mathcal{B}$ is a countable non-atomic Boolean subalgebra of the powerset $\mathcal{P}(\Theta)$ of all ultrafilters (or all binary epimorphisms) θ of $\mathcal{L}$. (If $\mathcal{L}$ is finite and has n generators then $\mathcal{L}$ is isomorphic to the whole of $\mathcal{P}(\Theta)$, which then has 2^{2^n} elements and is atomic, with exactly 2^n atoms.) $\mathcal{P}(\Theta)$ is the Stone space associated with $\mathcal{L}$, and each element or "point" θ corresponds to a complete theory of $\mathcal{L}$.

(c) The field of sets $\mathcal{B}$ mentioned above, where $\mathcal{B}$ is precisely characterizable as the class of clopens of the $\mathcal{P}(\Theta)$ Stone space of $\mathcal{L}$. Clopens correspond to the finitely axiomatizable theories of $\mathcal{L}$.

(d) A subdirect product of copies of $\{0, 1\}$.

(e) Any countable interval algebra such as the set of rationals in $[0, 1)$.

By the Normal Form Theorem ([13]), each sentence A in a free algebra $\mathcal{L}$ is expressible in normal form as a finite disjunction of finite conjunctions of literals. Also, by the above (equivalent) representations, A is the isomorphic image, respectively, of:

(a) The theory (or set of deductive consequences) derived from A.

(b) The set of complete theories that include A (as a derivable sentence).

(c)   A finitely axiomatizable theory that includes A.

(d)   A set of strings of 0's and 1's. If $\mathcal{L}$ has n generators, the strings have length n, and there are 2^n such strings: they correspond to the *interpretations* of the ordinary Model Theory (truth tables) of Sentential Logic.

(e)   A finite union of intervals of generators (given in some order).

All these properties are well-known (see e.g [13]) and can be considered elementary. What we are presently interested in is, simply, that, given the Boolean sentence algebra $\mathcal{L}$, there exist a set Θ (whatever the meaning we give its elements θ) *and* a 'representation' function that can be characterized as

a) a homomorphism of $\mathcal{L}$ into the powerset $\mathcal{P}(\Theta)$ of Θ,   i.e.

$$\rho : \mathcal{L} \longrightarrow \mathcal{P}(\Theta) : A \mapsto \mathbf{A} \quad (\mathbf{A} \subset \Theta)$$

b) an isomorphism of $\mathcal{L}$ into the Boolean subalgebra $\mathcal{B}$ of clopens in $\mathcal{P}(\Theta)$,   i.e.

$$\rho : \mathcal{L} \longleftrightarrow \mathcal{B} : A \mapsto \mathbf{A} \quad (\mathcal{B} \subset \mathcal{P}(\Theta), \mathbf{A} \subset \Theta)$$

which is just the restriction of ρ to $\mathcal{B}$ (though we also note it by "ρ"). (This is the *Representation theorem*)

So, every time we have a *Sentential Logic* we have also an inherent accompanying structure or *universe* that we make here *explicit* and name Θ; it is explicitly definable from its sentences $A \in \mathcal{L}$. (This *always* happens, even in strictly two-valued logics.)

Though clearly there is no need to name or qualify the members of Θ , we may indulge in calling them *possible worlds*, or *possibilities* (following Shafer [27]) or *eventualities* (to distinguish them from Probability's *outcomes* or elementary *events*) or *cases* (as in Laplace or Boole) or, metaphorically, even *observers* or *states*, *observations*, *instants* of time or *stages* of development, elementary *situations* or *contexts* in which things happen, and so on. Θ is thus configured as the real *universe of discourse* or *reference frame* (the set of *possible worlds*). It also coincides with Fenstad's [6] (or e.g. Miller's [17]) *model space*, and then the θs are also Fenstad's *models* or —after (d) above— all possible *interpretations* of A (2^n of them in the finite case).

We can establish a general, one-to-one correspondence between the two worlds (the language world $\mathcal{L}$ and the referential universe Θ, both made up of "propositions") and their constituent parts, thus:

$$
\begin{array}{ccc}
\mathcal{L} & \Longleftrightarrow & \mathcal{B} \\
A\ (A \in \mathcal{L}) & \Longleftrightarrow & \mathbf{A}\ (\mathbf{A} \subset \Theta) \\
A \wedge B & \Longleftrightarrow & \mathbf{A} \cap \mathbf{B} \\
A \vee B & \Longleftrightarrow & \mathbf{A} \cup \mathbf{B} \\
\neg A & \Longleftrightarrow & \mathbf{A}^c \\
\top & \Longleftrightarrow & \Theta \\
\bot & \Longleftrightarrow & \phi \\
A \vdash B & \Longleftrightarrow & \mathbf{A} \subset \mathbf{B} \\
A = B & \Longleftrightarrow & \mathbf{A} = \mathbf{B}
\end{array}
$$

If $\mathcal{L}$ has a finite number of generators, then it has 2^n atoms a and the following two bijective correspondences also hold:

$$
\begin{array}{ccc}
\mathcal{L} & \Longleftrightarrow & \mathcal{P}(\Theta) \\
a & \Longleftrightarrow & \{\theta\}
\end{array}
$$

(The arrows are meant to be read "corresponds isomorphically to"; also, "A^c" is the set-complement of A, a is an *atom* of $\mathcal{L}$, and θ is an element of Θ.)

As to the precise nature of the "possible worlds" $\theta \in \Theta$, nothing needs to be specified about them except, maybe, that each θ is just a *discernible unit*, merely something that can be isolated and told apart precisely from all other θs ; so Θ is —just— a *frame of discernment* (in Shafer's sense).

It is remarkable that when Boole, just after studying classes, set out to analyze *propositions* (in 1847), he conceived them by means of an alternative interpretation of his elective symbol x (already introduced for classes) which —he said— now stood for the *cases* (or "conceivable sets of circumstances") —out of a given hypothetical "universe"— in which the proposition was true. This is stunningly close to our set-extensional representation of propositions in the set Θ of *possible worlds*, to which the Stone theorem gives rigorous legitimacy. And, since set propositions are measurable —an idea we shall immediately turn to—, an easy step carries the picture into probability, a step Boole inevitably made when he, in the last chapters of his 1854 book [2], somewhat obscurely likened the product $x \cdot y$ of two propositions to the probability of simultaneaously having both.

Something similar occurred later to MacColl [16] when he distinguished between propositions that were *certain*, *variable* or *impossible* (meaning they were always, sometimes or never the case). Historically, Russell, when reviewing MacColl's work in 1906, argued that MacColl's "propositions" were not such but rather *propositional functions*. This is still the conventional answer today. However, we can reconcile both views by converting, through the Representation Theorem, a proposition A (a piece of the language $\mathcal{L}$) into a propositional function that takes values in $\{0,1\}$, the values of a characteristic function (i.e. Russell's "propositional function"), so defining a set A in a world of *cases* or *circumstances* (Boole's terms) corresponding to what we have called *possible worlds* and distinguishing MacColl's *certain* (or Peirce's [19] *necessary*) propositions (our case A = $\mathcal{B}$) from *variable* or *contingent* ones (the general A $\subset$ $\mathcal{B}$ case).

