

## A POLYNOMIAL BOUND FOR UNTANGLING GEOMETRIC PLANAR GRAPHS

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ABSTRACT. To untangle a geometric graph means to move some of the vertices so that the resulting geometric graph has no crossings. Pach and Tardos [*Discrete Comput. Geom.*, 2002] asked if every  $n$ -vertex geometric planar graph can be untangled while keeping at least  $n^\epsilon$  vertices fixed. We answer this question in the affirmative with  $\epsilon = 1/4$ . The previous best known bound was  $\Omega(\sqrt{\log n / \log \log n})$ . We also consider untangling geometric trees. It is known that every  $n$ -vertex geometric tree can be untangled while keeping at least  $\sqrt{n/3}$  vertices fixed, while the best upper bound was  $\mathcal{O}((n \log n)^{2/3})$ . We answer a question of Spillner and Wolff [<http://arxiv.org/abs/0709.0170>, 2007] by closing this gap for untangling trees. In particular, we show that for infinitely many values of  $n$ , there is an  $n$ -vertex geometric tree that cannot be untangled while keeping more than  $3(\sqrt{n} - 1)$  vertices fixed. Moreover, we improve the lower bound to  $\sqrt{n/2}$ .

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## 1 Introduction

Geometric reconfigurations consider the following fundamental problem. Given a starting and a final configuration of an object  $\mathcal{R}$ , determine if  $\mathcal{R}$  can move from the starting to the final configuration, subject to some set of movement rules. An object can be a set of disks in the plane, or a graph representing a protein, or a robot's arm, for example. Typical movement rules include maintaining connectivity of the object and avoiding collisions or crossings.

In this paper we study the problem where the object is a planar graph<sup>1</sup>  $G$ . The starting configuration is a drawing of  $G$  in the plane with vertices as distinct points and edges as straight-line segments (and possibly many crossings). Our goal is to relocate as few vertices of  $G$  as possible in order to remove all the crossings, that is, to reconfigure  $G$  to some straight line crossing-free drawing of  $G$ . More formally, a *geometric graph* is a graph whose vertices are distinct points in the plane (not necessarily in general position) and whose edges are straight-line segments between pairs of points. If the underlying combinatorial graph of  $G$  belongs to a class of graphs  $\mathcal{K}$ , then we say that  $G$  is a *geometric  $\mathcal{K}$  graph*. For example, if  $\mathcal{K}$  is the class of planar graphs, then  $G$  is a geometric planar graph. Where it causes no confusion, we do not distinguish between the geometric graph and its underlying combinatorial graph. Two edges in a geometric graph *cross* if they intersect at some point other than a common endpoint. A geometric graph with no pair of crossing edges is called *crossing-free*.

Consider a geometric graph  $G$  with vertex set  $V(G) = \{p_1, \dots, p_n\}$ . A crossing-free geometric graph  $H$  with vertex set  $V(H) = \{q_1, \dots, q_n\}$  is called an *untangling* of  $G$  if for all  $i, j \in \{1, 2, \dots, n\}$ ,  $q_i$  is adjacent to  $q_j$  in  $H$  if and only if  $p_i$  is adjacent to  $p_j$  in  $G$ . Furthermore, if  $p_i = q_i$  then we say that  $p_i$  is *fixed*, otherwise we say that  $p_i$  is *free*. If  $H$  is an untangling of  $G$  with  $k$  vertices fixed, then we say that  $G$  can be *untangled* while keeping  $k$  vertices fixed. Clearly only geometric planar graphs can be untangled. Moreover, since every planar graph is isomorphic to some crossing-free geometric graph [5, 13], trivially every geometric planar graph can be untangled while keeping at least 2 vertices fixed. For a geometric graph  $G$ , let  $\text{fix}(G)$  denote the maximum number of vertices that can be fixed in an untangling of  $G$ .

At the 5th Czech-Slovak Symposium on Combinatorics in Prague in 1998, Mamoru Watanabe asked if every geometric cycle (that is, all polygons) can be untangled while keeping at least  $\varepsilon n$  vertices fixed. Pach and Tardos [9] answered that question in the negative by providing an  $\mathcal{O}((n \log n)^{2/3})$  upper bound on the number of fixed vertices. Furthermore, they proved that every geometric cycle can be untangled while keeping at least  $\sqrt{n}$  vertices fixed.

