New results on metric-locating-dominating sets of graphs

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Abstract

A dominating set S of a graph is a metric-locating-dominating set if each vertex of the graph is uniquely distinguished by its distances from the elements of S, and the minimum cardinality of such a set is called the metric-location-domination number. In this paper, we undertake a study that, in general graphs and specific families, relates metric-locating-dominating sets to other special sets: resolving sets, dominating sets, locating-dominating sets and doubly resolving sets. We first characterize classes of trees according to certain relationships between their metric-location-domination number and their metric dimension and domination number. Then, we show different methods to transform metric-locating-dominating sets into locating-dominating sets and doubly resolving sets. Our methods produce new bounds on the minimum cardinalities of all those sets, some of them involving parameters that have not been related so far.

Keywords: metric-locating-dominating set, resolving set, dominating set, locating-dominating set, doubly resolving set.

1 Introduction and preliminaries

Let $G = (V(G), E(G))$ be a finite, simple, undirected, and connected graph of order $n = |V(G)| \geq 2$; the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest u-v path. We say that a subset $S \subseteq V(G)$ is a *resolving set* of G if for every $x, y \in V(G)$ there is a vertex $u \in S$ such that $d(u, x) \neq d(u, y)$ (it is said that S *resolves* $\{x, y\}$), and the minimum cardinality of such a set is called the *metric dimension* of G, written as $\dim(G)$. When S is also a *dominating set* of G (i.e., every $x \in V(G) \setminus S$ has a neighbor in S), then S is called a *metric-locating-dominating set* (MLD-set for short). The *metric-location-domination number* (resp., *domination number*), written

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as $\gamma_M(G)$ (resp., $\gamma(G)$), is the minimum cardinality of an MLD-set (resp., dominating set) of G.

MLD-sets were introduced in 2004 by Henning and Oellermann [9] combining the usefulness of resolving sets, that roughly speaking differentiate the vertices of a graph, and dominating sets, which cover the whole vertex set. Resolving sets were defined in the 1970s by Slater [16], and independently by Harary and Melter [7], whereas dominating sets were introduced in the 1960s by Ore [15]. Both types of sets have received much attention in the literature because of their many and varied applications in other areas; for example, resolving sets serve as a tool for robot navigation, and dominating sets are helpful to design and analyze communication networks (see [1] and [8] for more information on resolvability and domination). Although MLD-sets are hard to handle, for entailing the complexity of the other two concepts, they have been studied in several papers, for instance [3, 10], and further generalized in other works such as [13, 18].

This paper first focuses on the intrinsic relations between MLD-sets and resolving sets and dominating sets. Indeed, the corresponding parameters for all those sets satisfy by definition

$$\max\{\dim(G), \gamma(G)\} \leq \gamma_M(G) \leq \dim(G) + \gamma(G). \tag{1}$$

We consider here this chain restricted to trees; specifically, we characterize the trees for which equality occurs in (1), thereby continuing the work of Henning and Oellermann [9] that characterized the trees T with $\gamma_M(T) = \gamma(T)$. Analog characterizations of trees in terms of other related invariants can be found in [2, 6].

We also compare MLD-sets with other subsets of vertices defined by Slater [17] that are naturally connected to them: the *locating-dominating sets*. They are dominating sets that distinguish vertices by using neighborhoods instead of distances; more formally, a *locating-dominating set* (LD-set for brevity) of G is a dominating set $S \subseteq V(G)$ so that $N(x) \cap S \neq N(y) \cap S$ for every $x, y \in V(G) \setminus S$. The minimum cardinality of such a set, denoted by $\gamma_L(G)$, is the *location-domination number* of G. Clearly, an LD-set is an MLD-set, and so it is also a resolving set; consequently,

$$\dim(G) \leq \gamma_M(G) \leq \gamma_L(G). \tag{2}$$

Regarding the relation between $\gamma_M(G)$ and $\gamma_L(G)$, we propose a way to obtain LD-sets from MLD-sets which helps us to extend the following result due to Henning and Oellermann. (See [3, 10] for more properties of chain (2), and [8] for specific results on LD-sets.)

Theorem 1.1. [9] *For any tree T, it holds that $\gamma_L(T) < 2\gamma_M(T)$. However, there is no constant c such that $\gamma_L(G) \leq c\gamma_M(G)$ for all graphs G.*