3. TRUTH AS MEASURE

The valuation

$$v : \mathcal{L} \longrightarrow [0,1] : A \mapsto [\![A]\!]$$

and the representation isomorphism

$$\rho : \mathcal{L} \longleftrightarrow \mathcal{B} : A \mapsto \mathbf{A}$$

clearly induce a [0,1]-valued measure μ in $\mathcal{B} \subset \mathcal{P}(\Theta)$.

[*Definition:*] $\mu : \mathcal{B} \longrightarrow [0,1]$ is the valuation in $\mathcal{P}(\Theta)$ induced by the isomorphism
$$\rho : \mathcal{L} \longleftrightarrow \mathcal{B} \subset \mathcal{P}(\Theta) \quad \text{in such a way that } \mu = v \circ \rho^{-1}, \text{ i.e. } \mu(\mathbf{A}) = [\![A]\!].$$

Intuitively, the measure $\mu(\{\theta\})$ of each individual θ in a *finite* Θ universe should seemingly correspond to the relative importance or the relevance this individual has in that universe. Thus, in a reading of Θ where the θ are interpreted as *observers*, $\mu(\{\theta\})$ would represent the importance a "superobserver" assigns to each particular θ. In a *tests* or modal *"possible worlds"* reading, $\mu(\{\theta\})$ would be the relevance attributed to test θ or the degree of realizability of the given possible world. And so on. The μ measure corresponds to the weighing function λ in Fenstad's [6] *model space*.

As is known, μ (or λ) is not only additive but countably so; thus μ is eligible as a standard "probability" measure (in the technical sense), though we do not think countable additivity is anything we should require from such a measure (and we will not).

$$[\![B|A]\!] = \frac{[\![A \to B]\!]}{[\![A]\!]} - \frac{[\![A \to \neg A]\!]}{[\![A]\!]} = 1 - \frac{1 - [\![A \to B]\!]}{[\![A]\!]} \ .$$

Note that, in general,

$$[\![B|A]\!] \neq [\![A \to B]\!]$$

Particularly, we have *always*

$$[\![B|A]\!] \ < \ [\![A \to B]\!]$$

except when either $[\![A]\!] = 1$ or $[\![A \to B]\!] = 1$, in which cases (and they are the *only* ones) $[\![B|A]\!] = [\![A \to B]\!]$. (This has been noticed by many people, notably by Reichenbach [23], Stalnaker [28], Lewis [14] and Popper [20].)

The statement '$A \to B$' can have, among other readings, one logical ("A is sufficient for B" or "B is necessary for A"), another (loosely) "causal" ("A occurs and B follows"). Because $A \to B$ is valued in [0,1], its value $[\![A \to B]\!]$ (and the values $[\![B|A]\!]$ and $[\![A|B]\!]$) now mean only *degrees*, and so $B \to A$ may be —and usually is— read "evidentially" ("B is evidence for A"). Within such a frame of mind,

— $[\![B|A]\!]$ (or "σ_A" —or even "ν_B", see next paragraph—) could be termed "degree of *sufficiency* or *causality*" of A (or "*causal support* for B"), to be read as "degree in which A is sufficient for B" or "degree in which A is a cause of B". In view of (32), it is roughly a measure of how much of **A** is contained in **B**.

— $[\![A|B]\!]$ (or "ν_A" or "σ_B") could be termed "degree of *necessity*" or "*evidence*" of A (or "*evidential support* for A", to be read as "degree in which A is necessary for B" or "degree in which B is evidence (= support of hypothesis) for A (=the hypothesis)". With (32) in mind, it can be seen as how much of **B** overlaps with **A**.

Such measures may be directly estimated by experts, normally by interpreting the θs frequentially, in terms of *cases*, like Boole. ("Cases" may be statistically-based or simply imagined, presumably on the basis of past experience or sheer plausibility.) Thus, σ_A in a causal reading of "$A \to B$" would be determined by answering the question: "How many times (proportionally) —experience shows— A occurs and B follows?" For ν_A, the question would be: "How many times effect B occurs and A has occurred previously?" (Similarly for the evidential reading of "$A \to B$".) Once σ and ν have been guessed, they may be adjusted (via the

$$\frac{\sigma_A}{\nu_A} = \frac{[\![B]\!]}{[\![A]\!]}$$

relation) and then lead —by straightforward computation— to $[\![A \to B]\!]$, $[\![B \to A]\!]$ and the α_{AB} value (see below), which allows one to compute all other values for connectives and also to get a picture of the structural relations linking A and B.

4. CONNECTIVES AND SENTENTIAL STRUCTURE

The goal here is to find the truth value of composite propositions in $\mathcal{L}$. For the negation connective this is easy: it is given by formula (12). For the rest we have the three following formulas that are a direct spin-off of additivity (11) and the definitions (17) and (20):

$$[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] - [\![A \wedge B]\!]$$
$$[\![A \to B]\!] = 1 - [\![A]\!] + [\![A \wedge B]\!] \tag{33}$$

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Now suppose we want to express the conjunction value as a product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot \tau \qquad (\text{or } [\![A \wedge B]\!] = \tau' \cdot [\![B]\!])$$

We have (provided $[\![A]\!] \neq 0$):

$$\tau = \frac{[\![A \wedge B]\!]}{[\![A]\!]} = \frac{v(A \wedge B)}{v(A)} = v_A(B)$$

which yields on $\mathcal{L}$ a new valuation $v_A : \mathcal{L} \longrightarrow [0,1]$ with the same properties as the original valuation v (indeed v_A satisfies equations (9) and (11), as is easy to prove).

The $[\![A]\!] \neq 0$ proviso may be unnecessary if the τ function has been evaluated directly, as Popper proposed back in 1959 [20].

With the current $\mathcal{L}/\mathcal{P}(\Theta)$ representation in mind, we have:

$$\tau = \frac{\mu(\mathbf{A} \cap \mathbf{B})}{\mu(\mathbf{A})} \tag{32}$$

Thus, the new valuation takes in account, out of a subset of Θ, only the part contained in $\mathbf{A}$, and it gives it a value related only to that part. So we define τ as the *relative truth* " $[\![B|A]\!]$ " (i.e. the "truth of B relative to A"):

[*Definition:*] *Relative truth* of B with respect to A is the quotient

$$[\![B|A]\!] = \frac{[\![A \wedge B]\!]}{[\![A]\!]} \qquad ([\![A]\!] \neq 0)$$

[*Remark*: Here we break with our own implicit notational convention according to which double brackets separate the syntax (inside) from the outer world of semantics.]