Pach and Tardos [9] asked if every geometric planar graph can be untangled while keeping  $n^\varepsilon$  vertices fixed, for some  $\varepsilon > 0$ . In recent work, Spillner and Wolff [11] showed that geometric planar graphs can be untangled while keeping  $\Omega(\sqrt{\log n / \log \log n})$  vertices fixed. The best known bound before that was 3 [6]. In Section 4, we answer the question of Pach and Tardos [9] in the affirmative and provide the first polynomial lower bound for untangling geometric planar graphs. Specifically, our main result is that every  $n$ -vertex geometric planar graph can be untangled while keeping  $(n/3)^{1/4}$  vertices fixed.

There has also been considerable interest in untangling specific classes of geometric planar graphs. Spillner and Wolff [11] studied the untangling of geometric outerplanar graphs and showed that they can be untangled while keeping  $\sqrt{n/3}$  vertices fixed; and that for every sufficiently large  $n$ , there is an  $n$ -vertex outerplanar graph that cannot be untangled while keeping more than  $2\sqrt{n-1}-1$  vertices fixed. Thus  $\Theta(\sqrt{n})$  is the tight bound for outerplanar graphs. A  $\sqrt{n/3}$  lower bound for trees was shown by Goac *et al.* [6]. The best known upper bound for trees was  $\mathcal{O}((n \log n)^{2/3})$ , which was in fact proved for geometric paths, by Pach and Tardos [9]. In fact, Pach and Tardos [9] prove this upper bound for geometric cycles. However, their method readily applies for geometric paths. We answer a question posed by Spillner and Wolff [11] and close the gap for trees, by showing that for infinitely many values of  $n$ , there is a forest of stars that

<sup>1</sup>We consider graphs that are simple, finite, and undirected. The vertex set of a graph  $G$  is denoted by  $V(G)$ , and its edge set by  $E(G)$ . The subgraph of  $G$  induced by a set of vertices  $S \subseteq V(G)$  is denoted by  $G[S]$ .  $G \setminus S$  denotes  $G[V(G) \setminus S]$ .

cannot be untangled while keeping more than  $3(\sqrt{n} - 1)$  vertices fixed. This result is proved in Section 5. In addition, in Section 3, we demonstrate that every geometric tree can be untangled while keeping  $\sqrt{n/2}$  vertices fixed, thus slightly improving the  $\sqrt{n/3}$  lower bound of Goaoc *et al.* [6]. We conclude the paper with some open problems.

Untangling graphs has also been studied in [8, 12]. Goaoc *et al.* [6] also studied the computational complexity of the related optimization problems and showed various hardness results.

## 2 Lower bounds – a useful lemma

When proving lower bounds, our goal will be to show that given any geometric planar graph  $G$  we can find a large subset  $R$  of vertices of  $G$  such that  $G$  can be untangled while keeping  $R$  fixed. The following geometric lemma simplifies this task by allowing us to concentrate on the case in which all vertices of  $R$  are on the  $y$ -axis. This lemma will be useful both for untangling geometric trees in Section 3 and for untangling general geometric planar graphs in Section 4.

**Lemma 1.** *Let  $\overline{G}$  be an untangling of some geometric planar graph  $G$ . Let  $R$  be a set of vertices of  $G$  such that each vertex of  $R$  is on the  $y$ -axis in  $\overline{G}$  and has the same  $y$ -coordinate in  $\overline{G}$  as in  $G$ . Then there exists an untangling  $\overline{G'}$  of  $G$  in which the vertices in  $R$  are fixed.*

*Proof.* The proof uses the fact that it is possible to perturb the vertices of a crossing-free geometric graph without introducing crossings. More precisely, for any crossing-free geometric graph there exists a value  $\varepsilon > 0$  such that each vertex can be moved a distance of at most  $\varepsilon$ , and the resulting geometric graph is also crossing-free. The maximum value  $\varepsilon$  for which this property holds is called the *tolerance* of the arrangement of segments. This concept, both for the geometric realization and the combinatorial meaning of the graphs was systematically studied in [1, 10].