We finally find relationships between MLD-sets and other subsets for which, so far as we are aware, no direct connection is known: the *doubly resolving sets*. Cáceres et al. [4] introduced doubly resolving sets as a tool for computing the metric dimension of Cartesian products of graphs. These sets, that somehow distinguish vertices in two ways by means of distances, are formally defined as follows. Two vertices $u, v \in V(G)$ *doubly resolve* a pair $\{x, y\} \subseteq V(G)$ if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set $S \subseteq V(G)$ is a *doubly resolving set* of G if every pair $\{x, y\} \subseteq V(G)$ is doubly resolved by two vertices of S (it is said that S *doubly resolves* $\{x, y\}$), and the minimum cardinality of such a set is denoted by $\psi(G)$. Thus, a doubly resolving set is also a resolving set, and so

$$\dim(G) \leq \psi(G). \tag{3}$$

Although it is not straightforward to deduce any relation between $\psi(G)$ and $\gamma_M(G)$ from their definitions, we provide here bounds on $\psi(G)$ in terms of $\gamma_M(G)$ by generating doubly resolving sets from MLD-sets. We thus obtain, for specific classes and general graphs, similar chains to expression (2) that include $\psi(G)$. (See for example [11, 12, 14] for more results on doubly resolving sets and relations with other types of sets.)

The paper is organized as follows. In Section 2, we characterize the extremal graphs of expression (1) restricted to trees. We then show in Sections 3 and 4 how to construct LD-sets and doubly resolving sets from MLD-sets in arbitrary graphs and specific families, thus producing bounds on the corresponding parameters. Specifically, we prove in Section 3 that $\gamma_L(G) \leq \gamma_M^2(G)$ whenever G has no cycles of length 4 or 6 but, for arbitrary graphs, any upper bound on $\gamma_L(G)$ in terms of $\gamma_M(G)$ has growth at least exponential; in Section 4 we achieve the bounds $\psi(G) \leq \gamma_M(G)$ for graphs G with girth at least 5, and $\psi(G) \leq \gamma_M(G) + \gamma(G)$ for any graph G. We conclude the paper with some remarks and open problems in Section 5.

2 MLD-sets of trees

Henning and Oellermann [9] provided a formula for the metric-location-domination number of trees and characterized the trees T with $\gamma_M(T) = \gamma(T)$, giving both results in terms of support vertices (see Theorem 2.1 below). Recall that a vertex u of a tree T is a *support vertex* whenever it is adjacent to some *leaf* (i.e., a vertex of degree 1), and it is a *strong support vertex* if there are two or more leaves adjacent to u. We denote by $\mathcal{S}(T)$ (resp., $\mathcal{S}'(T)$) the set of support (resp., strong support) vertices of T; $\ell'(T)$ is the number of leaves adjacent to a strong support vertex.

Theorem 2.1. [9] *For any tree T, the following statements hold:*

(i) $\gamma_M(T) = \gamma(T) + \ell'(T) - |\mathcal{S}'(T)|$.

(ii) $\gamma_M(T) = \gamma(T)$ *if and only if* $\mathcal{S}'(T) = \emptyset$.

As the authors observed, any MLD-set must contain, for each support vertex u, either all the leaves adjacent to u or all but one of the leaves adjacent to u as well as vertex u. This observation leads us to see that $\ell(T) \leq \gamma_M(T)$, where $\ell(T)$ denotes the total number of leaves of any tree T. Hence, since $\dim(T) < \ell(T)$ (see [5]), expression (1) now becomes

$$\max\{\ell(T), \gamma(T)\} \leq \gamma_M(T) \leq \dim(T) + \gamma(T). \tag{4}$$

This section follows the same spirit as Henning and Oellermann [9], who characterized in statement (ii) of Theorem 2.1 the extremal trees for expression (4) with $\gamma_M(T) = \gamma(T)$. Indeed, we characterize the remaining extremal cases: $\gamma_M(T) = \dim(T) + \gamma(T)$ in Theorem 2.2, and $\gamma_M(T) = \ell(T)$ in Theorem 2.3. To do this, we first recall the following terminology extracted from [5]. A vertex $u \in V(T)$ of degree at least 3 is called a *major vertex* of T, and a leaf $x \in V(T)$ is a *terminal vertex* of u if the major vertex closest to x is u. The *terminal degree* of u, written as $\mathrm{ter}(u)$, is the number of its terminal vertices, and u is an *exterior major vertex* of T if it has positive terminal degree; we denote by $\mathrm{Ex}(T)$ the set of exterior major vertices of T.

Theorem 2.2. *Let T be a tree different from a path. Then, the following statements are equivalent:*

(i) $\gamma_M(T) = \dim(T) + \gamma(T)$.

(ii) $\dim(T) = \ell'(T) - |\mathcal{S}'(T)|$.

(iii) Every $u \in \mathrm{Ex}(T)$ with $\mathrm{ter}(u) \geq 2$ is the support vertex of each of its terminal vertices.

(iv) Any path joining two leaves of T at distance greater than 2 contains at least two major vertices.

