Geometric Tree Graphs of Points in Convex Position *

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Abstract

Given a set P of points in the plane, the geometric tree graph of P is defined as the graph $T(P)$ whose vertices are non-crossing rectilinear spanning trees of P, and where two trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges e and f. In this paper we concentrate on the geometric tree graph of a set of n points in convex position, denoted by G_n. We prove several results about G_n, among them the existence of Hamilton cycles and the fact that they have maximum connectivity.

1 Introduction

Given a connected graph G, the tree graph $T(G)$ is defined as the graph having as vertices the spanning trees of G, and edges joining two trees T_1, T_2 whenever $T_2 = T_1 - e + f$ for some edges e and f of G.

Tree graphs were introduced by Cummings [2] in connection with the study of electrical networks, showing that tree graphs are Hamiltonian. A simpler proof of the same fact was found later by Holzmann and Harary [6], and generalized to the base graph of a matroid. Liu [8] related the connectivity of $T(G)$ to the cyclotomic number of G. Later Liu showed that tree graphs have maximum connectivity, that is, connectivity equal to the minimum degree [9]. Additional results on tree graphs have been obtained recently [4].

Here we consider a geometric version of the problem. Given a set P of points in the plane, let $\mathcal{T}(P)$ be the set of non-crossing spanning trees of P (edges are straight line segments and do not cross). We define the *geometric tree graph* $T(P)$ as the graph having $\mathcal{T}(P)$ as vertex set and the same adjacencies as in combinatorial tree graph, that is, two non-crossing spanning

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trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$. Geometric tree graphs have appeared previously in the work of D. Avis and K. Fukuda [1] as a tool for enumerating spanning trees. They show that $T(P)$ is connected for any point set P in general position and has diameter bounded by $2n - 4$ if $n = |P|$.

In this paper we concentrate on the combinatorial properties of the graphs $T(P)$ in the case where P is a point set in convex position. For any $n \geq 2$, we denote G_n the geometric tree graph of a set of n points in convex position.

In this paper we obtain a number of new results about the graphs G_n. In Section 2 we give definitions and preliminary results as the minimum and the maximum degree of G_n. In Sections 3 and 4 we determine the radius, the center and the group of automorphisms of G_n. We also show that the diameter of G_n is at least $3n/2 - 5$. In Section 6 we present a *tree of geometric trees*, a recursive construction of the graphs G_n in which a tree T in G_n gives rise to $\binom{d+2}{2}$ different trees in G_{n+1}, where d is the degree of the n-th vertex in T. This tool is then used to produce inductive proofs of two main results: G_n is a Hamiltonian graph for every $n \geq 3$, and G_n has connectivity equal to the minimum degree $2n - 4$. We remark that this kind of construction has proved useful in solving similar problems for graphs of triangulations instead of tree graphs [7].

To determine the exact value of the diameter is the main open problem left in this paper. In the case of combinatorial tree graphs, the diameter is obviously bounded by $n - 1$, because spanning trees satisfy the exchange property of the set of basis of a matroid. But this ceases to be true in the geometric case.

2 Definitions and preliminaries

2.1 Geometric tree graphs

Let $P = \{1, \ldots, n\}$ be a set of points in the plane, no three of them collinear. A *non-crossing spanning tree* for P is a spanning tree of P with edges given by straight line segments that do not cross. Let $\mathcal{T}(P)$ be the set of non-crossing spanning trees of P. The *geometric tree graph* $T(P)$ of the set P has a vertex for every element in $\mathcal{T}(P)$ and two trees $T_1, T_2 \in \mathcal{T}(P)$ are *adjacent*, and we write $T_1 \sim T_2$, when there are edges $e \in T_1 \setminus T_2$ and $f \in T_2 \setminus T_1$ such that $T_2 = T_1 + f - e$.

An example is shown in Figure 1.

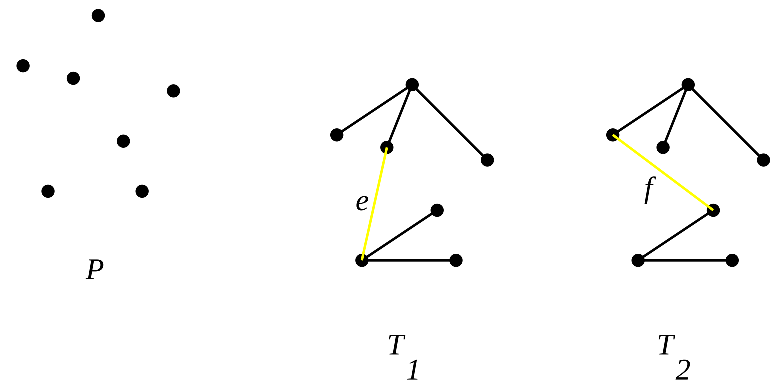


Figure 1: Two trees adjacent in $T(P)$

2.2 The graph G_n

Since any two sets of points, both in convex position, are equivalent with respect to their non-crossing spanning trees, all sets of n points in convex position have the same geometric tree graph, denoted simply by G_n. So we are free to work with the set P_n of vertices of a regular polygon. We assume, without less of generality, that its vertices are labelled by integers 1 to n, sorted counter-clockwise, and that 1 is the vertex with minimum x-coordinate. The arithmetic of the indices is done modulo n. Let us denote by $\mathcal{T}_n$ the set of all non-crossing spanning trees of P_n, that is, the vertex set of the graph G_n.