Consider the untangling  $\overline{G}$  of  $G$  and let  $\varepsilon > 0$  be the value obtained when the above perturbation fact is considered for to  $\overline{G}$ . Let  $X$  denote the maximum absolute value of an  $x$ -coordinate in  $G$  of a vertex in  $R$ . Let  $\overline{G''}$  be the geometric graph obtained from  $\overline{G}$  as follows. For each vertex  $v \in R$  positioned at  $(x, y)$  in  $G$ , move  $v$  from  $(0, y)$  in  $\overline{G}$  to  $(x\varepsilon/X, y)$  in  $\overline{G''}$ . The vertices not in  $R$  are unmoved. So each vertex moves a distance of at most  $\varepsilon$ , and  $\overline{G''}$  is crossing-free. Scale  $\overline{G''}$  by multiplying the  $x$ -coordinates of all vertices in  $\overline{G''}$  by  $X/\varepsilon$  to obtain a crossing-free geometric graph  $\overline{G'}$ . Then every vertex of  $R$  has the same location in  $\overline{G'}$  as it does in  $G$ . Thus  $\overline{G'}$  is an untangling of  $G$  that keeps the vertices of  $R$  fixed.  $\square$

## 3 Trees – lower bound

Goaoc *et al.* [6] proved a lower bound of  $\text{fix}(T) \geq \sqrt{n/3}$  for every  $n$ -vertex geometric tree  $T$ . We now give a different construction that yields a slightly better constant. In addition to an improved constant, our motivation for including this result is that it provides a warm up to our main result, the polynomial lower bound for planar graphs.

**Theorem 1.** *Every  $n$ -vertex geometric tree  $T$  can be untangled while keeping at least  $\sqrt{n/2}$  vertices fixed. That is,  $\text{fix}(T) \geq \sqrt{n/2}$ .*

In a vertex 2-colouring of  $T$ , the largest of the two colour classes has at least  $n/2$  vertices. Therefore, the following lemma, coupled with Lemma 1, implies Theorem 1.

**Lemma 2.** Let  $T$  be an  $n$ -vertex geometric tree  $T$  whose vertices are 2-coloured. Let  $S$  be one of the two colour classes. Then there exists a set  $R$  of vertices in  $T$  such that  $|R| \geq \sqrt{|S|}$  and there is an untangling  $T'$  of  $T$  in which each vertex in  $R$  is on the  $y$ -axis and has the same  $y$ -coordinate in  $T'$  as in  $T$ .

*Proof.* Root  $T$  at any vertex and label its vertices  $(v_1, \dots, v_n)$  based on a postorder traversal of  $T$ .

While we make no general position assumption on the vertices of  $T$ , we may assume, by a suitable rotation, that no pair of vertices of  $T$  have the same  $y$ -coordinate. Let  $R$  be a largest ordered subset  $R \subseteq S$  such that the  $y$ -coordinates of the vertices of  $R$  are either monotonically increasing or monotonically decreasing when considered in the order  $(v_1, \dots, v_n)$ . By the Erdős-Szekeres Theorem [4],  $|R| \geq \sqrt{|S|}$ . Without loss of generality, assume  $R$  is monotonically increasing.

Let  $T'$  be a geometric tree obtained from  $T$  as follows. For each vertex  $v \in R$  positioned at  $(x, y)$  in  $T$ , move  $v$  to  $(0, y)$  in  $T'$ . Move each vertex in  $S \setminus R$  from its position in  $T$  to the  $y$ -axis, such that all the vertices of  $S$  in  $T'$  appear in the order  $\sigma$  on the  $y$ -axis. The vertices in  $V(T) \setminus S$  remain unmoved. To complete the proof of the lemma, it remains to show how to untangle  $T'$  while keeping  $S$  fixed. We prove that by induction on  $i$ , with the following induction hypothesis.