Proof. ($i \iff ii$) This equivalence is guaranteed by statement (i) of Theorem 2.1.
($ii \iff iii$) It is known that any set $S \subseteq V(T)$ composed by all but one of the terminal vertices of each $u \in \mathrm{Ex}(T)$ is a minimum resolving set of T (see Theorem 5 of [5] and its proof). Thus, let $S \subseteq V(T)$ be such a set, and note that, as any strong support vertex u belongs to $\mathrm{Ex}(T)$, then S must contain all but one of the leaves adjacent to u; consequently, $\dim(T) \geq \ell'(T) - |\mathcal{S}'(T)|$. Hence, if $\dim(T) = \ell'(T) - |\mathcal{S}'(T)|$ then S is only formed by all but one of the leaves adjacent to each strong support vertex.

Let $u \in \mathrm{Ex}(T)$ with $\mathrm{ter}(u) \geq 2$, and let x and y be two terminal vertices of u. On the contrary, let us assume for instance that $x \notin N(u)$, and let v be the only neighbor of u in the u-x path. Clearly, u is the support vertex of y because otherwise there is $v' \in N(u) \setminus \{y\}$ in the u-y path but $\{v, v'\}$ would not be resolved by S since no vertex in either the u-x path or the u-y path is in S (by construction of set S). Further, one of the leaves adjacent to u, say z, is not in S but reasoning as before yields that S does not resolve $\{v, z\}$; a contradiction since S is a resolving set of T. Therefore, u is the support vertex of each of its terminal vertices.

Reciprocally, let us suppose that each $u \in \mathrm{Ex}(T)$ with $\mathrm{ter}(u) \geq 2$ is the support vertex of its terminal vertices, which implies that $u \in \mathcal{S}'(T)$. Thus, any set $S \subseteq V(T)$ composed by all but one of the leaves adjacent to each strong support vertex is a resolving set since it contains all but one of the terminal vertices of each exterior major vertex. Moreover, $|S| = \ell'(T) - |\mathcal{S}'(T)|$ is minimum since we have noticed in the beginning of this proof that $\dim(T) \geq \ell'(T) - |\mathcal{S}'(T)|$. Therefore, $\dim(T) = \ell'(S) - |\mathcal{S}'(T)|$.

($iii \iff iv$) Let us suppose that any $u \in \mathrm{Ex}(T)$ with $\mathrm{ter}(u) \geq 2$ is adjacent to each of its terminal vertices. Given two leaves x and y at distance greater than 2, there is at least one major vertex u in the x-y path since G is not isomorphic to a path. Further, this vertex u cannot be unique since otherwise x and y are terminal vertices of u with either $d(u,x) \geq 2$ or $d(u',y) \geq 2$, that is, u is not the support vertex of either x or y, which is impossible. Reciprocally, let us assume statement (iv) to hold true, and let $u \in \mathrm{Ex}(T)$ with $\mathrm{ter}(u) \geq 2$. It is easily seen that no terminal vertex x of u verifies $d(u,x) \geq 2$ since otherwise $d(x,y) \geq 3$ for any other terminal vertex y of u, setting u as the only major vertex in the x-y path, which contradicts statement (iv). □

We want to remark that, although the preceding result does not consider paths, it is easy to check that P_3 is the only path satisfying $\gamma_M(P_n) = \gamma(P_n) + \dim(P_n)$ since $\gamma_M(P_n) = \gamma(P_n)$ whenever $n \neq 3$ (by statement (i) of Theorem 2.1), and $\dim(P_n) = 1$ for all $n \geq 1$ (see for instance [5]).

Theorem 2.3. *Let T be a tree different from the path P_2. Then, the following statements are equivalent:*

(i) $\gamma_M(T) = \ell(T)$.

(ii) $\gamma(T) = |\mathcal{S}(T)|$.

(iii) For every $u \in V(T)$, there exists a leaf at distance at most 2 from u.

Proof. ($i \iff ii$) By statement (i) of Theorem 2.1, $\gamma_M(T) = \ell(T)$ is equivalent to $\gamma(T) = \ell(T) - \ell'(T) + |\mathcal{S}'(T)|$ but $\ell(T) - \ell'(T)$ is the number of non-strong support vertices, i.e. $|\mathcal{S}(T)| - |\mathcal{S}'(T)|$, which gives $\gamma(T) = |\mathcal{S}(T)|$.
($ii \iff iii$) Let $S \subseteq V(T)$ be a minimum dominating set of T. Observe that, when replacing any leaf of S by its corresponding support vertex, the resulting set is still a dominating set without leaves and containing each support vertex of T (since all leaves must be dominated by vertices of S). Thus, let us assume $\mathcal{S}(T) \subseteq S$, which implies that $\gamma(T) \geq |\mathcal{S}(T)|$ since S has minimum cardinality. Therefore, $\gamma(T) = |\mathcal{S}(T)|$ if and only if $\mathcal{S}(T)$ is a minimum dominating set of T, i.e., every $u \in V(T)$ is either in $\mathcal{S}(T)$ or has a neighbor in $\mathcal{S}(T)$. Equivalently, u is either a support vertex ($d(u,x) = 1$ for some leaf $x \in N(u)$) or a leaf ($d(u,u) = 0$) or adjacent to a support vertex, say v ($d(u,x) = 2$ for some leaf $x \in N(v)$). □

