The Parallel Approximability of the False and True Gates Problems for Nor Circuits

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Abstract

We study the parallel approximability of computing the number of true gates and false gates for circuits with only NOR gates, that is the NOR-FALSE GATES and NOR-TRUE GATES problems, respectively. We show that the parallel approximability of these problems depends on restrictions on the topology of the circuit. More precisely, for circuits with fan-in and fan-out bounded by a constant and having a constant number of output gates both problems exhibit a threshold behavior in their parallel approximability. Bounding only the number of outputs gives threshold results for the FALSE GATES problem but non-approximability (for any constant) for the number of true gates. Finally for the case of unbounded number of outputs none of the two problems can be approximated in parallel within any constant. The non-approximability results are obtained through the technique of *logspace gap-reduction*. Finally, we use the threshold result of FALSE GATES to isolate a subclass of linear programming that also exhibits a threshold behavior in its parallel approximability.

1 Introduction

The CIRCUIT VALUE PROBLEM has been the starting point for the definition of other function problems which represent interest to the parallel complexity and approximability theory. Among others, there is the TRUE GATES PROBLEM in which we are given a circuit instance together with an input assignment and we want to compute the number of true gates in the circuit. This problem have been shown P-complete and even non-approximable in parallel [Ser90].

For the theory of the approximations problems that represent a threshold in their approximabilit are of special interest. In the sequencial case this behavior

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is captured in the classes above MaxSNP [PY91, ALM+92]. In the parallel setting there are few examples [DSST97], our notion of parallel approximability correspond to that of finding an NC algorithm that r-approximates a given problem, i.e. find a feasible solution to the problem whose measure is guaranteed to be within a multiplicative factor r of the optimum. Anderson and Mayr [AM86] showed a threshold for the High Degree Subgraph problem, namely the problem can be approximated within a factor of c for any $c < 1/2$ in NC, but cannot be approximated by a factor c, $c > 1/2$ unless P=NC. Another interesting problem exhibiting the same threshold behavior is the High Connected Subgraph problem given by Kirousis, Serna and Spirakis [KSS93]. We will address this question for the problem of computing the number of FALSE GATES and that of computing the number of TRUE GATES for circuits with only NOR (NAND) gates. We will focus on the first family, that is to a circuit all whose gates are fan-in two NOR gates, it is also usual to assume that for such circuits all the inputs are set to one (see, e.g. [GHR95]).

For general circuits the non approximability of false/true gates is maintained, even for bounded fan-out or fan-in gates [Ser91]. We will show that for NOR circuits the parallel approximability changes when we bound (by a constant) or not some parameters. We will consider two parameters, the number of *output* gates and the *fan-out* of the gates. When the two parameters are constant both FALSE GATES and TRUE GATES problems present a threshold behavior, that is there is a constant ratio for which the problem can be approximated in parallel (NC) and a constant ratio for which the problem becomes non-approximable in parallel. Bounding only the number of outputs gives threshold results for the FALSE GATES problem but non-approximability (for any constant) for the number of true gates. Finally for the case of unbounded number of outputs none of the two problems can be approximated within any constant. The non-approximability results are obtained through the technique of *logspace gap-reduction*.

The existence of a gap in the parallel approximability of a subclass of the FALSE GATES problem allow us to define a subclass of Linear Programming (LP) which has a threshold in the parallel approximability. We call this class as *Linear Programming with Triplets*. This class is in fact a bit artificial and is defined having in mind the NOR-FALSE GATES problem. However, it is interesting because it fills a gap on the parallel approximability of LP. The parallel complexity of (subclasses of) LP can be summarized as follows: (a) LP is hard to solve in parallel [DLR79]; (b) LP is hard to approximate in parallel [Ser91]; (c) the subclass of Positive LP can be approximated in parallel within any constant [LN93]; (d) the subclass of Positive LP cannot be exactly solved in parallel [TX98]; (e) the generalization of Positive LP in which are allowed also equality constraints cannot be approximated in parallel within any constant [TX98]. Our class fills the gap of LP between the Positive Linear Programming with Equalities which cannot be approximated in parallel within any constant, and that of Positive Linear Programming that is known to be approximable in parallel within any constant. Though it is very restricted, it may well serve as a starting point to isolate a more general class of LP having such a property.