For a point  $p$  with coordinates  $(x, y)$ , a *right ray* at  $p$  is the open half-line containing all the points  $(x', y)$  where  $x' > x$ . Let  $T_i := T'[\{v_1, \dots, v_i\}]$ . For each  $i \in \{1, \dots, n\}$ , there is an untangling  $\overline{T}_i$  of  $T_i$  such that  $S \cap V(T_i)$  is fixed, and

- (1) for all  $j \in \{2, \dots, i\}$ , the  $y$ -coordinate of  $v_j$  is greater than the  $y$ -coordinate of  $v_{j-1}$ , and
- (2) for each vertex  $v$  whose parent in  $T$  is not in  $\{v_1, \dots, v_i\}$ , the right ray at  $v$  does not intersect  $\overline{T}_i$ .

Figure 1 depicts such an untangling of the complete binary tree of depth 4.

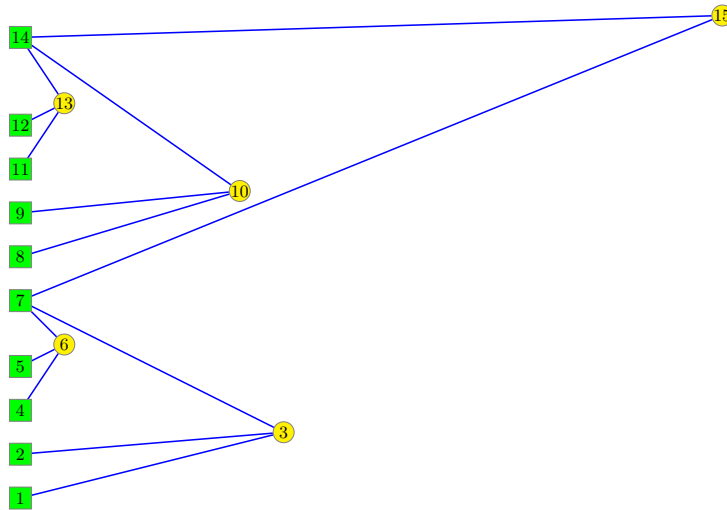


Figure 1: An untangling of the complete binary tree of depth 4. Vertices of  $S$  are depicted by squares.

For  $i = 1$ , the statement is true trivially. Assume now that  $i > 1$ , and that the statement is true for  $i - 1$ . There are two cases to consider:  $v_i \in S$  and  $v_i \notin S$ .

Consider first the case that  $v_i \notin S$ . Since  $S$  is a colour class in a 2-colouring of  $T$ , each child of  $v_i$ , if any, is in  $S$  and thus is on the  $y$ -axis. Assign an  $y$ -coordinate to  $v_i$  that is greater than the  $y$ -coordinate of each vertex in  $\overline{T}_{i-1}$  and less than the  $y$ -coordinate of each vertex in  $S \setminus V(T_{i-1})$ . This ensures that condition (1) is maintained in  $\overline{T}_i$ . Assign a positive  $x$ -coordinate to  $v_i$  such that  $\overline{T}_i$  is crossing-free. Condition (2) on

$T_{i-1}$  guarantees that this is always possible. Condition (2) is clearly maintained for  $v_i$  in  $\overline{T_i}$ . The only other vertices of  $\overline{T_i}$  which may violate condition (2), are vertices whose y-coordinates in  $\overline{T_i}$  are between that of  $v_i$  and its smallest indexed child. However, since the vertices are indexed by the postorder traversal of  $T$ , all such vertices are in the subtree of  $T$  rooted at  $v_i$ , and thus each of their parents is in  $T_i$ . Therefore condition (2) is maintained in  $\overline{T_i}$ .

