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AN INTERIOR - POINT ALGORITHM FOR HYPER-PLASTIC MODELS FOR SOILS

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Summary. In numerical analysis of geotechnical problems it is often necessary to use sophisticated elastoplastic constitutive models. Particularly, the presence of friction as the fundamental mechanism to dissipate energy provides most of the models used in this field with a non-associated character. This characteristic introduces theoretical and numerical complexity at time of deriving the models from thermo-mechanical principles and implementing them. The paper presents the formulation and implementation for a family of non-associated models based on the hyper-plasticity approach. The paper starts with the hyper-plastic framework, that proves to be a powerful technique to derive evolution equations using standard thermo-mechanical procedures based on the differentiation of the energy expressions (free Helmholtz energy f_s, Gibbs energy g_s, and dissipation d), even for non-associated laws. The characterization of the variational structure behind the discrete equations of the Interior-Point projection of approximation is discussed in the second part. Thirdly is described the implementation of the models using convex programming theory. The integration of the incremental response is achieved following a classical primal-dual interior-point algorithm which presents good performance to resolve saddle-point problems. This algorithm has the particularity to solve the optimality conditions associated with a suitably penalized mathematical program taking Newton steps or damped Newton steps toward points on the central path or quasi-central path previously defined. The global convergence characteristic of the algorithm is achieved introducing an appropriate line search scheme. The algorithm is finally tested on initial-boundary-value problem for an elastoplastic continuum of Cam clay type. The performance of the algorithm is discussed on the basis of the results obtained.

1 INTRODUCTION

The hyper-plastic approach allows the development of constitutive models from hypotheses on the form of an energy potential (g_s) and dissipation function (d). Those functions allow the integration of the models to be performed using convex programming theory. The Primal-Dual interior-point methods based on the last theory find primal-dual variables solution $(\epsilon^e, \rho, \Delta\gamma, z)$ by applying variants of the Newton's method to the standard form problem.

2 A VARIATIONAL FORMULATION FOR HYPERPLASTICITY

A hyper-plastic constitutive model can be entirely defined by two scalar value functions, the Gibbs energy function $g_s(\sigma,\alpha)$ and the dissipation energy $d(\dot{\alpha})$ or the alternative complementary expressions obtained through the classical *Legendre transformation*. The former energy functions for pressure dependent shear modulus (Houlsby[1]) can be expressed as

$$\begin{array}{rcl} g_s & = & \kappa^\star \frac{\sigma_{ii}}{3}\left[\ln\left(\frac{\sigma_{jj}}{3p^o}\right)-1\right]+\frac{3s_{ij}s_{ij}}{4\alpha_G\sigma_{kk}}-(\sigma_{ij}\alpha_{ij})+\frac{1}{2}p^c\left(\lambda^\star-\kappa^\star\right)\exp\left(\frac{\alpha_{jj}}{\lambda^\star-\kappa^\star}\right) \\ d & = & \rho_{kk}\left(\alpha_{kk}\right)\left[\left(\dot{\alpha}_{jj}\right)^2+M^2\left(\dot{\alpha'}_{ij}\right)^2\right]^{1/2} \end{array} \tag{1}$$

The yield function $(f^y(\chi))$ is obtained as a Legendre transformation of the dissipation energy (d) function. The convex characteristics of those functions (g_s) and (f^y) in their arguments and their twice differentiability property induce to the positive definition of the Hessian matrix $\nabla^2\mathcal{L}^{\bar{g}}$ of the Lagrangian defined later. Consider the following minimization problem

$$(\text{CCM})\left\{\begin{array}{ll} \min_{(\sigma,\alpha)} & -g_s(\sigma,\alpha)-(\sigma\epsilon_{n+1}+\chi_{n+1}\alpha)+\mu\log z \\ \text{sub. to} & f^y(\chi)+z=0 \end{array}\right. \tag{2}$$

where the slack variable (z) has been introduced in order to transform the system into the standard form and (μ) is a barrier parameter usually called "duality gap" variable, (Byrd[2]). From standard arguments in constrained optimization the Lagrangian function associated to the elastoplastic variational problem (2) (Krabbenhoft[3]) is,

$$\mathcal{L}^{\bar{g}}(\sigma,\chi,\gamma)=-g_s(\sigma,\alpha)-\left(\sigma\epsilon^{\text{trial}}_{n+1}+\chi^{\text{trial}}_{n+1}\alpha\right)+\Delta\gamma\left[f^y(\chi)+z\right]+\mu\log(z) \tag{3}$$

Using the standard state equations of hyper-plasticity, the neccesary first order optimality conditions on the last equation yields to an extension of the KKT conditions, expressed as

$$\begin{array}{rcl} \epsilon^e_{n+1}-\epsilon^{e^{\text{trial}}}_{n+1}+\Delta\gamma\partial_\chi f^y_{n+1}(\chi)\,\partial_\sigma\chi & = & 0 \\ -\rho_{n+1}+\rho^{\text{trial}}_{n+1}+\Delta\gamma\partial_\chi f^y_{n+1}(\chi)\,\partial_\rho\chi\partial^2_{\alpha\alpha}g_{s_2} & = & 0 \\ f^y_{n+1}(\chi)+z & = & 0 \\ \Delta\gamma-\frac{\mu}{z} & = & 0 \\ (\Delta\gamma,z) & \geq & 0 \end{array} \tag{4}$$