Recall that a similar approach was used to isolate a subclass of linear programming starting from the MaxSat problem [Tre96], but in this case the instances can be approximated in parallel up to any constant.

2 NOR-FALSE (TRUE) GATES PROBLEM

A circuit is a directed acyclic graph whose nodes are labeled from a given set: the nodes with fan-in zero are called inputs and all the remaining nodes are gates. When the inputs are labeled by Boolean variables and gates by Boolean functions, then Boolean circuits are obtained. We will consider the case when the Boolean functions are from the *standard basis*. A description of a Boolean circuit together with an input assingment is $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_k, \ldots, \alpha_n)$ where each α_k is either an input with label 0 or 1, or is a gate with α_i, α_j as inputs, $j, i < k$, the gate α_n is the circuit's output. For a boolean circuit, we will refer to a node with fan-out zero as an *output gate*.

Definition 1 [NOR-CIRCUIT VALUE PROBLEM] *Given a Boolean circuit with* NOR *gates of fan-in two, decide whether the output of the circuit is* 1 *when we assign the value one to all inputs.*

It is well known that the NOR-CIRCUIT VALUE PROBLEM is P-complete even when the gates have bounded fan-out (see, e.g., [GHR95]). We can also extend this result to circuits with a bounded number of outputs.

Theorem 1 NOR-CIRCUIT VALUE PROBLEM *restricted to circuits with only one output gate is* P*-complete.*

PROOF: We reduce the general NOR-CIRCUIT VALUE PROBLEM to the restricted version, by constructing a new circuit, in which all output gates that are not the circuit's output are inputs to a binary tree of NOR gates. The output of this tree is connected to the rest of the circuit as shown in Figure 1. Starting from a circuit C, the so constructed circuit C' has only one output node, and can be constructed in NC. Furthermore C outputs one if and only if C' outputs one. □

For a circuit C, we will denote by $FG(C)$ (resp. $TG(C)$) the number of False Gates (resp. True Gates) of the circuit corresponding to the assignment. We define the NOR-FALSE GATES PROBLEM and NOR-TRUE GATES PROBLEM as follows.

Definition 2 [NOR-FALSE GATES PROBLEM]
Given a boolean circuit C consisting of only NOR *gates of fan-in two, and inputs assigned to one, compute $FG(C)$.*

Definition 3 [NOR-TRUE GATES PROBLEM]
Given a boolean circuit C consisting of only NOR *gates of fan-in two, and inputs assigned to one, compute $TG(C)$.*

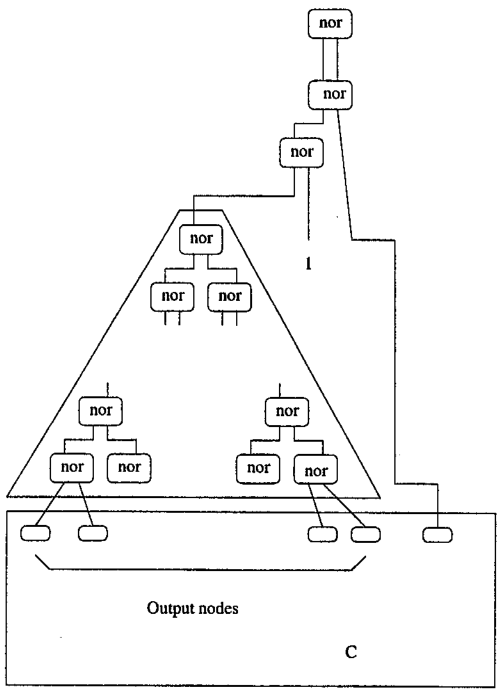

Figure 1: The reduction from CVP to CVP with only one output gate.

Of course the two problems are in P. We will show that the parallel approximability of both problems changes when we consider different restrictions on the topology of the circuit. Let us start with circuits with only one output gate having fan-out two (and fan-in two) NOR gates. We show non-approximability result for the NOR-FALSE GATES and NOR-TRUE GATES restricted to such circuits.

Theorem 2 *The* NOR-FALSE GATES PROBLEM *restricted to circuits with only one output gate, and gates with fan-out two and fan-in two, cannot be approximated in* NC *within* $\frac{2}{1+n^{-\epsilon}}$ *for any* $\epsilon > 0$*, unless* P=NC.