If $[\![B|A]\!] = [\![B]\!]$, then we say that A and B are *independent* (because the valuation $v_A(B) = v(B)$ remains unaffected by A). In that case, the conjunction can be expressed as the product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!]$$

(and then we have also:

$$[\![A \to B]\!] = 1 - [\![A]\!] + [\![A]\!] \cdot [\![B]\!] = 1 - [\![A]\!] \cdot [\![\neg B]\!] \,.)$$

In any other case we say that A and B are mutually *dependent* and speak of the *relative truth* of one with respect to the other. Note the dependence goes both ways and the two situations are symmetrical. We have (always assuming $[\![A]\!] \neq 0$ and, in the third and seventh formulas, A not binary):

$$[\![A]\!] \cdot [\![B|A]\!] = [\![B]\!] \cdot [\![A|B]\!] = [\![A \wedge B]\!] \quad \textit{(Bayes formula)}$$

$$[\![A|A]\!] = 1$$

$$[\![A|\neg A]\!] = [\![\neg A|A]\!] = 0$$

$$[\![A \wedge B|A]\!] = [\![B|A]\!] = [\![A \to B|A]\!]$$

$$[\![A \vee B|A]\!] = 1 = [\![A \to B|B]\!]$$

$$[\![B|A]\!] + [\![\neg B|A]\!] = 1 \quad \textit{(Complementation)}$$

$$[\![B]\!] = [\![A]\!] \cdot [\![B|A]\!] + [\![\neg A]\!] \cdot [\![B|\neg A]\!] \quad \textit{(Distribution)}$$

$$[\![A \to B]\!] = 1 - [\![A]\!] \cdot [\![\neg B|A]\!]$$

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$$[\![A \leftrightarrow B]\!] = 1 - [\![A]\!] - [\![B]\!] + 2 \cdot [\![A \wedge B]\!] \tag{34}$$

(formula (33) is (18) again, and (34) immediately derives from (21)).

So the problem now reduces to finding the numerical expression $[\![A \wedge B]\!]$ of the conjunction $A \wedge B$ as a function of the (numerical) "truth" values $[\![A]\!]$ and $[\![B]\!]$ of the component propositions A and B.

For any $\mathbf{A} = \rho(A)$ and $\mathbf{B} = \rho(B)$ we have, obviously:

$$\phi \subset \mathbf{A} \cap \mathbf{B} \subset \mathbf{A} \subset \mathbf{A} \cup \mathbf{B} \subset \Theta \tag{35}$$

and, because of the induced monotonicity of μ:

$$0 \leq \mu(\mathbf{A} \cap \mathbf{B}) \leq \mu(\mathbf{A}) \leq \mu(\mathbf{A} \cup \mathbf{B}) \leq 1$$

In (35) there is a smooth, comprehensive gradation of possible cases. We distinguish four extreme cases; the first two concern intersection, the third and the fourth union:

- Case 1: $\mathbf{A} \cap \mathbf{B} = \phi$. We have:
$$\mu(\mathbf{A} \cap \mathbf{B}) = 0$$

- Case 2: Subcase (a): $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$ (i.e. $\mathbf{A} \subset \mathbf{B}$). We have:
$$\mu(\mathbf{A} \cap \mathbf{B}) = \mu(\mathbf{A}) \leq \mu(\mathbf{B})$$
or Subcase (b): $\mathbf{A} \cap \mathbf{B} = \mathbf{B}$ (i.e. $\mathbf{B} \subset \mathbf{A}$). We have:
$$\mu(\mathbf{A} \cap \mathbf{B}) = \mu(\mathbf{B}) \leq \mu(\mathbf{A})$$

In either subcase we have:
$$\mu(\mathbf{A} \cap \mathbf{B}) = \min\,[\mu(\mathbf{A}), \mu(\mathbf{B})] \tag{36}$$

- Case 3: Subcase (a): $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$ (i.e. $\mathbf{B} \subset \mathbf{A}$). We have:
$$\mu(\mathbf{A} \cup \mathbf{B}) = \mu(\mathbf{A}) \geq \mu(\mathbf{B})$$
or Subcase (b): $\mathbf{A} \cup \mathbf{B} = \mathbf{B}$ (i.e. $\mathbf{A} \subset \mathbf{B}$). We have:
$$\mu(\mathbf{A} \cup \mathbf{B}) = \mu(\mathbf{B}) \geq \mu(\mathbf{A})$$

In either subcase we have:
$$\mu(\mathbf{A} \cup \mathbf{B}) = \max\,[\mu(\mathbf{A}), \mu(\mathbf{B})] \tag{37}$$

- Case 4: $\mathbf{A} \cup \mathbf{B} = \Theta$. We have:
$$\mu(\mathbf{A} \cup \mathbf{B}) = 1$$

Cases 2 and 3 describe a single situation —that we shall call 'case $\oplus$'— because, thanks to the additivity of μ:

$$\begin{aligned} \mu(\mathbf{A} \cup \mathbf{B}) &= \mu(\mathbf{A}) + \mu(\mathbf{B}) - \mu(\mathbf{A} \cap \mathbf{B}) = \\ &= \mu(\mathbf{A}) + \mu(\mathbf{B}) - \min\,[\mu(\mathbf{A}), \mu(\mathbf{B})] = \\ &= \max\,[\mu(\mathbf{A}), \mu(\mathbf{B})] \end{aligned}$$

and vice versa. On the other hand, cases 1 and 4 can be treated as one, because we have, respectively:

$$\mu(\mathbf{A} \cap \mathbf{B}) = 0 \qquad\qquad \text{(for case 1),} \qquad \text{and}$$
$$\mu(\mathbf{A} \cup \mathbf{B}) = 1 \qquad\qquad \text{(for case 4),}$$

that by the additivity of μ yield, respectively:

$$\mu(\mathbf{A} \cup \mathbf{B}) = \mu(\mathbf{A}) + \mu(\mathbf{B}) \qquad\qquad \text{(for case 1),} \qquad \text{and}$$

$$\mu(\mathbf{A} \cap \mathbf{B}) = \mu(\mathbf{A}) + \mu(\mathbf{B}) - 1 \qquad \text{(for case 4).}$$

We can cover the situations described by the above four formulas with a single name —we shall call it 'case $\ominus$'— and a pair of formulas:

$$\mu(\mathbf{A} \cap \mathbf{B}) = \max [0, \mu(\mathbf{A}) + \mu(\mathbf{B}) - 1], \qquad (38)$$

that summarizes in a single statement the first and fourth formulas, and

$$\mu(\mathbf{A} \cup \mathbf{B}) = \min [1, \mu(\mathbf{A}) + \mu(\mathbf{B})], \qquad (39)$$

that does the same for the second and third formulas.

Thus, by tracking what happens with measure μ when $\mathbf{A}$ (as $\mathbf{B}$) goes all the way —in smooth gradation— from ϕ to Θ (see (35)), it is easy to see that not one but *many* values are possible for $\mu(\mathbf{A} \cap \mathbf{B})$ and $\mu(\mathbf{A} \cup \mathbf{B})$, and that those values are strictly *bounded* as prescribed by formulas (36) and (37) in one extreme case, and by (38) and (39) in the other. This has a straightforward translation into *truth values* and *composite propositions*. The first thing we learn is that the binary connectives —as propositional functions— are *not* functional, i.e. they yield different values for a proposition despite the fact that the operands may have stable values. (We shall see below, however, that the binary connectives are actually *functional*, but in *three* —not two— variables.) The second is that the range of *values* of composite propositions has, nevertheless, strict and prescribable *bounds*. We analyze that, and distinguish the two extreme cases we mentioned:

A) **Case $\oplus$** : It summarizes cases 2 and 3 above. This situation is what we call *maximum compatibility* between two propositions A and B. The value of the connectives is given by:

$$[\![A \wedge B]\!] = \min ([\![A]\!], [\![B]\!])$$
$$[\![A \vee B]\!] = \max ([\![A]\!], [\![B]\!])$$

(which are (36) and (37) in truth-value notation).