To complete the proof we consider the case that  $v_i \in S$ . We start with an observation. Consider a vertex  $v \in V(\overline{T_{i-1}}) \setminus S$  whose parent is not in  $\overline{T_{i-1}}$ . Let the coordinates of  $v$  in  $\overline{T_{i-1}}$  be  $(x, y)$ . Each child of  $v$  is in  $S$  and thus lies on the y-axis. Denote their y-coordinates by  $y_1, \dots, y_d$ . By condition (1), for each  $i \in \{1, \dots, d\}$ , the right ray at  $(0, y_i)$  can only be intersected by an edge incident to  $v$  in  $\overline{T_{i-1}}$ . Thus  $v$  can be moved to any position  $(x', y)$ ,  $x' \geq 0$ , and the resulting untangling of  $T_{i-1}$  still satisfies the two conditions. We are now ready to untangle  $T_i$ . Vertex  $v_i$  is fixed, and thus its position in  $\overline{T_i}$  is predetermined. None of its children are in  $S$ . Thus we are allowed to move any child of  $v_i$  from its position in  $\overline{T_{i-1}}$  to a new position. By the above observation it is possible to move each child  $w$  of  $v_i$  (one by one, in the decreasing order of their y-coordinates), such that the resulting untangling  $\overline{T'_{i-1}}$  of  $T_{i-1}$  satisfies conditions (1) and (2), and such that the open segment  $\overline{wv_i}$  does not intersect  $\overline{T'_{i-1}}$ . Connect  $v_i$  by a segment to each of its children in  $\overline{T'_{i-1}}$ . Then the resulting untangling  $\overline{T_i}$  is crossing-free. Condition (1) is maintained since all the vertices of  $T_{i-1}$  have smaller y-coordinate than  $v_i$  in  $\overline{T_i}$ . Condition (2) is maintained in  $\overline{T_i}$  by the same arguments used when  $v_i \notin S$ .  $\square$

#### 4 Planar graphs - lower bound

Let  $G$  be an  $n$ -vertex geometric planar graph. In this section we prove that  $G$  can be untangled while keeping  $(n/3)^{1/4}$  vertices fixed (as stated in Theorem 2 below). It suffices to prove this theorem for edge-maximal geometric planar graphs. Thus for the remainder of this section assume that  $G$  is edge-maximal.<sup>2</sup>

Let  $\mathcal{E}$  be an embedded planar graph isomorphic to  $G$ . Each face of  $\mathcal{E}$  is bounded by a 3-cycle. Canonical orderings of embedded edge-maximal planar graphs were introduced by de Fraysseix *et al.* [2], where they proved that  $\mathcal{E}$  has a vertex ordering  $\sigma = (v_1 := x, v_2 := y, v_3, \dots, v_n := z)$ , called a *canonical ordering*, with the following properties. Define  $G_i$  to be the embedded subgraph of  $\mathcal{E}$  induced by  $\{v_1, v_2, \dots, v_i\}$ . Let  $C_i$  be the subgraph of  $\mathcal{E}$  induced by the edges on the boundary of the outer face of  $G_i$ . Then

- $x, y$  and  $z$  are the vertices on the outer face of  $\mathcal{E}$ , and
- For each  $i \in \{3, 4, \dots, n\}$ ,  $C_i$  is a cycle containing  $xy$ .
- For each  $i \in \{3, 4, \dots, n\}$ ,  $G_i$  is biconnected and *internally 3-connected*; that is, removing any two interior vertices of  $G_i$  does not disconnect it.
- For each  $i \in \{3, 4, \dots, n\}$ ,  $v_i$  is a vertex of  $C_i$  with at least two neighbours in  $C_{i-1}$ , and these neighbours are consecutive on  $C_{i-1}$ .

For example, the ordering in Figure 2(a) is a canonical ordering of the depicted embedded graph  $\mathcal{E}$ .

We now introduce a new combinatorial structure that is critical to the proof of Theorem 2. The *frame*  $\mathcal{F}$  of  $\mathcal{E}$  is the oriented subgraph of  $\mathcal{E}$  with vertex set  $V(\mathcal{F}) := V(\mathcal{E})$ , where:

- $xy$  is in  $E(\mathcal{F})$  and is oriented from  $x$  to  $y$ .

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<sup>2</sup>A planar graph  $H$  is edge-maximal (also called, a *triangulation*), if for all  $vw \notin E(H)$ , the graph resulting from adding  $vw$  to  $H$  is not planar.