3 MLD-sets versus LD-sets

In view of the relationship $\gamma_M(G) \leq \gamma_L(G)$ given in expression (2), it is natural to look for upper bounds on $\gamma_L(G)$ in terms of $\gamma_M(G)$ as Henning and Oellermann [9] did in Theorem 1.1. In this section, we extend this result by first providing a wide class of graphs G (that includes trees) whose MLD-sets can be transformed into LD-sets; this leads us to the upper bound $\gamma_L(G) \leq \gamma_M^2(G)$. We also prove, for arbitrary graphs G, that broadly speaking any upper bound on $\gamma_L(G)$ as a function of $\gamma_M(G)$ has growth at least exponential.

Let G be a graph not having the cycles C_4 or C_6 as a subgraph, and let S be any subset of $V(G)$. We assign to every pair $u, v \in S$ a set of vertices $\pi(u,v)$ given by $\{u', v'\}$ whenever there exists a u-v path (u, u', v', v) (that is unique because of the C_4- and C_6-free condition), and $\emptyset$ otherwise. Let $\pi(S) = \bigcup_{u,v\in S} \pi(u,v)$.

Proposition 3.1. *Let G be a graph not containing C_4 or C_6 as a subgraph. For every MLD-set $S \subseteq V(G)$, the set $S \cup \pi(S)$ is an LD-set of G. Consequently, $\gamma_L(G) \leq \gamma_M^2(G)$.*

Proof. Let $\overline{S} = S \cup \pi(S)$. We need to check that $N(x) \cap \overline{S} \neq N(y) \cap \overline{S}$ for every $x, y \in V(G) \setminus \overline{S}$ (since $S \subseteq \overline{S}$ is a dominating set of G). On the contrary, let us assume the existence of two vertices $x, y \in V(G) \setminus \overline{S}$ so that $N(x) \cap \overline{S} = N(y) \cap \overline{S}$. Since S is a dominating set, then there are $u, v \in S$ such that $x \in N(u)$ and $y \in N(v)$. If $u \neq v$ then either $x \notin N(v)$ or $y \notin N(u)$ (otherwise (u, x, v, y) would be a cycle on 4 vertices of G, which is impossible), and so $N(x) \cap \overline{S} \neq N(y) \cap \overline{S}$. Hence, $u = v$. Moreover, since the existence of a vertex $v \in N(x) \cap N(y)$ different from u produces the cycle (u, x, v, y), which cannot exist, then we have that $N(x) \cap N(y) = \{u\}$, and so $N(x) \cap S = N(y) \cap S = \{u\}$.

Let $z \in V(G) \setminus S$ be a neighbor of either x or y (that exists because otherwise $N(x) = N(y) = \{u\}$ and so the pair $\{x, y\}$ is not resolved by S, which is impossible since S is a resolving set). Assuming without loss of generality $z \in N(x)$, we have that $z \notin N(y)$ since we have seen that $N(x) \cap N(y) = \{u\}$. On the other hand, there is a vertex $u' \in S$ dominating z. If $u' \neq u$ then $\pi(u, u') = \{x, z\} \subseteq \pi(S)$; a contradiction since $x \notin \pi(S)$. Therefore, $u = u'$ and $N(z) \cap S = \{u\}$.

Let $z' \in V(G) \setminus S$ be such that either $z' \in N(x) \setminus N(z)$ or $z' \in N(z) \setminus N(x)$ (which exists since otherwise $N[x] = N[z] = \{x, z, u\}$ and so the pair $\{x, z\}$ is not resolved by S). If $z' \in N(x) \setminus N(z)$ (analogous for the case $z' \in N(z) \setminus N(x)$) then there is $u'' \in S$ dominating z', different from u since otherwise we could obtain the cycle (u, z', x, z). Hence, $\pi(u, u'') = \{x, z'\} \subseteq \pi(S)$, a contradiction with $x \notin \overline{S} = S \cup \pi(S)$.

We have thus proved that $\overline{S}$ is an LD-set of G. To complete the proof, observe that, for any pair $u, v \in S$, the set $\pi(u, v)$ may intersect either S or $\pi(u', v')$ for another pair $u', v' \in S$. Consequently, $|\pi(S)| \leq 2\binom{|S|}{2}$ and so $|\overline{S}| \leq |S| + |\pi(S)| \leq |S|^2$. Therefore, choosing S with minimum cardinality yields $\gamma_L(G) \leq |\overline{S}| \leq |S|^2 = \gamma_M^2(G)$, as required. □

Henning and Oellermann [9] showed that there is no linear upper bound on $\gamma_L(G)$ in terms of $\gamma_M(G)$ by building up an appropriate family of graphs G with $\gamma_L(G) > c\gamma_M(G)$ for any constant c. However, their construction satisfied $\gamma_L(G) < \gamma_M^2(G)$, and so to extend their result to other polynomial orders a new family of graphs is required. We next provide such a family of graphs in the proof of the following theorem that, in particular, shows that there is no polynomial upper bound on $\gamma_L(G)$ depending on $\gamma_M(G)$.