We shall often abbreviate the right-hand members as " $[\![A \wedge B]\!]^+$" and " $[\![A \vee B]\!]^+$ ", respectively.

Case $\oplus$ obviously corresponds to any of those situations:

$$\mathbf{A} \subset \mathbf{B} \quad (\text{ or, equivalently, } A \vdash B)$$
$$\mathbf{B} \subset \mathbf{A} \quad (\text{ or, equivalently, } B \vdash A)$$

all of which justifies our speaking of "*maximum compatibility*". We could have called this case also simply *compatibility* or *coherence* (because of lack of incoherence, see case $\ominus$) or *mutual implication* (because here either $A \vdash B$ or $B \vdash A$). The situation here is one of [mutual] *dependence*, as $[\![B|A]\!]$ —or $[\![A|B]\!]$— equals one. (We could speak of *correlation* as well.)

From $[\![A \wedge B]\!]$ and (33)-(34) we obtain at once:

$$[\![A \rightarrow B]\!] = \min (1, 1 - [\![A]\!] + [\![B]\!])$$
$$[\![A \leftrightarrow B]\!] = 1 - |[\![A]\!] - [\![B]\!]|$$

We shall abbreviate the right-hand members as " $[\![A \rightarrow B]\!]^+$ " and " $[\![A \leftrightarrow B]\!]^+$ ".

B) **Case $\ominus$** : It summarizes cases 1 and 4 above. There is what we call *minimum compatibility* between two propositions A and B. The value of the connectives is given by:

$$[\![A \wedge B]\!] = \max (0, [\![A]\!] + [\![B]\!] - 1)$$
$$[\![A \vee B]\!] = \min (1, [\![A]\!] + [\![B]\!])$$

(which are (38) and (39) in truth-value notation).

We shall often abbreviate the right-hand members as " $[\![A \wedge B]\!]^-$ " and " $[\![A \vee B]\!]^-$ ", respectively.

Case $\ominus$ obviously corresponds to any of the situations described next:

$$\mathbf{A} \cap \mathbf{B} = \phi \quad (\text{ or, equivalently, } A \wedge B = \perp)$$

$$\mathbf{A} \cup \mathbf{B} = \Theta \quad (\text{ or, equivalently, } \vdash A \vee B)$$

all of which justifies our speaking of "*minimum compatibility*". We could have called this case also simply *incompatibility* or *incoherence* (because either $A \wedge B = \perp$ or $\neg A \wedge \neg B = \perp$), or *mutual contradiction* (because here either $A \vdash \neg B$ or $\neg A \vdash B$).

From $[\![A \wedge B]\!]$ and (33-34) we obtain at once:

$$[\![A \to B]\!] = \max (1 - [\![A]\!], [\![B]\!])$$

$$[\![A \leftrightarrow B]\!] = |[\![A]\!] + [\![B]\!] - 1|$$

We shall abbreviate the right-hand members as " $[\![A \to B]\!]^-$ " and " $[\![A \leftrightarrow B]\!]^-$ ".

So, in summary, the value of the connectives is always inside a slack interval, with bounds $\ominus$ and $\oplus$:

Minimum value	Actual value	Maximum value
$\max (0, [\![A]\!] + [\![B]\!] - 1)$	$[\![A \wedge B]\!]$	$\min ([\![A]\!], [\![B]\!])$
$\max ([\![A]\!], [\![B]\!])$	$[\![A \vee B]\!]$	$\min (1, [\![A]\!] + [\![B]\!])$
$\max (1 - [\![A]\!], [\![B]\!])$	$[\![A \to B]\!]$	$\min (1, 1 - [\![A]\!] + [\![B]\!])$
$|[\![A]\!] + [\![B]\!] - 1|$	$[\![A \leftrightarrow B]\!]$	$1 - |[\![A]\!] - [\![B]\!]|$

We have, perhaps more graphically:

$$
\begin{array}{ccccccc}
\ominus & [\![A \wedge B]\!]^- & \leq & [\![A \wedge B]\!] & \leq & [\![A \wedge B]\!]^+ & \oplus \\
\oplus & [\![A \vee B]\!]^+ & \leq & [\![A \vee B]\!] & \leq & [\![A \vee B]\!]^- & \ominus \\
\ominus & [\![A \to B]\!]^- & \leq & [\![A \to B]\!] & \leq & [\![A \to B]\!]^+ & \oplus \\
\ominus & [\![A \leftrightarrow B]\!]^- & \leq & [\![A \leftrightarrow B]\!] & \leq & [\![A \leftrightarrow B]\!]^+ & \oplus \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rule{1cm}{0.4pt} & c \rule{1cm}{0.4pt} & a & \rule{1cm}{0.4pt} d & \rule{1cm}{0.4pt} & 1
\end{array}
$$
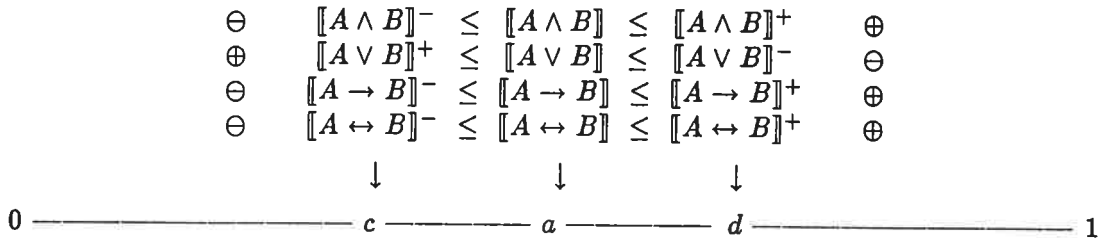
The above diagram is actually oversimplified: each connective would have a different projection on the [0,1] line, and each would have a different triple of values c, a and d; in each triple, c and d are the minimum (c) and maximum (d) values, respectively, that the actual value (a) of the corresponding composite proposition can reach, as the swing along [0,1] takes a from the $\ominus$ to the $\oplus$ case.

A rather stunning fact about the above graph is that the width $d - c$ is *constant* for the three first connectives (and exactly double that length for the biconditional). Indeed,

$$[\![A \wedge B]\!]^+ - [\![A \wedge B]\!]^- = [\![A \vee B]\!]^- - [\![A \vee B]\!]^+ = [\![A \to B]\!]^+ - [\![A \to B]\!]^- =$$

$$= ([\![A \leftrightarrow B]\!]^+ - [\![A \leftrightarrow B]\!]^-)/2 = \min ([\![A]\!], [\![B]\!], 1 - [\![A]\!], 1 - [\![B]\!]) ,$$

a quadruple minimum that only depends on the values of $[\![A]\!]$ and $[\![B]\!]$ and is always $\leq 1/2$. We note this value by " $\Delta(A, B)$" or " Δ_{AB}" and notice that $\Delta(A, B) = \Delta(\neg A, B) = \Delta(A, \neg B) = \Delta(\neg A, \neg B)$.