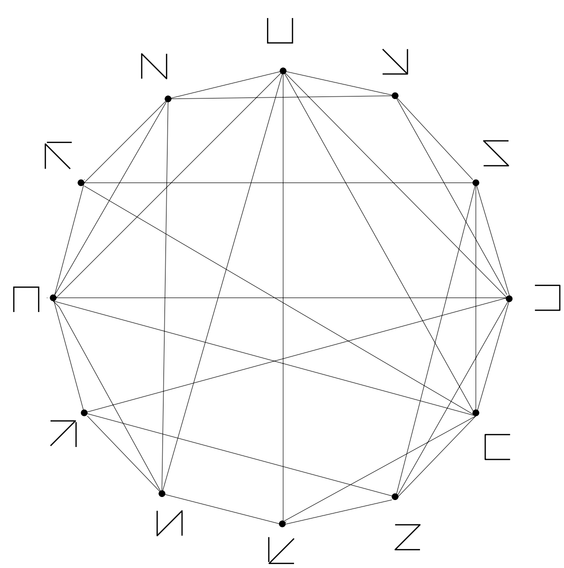


Figure 2: The graph G_4

We summarize next what is known about the graphs G_n.

i) G_n is connected and has diameter bounded above by $2n-4$ [1].

ii) The number of vertices of G_n is $t_n = \frac{1}{2n-1}\binom{3n-3}{n-1}$ [3, 10], and every geometric tree graph of a set of n points has at least this number of vertices [5].

iii) The chromatic number of G_n is in $\Theta(n^2)$ [4].

We finally remark a very useful property that will be used in the following sections.

Remark 2.1 Any tree $T \in \mathcal{T}_n$, $n \geq 3$, has at least two edges on the boundary of P_n, that is, two edges of the type $(i, i+1)$, and such that either the vertex i or the vertex $i+1$ is a leaf of T.

2.3 Maximum and minimum degree

The degrees of the vertices of G_n can be quite different. There are vertices with degree $\Theta(n)$ and vertices with degree $\Theta(n^3)$, as shown below.

There are some trees with a specially simple structure called *stars*. The star S_i is obtained by joining the vertex i to all the other vertices. Note that for $n = 2, 3$ all trees are stars. In order to obtain a tree of $\mathcal{T}_n$ from a star S_i we can only add an edge of the boundary of P_n that is not in S_i. There are $n-2$ edges of this kind. If $(k, k+1)$ is one of these edges, when it is added we must remove either the edge (i, k) or the edge $(i, k+1)$ of the cycle that appears in $S_i \cup (k, k+1)$. Then we conclude that the degree of a star in G_n is $2(n-2)$. Let $d_G(i)$ denote the degree of a vertex i in a graph G and $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree respectively. An easy induction proves the next result.

Proposition 2.2 $\delta(G_n) = 2n-4$ *and only the stars have this degree.*

Other special trees are the *chains*. The chain C_i is obtained by taking all the edges in the boundary of P_n, except $(i, i+1)$. The next result can also be proved by induction.

Proposition 2.3 $\Delta(G_n) = \binom{n+1}{3} - n + 1$ *and only the chains have this degree.*

3 Center, radius and diameter

In this Section we continue the study of properties of the graph G_n. We will denote by $d(T, T')$ the distance in G_n between two trees T and T' of $\mathcal{T}_n$, that is, the minimum number of edges we have to change from one of these trees in order to obtain the other one, so as at each exchange the resulting tree is non-crossing. The eccentricity $e(T)$ of $T \in \mathcal{T}_n$, is defined as the maximum distance between T and any other tree in $\mathcal{T}_n$. The radius of the graph G_n is the minimum of the eccentricities of the vertices of G_n, and the center of G_n is the set of all vertices that have eccentricity equal to the radius.

Remark 3.1 Let $T \in \mathcal{T}_n$ and let d_i be the degree of i in T, for $1 \leq i \leq n$. Then $d(T, S_i) = n - 1 - d_i$ (see [1]).

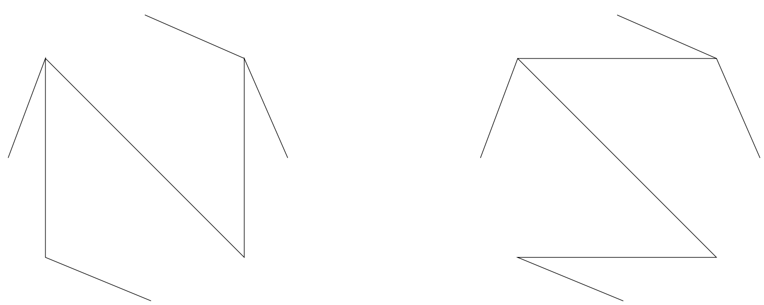
Remark 3.2 Let $T \in \mathcal{T}_n$ and let $ch(T)$ be the number of edges of T in the boundary of P_n. Then

$$d(T, C_i) = \begin{cases} n - ch(T) & \text{if } (i, i+1) \in T, \\ n - 1 - ch(T) & \text{if } (i, i+1) \notin T. \end{cases}$$

The following result shows that the stars and the chains play a special role in the graph.