Theorem 3.2. *Let $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $\gamma_L(G) \leq \Phi(\gamma_M(G))$ for every graph G. Then, $\Phi(x) \geq 2^{x-2} - 1$ for all $x \geq 0$.*

Proof. To prove the result, we construct a family of graphs G_s such that $\gamma_L(G_s) \geq 2^{\gamma_M(G_s)-2} - 1$ as follows. For a positive integer s, let $A = \{a_i : i = 0, ..., 2^{s+1} - 1\}$, $B = \{b_i : i = 0, ..., 2^{s+1} - 1\}$ and $C = \{c_i : i = 0, ..., s\}$. The graph G_s has vertex set $V(G_s) = \{p\} \cup A \cup B \cup C$ and edge set given by the pairs pa_i and a_ib_i for every $i \in \{0, \ldots, 2^{s+1} - 1\}$, and b_ic_j whenever the binary representation of i has a 1 in its j-th position (Figure 1 illustrates the case for $s = 2$).

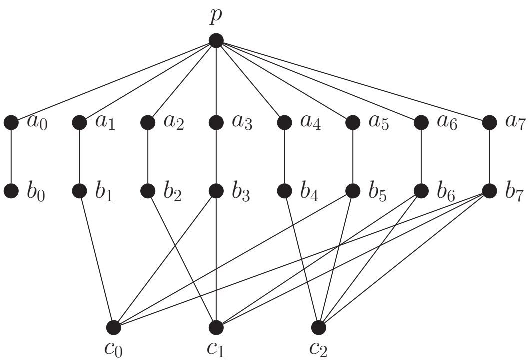

Figure 1: The graph G_2.

It is easy to check that the set $\{C\} \cup \{p\} \cup \{b_0\}$ is an MLD-set of G_s, which implies that

$$\gamma_M(G_s) \leq s + 3. \tag{5}$$

On the other hand, each LD-set S of G_s must contain, for each $i \in \{0, \ldots, 2^{s+1} - 1\}$ but at most one, either vertex a_i or vertex b_i. Indeed, if there exist i and j such that $a_i, b_i, a_j, b_j \notin S$ then

$$N(a_i) \cap S = N(a_j) \cap S = \begin{cases} \{p\} & if \quad p \in S \\ \emptyset & \quad otherwise \end{cases}$$

But this is impossible since S is an LD-set of G. Thus,

$$\gamma_L(G_s) \geq 2^{s+1} - 1. \tag{6}$$

Therefore, combining inequalities (5) and (6) yields $\gamma_L(G_s) \geq 2^{\gamma_M(G_s)-2} - 1$, which gives the result. □

4 Doubly resolving sets from MLD-sets

In this section, we show how useful MLD-sets can be for constructing doubly resolving sets of graphs. Indeed, we design a method that, given an MLD-set of a graph G with

$g(G) \geq 5$, produces a doubly resolving set of the same size (recall that g(G) is the *girth* of G, i.e., the length of a shortest cycle of G); when G is any graph, our method also implies the use of dominating sets. In both cases we obtain bounds that involve $\psi(G)$ and $\gamma_M(G)$ (Proposition 4.4 and Theorem 4.6), giving rise to chains similar to expression (2) but including the invariant $\psi(G)$ (Corollaries 4.5 and 4.7). We start with the following two lemmas that are the key to relate MLD-sets to doubly resolving sets.

Lemma 4.1. *Let G be a graph and let S be an MLD-set of G. Then, every pair $x, y \in V(G) \setminus S$ is doubly resolved by S.*

Proof. Given any two vertices $x, y \in V(G) \setminus S$, we shall prove that there exist $u, v \in S$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Indeed, let $u', v' \in S$ such that $x \in N(u')$ and $y \in N(v')$, which exist since S is a dominating set of G. We distinguish two cases:

1. $u' = v'$. We have $d(u', x) - d(u', y) = 1 - 1 = 0$ and, since S is a resolving set, there is a vertex $w \in S \setminus \{u'\}$ such that $d(w, x) \neq d(w, y)$, which implies that $d(w, x) - d(w, y) \neq 0$. Therefore, we set $\{u, v\} = \{u', w\}$.