Another striking fact about connectives is that we can parameterize their values through a unique parameter we name " $\alpha(A, B)$ " or " α_{AB} " and we call "*degree of compatibility* between propositions A and B" or "relative position of propositions **A** and **B** (inside Θ)". Its value is:

$$\alpha_{AB} = \frac{[\![A \wedge B]\!] - [\![A \wedge B]\!]^-}{[\![A \wedge B]\!]^+ - [\![A \wedge B]\!]^-} \tag{40}$$

Symmetrically we define a second parameter we name " $\beta(A, B)$ " or " β_{AB} " —that we call "*degree of incompatibility* [between propositions A and B]"— through the formula

$$\beta_{AB} =_{df} 1 - \alpha_{AB}$$

Naturally, $0 \leq \alpha_{AB} \leq 1$ and, simultaneously, $1 \geq \beta_{AB} \geq 0$ —where the leftmost and rightmost bounds refer to cases $\ominus$ and $\oplus$, respectively, so that both cases are completely determined by one parameter (or both of them):

Case $\oplus$ (*Maximum compatibility*): $\alpha_{AB} = 1$ (or $\beta_{AB} = 0$).

(Note that then —and only then— $[\![A \wedge B]\!] = [\![A \wedge B]\!]^{+}$.)

Case $\ominus$ (*Minimum compatibility*): $\alpha_{AB} = 0$ (or $\beta_{AB} = 1$).

(Note that then —and only then— $[\![A \wedge B]\!] = [\![A \wedge B]\!]^{-}$.)

Both cases coincide if —and only if— at least one of the propositions A or B is valued binarily. In this situation —which is equally well described by both case profiles— α_{AB} and β_{AB} are *undetermined*, and the actual value of the connectives is given by *any* of the formulas above.

In the general case, the parameter α_{AB} acts as an indicator or measure of "relative position" of propositions A and B inside $\ominus$, and also as a cursor ranging inside the (fixed) interval between bounds, pointing to the actual value of the connective. We can formulate each connective as a linear function (a convex combination of case $\oplus$ and case $\ominus$ values) "interpolating" between bounds (=the extreme $\oplus$ and $\ominus$ values), so that its effective value is given by the values $[\![A]\!]$ and $[\![B]\!]$ *and* the parameter α. (Thus each connective is functional in *three* variables, the third being α.)

Indeed we can, and get the following set of formulas (where (41) derives directly from (40) while (42)-(44) are obtained from (41) via (11) and (33-34)):

$$[\![A \wedge B]\!] = \alpha_{AB} \cdot [\![A \wedge B]\!]^{+} + \beta_{AB} \cdot [\![A \wedge B]\!]^{-} \qquad (41)$$

$$[\![A \vee B]\!] = \alpha_{AB} \cdot [\![A \vee B]\!]^{+} + \beta_{AB} \cdot [\![A \vee B]\!]^{-} \qquad (42)$$

$$[\![A \rightarrow B]\!] = \alpha_{AB} \cdot [\![A \rightarrow B]\!]^{+} + \beta_{AB} \cdot [\![A \rightarrow B]\!]^{-} \qquad (43)$$

$$[\![A \leftrightarrow B]\!] = \alpha_{AB} \cdot [\![A \leftrightarrow B]\!]^{+} + \beta_{AB} \cdot [\![A \leftrightarrow B]\!]^{-} \qquad (44)$$

So by knowing a single value (either of $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$, $[\![A \rightarrow B]\!]$, $[\![A \leftrightarrow B]\!]$, α_{AB} or β_{AB}) we can compute the other five.

Alternatively, formulas (41)-(44) can be replaced by this set:

$$[\![A \wedge B]\!] \;=\; \min\left([\![A]\!], [\![B]\!]\right) \;-\; \beta_{AB} \cdot \Delta_{AB} \tag{45}$$

$$[\![A \vee B]\!] \;=\; \max\left([\![A]\!], [\![B]\!]\right) \;+\; \beta_{AB} \cdot \Delta_{AB} \tag{46}$$

$$[\![A \rightarrow B]\!] \;=\; \min\left(1, 1 - [\![A]\!] + [\![B]\!]\right) \;-\; \beta_{AB} \cdot \Delta_{AB} \tag{47}$$

$$[\![A \leftrightarrow B]\!] \;=\; 1 - \left|\,[\![A]\!] - [\![B]\!]\,\right| \;-\; 2 \cdot \beta_{AB} \cdot \Delta_{AB} \tag{48}$$

where it is prominent that the value of the connectives is the value for *case* $\oplus$ plus a *negative correction* (except for $\vee$, whose correction is *positive*) of size proportional to the *incompatibility* β_{AB} and the (constant) *interval length* Δ_{AB} (which is a function of $[\![A]\!]$ and $[\![B]\!]$ only).

Incidentally, connectives for case $\oplus$ —(generally) upper *bounds* for actual values, in fact— coincide with values (functionally) assigned by Lukasiewicz-Tarski to the connectives in their L_∞ logic [15]. On the other hand, connectives for case $\ominus$ —(generally) lower bounds for actual values, in fact— coincide with values (functionally) assigned to them by *threshold logic*. The difference here is that those connectives are no longer *functional* in the truth values of the operands, but act as mere *bounds* for actual values. These depend not only on the truth-values of the component propositions but also on a third term indicating their relative position as well.

The α parameter satisfies the following relations:

$$\alpha(A, A) = 1 \qquad \text{and} \qquad \alpha(A, \neg A) = 0$$

$$\alpha(A, B) = \alpha(B, A)$$

$$\alpha(A, B) = \alpha(\neg A, \neg B) = 1 - \alpha(A, \neg B) = 1 - \alpha(\neg A, B) \quad [\; = \beta(\neg A, B) \,, \text{ etc. }]$$

(and analogously for β). With those relations in mind various properties of ordinary logic can be easily proved. For instance:

$$[\![A \wedge \neg A]\!] = \alpha(A, \neg A) \cdot [\![A \wedge \neg A]\!]^+ + \beta(A, \neg A) \cdot [\![A \wedge \neg A]\!]^- = [\![A \wedge \neg A]\!]^- = 0$$

$$[\![A \vee \neg A]\!] = \alpha(A, \neg A) \cdot [\![A \vee \neg A]\!]^+ + \beta(A, \neg A) \cdot [\![A \vee \neg A]\!]^- = [\![A \vee \neg A]\!]^- = 1$$

$$[\![A \rightarrow B]\!] = \alpha(A, B) \cdot \min\left(1, 1 - [\![A]\!] + [\![B]\!]\right) + \beta(A, B) \cdot \max\left(1 - [\![A]\!], [\![B]\!]\right) =$$
$$= \beta(\neg A, B) \cdot \min\left(1, [\![\neg A]\!] + [\![B]\!]\right) + \alpha(\neg A, B) \cdot \max\left([\![\neg A]\!], [\![B]\!]\right) = [\![\neg A \vee B]\!]$$