Theorem 3.3 *The radius of G_n is equal to $n - 2$, and the center consists of the n stars $S_1, \ldots, S_n$ and the n chains $C_1, \ldots, C_n$.*

Proof. From Remark 3.1 it is obvious that the eccentricity of a star is equal to $n - 2$, because any tree has at least one edge in common with any star. Since all trees $T \in \mathcal{T}_n$ have two edges on the boundary of P_n, the eccentricity of a chain is also $n - 2$. It remains to show that if a tree T is neither a star nor a chain, then $e(T) > n - 2$. It is sufficient to show that, for any of these trees T there is another T' disjoint with T, that is, T' does not have any edge in common with T, because then it is clear that $d(T, T') \geq n - 1$. The existence of T' is easily proved by induction. $\square$



T T'

Figure 3: $ch(T) = ch(T'),\ d_T(i) = d_{T'}(i)\ \forall i$

Remark 3.4 From Remarks 3.1 and 3.2 the distances from a tree to the vertices of the center are easily computed. One could think that these distances determine the tree. This is not so, moreover, as shown in Figure 3, one can find two different trees with the same degree sequences and even the same edges on the boundary.

After having established the value of the radius, it is natural to ask about the diameter, that is, the maximum of the eccentricities of the vertices of G_n. An obvious upper bound for the diameter is twice the radius, i.e. $2(n-2)$, and a trivial lower bound is $n-1$. We give here a more precise lower bound.

Let n be even, and let T_1 and T_2 be the following trees (see Figure 4) :

$$\begin{aligned} T_1 &= \{(n,k)|\ 1 \le k \le \frac{n}{2}\} \cup \{(\frac{n}{2},k)|\ \frac{n}{2}+1 \le k \le n-1\} \\ T_2 &= \{(\frac{n}{2}+1,k)|\ 1 \le k \le \frac{n}{2}\} \cup \{(1,k)|\ \frac{n}{2}+2 \le k \le n\}. \end{aligned}$$

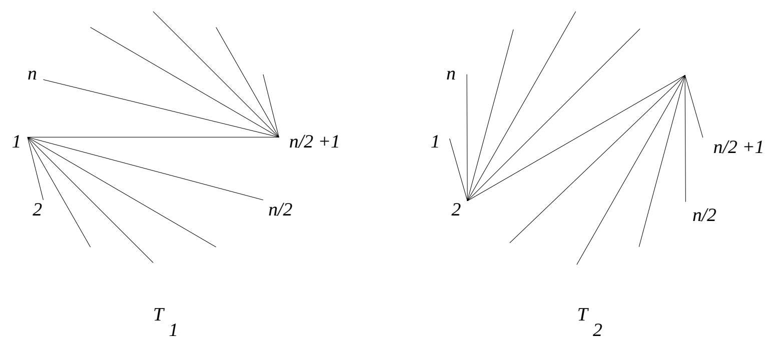


Figure 4: Two trees with $d(T_1,T_2) = \dfrac{3n}{2} - 5$

Theorem 3.5 *The diameter of G_n is at least $\lfloor 3n/2 \rfloor - 5$.*

Proof. We prove here the case n even. The case n odd is handled with a slight modification of the trees T_1 and T_2 defined above.

The edge $e = (2,n) \in T_2$ is an edge of T_2 that has the minimum number of crossings with the edges of T_1, and e intersects $n/2-1$ edges of T_1. To add an edge of T_2 we need at least $n/2$-1 changes. In the best case, we need a change for introducing each of the remaining edges of T_2 that do not appear in T_1 . As T_1 and T_2 have in common two edges, we obtain:

$$d(T_1,T_2) \ge (\frac{n}{2}-1) + (n-4) = \frac{3n}{2} - 5.$$

Finally, it is easy to find a path between T_1 and T_2 that has exactly this length. □

For the upper bound of the diameter, besides the trivial value $2n-4$, we have only obtained partial results, bounding the eccentricity of certain families of trees. From Remarks 3.1 and 3.2 we can conclude:

Lemma 3.6 *Let $T \in \mathcal{T}_n$. If $ch(T) \geq \frac{n}{2}$ or there is some vertex i such that $d_T(i) \geq \frac{n}{2}$ then $e(T) \leq \frac{3n}{2} - 2$.*

Taking into account this result, in order to maximize the eccentricity, it is natural to consider those trees that have few edges in the boundary of P_n and vertices with low degree. But we have obtained the next result, whose proof is omitted.

Lemma 3.7 *If $T \in \mathcal{T}_n$ is such that $ch(T) = 2$, then $e(T) \leq 3n/2 - 2$.*

4 Group of automorphisms

Let us denote by $\Gamma(G_n)$ the automorphism group of G_n. It is clear that any symmetry of the regular n-polygon will induce a corresponding automorphism on G_n. No more automorphisms are possible, as proved next.