PROOF: We give a reduction from the NOR-CIRCUIT VALUE PROBLEM that creates a gap in the number of false gates. The reduction is given in Figure 2 and consists in adding a gadget that expands the circuit output mantaining the condition on the fan-out. We add a gadget formed by k blocks of three NOR gates. Let us denote by C' the resulting circuit starting from a circuit C. Notice that when the output of the circuit C is one then in any gadget two gates become false and the third one becomes true, but when the circuit ouputs 0, the proportion changes, so two gates become true and only one gate is false.

Therefore assuming that the total number of gates in the circuit C is n and that we add k gadgets formed by three gates, we have that in the new circut C',

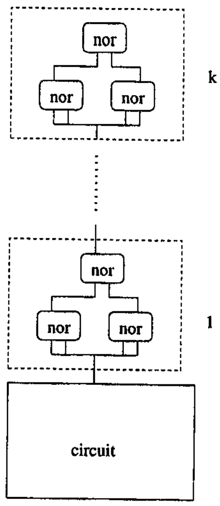

Figure 2: The reduction for circuits with bounded fan-in, fan-out gates, and only one output gate.

$FG(C') \geq 2k$ if the circuit C outputs one and $FG(C') \leq n + k$ if the circuit C outputs zero. Therefore taking $k = n^{\epsilon+1}$ the circuit can be constructed in NC and we get the desired bound. □

The same result holds when we compute the number of true gates, but now notice that when the circuit outputs 0 then the number of true gates is at least $2k$ and when the output is 1 then the number of true gates is at most $m + k$. Therefore choosing $k = n^{\epsilon+1}$ we get again the desired bound.

Theorem 3 *The* NOR-TRUE GATES PROBLEM *restricted to circuits with only one output gate, and gates with fan-out two and fan-in two, cannot be approximated in* NC *within* $\frac{2}{1+n^{-\epsilon}}$ *for any* $\epsilon > 0$, *unless* P=NC.

The approximability results for the NOR-TRUE GATES and the NOR-FALSE GATES problems come from simple combinatorial properties of the circuits, as given by the following theorems.

Theorem 4 *The* NOR-TRUE GATES PROBLEM *restricted to circuits of a unique output gate, and gates with fan-in two and fan-out two, can be approximated in* NC *with ratio 3.*

PROOF: Observe that in any instance C of the NOR-TRUE GATES PROBLEM all gates connected to an input gate evaluate to false. Further, the maximum number of false gates that can produce a true gate is two, so we have that the number of false gates, not connected to an input gate, is at most twice the number of true gates. Let l be the number of gates connected to an input

3 Linear Programming with Triplets

The results in the previous section, and the relationship between evaluating a circuit and the Linear Programming problem, suggests that there might be subclasses of LP having a threshold in the parallel approximability. In this section we isolate one such subclass of LP. It is in essence the linear programming of NOR-FALSE GATES PROBLEM. Informally speaking, in this subclass the set of linear restrictions contains *triplets* of linear restrictions that correspond to a NOR gate.

Definition 4 *A triplet of linear restrictions on 0/1 variables x_i, x_j, x_k and f_i, f_j, f_k has the following form:*

$$\begin{aligned} x_k + (1 - f_i) &\leq 1 \\ x_k + (1 - f_j) &\leq 1 \\ f_k + f_i + f_j &\leq 2 \end{aligned} \tag{3}$$

Notice that the two first restrictions are *dominating* ones, in the sense that if f_i is 1, the restriction is satisfied independently of the value of x_k and when $f_i = 0$ then this forces $x_k = 0$. (Recall that the linear integer programs introduced by Barland et al. [BKT96] have all their restrictions that are dominating ones. In this sense, an instance of linear programming containing triplets can be seen as a slight generalization of the linear programs introduced by Barland et al. [BKT96] whose relaxations can be approximated in parallel within any constant [STX98].)