We said that two propositions A and B were *independent* when their conjunction could be expressed —in value— as the product of $[\![A]\!]$ and $[\![B]\!]$:

$$[\![A \wedge B]\!] \;=\; [\![A]\!] \cdot [\![B]\!]$$

It is easily shown that the necessary and sufficient condition for that to happen is:

$$\alpha_{AB} \;= \max\left([\![A]\!], [\![B]\!]\right) \quad \text{if } [\![A]\!] + [\![B]\!] \leq 1$$
$$= \max\left([\![\neg A]\!], [\![\neg B]\!]\right) \quad \text{if } [\![A]\!] + [\![B]\!] \geq 1$$

(the expression in the second row is equivalent to $1 - \min\left([\![A]\!], [\![B]\!]\right)$). Analogously,

$$\beta_{AB} \;= \min\left([\![\neg A]\!], [\![\neg B]\!]\right) \quad \text{if } [\![A]\!] + [\![B]\!] \leq 1$$
$$= \min\left([\![A]\!], [\![B]\!]\right) \quad \text{if } [\![A]\!] + [\![B]\!] \geq 1$$

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(the expression in the first row is equivalent to $1- \max ([\![A]\!], [\![B]\!])$). Note that A —always compatible with itself— is never independent from itself (except when its truth value is binary). Similarly, A and $\neg A$ —always incompatible— are never mutually independent (except when their truth value is binary).

When two propositions are *independent*, the connectives can be expressed —in value— in this way:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!]$$
$$[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] - [\![A]\!] \cdot [\![B]\!]$$
$$[\![A \rightarrow B]\!] = 1 - [\![A]\!] + [\![A]\!] \cdot [\![B]\!]$$
$$[\![A \leftrightarrow B]\!] = 1 - [\![A]\!] - [\![B]\!] + 2 \cdot [\![A]\!] \cdot [\![B]\!]$$
$$[\![A|B]\!] = [\![A]\!] \quad \text{and} \quad [\![B|A]\!] = [\![B]\!]$$

In the general case, the values of α_{AB} (or β_{AB}) are usually not known, but two considerations stand out: first, all computations can proceed if we know $[\![A]\!]$, $[\![B]\!]$ and —just— *one* of these eight values: $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$, $[\![A \rightarrow B]\!]$, $[\![A \leftrightarrow B]\!]$, $[\![A|B]\!]$, $[\![B|A]\!]$, α_{AB} or β_{AB} —from which all others are derivable at once by the above formulas (40)-(48). Also, by being given $[\![B|A]\!]$ and $[\![A|B]\!]$ (i.e. the σ_A and ν_A easily elicited from experts) we can compute every value, e.g.:

$$\alpha_{AB} = \tfrac{1}{\Delta_{AB}} [\sigma_A \cdot [\![A]\!] - \max (0, [\![A]\!] + [\![B]\!] - 1)] \quad (\text{or else} = 1 - \beta_{AB})$$
$$\beta_{AB} = \tfrac{1}{\Delta_{AB}} [\min ([\![A]\!], [\![B]\!]) - \sigma_A \cdot [\![A]\!]] \quad (\text{or else} = 1 - \alpha_{AB})$$
$$[\![A \wedge B]\!] = \sigma_A \cdot [\![A]\!] = \nu_A \cdot [\![B]\!]$$
$$[\![A \vee B]\!] = (1 - \sigma_A) \cdot [\![A]\!] + [\![B]\!] = [\![A]\!] + (1 - \nu_A) \cdot [\![B]\!]$$
$$[\![A \rightarrow B]\!] = 1 - [\![A]\!] \cdot (1 - \sigma_A) = 1 - [\![A]\!] + \nu_A \cdot [\![B]\!]$$
$$[\![B \rightarrow A]\!] = [\![A \rightarrow B]\!] + ([\![A]\!] - [\![B]\!])$$
$$[\![A \leftrightarrow B]\!] = [\![A \rightarrow B]\!] + [\![B \rightarrow A]\!] - 1$$

In certain cases the value of α is either irrelevant (e.g. when one of the propositions is binary-valued) or immediate. This last happens, for instance, when $A \vdash B$ or $B \vdash A$ (so $\alpha_{AB} = 1$) or when $A \vdash \neg B$ or $\neg B \vdash A$ (so $\alpha_{AB} = 0$). And also, in particular, when $B = A$ or when $B = \neg A$, because then $\alpha(A, A) = 1$ and $\alpha(A, \neg A) = 0$, whence we can easily derive, e.g., the principles of identity ($\vdash A \rightarrow A$), non-contradiction ($\vdash \neg(A \wedge \neg A)$) and excluded middle ($\vdash A \vee \neg A$) —by verifying that, indeed, $[\![A \rightarrow A]\!] = 1$, $[\![A \wedge \neg A]\!] = 0$ and $[\![A \vee \neg A]\!] = 1$ for any valuation.

Knowing the truth values $[\![A]\!]$ and $[\![B]\!]$ is the only requisite demanded by logics based on *functional* connectives such as Lukasiewicz's L_∞, where the values of composite propositions are given directly by our case $\oplus$-formulas (and so $\alpha_{AB} = 1$ for every A and B), which means that —viewed from our perspective— in L_∞ all *propositions are compatible*. In our treatment of logic —incidentally— assuming such a thing would be clearly unrealistic or utterly impossible, because in $\mathcal{L}$ every proposition A has a negation, and a negation $\neg A$ automatically stands as *incompatible* with A, thus contradicting the assumption (that all propositions are compatible).

The need to know the α_{AB} (or β_{AB}) parameter is a new and peculiar requisite to the present "probability logic" formalization (which has yielded us a very general Boolean multi-valued non-truth-functional logic). The α parameter roughly corresponds to what Reichenbach in his 'probability logic' called '*Kopplungsgrad*' [23]; it acts as a sort of descriptor or "memory"

of the (relative) structural disposition of propositions A and B "inside" $\mathcal{L}$ (or of $\mathbf{A}$ and $\mathbf{B}$ inside Θ). This situation parallels the one familiar in Probability.

5. THREE-VALUED LOGIC AS A SPECIAL CASE

Classical three-valued logics, especially Kleene's system of strong connectives [11] and Lukasiewicz's L_3, give the following tables for the values of connectives (where U stands for "undetermined"):

$\wedge$	0	U	1		$\vee$	0	U	1		$\rightarrow$	0	U	1
0	0	0	0		0	0	U	1		0	1	1	1
U	0	X	U		U	U	Y	1		U	U	Z	1
1	0	U	1		1	1	1	1		1	0	U	1

where $X = Y = U$ in both Kleene's and Lukasiewicz's tables, and $Z = U$ in Kleene's but $Z = 1$ in Lukasiewicz's. Clearly, the three classical Aristotelian principles do not hold in these logics (except that $\vdash A \rightarrow A$ does in Lukasiewicz's), for if A (and so $\neg A$ too) is "undetermined" (i.e. its value is U) there is no way that $\neg(A \wedge \neg A)$ and $A \vee \neg A$ can be assigned in them the 1 value they should have (and, so, these expressions can never be theorems in those logics —unless, that is, theoremhood is extended to designated values other than 1).