Theorem 4.1 *The automorphism group $\Gamma(G_n)$ is isomorphic to the dihedral group D_n of the symmetries of a regular polygon with n sides.*

Proof. We know from Theorem 3.3 that the center C of G_n is equal to the set of stars and chains

$$C = \{S_1, \ldots, S_n, C_1, \ldots, C_n\}.$$

Moreover, it is straightforward to see that

$$\begin{aligned} d(S_i, S_j) &= n - 2; \\ d(C_i, C_j) &= 1; \\ d(C_i, S_j) &= \begin{cases} n-2 & \text{if } j = i \text{ or } j = i+1; \\ n-3 & \text{otherwise.} \end{cases} \end{aligned}$$

Now let γ be in $\Gamma(G_n)$. The center of the graph is invariant by γ, more precisely, because of the above relations, we see that

$$\begin{aligned} \gamma(\{S_1, \ldots, S_n\}) &= \{S_1, \ldots, S_n\}, \\ \gamma(\{C_1, \ldots, C_n\}) &= \{C_1, \ldots, C_n\}. \end{aligned}$$

If $\gamma(S_1) = S_j$, as $d(S_1, C_1) = d(S_j, \gamma(C_1))$, then either $\gamma(C_1) = C_j$ or $\gamma(C_1) = C_{j-1}$. In the first case it follows that $\gamma(C_2) = C_{j+1}$, and in the second case that $\gamma(C_2) = C_{j-2}$. Proceeding in this way we see that γ is either a rotation or a reflection of the index set $\{1, \ldots, n\}$. This shows that the restriction of $\Gamma(G_n)$ to the center is equivalent to the dihedral group D_n.

That action on the center is crucial, as we conclude next by proving that $\gamma_{|C} = \mu_{|C}$ implies $\gamma = \mu$. Equivalently, we are going to show that if $\gamma_{|C} = 1_{|C}$ then $\gamma = 1$. Before proving this, assuming $\gamma_{|C} = 1_{|C}$, we make two remarks.

i) Let T be any tree and $d_i = d_T(i)$ the degree of T on the vertex i. From Remark 3.1 we know that $d(T, S_i) = n - 1 - d_i$. But, by hypothesis, γ is trivial on the stars, and an automorphism preserves distances, hence

$$d(\gamma(T), S_i) = d(\gamma(T), \gamma(S_i)) = d(T, S_i) = n - 1 - d_i$$

and the vertex i has the same degree in T and $\gamma(T)$ for any i, $1 \leq i \leq n$.

ii) On the other hand, if $ch(T)$ is the number of edges that T has in the boundary, then

$$d(\gamma(T), C_i) = d(\gamma(T), \gamma(C_i)) = d(T, C_i) = \begin{cases} n - ch(T) & \text{if } (i, i+1) \in T, \\ n - 1 - ch(T) & \text{if } (i, i+1) \notin T \end{cases}$$

hence T and $\gamma(T)$ have the same edges in the boundary.

We prove now that if $\gamma_{|C} = 1$ then $\gamma = 1$, by induction on the number of vertices. Let T be any tree of $\mathcal{T}_n$ and let i be a vertex such that $d_T(i) = 1$ and $(i, i+1) \in T$. We consider $T_* = T \setminus i$. By i) and ii) we can affirm that $d_{\gamma(T)} = 1$ and $(i, i+1) \in \gamma(T)$. We can consider the restriction γ_* of γ to $P_n \setminus \{i\}$ which is an automorphism of G_{n-1} that is trivial on the center. By induction $\gamma(T_*) = T_*$, hence T and $\gamma(T)$ must be the same and we conclude that $\gamma = 1$. $\square$

5 Tree of geometric trees

In this section we describe the main tool for proving the results that appear in the next two sections, on the Hamiltonicity and the connectivity of the graph G_n. This tool is a recursive construction of the graphs G_n in which a tree T of G_n gives rise to $\binom{d+2}{2}$ different trees of G_{n+1}. In this way we obtain an infinite tree, whose vertices are the trees in $\mathcal{T}_n$, for all n. This kind of construction has proved useful in solving similar problems for graphs of triangulations [7]. In this infinite tree, every $T \in \mathcal{T}_n$ has one father, belonging to $\mathcal{T}_{n-1}$, and some sons, belonging to $\mathcal{T}_{n+1}$.

If $T \in \mathcal{T}_n$ is such that $i_1 < i_2 < \ldots < i_d$ are the vertices adjacent to n in T, we construct its sons $S_{i,j}(T)$ as the trees of $\mathcal{T}_{n+1}$ defined as follows. We distinguish three kinds of sons:

Type 0: We add the edge $(n, n+1)$ to T and distribute between n and $n+1$ the edges (i_k, n) of T:

$$S_{0,0}(T) = T \cup \{(n, n+1)\};$$

and for k, $1 \leq k \leq d$,

$$\begin{aligned} S_{0,k}(T) &= \{(a,b) | a, b \neq n, (a,b) \in T\} \cup \{(n+1, i_p) |\ 1 \leq p \leq k\} \cup \\ &\quad \cup \{(n, i_p) |\ k+1 \leq p \leq d\} \cup \{(n, n+1)\}. \end{aligned}$$

Type 1: We split the edge (i_k, n) into the two edges (i_k, n) and $(i_k, n+1)$.

$$\begin{aligned} S_{1,k}(T) &= \{(a,b) | a, b \neq n, (a,b) \in T\} \cup \{(n+1, i_p) |\ 1 \leq p \leq k\} \cup \\ &\quad \cup \{(n, i_p) |\ k \leq p \leq d\}, \end{aligned}$$

for $1 \leq k \leq d$.