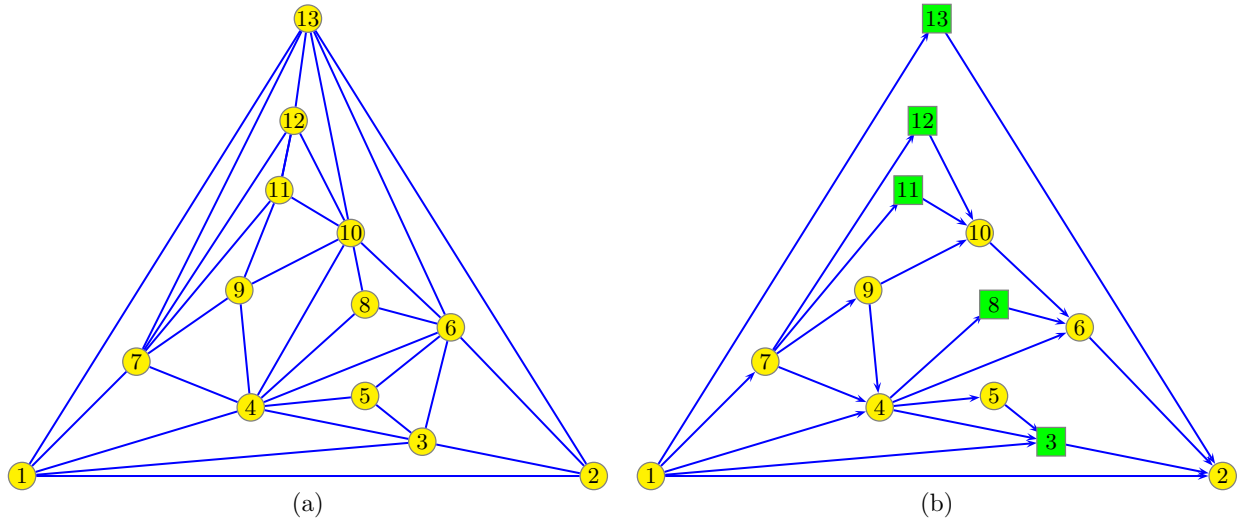


Figure 2: (a) Canonical ordering of  $\mathcal{E}$ , (b) Frame  $\mathcal{F}$  of  $\mathcal{E}$ . Vertices forming a largest antichain in  $\langle_{\mathcal{F}}$ , that is the vertices in  $S$ , are depicted by squares.

- For each  $i \in \{3, 4, \dots, n\}$  in the canonical ordering  $\sigma$  of  $\mathcal{E}$ , edges  $pv_i$  and  $v_i p'$  are in  $E(\mathcal{F})$ , where  $p$  and  $p'$  are the first and the last neighbour, respectively, of  $v_i$  along the path in  $C_{i-1}$  from  $x$  to  $y$  not containing edge  $xy$ . Edge  $pv_i$  is oriented from  $p$  to  $v_i$ , and edge  $v_i p'$  is oriented from  $v_i$  to  $p'$ , as illustrated in Figure 2(b). We call  $p$  the *left predecessor* of  $v$  and  $p'$  the *right predecessor* of  $v$ .

We also say that  $\mathcal{F}$  is a frame of  $G$ . By definition,  $\mathcal{F}$  is a directed acyclic graph with one source  $x$  and one sink  $y$ .  $\mathcal{F}$  defines a partial order  $\langle_{\mathcal{F}}$  on  $V(\mathcal{F})$ , where  $v \langle_{\mathcal{F}} w$  whenever there is a directed path from  $v$  to  $w$  in  $\mathcal{F}$ .

The remainder of this section is dedicated to proving the following two lemmas, which readily imply the desired result, as shown in the proof of Theorem 2 below.