Now, if we abbreviate "$[\![A]\!] \in (0,1)$" by "$[\![A]\!] = U$" (as in Kleene o Lukasiewicz) the values given in the above tables coincide exactly with those that would have been computed by our formulas, *except* that X, Y and Z would remain undetermined until we knew α_{AB}. In general, our values would match Kleene's, but in certain cases they would yield differing results:

1) If $[\![A]\!] + [\![B]\!] \leq 1$ and A and B are *incompatible* (typically because $\mathbf{A} \cap \mathbf{B} = \phi$) then $X = 0$.

2) If $[\![A]\!] + [\![B]\!] \geq 1$ and A and B are *incompatible* (typically because $\mathbf{A} \cup \mathbf{B} = \Theta$) then $Y = 1$.

3) If $[\![A]\!] \leq [\![B]\!]$ and A and B are *compatible* (typically because $\mathbf{A} \subset \mathbf{B}$) then $Z = 1$.

Note that in the particular case in which B is $\neg A$ we have *always* $[\![A \wedge \neg A]\!] = 0$ (from 1) and $[\![A \vee \neg A]\!] = 1$ (from 2). We have also $[\![A \rightarrow A]\!] = 1$ (from 3). These three results are perfectly classical and in full agreement with what is to be expected from a Boolean logic.

The above analysis strongly suggests that if, for some reason, upholding classical Aristotelian principles is worthwhile, then any three-valued logic should *not* be strictly functional, but allow some *non*-functionally-determined values available in predictable cases. Something in this spirit is what Van Fraassen suggested when he advocated '*supervaluations*' [30]. Now, we contend, the same results are obtained *automatically* through our formulas.

6. THE GEOMETRY OF LOGIC: ORDER, DISTANCE, TRUTH LIKELIHOOD, INFORMATIVENESS, IMPRECISION AND ENTROPY IN $\mathcal{L}$

[*Definition:*] *Distance* (or *Boolean distance*) between two propositions or theories A and B is:

$$d(A, B) =_{df} 1 - [\![A \leftrightarrow B]\!] = [\![A \vee B]\!] - [\![A \wedge B]\!] = |[\![A]\!] - [\![B]\!]| + 2 \cdot \beta_{AB} \cdot \Delta_{AB}$$

(Naturally, if $A \vdash B$ then $d(A, B) = [\![B]\!] - [\![A]\!]$.)

[*Definition:*] *Compatible distance* between two propositions or theories A and B is:

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$$d^+(A, B) = |[\![A]\!] - [\![B]\!]| = 1 - [\![A \leftrightarrow B]\!]^+$$

This distance can also be expressed in this way:

$$d^+(A, B) = [\![A]\!] + [\![B]\!] - 2 \min([\![A]\!], [\![B]\!]) \tag{49}$$

It is interesting to notice that:

– both d and d^+ are really distances (in the mathematical sense)

– the distance between two propositions or theories is the same as the distance between their negations or antitheses (i.e. $d(A, B) = d(\neg A, \neg B)$, and the same holds for d^+)

– the Boolean distance $d(A, B)$ equals the value of the *symmetric difference* between A and B [defined by $A \bigtriangleup B =_{df} (A \wedge \neg B) \vee (\neg A \wedge B)$] so we have:

$$d(A, B) = [\![A \bigtriangleup B]\!] \quad \text{and} \quad d(A, \bot) = [\![A]\!]$$

Moreover, we have the following miscellaneous properties —that hold in any metric Boolean algebra and were comprehensively studied by David Miller in his 'Geometry of Logic' paper [18]:

$$d(A, B) = d(A \bigtriangleup B, \bot)$$
$$d(A, B) + d(\neg A, B) = d(A, \neg A) = 1$$
$$d(A \wedge B, A) = d(B, A \vee B)$$
$$d(A, B) = d(A \wedge B, A \vee B)$$

(Incidentally, the last two formulas show the quadrilateral ACBD —where $C \equiv A \wedge B$ and $D \equiv A \vee B$— is a rectangle.)

In view of the previous relations, we could define a *truth likelihood* value for A —approximating Popper's (and Miller's) *verisimilitude* measure— by making it to equal the distance between A and falsehood, i.e. $d(A, \bot)$. We obtain, immediately:

$$d(A, \bot) = d(\top, \bot) - d(\top, A) = 1 - d(A, \top) = 1 - d(A \bigtriangleup \top, \bot) = 1 - d(\neg A, \bot) = 1 - [\![\neg A]\!] = [\![A]\!]$$

So here we have a further interpretation of our "truth values" $[\![A]\!]$ in terms of *truth likelihood* or Popper's *verisimilitude* (a promising concept —only partially developed— which initially led this author to suggest it as a means to measure the *truth content* of a theory and to evaluate the distance to other theories). We remark that we might as well consider $[\![A]\!]$ as a rough measure of *partial truth* or "truth content" of A. In a similar spirit, we are reminded that a rather puzzled Scott [25] once suggested the "truth value" $[\![A]\!]$ of many-valued logics could be interpreted as one (meaning truth) less the *error* of A (or rather of a measure settling the truth of A) or the *inexactness* of A (as a theory); in this framework, it comes out that, in our terms, $[\![A]\!] = 1 - \varepsilon_A$ and $\varepsilon_A = 1 - [\![A]\!] = d(A, \top)$.

Further along the line of exploring related ideas, we now observe that, for any propositional letters P and Q, any uniform truth valuation yields $[\![P]\!] = [\![\neg P]\!] = .50$, $[\![P \wedge Q]\!] = .25$ and $[\![P \vee Q]\!] = .75$, which is like saying that, if all letters are equiprobable, the given values are the probability of the given proposition being true (a number that Bar-Hillel and Hintikka call, appropriately, "truth-table probability" [10]). So this value's complement to one should seemingly correspond to the amount of information —in a loose sense— we have when the proposition is true. This is precisely what Bar-Hillel and Hintikka define as "degree of information", *semantical information* or *informativeness* $I(A)$ of a proposition A. (Viewed in our terms, $I(A)$ equals $1 - [\![A]\!]$, or $[\![\neg A]\!]$.) These authors use the concept to model the reasoning process, assumed to be driven by an increase both of informativeness and parsimony. What is interesting is that they compute the informativeness of composite propositions by combining

them according to rules which amount to a multi-valued truth-value non-functional calculus of the functional-cum-supervaluation type. So, they suggest that equalities

$$I(A \lor B) = I(A) \cdot I(B) \qquad \text{and}$$
$$I(A \land B) = I(A) + I(B) - I(A) \cdot I(B)$$

are appropriate, but that $I(A \land \neg A)$ and $I(A \lor \neg A)$ should be assigned, respectively, the 1 and 0 values (supervalues, in fact, in Van Fraassen's sense [30]). All these values coincide with those predicted by our formulas: indeed, if A and B are assumed *independent* the above equations follow automatically from our formulas, while if $B = \neg A$ the 1 and 0 are a consequence of the zero compatibility between A and $\neg A$.