Type *j*, *j* ≥ 2: For every subset S of cardinal j of $\{i_1, \ldots, i_d\}$, $S = \{i_k, i_{k+1}, \ldots, i_{k+j-1}\}$, we build the chain $n+1, i_k, i_{k+1}, \ldots, i_{k+j-1}, n$:

$$\begin{aligned} S_{j,k}(T) &= \{(a,b) | a, b \neq n, (a,b) \in T\} \cup \{(n+1, i_p) |\ 1 \leq p \leq k\} \\ &\quad \cup \{(n, i_{p+j-2}) |\ k+1 \leq p \leq d-j+2\} \\ &\quad \cup \{(i_k, i_{k+1}), (i_{k+1}, i_{k+2}), \ldots, (i_{k+j-2}, i_{k+j-1})\}. \end{aligned}$$

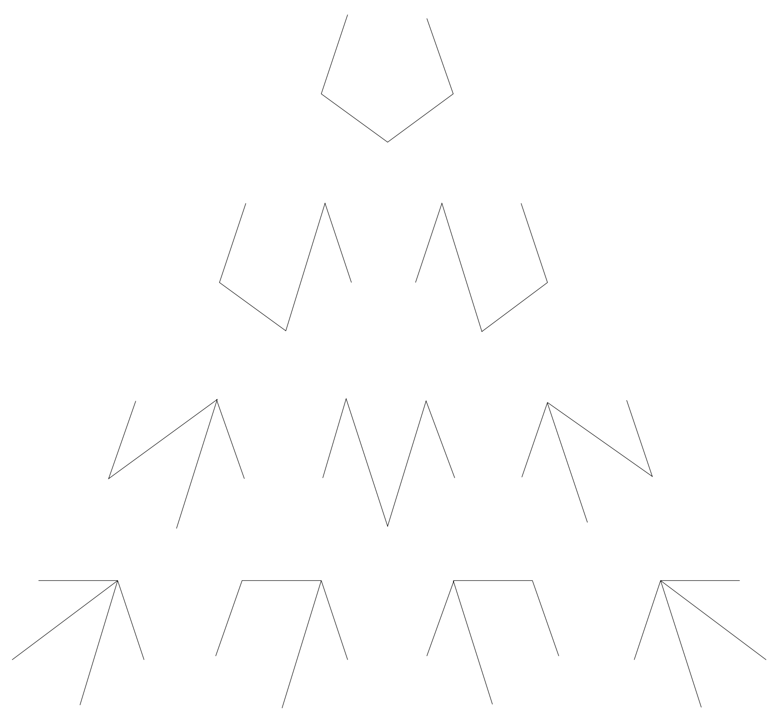


Figure 5: Construction of the sons $S_{j,k}(S_4)$ of the star $S_4 \in \mathcal{T}_4$

In Figure 5 we show all the sons of the star $S_4 \in \mathcal{T}_4$. The sons of type 0 are at the first floor, those of type 1 at the second one, and so on.

The number of sons of a tree $T \in \mathcal{T}_n$ depends on the degree of n in T. More exactly, if this degree is d, then the number of sons of T is

$$(d+1) + d + (d-1) + \ldots + 1 = \frac{(d+2)(d+1)}{2} = \binom{d+2}{2}.$$

We observe that any T has always the sons $S_{0,0}(T)$ and $S_{0,d}(T)$. This sons are a copy of T with a pending edge $(n, n+1)$, and will play an important role later. We denote them by $F(T)$ and $L(T)$, respectively. F and L stand for *first* and *last*, a name that will become clear later.

If $T \in \mathcal{T}_n$ is a son of $T_* \in \mathcal{T}_{n-1}$, we say that T_* is a father of T and we write $T_* = f(T)$. If T_1, T_2 have the same father, we say that T_1 and T_2 are *brothers*. The father $f(T)$ of T is easily obtained by reversing the process. The father is unique, hence we have an (infinite) tree as follows. Taking the unique vertex of G_2 as the root of this tree, at level $n-1$ we have all the trees of $\mathcal{T}_n$, that is, the vertices of G_n. In Figure 6 we can see the first three levels of the tree.

The adjacencies in the graphs G_n are lifted up and down through the tree just constructed in a way we describe in the next two Lemmas, which are immediate.

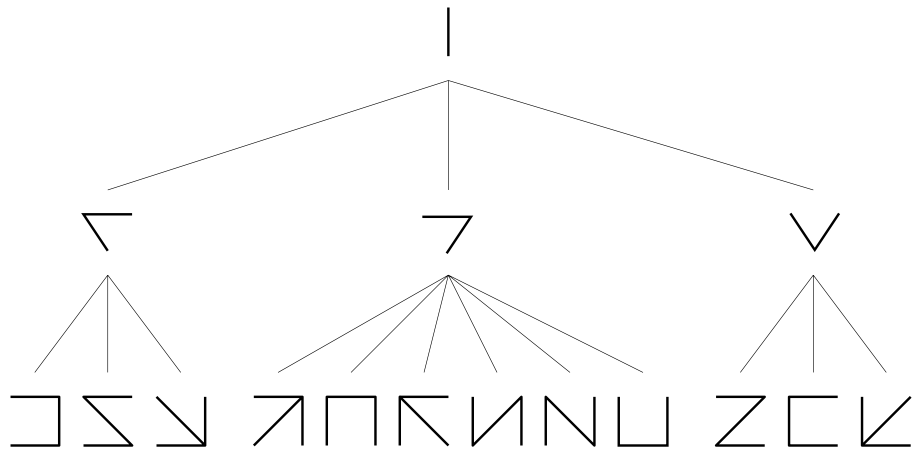


Figure 6: First levels of a tree of geometric trees

Lemma 5.1 *Let T_1, $T_2 \in \mathcal{T}_n$. The following properties hold.*

(a) If T_1, T_2 are adjacent, then $F(T_1), F(T_2)$ are adjacent and $L(T_1), L(T_2)$ are also adjacent.