All primal-dual methods generate iterates $(\epsilon^e,\rho,\Delta\gamma,z)$ that satisfy the bounds (4e) strictly, that is $(\Delta\gamma^{(k)}>0)$ and $(z^{(k)}>0)$. The duality measure variable $(\mu=\Delta\gamma z/m)$ fulfill the role of keeping the variables $(\Delta\gamma,z)$ from moving too close to the boundary of the nonnegative orthant. The biased search direction is described introducing the centering parameter (σ_c), accompanied by the former variable. The generic step equations $(d_{\epsilon^e},d_\rho,d_{\Delta\gamma},d_z)$ are derived from the modify residual vector

$$\mathbf{r}(\mathbf{x})=\left\{\begin{array}{c} \epsilon^e_{n+1}-\epsilon^{e^{\text{trial}}}_{n+1}+\Delta\gamma\partial_\chi f^y_{n+1}(\chi)\,\partial_\sigma\chi \\ -\rho_{n+1}+\rho^{\text{trial}}_{n+1}+\Delta\gamma\partial_\chi f^y_{n+1}(\chi)\,\partial_\rho\chi\partial^2_{\alpha\alpha}g_{s_2} \\ f^y_{n+1}(\chi)+z \\ \Delta\gamma z-\sigma_c\mu \end{array}\right\} \tag{5}$$

If ($\sigma_c = 1$) the eqs.[5] define a centering direction while ($\sigma_c = 0$) gives a standard Newton's step (Ralph[4]), which involve the corresponding Hessian matrix

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \mathrm{I} + \Delta\gamma\partial^2_{\chi\chi} f^y \left(\partial_\sigma \chi\right)^2 \partial^2_{\epsilon\epsilon} f_s & \Delta\gamma\partial^2_{\chi\chi} f^y \partial_\sigma\chi\partial_\rho\chi\partial^2_{\alpha\alpha} g_{s_2} & \partial_\chi f^y \partial_\sigma\chi & 0 \\ \Delta\gamma\partial^2_{\chi\chi} f^y \partial_\rho\chi\partial_\sigma\chi\partial^2_{\alpha\alpha} g_{s_2}\partial^2_{\epsilon\epsilon} f_s & -\mathrm{I} + \Delta\gamma\partial^2_{\chi\chi} f^y \left(\partial_\rho\chi\right)^2 \partial^2_{\alpha\alpha} g_{s_2} & \partial_\chi f^y \partial_\rho\chi\partial^2_{\alpha\alpha} g_{s_2} & 0 \\ \partial_\chi f^y \partial_\sigma\chi\partial^2_{\epsilon\epsilon} f_s & \partial_\chi f^y \partial_\rho\chi & 0 & 1 \\ 0 & 0 & z & \Delta\gamma \end{bmatrix} \tag{6}$$

being a positive definite and non-symmetric matrix.

3 PRIMAL-DUAL INTERIOR-POINT ALGORITHM

A summary of the algorithm is presented in "box 1". The **fast-step** algorithm refers to the pure Newton's step ($\sigma_c = 0$) while **safe-step** algorithm is a centering step imposing ($\sigma_c = 1$)

box 1: "primal-dual IPM" for biphasic continuum

1. Input data: (η, ω), $\left(\sigma_c^{\min}, \sigma_c^{\max}\right)$, $\left(\epsilon_{n+1}, \epsilon_{n+1}^{e^{\text{trial}}}\right)$ with $\left(\sigma_{n+1}^{\text{trial}}, \rho_{n+1}^{\text{trial}}\right)$ and $f_{n+1}^{y^{\text{trial}}} > 0$
2. Initialize: set $k = 0$

$$\mathbf{x}^{(0)} = \left\{ \begin{array}{c} \epsilon_{n+1}^{\text{trial}} \\ \rho_{n+1}^{\text{trial}} \\ \Delta\gamma^{(0)} \\ z^{(0)} \end{array} \right\} \in \mathcal{E}_{-\infty}\left(\eta^{\max}, \omega^{\min}\right) \quad \text{and} \quad \left\{\mathbf{r}\left(\mathbf{x}\right)^{(0)}\right\}$$

$$t^{(0)} = 0,\ \mu^{(0)},\ \eta^{(0)} \leftarrow \eta^{\max},\ \omega^{(0)} \leftarrow \omega^{\min}$$

3. Check overall convergence: $\left(\mathrm{e}_{\mathbf{x}}^{(k)} \ \leq\ \text{itol}\right)$
4. Comupute the Jacobian matrix $\mathbf{J}^{(k)}$ evaluating the variables at

$$\mathbf{x}^{(k)} = \left\{\epsilon_{n+1}^{(k)}, \rho_{n+1}^{(k)}, \Delta\gamma^{(k)}, z^{(k)}\right\}^T$$