2. $u' \neq v'$. We can assume that $x \notin N(v')$ and $y \notin N(u')$ (otherwise we proceed as in the previous case) and so $d(u', y), d(v', x) > 1$. Hence, $d(u', x) - d(u', y) < 0$ and $d(v', x) - d(v', y) > 0$, so we can take $\{u, v\} = \{u', v'\}$.

□

Lemma 4.2. *Let S be an MLD-set of a graph G, and let $u \in S$ and $x \in V(G) \setminus S$ such that $\{u, x\}$ is not doubly resolved by S. Then, $N(x) \cap S = \{u\}$. Furthermore, x is the only vertex of $V(G) \setminus S$ so that $\{u, x\}$ is not doubly resolved by S.*

Proof. First, we prove that $N(x) \cap S = \{u\}$. Observe that, for every $v \in S$,

$$d(u, u) - d(u, x) = d(v, u) - d(v, x) \tag{7}$$

since pair $\{u, x\}$ is not doubly resolved by S (in particular by u and v). As S is a dominating set, there is some vertex $v^* \in S$ with $x \in N(v^*)$, and so setting $v = v^*$ in (7) yields $-d(u, x) = d(v^*, u) - 1$. Necessarily, $d(v^*, u) = 0$ and $d(u, x) = 1$, which implies that $v^* = u$ and (7) becomes

$$d(v, x) = d(v, u) + 1 \tag{8}$$

for each $v \in S$. Hence, $d(v, x) > 1$ whenever $v \in S \setminus \{u\}$, and so $N(x) \cap S = \{u\}$.

Now, we show that there is no other vertex $x' \in V(G) \setminus S$ different from x so that $\{u, x'\}$ is not doubly resolved by S. Let us assume on the contrary the existence of such a vertex x'. Reasoning as above with vertex x, we easily get $d(v, x') = d(v, u) + 1$ for any $v \in S$, which combined with (8) gives $d(v, x) = d(v, x')$; a contradiction since S is a resolving set of G. □

Observation 4.3. *For any subset of vertices S of a graph G, it is obvious that any pair $\{u, v\} \subseteq S$ is doubly resolved by u and v.*

Regarding Lemmas 4.1 and 4.2 and Observation 4.3, it is natural to ask whether MLD-sets S doubly resolve pairs $\{u, x\}$ with $u \in S$ and $x \in V(G) \setminus S$, thus implying that MLD-sets would be doubly resolving sets (and so $\psi(G) \leq \gamma(G)$). Unfortunately, this is not true in general as graph H_t depicted in Figure 2 shows because the set $\{a_1, \ldots, a_t\}$ is an MLD-set of H_t but it does not doubly resolve any pair $\{a_i, c_i\}$; also, $\psi(H_t) = 2t = 2\gamma_M(G)$. Furthermore, even adding the extra condition $\mathrm{g}(G) \geq 5$, MLD-sets are not necessarily doubly resolving sets (see the graph H'_t of Figure 3). However, for this class of graphs, we next describe how to modify the elements of any MLD-set to obtain a doubly resolving set, thereby producing the bound $\psi(G) \leq \gamma_M(G)$.

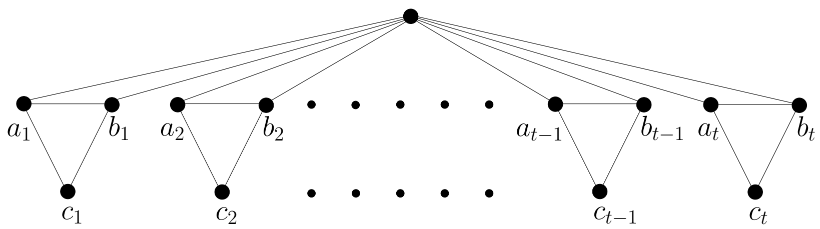

Figure 2: The graph H_t, with $t \geq 2$, for which $\{a_1, \ldots, a_t\}$ and $\{b_1, \ldots, b_t, c_1, \ldots, c_t\}$ are, respectively, a minimum MLD-set and a minimum doubly resolving set.

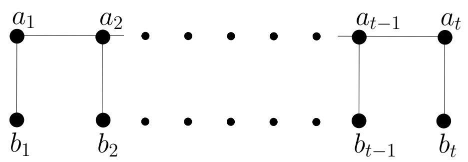

Figure 3: The graph H'_t, with $t \geq 1$, where $\{a_1, \ldots, a_t\}$ is a minimum MLD-set but it does not doubly resolve any pair $\{a_i, b_i\}$.

Let S be an MLD-set of a graph G with $\mathrm{g}(G) \geq 5$, and observe that every $u \in S$ has at most one neighbor of degree 1 in $V(G) \setminus S$ (otherwise S would not be a resolving set). Thus, let $\overline{u}$ be such a neighbor if it exists, and u otherwise. Note that, by construction, $\overline{u} \neq \overline{v}$ for any two different vertices $u, v \in S$. The following result proves that changing each $u \in S$ to $\overline{u}$ yields a doubly resolving set of G whenever $g(G) \geq 5$.