(b) If T_1, T_2 are adjacent and $(i, n) \in T_1 \cap T_2$, then $S_{j,i}(T_1), S_{j,i}(T_2)$ are adjacent for $j = 1, 2$.

(c) If T_1, T_2 are adjacent and have in common all the edges adjacent to n, then $S_{j,k}(T_1)$ and $S_{j,k}(T_2)$ are adjacent for all j, k.

Lemma 5.2 *The sons of T induce a subgraph $\mathcal{S}_T$ in G_{n+1} that has the following properties.*

(a) $\mathcal{S}_T$ is 2-connected.

(b) $\mathcal{S}_T$ has a Hamiltonian path with extremes $F(T)$ and $L(T)$.

(c) *The degree of the vertices of* $\mathcal{S}_T$ *is between 2 and 6 (in* $\mathcal{S}_T$*).*

In the rest of the paper we will refer to the subgraph of G_{n+1} induced by the set of sons of T as $\mathcal{S}_T$, and to the Hamiltonian path of Lemma 5.2(b) as a *brother-path* (from F(T) to L(T)). Figure 7 illustrates the last lemma. Each vertex of the figure represents a son of the tree T. Sons of type $S_{j,k}$ are at the $(j+1)^{th}$- floor (bottom to top).

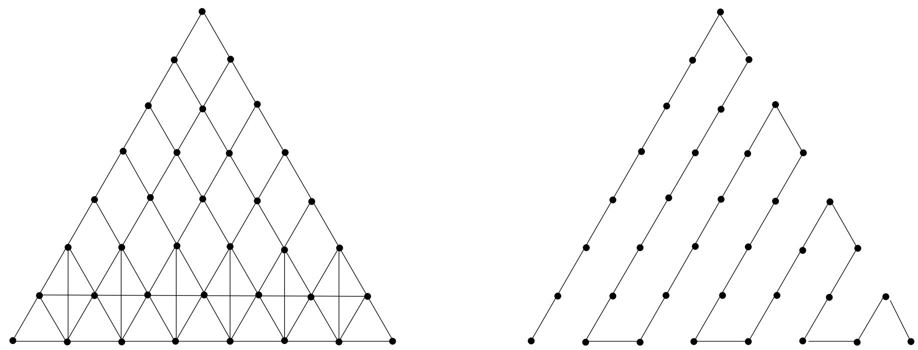


Figure 7: The sons subgraph $\mathcal{S}_T$ and the brother-path having extremes $L(T)$ and $F(T)$

Because of Lemma 5.1 any substructure of G_n has an isomorphic copy in G_{n+1} via $F = \{F(T) = S_{0,0}(T) \mid T \in \mathcal{T}_n\}$ or via $L = \{L(T) = S_{0,d}(T) \mid T \in \mathcal{T}_n\}$. For this reason we can say that F and L are copies of G_n in G_{n+1}. We can obtain all the vertices of G_{n+1} from these two copies of G_n, joining the two copies $F(T), L(T)$ of each vertex T of G_n through the Hamiltonian path in $\mathcal{S}_T$ (see Lemma 5.2).

6 Hamiltonicity and connectivity

As a first application of the tree introduced in the preceding section, we prove that the graph G_n is Hamiltonian by means of an inductive construction. We have to consider two special kind of trees, C_n and B_n, defined as follows. C_n is the chain having all the edges of the boundary except $(1,n)$, and B_n is the tree having all its edges in common with C_n except the edge $(1,2)$ that is replaced by $(1,3)$ instead. It is clear that B_n and C_n are adjacent in G_n, and that the next properties are also satisfied.

Lemma 6.1 *The sons of* C_n *and* B_n *have the following properties:*

(a) C_n *has exactly three sons and they are connected through the path* $F(C_n) = S_{0,0}(C_n) \sim S_{1,1}(C_n) \sim S_{0,1}(C_n) = L(C_n)$.

(b) B_n *has exactly three sons and they are connected through the path* $F(B_n) = S_{0,0}(B_n) \sim S_{1,1}(B_n) \sim S_{0,1}(B_n) = L(B_n)$.

(c) $F(C_n) = C_{n+1}, F(B_n) = B_{n+1}$.

(d) $S_{i,j}(C_n) \sim S_{i,j}(B_n)$.