**Lemma 3.** *Every  $n$ -vertex geometric planar graph  $G$  whose partial order  $\langle_{\mathcal{F}}$  associated with its frame  $\mathcal{F}$  has a chain of size  $\ell$  can be untangled while keeping  $\sqrt{\ell/3}$  vertices fixed.*

**Lemma 4.** *Every  $n$ -vertex geometric planar graph  $G$  whose partial order  $\langle_{\mathcal{F}}$  associated with its frame  $\mathcal{F}$  has an antichain of size  $t$  can be untangled while keeping  $\sqrt{t}$  vertices fixed.*

**Theorem 2.** *Every  $n$ -vertex geometric planar graph  $G$  can be untangled while keeping at least  $(n/3)^{1/4}$  vertices fixed. That is,  $\text{fix}(G) \geq (n/3)^{1/4}$ .*

*Proof.* Let  $\mathcal{F}$  be a frame of  $G$  and let  $\langle_{\mathcal{F}}$  be its associated partial order. If  $\langle_{\mathcal{F}}$  has a chain of size at least  $\sqrt{3n}$  then we are done by Lemma 3. Otherwise, by Dilworth's theorem [3],  $\langle_{\mathcal{F}}$  has a partition into  $\sqrt{3n}$  antichains. By the pigeon-hole principle there is an antichain in that partition that has at least  $\frac{n}{\sqrt{3n}}$  vertices, which completes the proof, by Lemma 4.  $\square$

The remainder of this section is dedicated to proving Lemma 3 and Lemma 4.

#### 4.1 Big chain - Proof of Lemma 3

A *chord* of a cycle  $C$  is an edge that has both endpoints in  $C$ , but itself is not an edge of  $C$ . Consider a cycle  $C$  in an embedded planar graph  $\mathcal{E}$ .  $C$  is called *externally chordless* if each chord of  $C$  is embedded inside of  $C$  in  $\mathcal{E}$ . The following theorem is by Spillner and Wolff [11].

**Theorem 3.** [11] *Let  $G$  be a geometric planar graph and  $\mathcal{E}$  an embedding planar graph isomorphic to  $G$ . If  $\mathcal{E}$  has an externally chordless cycle on  $\ell$  vertices, then  $G$  can be untangled while keeping at least  $\sqrt{\ell/3}$  vertices fixed. Note that this result is expressed in slightly different form in [11] (see Theorem 2 in [11]).*

**Lemma 5.** *Consider any directed path on at least three vertices from  $x$  to  $y$  in  $\mathcal{F}$ . The cycle comprised of that path and edge  $xy$  is externally chordless in  $\mathcal{E}$ .*

*Proof.* Denote the cycle in question by  $C$ , and denote the directed path between  $x$  and  $y$  in  $C$  not containing edge  $xy$  by  $P$ . Consider a chord  $v_i v_j$  of  $C$ . Without loss of generality,  $v_i <_\sigma v_j$  in the canonical ordering  $\sigma$ . Thus  $v_i$  is in  $G_{j-1}$  and  $v_i v_j$  is an edge of  $G_j$ . The neighbours of  $v_j$  in  $G_{j-1}$  appear consecutively along the boundary  $C_{j-1}$  of  $G_{j-1}$ . Let  $x_1, \dots, x_d$  be the neighbours of  $v_j$  in left-to-right order on  $C_{j-1}$ . Thus  $x_1 v_j$  and  $v_j x_d$  are arcs in  $\mathcal{F}$ . Let  $uv_j$  and  $v_j w$  be the incoming and outgoing arcs in  $P$  at  $v_j$ . Then the counterclockwise order of edges incident to  $v_j$  in  $\mathcal{E}$  is  $(u, \dots, x_1, \dots, x_d, \dots, w, \dots)$ . In particular, each edge  $v_j x_\ell$  is contained in the closure of the interior of  $C$ . Now  $v_i = x_\ell$  for some  $\ell \in [1, d]$ . Thus  $v_i v_j$  is an internal chord of  $C$ .  $\square$

This lemma, coupled with Theorem 3, implies Lemma 3, as demonstrated below.