Definition 5 *An instance of Linear Programming with Triplets (LPT) is as follows:*

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{n} f_i \\ \text{s.t.} \quad & \\ & x_k + (1 - f_i) \leq 1 \\ & x_k + (1 - f_j) \leq 1 \qquad 1 \leq i, j, k \leq n \\ & f_k + f_i + f_j \leq 2 \\ & A^{(1)}\mathbf{x} = \mathbf{b}_1 && (4) \\ & A^{(2)}\mathbf{x} + A^{(3)}\mathbf{f} = \mathbf{b}_2 && (5) \\ & \mathbf{0} \leq \mathbf{x}, \mathbf{f} \leq \mathbf{1} \end{aligned} \tag{LPT}$$

and furthermore:

1) *the entries of $A^{(1)}, A^{(2)}, A^{(3)}$ are from $\{0, 1\}$ and the coefficients of $\mathbf{b}_1$, $\mathbf{b}_2$, c are non-negative;*

2) *For any row i, there exist at least a pair k, j such that $A^{(1)}_{i,j} \neq 0$ and $A^{(2)}_{i,k} \neq 0$;*

3) *there exists a constant* $\gamma < 1$ *such that* $C^* \geq \gamma \sum c_i$, *where* C^* *is the optimal value;*

The following theorem shows a relation between NOR-FALSE GATES PROBLEM and LPT.

Proposition 1 *The* NOR-FALSE GATES PROBLEM *with a unique output can be written as a Linear Program with Triplets.*

PROOF: We assume, without loss of generality, that there is only one input to the circuit and moreover that it equals 1. Let $g_1, \ldots, g_n$ be the gates of the circuit, they are of the form $g_k = \neg(g_i \vee g_j)$, and let NOR be the set of such gates (we will use the notation $(i,j,k) \in NOR$). Let f_i be the 0/1 variable associated to gate g_i with the usual meaning $f_i = 1$ iff the output of g_i is false and $x_i = 1 - f_i$. To any gate that is not an input there is associated a triplet of linear constraints such that only one feasible solution exists, the solution in which the values of x_i/f_i are consistent with their intended meaning.

We express the instance of FALSE GATES as a linear program in which we want to maximize the number of False Gates.

$$
\begin{array}{lll}
\text{maximize} & \sum_{i=1}^{n} f_i & \\
\text{subject to} & & \\
 & x_0 = 1 & g_0 \text{ input} \\
 & & \\
 & x_k + (1 - f_i) \leq 1 & \\
 & x_k + (1 - f_j) \leq 1 & \forall (i,j,k) \in NOR \\
 & f_k + f_i + f_j \leq 2 & \\
 & & \\
 & x_i + f_i = 1 & \forall i \in \{1, \ldots, n\} \\
 & x_i, f_i \geq 0 & \forall i \in \{1, \ldots, n\}\ ,
\end{array}
\qquad \text{(FGP)}
$$

Clearly, any optimal solution (there is only one feasible solution!) of (FGP) program computes *exactly* the number of False Gates of the given instance.

We observe that the linear program (FGP) is in LPT form according to Definition 5 with $\gamma = 1/3$ (recall Eq. (2)). □

Remark 1 *Our assumption on the unique output gate is used implicitely to assure the lower bound on the optimal value* C^* *of (FGP) program. Indeed, if several output gates are allowed then the resulting linear program has the objective function unbounded from below (see reduction given in Theorem 7.)*

From the above theorem we have the following corollary:

Corollary 1 *Linear programming with Triplets cannot be approximated in* NC *within* $\frac{2}{1+n^{-\epsilon}}$ *for any constant* $\epsilon > 0$ *unless* P=NC.

On the other hand, notice that the optimal value of an LPT can be trivially constant approximated (in NC) by the assumption **3)** of the Definition 5.

Notice that the condition **3)** of the Definition 5 may seem restrictive and unnatural. However, if we look closely the threshold results for the other known problems (High Degree Subgraph [AM86] and High Connected Subgraph [KSS93]) we will notice that in both cases, there has been proven –via an extremal graph result– a lower bound for the computed parameter and then the threshold behavior.

The class of Linear Programming with Triplets can be generalized to Linear Programming containing constraints of the form:

$$\begin{array}{l} x_k + (1 - f_{i_1}) \leq 1 \\ x_k + (1 - f_{i_2}) \leq 1 \\ \dots \quad \dots \quad \dots \\ x_k + (1 - f_{i_l}) \leq 1 \\ f_k + f_{i_1} + f_{i_2} + \dots + f_{i_l} \leq l \end{array} \qquad (6)$$

Notice that such a set of constraints correspond to a NOR gate of fan-in l. We can, similarly as in the case of LPT, define linear programs containg constraints of type (6) and extend directly the results of LPT to LP with such constraints, namely such linear programs cannot be approximated in NC within $l/(1+n^{-\epsilon})$ for any constant $\epsilon > 0$, unless P=NC.

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