Theorem 6.2 *G_n is a Hamiltonian graph for all $n \geq 3$. Moreover, there is a Hamiltonian cycle in which C_n and B_n are adjacent.*

Proof. We proceed by induction on n. G_3 is K_3, and the basis of the induction is clear. Let us assume now that G_n has a Hamiltonian cycle C as in the statement. We obtain a copy of C in G_{n+1} via L, and a second and disjoint copy via F. For every tree T_n of G_n the vertices $L(T_n)$ and $F(T_n)$ are connected through the path formed by the sons of T_n, and all the vertices of G_{n+1} belong to some of these paths. By Lemma 6.1 we have $F(C_n) = C_{n+1}$ and $F(B_n) = B_{n+1}$. Taking into account these facts and Lemma 6.1 we construct a Hamiltonian cycle in G_{n+1} in the way depicted in Figure 8. The case where G_n has an even number of vertices is shown in the middle of the figure, and the case where this number is odd is shown on the right. $\square$

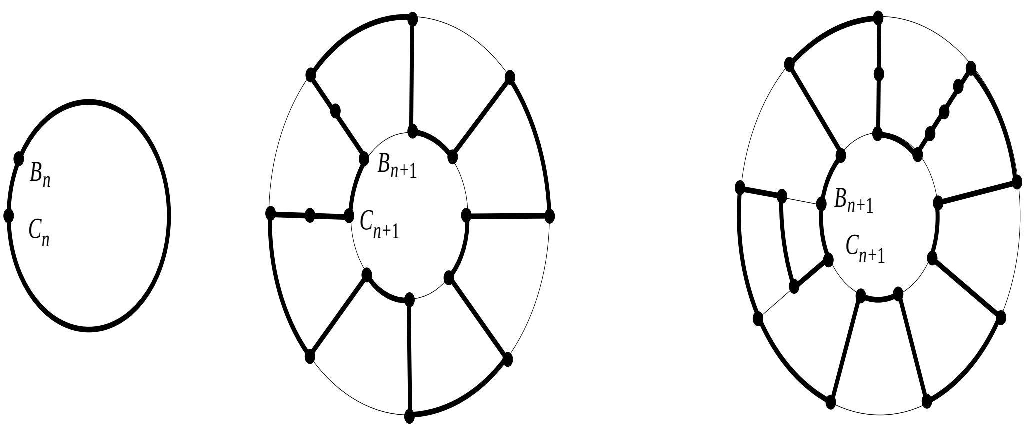


Figure 8: Constructing a Hamiltonian cycle in G_{n+1} given a cycle in G_n

As a second example of application of the tree introduced in Section 5, we compute the connectivity of the graph G_n. We need the following lemmas.

Lemma 6.3 *Let $T, Y \in \mathcal{T}_n$ be two adjacent vertices of G_n. If T, Y are not brothers, then the number of vertices in $\mathcal{S}_{f(Y)}$ which are adjacent to T is at most two.*

Proof. If T is not a brother of Y then their fathers $f(T)$ and $f(Y)$ are different vertices of G_{n-1}. In particular, they differ in at least one edge.

If all the edges of $f(Y)$ not appearing in $f(T)$ are not incident with the vertex $n-1$, then the only vertex of $\mathcal{S}_{f(Y)}$ adjacent to T is Y. Otherwise $f(T)$ and $f(Y)$ differ in an edge incident with the vertex $n-1$. Let be $d = d_{f(Y)}(n-1)$ and let $i_1 < i_2 < \ldots < i_d$ be the vertices adjacent to $n-1$ in the tree $f(Y)$. We have two cases.

Case 1: $(n-1, i_1) \notin f(T)$. Every son of $f(Y)$ has either the edge $(n-1, i_1)$ or the edge (n, i_1), but none of them appear in any son of $f(T)$. In particular, they cannot appear in T. Since T and Y are adjacent, they differ only in an edge and T cannot be adjacent to any other son of $f(T)$ because they differ in more than two edges. The same proof applies if $(n-1, i_d) \notin f(T)$.

Case 2: $(n-1, i_k) \notin f(T)$, with $1 < k < d$. In this case $(n-1, i_k)$ and (n, i_k) are not edges of any son of $f(T)$, but they may be edges of Y. We observe that if $(n-1, i_k)$ and (n, i_k) are both edges of Y, then T and Y differ in more than two edges, so they cannot be adjacent in G_n.

Case 2.1: Either (n, i_k) or $(n-1, i_k)$ is an edge of Y. We suppose that $(n, i_k) \in Y$ (the proof is analogous if $(n-1, i_k) \in Y$). All edges of the tree T are shared with Y, but (n, i_k). Let e be the edge of T not belonging to Y. The edge e is not contained in any brother of Y, as otherwise T would be a brother of Y, against the hypothesis. Only the brothers of Y which differ of Y exactly by e can be adjacent to T and there is a unique brother of Y verifying this condition. Hence, T can be adjacent to at most two sons of $f(Y)$.

Case 2.2.: Neither $(n-1, i_k)$ nor (n, i_k) are edges of Y. In this case, the edge $e \in T$ such that $e \notin Y$ is an edge which does not belong to any brother of Y. As in the preceding case, we can conclude that T is adjacent to at most two of the sons of $f(Y)$. □

A tree of $\mathcal{T}_n$, $(n \geq 5)$ containing a path $i_1, \ldots, i_k$, where $k \geq 4$ and the first and last vertices are consecutive on the boundary, will be called a P_4-tree. A simple induction gives the next lemma.

Lemma 6.4 *If $T \in \mathcal{T}_n$ is a P_4-tree, then $d_{G_n}(T) \geq 2n+1$.*

Now we are ready for proving our last theorem.