Proposition 4.4. *Let G be a graph with $\mathrm{g}(G) \geq 5$ and $G \not\cong P_2$. For any MLD-set $S \subseteq V(G)$, the set $\overline{S} = \{\overline{u} : u \in S\}$ is a doubly resolving set of G. Consequently,*

$$\psi(G) \leq \gamma_M(G)$$

and this bound is tight.

Proof. We begin by noticing that, given a vertex $u \in S$, we have that

$$d(\overline{u}, x) - d(\overline{u}, y) = d(u, x) - d(u, y) \tag{9}$$

for any two vertices $x, y \neq \overline{u}$ (because if $\overline{u} \neq u$ then $N(\overline{u}) = \{u\}$, and so $d(\overline{u}, x) - d(\overline{u}, y) = (d(u, x)+1)-(d(u, y)+1)$). To prove that $\overline{S}$ is a doubly resolving set, we first show that $\overline{S}$ doubly resolves at least the same pairs as S. Indeed, let $\{x, y\} \subseteq V(G)$ be a pair that is doubly resolved by S, i.e., there exist $u, v \in S$ such that $d(u, x)-d(u, y) \neq d(v, x) - d(v, y)$. We shall see that $\overline{u}$ and $\overline{v}$ (which must be different) doubly resolve $\{x, y\}$. Clearly, if $x, y \notin \{\overline{u}, \overline{v}\}$ then $\overline{u}$ and $\overline{v}$ doubly resolve $\{x, y\}$, by (9); otherwise, either $\{x, y\} = \{\overline{u}, \overline{v}\}$ (and so $\overline{u}$ and $\overline{v}$ doubly resolve $\{x, y\}$, by Observation 4.3) or $|\{\overline{u}, \overline{v}\} \cap \{x, y\}| = 1$. Thus, let us assume without loss of generality that $x = \overline{u}$ and $y \neq \overline{v}$. We distinguish two cases.

Case 1. $u = \overline{u}$: We have $d(\overline{u}, x) - d(\overline{u}, y) = d(u, x) - d(u, y)$ and, by (9), $d(\overline{v}, x) - d(\overline{v}, y) = d(v, x) - d(v, y)$. But $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ by assumption, which implies that $\overline{u}$ and $\overline{v}$ doubly resolves $\{x, y\}$.

Case 2. $u \neq \overline{u}$: By the triangle inequality, $d(\overline{v}, y) \leq d(\overline{v}, u) + d(u, y) \leq (d(\overline{v}, \overline{u}) - 1) + (d(\overline{u}, y) - 1)$, or equivalently $-d(\overline{u}, y) \leq d(\overline{v}, x) - d(\overline{v}, y) - 2$ since $\overline{u} = x$. Thus, $d(\overline{u}, x) - d(\overline{u}, y) \leq d(\overline{v}, x) - d(\overline{v}, y) - 2 < d(\overline{v}, x) - d(\overline{v}, y)$, and then $\{x, y\}$ is doubly resolved by $\overline{u}$ and $\overline{v}$.

Now, we show that $\overline{S}$ also doubly resolves the pairs $\{x, y\}$ not being doubly resolved by S. By Lemmas 4.1 and 4.2 and Observation 4.3, we can assume that $x \in S$, $y \in V(G) \setminus S$ and $N(y) \cap S = \{x\}$, being y the only vertex in $V(G) \setminus S$ so that $\{x, y\}$ is not doubly resolved by S. Furthermore, we can prove that $N(y) = \{x\}$ (this is shown below), which implies that $\overline{x} = y \in \overline{S}$. Thus, pair $\{x, y\}$ is doubly resolved by $y \in \overline{S}$ and any vertex $v \in \overline{S} \setminus \{y\}$ since $d(y, x) - d(y, y) = 1 \neq -1 = d(v, x) - d(v, y)$ (note that such a vertex v exists since $|S| \geq \gamma_M(G) \geq 2$ as $G \ncong P_2$; see for instance [3]). Therefore, we have proved that $\overline{S}$ is a doubly resolving set of G of cardinality $|S|$, which gives $\psi(G) \leq \gamma_M(G)$ by choosing S of minimum cardinality. Moreover, this bound is tight because the graph H'_t of Figure 3 satisfies $\psi(H'_t) = \gamma_M(H'_t) = t$ (it is easy to see that $\{b_1, \ldots, b_t\}$ is the unique minimum doubly resolving set of H'_t; see [4] for details).