We now define a new measure in $\mathcal{L}$:

[*Definition:*] *Imprecision* of a proposition (or theory) A in $\mathcal{L}$ (or $\hat{\mathcal{L}}$) is the value for A of the function

$$f : \mathcal{L} \longrightarrow [0,1] \quad \text{such that} \quad f(A) = [\![A \leftrightarrow \neg A]\!]^{+} \tag{50}$$

(We could as well have used the word "fuzziness" instead of *imprecision*, but the former term — aside from its technical adequacy— is so much tied to Fuzzy Set Theory that it is better to leave it there; moreover, "imprecision" is a rather neutral descriptor, aptly covering *vagueness* —a formal imprecision built *inside* the language— as well as *uncertainty* —of a more epistemological nature.) It is immediate that:

$$f(A) = 1 - d^{+}(A, \neg A)$$

(and note that, had we used $[\![A \leftrightarrow \neg A]\!]$ in (50) or $d(A, \neg A)$ in the last formula, we would have had an identically zero function f —and a useless definition—).

As is easily deducible —from (49)—, we have:

$$f(A) = 2 \min \left([\![A]\!], 1 - [\![A]\!] \right) \tag{51}$$

which is equivalent to saying that the imprecision of a proposition A equals twice the error we make when we evaluate on A the truth of the law of non-contradiction or of the excluded middle by considering there really *is* maximum compatibility between A and $\neg A$ (This equivalence is trivial to verify, since the right-hand expression in (51) equals $2 \times [\![A \land \neg A]\!]^{+}$ or, equivalently, $2 \times (1 - [\![A \lor \neg A]\!]^{+})$.)

Following this line of reasoning, we observe:

– Classical (two-valued) logic is the special case of ours in which *all* propositions in $\mathcal{L}$ have *zero* imprecision; if there exists just one imprecise proposition then we are in our general logic.

– Ordinary multi-valued logics —like L_{∞}— are the ones where at least one proposition has non-zero imprecision. This measure being —as it is— an error function, imprecision is here just the degree in which these logics fail to distinguish contradictions (their lack of "resolving power", so to speak) affecting the proposition involved.

– Measure f fulfills a number of conditions that suggest it can be used as an *entropy* measure (in the sense of De Luca & Termini [5]); these conditions are:

    – $f(A) = f(\neg A)$ (i.e. affirmation and negation of A are both equally imprecise)

    – There are in $\mathcal{L}$ 'minimal' propositions (i.e. A such that $f(A) = 0$) as well as 'maximal' propositions (i.e. A such that $f(A) \in (0,1]$)

    – If $[\![A]\!] < [\![B]\!] < 1/2$ or $[\![A]\!] > [\![B]\!] > 1/2$ then $f(A) < f(B)$

    – A proposition reaches its highest imprecision value when its truth value is $1/2$.

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(Note, however, that the f measure sometimes plays —often forcedly— the role of a measure of *ignorance*. But, if so, assimilating ignorance to maximum imprecision seems too simplistic (ignorance would then reduce to the $[\![A]\!] = 1/2$ case). Instead, a situation of ignorance seems to be better and more realistically modeled not just inside a proposition, with no concern for alternatives, but rather in the form of a certainty function —plausibly of the sub-additive type— like the one we mention elsewhere [24] and that we call —after Shafer [27]— 'vacuous'.)

7. THE SPECIAL CASES

The valuation logic we have defined in this paper is general and powerful enough to include familiar, well-known logics as particular instances of itself. This is the case of (a) standard *classical* logic —normally considered to be necessarily two-valued— and of (b) standard (but *non-classical*) many-valued logics like Łukasiewicz-Tarski's $Ł_\infty$ infinite-valued logic [15] —or, more simply, the three-valued formalisms by Kleene, Bochvar, Łukasiewicz (and others) we already treated above as a special case of ours. But our logic may also help to explain certain scattered results or ideas that are not normally part of the logic mainstream. We have mentioned already Boole's [2], MacColl's [16] and Peirce's [19] views on propositional functions, or Scott's view of error [25], among others. All of them can be naturally explained inside our frame. Thus:

Classical two-valued logic is the special case of our general logic in which *every* proposition is *binary* (i.e. $\forall A \in \mathcal{L}$ $[\![A]\!] \in \{0, 1\}$) or, equivalently (by (51)), in which *every* proposition has *zero* imprecision (i.e. $\forall A \in \mathcal{L}$ $f(A) = 0$).

Łukasiewicz-Tarski's $Ł_\infty$ logic [15], as ours, generalizes classical (two-valued) logic in the sense that it allows the members of the lattice of propositions to take values in [0,1] other than 0 or 1. Both systems of many-valued logic include classical two-valued logic as a special case. Nevertheless, while we give up *functionality*, $Ł_\infty$ gives up *Booleanity*. In fact, if the $\{0, 1\}$ truth set is extended to [0,1] those two properties of classical logic cannot be both upheld, and either must be given up.

Our logic admits $Ł_\infty$ as a special case since, indeed, $Ł_\infty$ behaves exactly as ours would do if *no* proposition in $\mathcal{L}$ could be recognized as a *negation* of some other and, then, it would be assigned systematically the ⊕-case connective formulas. Naturally that would give an error in the values of composite propositions involving non-fully-compatible propositions, but it would also restore the lost truth-functionality of two-valued logic. Recalling what the ⊕-case stands for, $Ł_\infty$ amounts —from our perspective— to viewing *all* propositions as having *always* maximal mutual compatibility. That means that $Ł_\infty$ conceives *all* propositions as *nested* (i.e. for every A and A', either A ⊂ A' or A' ⊂ A where A is $\rho(A)$). (Such a picture is strongly reminiscent of Shafer's [27] description of conditions present in what he calls *consonant* valuations.)

Thus, in our terms, $Ł_\infty$ is a special case of our general logic (or, better, an approximation to it under special assumptions) characterized by systematic usage of the ⊕-formulas. This approximation (a simplification, actually) implies maximum compatibility between all pairs of propositions of $\mathcal{L}$, which entails the fiction of a *total*, linear order ⊢ in $\mathcal{L}$ —and ⊂ in $\mathcal{P}(\Theta)$— (that means a coherent, negationless universe) and which is probably the best assumption we can make when information on propositions is lacking and negations are not involved —or cannot be identified as such. (Such an option is as legitimate as that of assuming, in the absence of information on propositions, that these are independent.) In $Ł_\infty$ obvious negations may be a problem, but it can be solved by applying error-correcting *supervaluations* —that our logic supplies automatically. (And note that, in particular, the error incurred in by $Ł_\infty$ when failing to distinguish between a proposition and its negation —thus not being able to recognize a contradiction— is just the quantity we called *imprecision*.)

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