Theorem 6.5 *The connectivity of the graph G_n is equal to $2n-4$.*

Proof. As the minimum degree is $2n-4$ we only have to prove that the graph remains connected when any $2n-5$ vertices are suppressed. This is clear for $n=3$. The case $n=4$ is easily proved by direct inspection. We assume now that the property holds for some $n \geq 4$ and proceed by induction: we will prove that G_{n+1} remains connected after the removal of any set W of $2n-3$ vertices. We distinguish three cases. Recall that F and L are isomorphic copies of G_n in G_{n+1}.

Case 1: $W \subset F$ or $W \subset L$. If $W \subset F$, we can construct a path between any two given nodes T and Y as follows: from T to $L(f(T))$, then from $L(f(T))$ to $L(f(Y))$, and finally to Y. The same proof applies when $W \subset L$.

Case 2: $|W \cap F| = 2n-4$ or $|W \cap L| = 2n-4$. If $|W \cap F| = 2n-4$, there is only one vertex Z in W that is not in F. If $Z \neq f(T)$ and $Z \neq f(Y)$ then, because of the 2-connectivity of the subgraphs of sons, we can construct a path as in the preceding case. If $Z = f(T)$ or $Z = f(Y)$, it is easy to see that T has, at least, one adjacent vertex outside of $\mathcal{S}_{f(T)}$ and that it is not in L. The same proof applies when $|W \cap L| = 2n-4$.

Case 3: $|W \cap L| \leq 2n-5$ and $|W \cap F| \leq 2n-5$. Because of the induction, the subgraphs $L-W$ and $F-W$ of $G_{n+1}-W$ are connected. On the other hand, we know that the number of trees of G_n is $t_n > 2n-5$, $(n \geq 4)$. Hence we can assure the existence of at least one complete brother-path in $G_{n+1}-W$. So, it is enough to prove that from any vertex of $G_{n+1}-W$ we can reach some vertex in F, or in L. Let T be a vertex of $G_{n+1}-W$ and suppose that both paths from T to $F(f(T))$ and from T to $L(fT))$ through $\mathcal{S}_{f(T)}$ are broken in $G_{n+1}-W$. We have two subcases.

Case 3.1: T is a star in G_{n+1}.

If T is S_n or S_{n+1}, we know that T is, respectively, $F(S_n)$ or $L(S_n)$, considering now S_n as a star in G_n. Therefore, T is already in F or L.

If T is the star S_i, $1 \leq i \leq n-1$, the father of S_i is the star S_i as a vertex of G_n, and $\mathcal{S}_{f(S_i)}$ has only three vertices. Hence, the number of vertices in G_{n+1} adjacent to T that are not in $\mathcal{S}_{f(S_i)}$ is greater or equal than $d_{G_{n+1}}(S_i) - 2 = 2n-4$. By Lemma 6.3 these vertices are distributed in at least $n-2$ different brother-paths. The minimum number of vertices we would have to remove from G_{n+1} in order to separate S_i from F and L would be at least $2(n-2)+2 = 2n-2$ which is strictly greater than $2n-3$, the cardinal of the set W actually removed.

Case 3.2: T is not a star in G_{n+1} (and, as a consequence, $d_{G_{n+1}}(T) \geq 2n-1$).

Case 3.2.1: Suppose first that T is a son of type 0 not in $F \cup L$, or of type j, $j \geq 2$. These trees are P_4-trees and have at most 5 neighbors belonging to $\mathcal{S}_{f(T)}$. Hence, by Lemma 6.4, T is adjacent to at least $2n+3-5 = 2n-2$ vertices not in $\mathcal{S}_{f(T)}$, and we can conclude as in case 3.1.

Case 3.2.2: If T is $S_{1,1}(f(T))$ (or T is $S_{1,d}(f(T))$, which is handled similarly), then T is adjacent to $F(f(T))$ and to two more trees of $\mathcal{S}_{(f(T))}$ one of type 0, not in $F \cup L$, and one having type j, $j \geq 2$, which are connected to F or to L, as we have seen in case 3.1.1. T has 4 neighbors in $\mathcal{S}_{f(T)}$, so the number of trees adjacent to T that are not in $\mathcal{S}_{f(T)}$ is at least $2n-1-4 = 2n-5$, and they are distributed in at least $n-2$ brother-paths. Again, the minimum number of vertices we have to remove for separating T from F and L is $2(n-2)+3 = 2n-1 > 2n-3$.

Case 3.2.3: If $T = S_{1,k}(f(T))$, $2 \leq k \leq d-1$, T is adjacent to 6 trees in $S_{f(T)}$, four of them of type 0 or j, $j \geq 2$, which are connected to F or to L. On the other hand, T is adjacent to not less than $2n-1-6 = 2n-7$ vertices not in $S_{f(T)}$, distributed in at least $n-3$ brother-paths. Hence, the minimum number of vertices we have to remove in G_{n+1} for separating T from F and L is $2(n-3)+4 = 2n-2 > 2n-3$. $\square$

7 Conclusions and open problems

Many basic properties of the graphs G_n have been obtained in this work. We consider that to exactly determine the diameter is the main open problem left in this paper. An efficient algorithm for finding shortest paths in G_n would be also interesting.

On the other hand, when the set P of points is not in convex position it would be interesting to relate the position of the points to the properties of $T(P)$, as well as to try to characterize the graphs which are geometric tree graphs for some P.

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