To finish the proof, it only remains to check that any pair $\{x, y\}$ that is not doubly resolved by S satisfies $N(y) = \{x\}$. On the contrary, let us suppose the existence of a vertex $z \in N(y) \setminus \{x\}$, which is not in S since $N(y) \cap S = \{x\}$. As S is an MLD-set, there is a vertex $w \in S \cap N(z)$, which must be different from x since $g(G) \geq 5$. For the same reason, we have that $d(w, x) \geq 2$, and also that $d(w, y) = 2$. Thus, $d(w, x) - d(w, y) \geq 0$ but $d(x, x) - d(x, y) = -1$, so $w, x \in S$ doubly resolve $\{x, y\}$; a contradiction. □

This last result, together with expression (3), allows us to place $\psi(G)$ into the chain of expression (2) as the following corollary shows.

Corollary 4.5. *Let G be a graph with $g(G) \geq 5$. Then,*

$$\dim(G) \leq \psi(G) \leq \gamma_M(G) \leq \gamma_L(G).$$

Now, we provide a bound on $\psi(G)$ for arbitrary graphs G. To do this, we follow a similar process than in the proof of Proposition 4.4 but using also dominating sets.

Theorem 4.6. *For every graph G, it holds that*

$$\psi(G) \le \gamma_M(G) + \gamma(G)$$

and this bound is tight.

Proof. Let S_1 and S_2 be a minimum MLD-set and a minimum dominating set, respectively, of G. Also, let $\{x, y\} \subseteq V(G)$ be a pair that is not doubly resolved by $S = S_1 \cup S_2$. Since set S is in particular an MLD-set, by Lemmas 4.1 and 4.2 and Observation 4.3, we can assume $x \in S$, y to be the only vertex of $V(G) \setminus S$ so that $\{x, y\}$ is not doubly resolved by S, and $N(y) \cap S = \{x\}$. Furthermore, $x \in S_1 \cap S_2$ because $x \in S_1 \setminus S_2$ (analogous for $x \in S_2 \setminus S_1$) implies that there is $u \in S_2$ dominating y since S_2 is a dominating set, which contradicts $N(y) \cap S = \{x\}$.

Let S' be the set of vertices $y \in V(G) \setminus S$ so that $\{x, y\}$ is not doubly resolved by S for some $x \in S_1 \cap S_2$ (note that $|S'| \le |S_1 \cap S_2|$ by the uniqueness of each vertex $y \in V(G) \setminus S$). Clearly, $S \cup S'$ is a doubly resolving set of G and has cardinality $|S| + |S'| \le |S_1 \cup S_2| + |S_1 \cap S_2| = \gamma_M(G) + \gamma(G)$, which yields the expected bound. To prove tightness, we consider the graph H_t of Figure 2 which verifies $\psi(H_t) = 2t$ and $\gamma_M(H_t) = \gamma(H_t) = t$ for each $t \ge 2$. ☐

Combining Theorem 4.6 and the fact that $\gamma(G) \le \gamma_M(G)$ for any graph G, we achieve the following chain that is similar to expression (2) and includes $\psi(G)$.

Corollary 4.7. *For every graph G, it holds that*

$$\dim(G) \le \psi(G) \le 2\gamma_M(G) \le 2\gamma_L(G).$$

We remark that tightness in the bound $\psi(G) \le 2\gamma_M(G)$ is guaranteed by the graph H_t of Figure 2.

Finally, we propose the following conjecture that is supported by Proposition 4.4 since $\gamma_M(G) \le \dim(G) + \gamma(G)$.

Conjecture 1. *For every graph G, it holds that*

$$\psi(G) \le \dim(G) + \gamma(G).$$

5 Concluding remarks and open questions

In this paper, we have first characterized the trees T in the cases $\gamma_M(T) = \dim(T) + \gamma(T)$ and $\gamma_M(T) = \ell(T)$. We have then shown how to obtain LD-sets from MLD-sets of graphs G without C_4 or C_6, giving rise to the polynomial bound $\gamma_L(G) \le \gamma_M^2(G)$. For arbitrary graphs G, we have proved that any upper bound on $\gamma_L(G)$ as a function of

$\gamma_M(G)$ has growth at least exponential. Finally, we have constructed doubly resolving sets from MLD-sets in order to show that $\psi(G) \leq \gamma_M(G)$ whenever $\mathrm{g}(G) \geq 5$, and $\psi(G) \leq \gamma_M(G) + \gamma(G)$ for any graph G.

It would be interesting to characterize the trees T with $\dim(T) = \gamma(T)$. Also, we could find new polynomial upper bounds on $\gamma_L(G)$ in terms of $\gamma_M(G)$ for other specific families of graphs. For arbitrary graphs, Theorem 3.2 could be improved by providing either a new construction (better than the graph G_s) or an upper bound on $\gamma_L(G)$ depending on $\gamma_M(G)$. Concerning $\psi(G)$, it would be of interest to find other classes of graphs G with $\psi(G) \leq \gamma_M(G)$, as well as to settle Conjecture 1.

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