5. Attempt a fast step:

$\left(\mathbf{x}^{(k+1)}, \mathbf{r}^{(k+1)}, \mu^{(k+1)}\right) = \textbf{fast-step}\left(\ \mathbf{d}^{(k)} = -\left(\mathbf{J}^{(k)}\right)^{-1} \mathbf{r}_{(\sigma_c=0)}^{(k)}\ ; \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^* \mathbf{d}^{(k)}\ \right)$

if $\mu^{(k+1)} \leq \rho_c \mu^{(0)}$

$\eta^{(k+1)} \leftarrow \eta^{\min} + \bar{\eta}^{t^{(k)}}\left(\eta^{\max} - \eta^{\min}\right);\ \omega^{(k+1)} \leftarrow \left(1 + \bar{\eta}^{t^{(k)}+1}\right)\omega^{(k)};\ t^{(k+1)} \leftarrow t^{(k)} + 1$

else

6. Revert to a safe step:

$\left(\mathbf{x}^{(k+1)}, \mathbf{r}^{(k+1)}, \mu^{(k+1)}\right) = \textbf{safe-step}\left(\ \mathbf{d}^{(k)} = -\left(\mathbf{J}^{(k)}\right)^{-1} \mathbf{r}_{(\sigma_c>0)}^{(k)}\ ; \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^* \mathbf{d}^{(k)}\ \right)$

$\eta^{(k+1)} \leftarrow \eta^{(k)};\quad \omega^{(k+1)} \leftarrow \omega^{(k)};\quad t^{(k+1)} \leftarrow t^{(k)}$

end if

7. set $k \leftarrow k+1$ and go to 3.
8. set $(\epsilon^e, \rho, \sigma, \alpha)_{n+1} \leftarrow (\epsilon^e, \rho, \sigma, \alpha)_{n+1}^{(k+1)}$

where (α^*) is the so-called line search parameter, determined such that at each iteration $\mathbf{x} \in \mathcal{E}_{-\infty}(\eta, \omega)$, assuring the global convergence of the algorithm.

4 RESULTS AND CONCLUDING REMARKS

The performance of the algorithm is tested through modeling a conventional triaxial test of a soil sample with the corresponding material's parameters ($\kappa = 0.018$, $\lambda = 0.09$, $e_o = 0.63$, M=1, $p^c = 100kPa$, $p^o = 1kPa$). The "Interior-Point" algorithm presented is both efficient and robust and presents much in common with the classical well known "return-mapping" method.

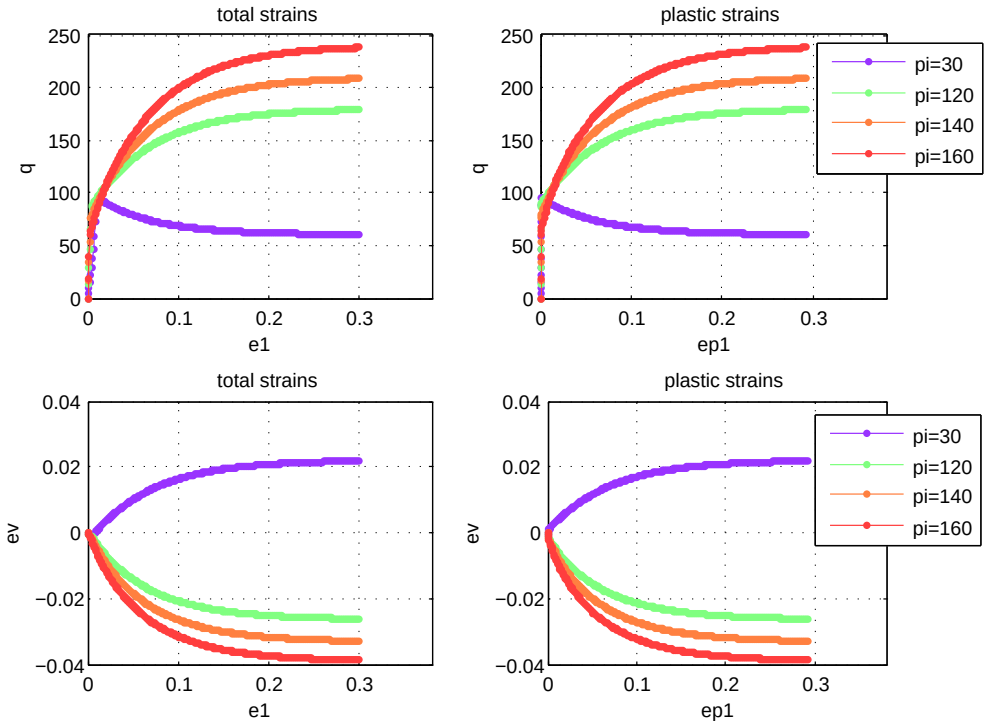

Figure 1: Conventional Drained Triaxial Test on normally and overconsolidated compressed soil samples.

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