*Proof of Lemma 3.* If  $\ell < 3$ , the claim follows trivially. Assume now that  $\ell \geq 3$ . Since  $<_{\mathcal{F}}$  has a chain of size  $\ell$ ,  $<_{\mathcal{F}}$  has a maximal chain of size  $\ell' \geq \ell$ . Every maximal chain in  $<_{\mathcal{F}}$ , is a path from  $x$  to  $y$  in  $\mathcal{F}$ . Therefore, Lemma 5 implies that  $\mathcal{E}$  contains an externally chordless cycle on  $\ell'$  vertices, and the result follows from Theorem 3.  $\square$

#### 4.2 Big Antichain - Proof of Lemma 4

For each vertex  $v \in V(\mathcal{F})$ , we define  $\text{Lroof}(v)$  and  $\text{Rroof}(v)$ , as the following directed paths in  $\mathcal{F}$ .

$$\begin{aligned} \text{Lroof}(v_1) &:= \emptyset \text{ and } \text{Rroof}(v_1) := \emptyset, \\ \text{Lroof}(v_2) &:= \emptyset \text{ and } \text{Rroof}(v_2) := \emptyset. \end{aligned}$$

For each  $i \in \{3, \dots, n\}$ , define  $\text{Lroof}(v_i)$  and  $\text{Rroof}(v_i)$  recursively, as follows.

$$\begin{aligned} \text{Lroof}(v_i) &:= \text{Lroof}(p) \cup \{pv_i\}, \text{ and} \\ \text{Rroof}(v_i) &:= \{v_i p'\} \cup \text{Rroof}(p), \end{aligned}$$

where  $p$  is the left and  $p'$  the right predecessor of  $v_i$ . Finally, define the *roof* of  $v_i$  to be  $\text{roof}(v_i) := \text{Lroof}(v_i) \cup \text{Rroof}(v_i)$ .

Note that for each  $i \in \{3, \dots, n\}$ ,  $\text{roof}(v_i)$  is a directed path in  $\mathcal{F}$  from  $x$  to  $y$  containing  $v_i$ , where the sub-path ending at  $v_i$  is  $\text{Lroof}(v_i)$ , and the sub-path starting  $v_i$  is  $\text{Rroof}(v_i)$ .

Let  $S$  be the set of vertices that comprise a largest antichain in  $<_{\mathcal{F}}$ , as illustrated in Figure 2(b) with squares. Now consider the given geometric graph  $G$ . We may assume, by a suitable rotation, that no pair of vertices of  $G$  have the same y-coordinate. Let  $R$  be a largest ordered subset  $R \subseteq S$  such that the y-coordinates of the vertices of  $R$  are either monotonically increasing or monotonically decreasing when considered in the order given by  $\sigma$ . By the Erdős-Szekeres Theorem [4],  $|R| \geq \sqrt{|S|}$ . Without loss of generality, assume  $R$  is monotonically increasing. In what follows, we untangle  $G$  while keeping  $R$  fixed.

Let  $\mathcal{H}$  be the graph induced in  $\mathcal{E}$  by the following set of vertices:  $V(\mathcal{H}) := \cup\{\text{roof}(w) : w \in R\}$ ; that is,  $\mathcal{H} = \mathcal{E}[V(\mathcal{H})]$ . Note that  $\mathcal{H}$  is not necessarily a subgraph of  $\mathcal{F}$ , as illustrated in Figure 3.

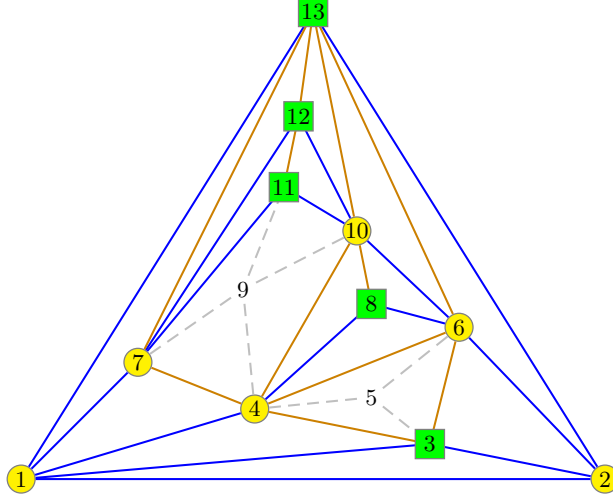


Figure 3: The graph  $\mathcal{H}$ . The vertices in  $R \subseteq S$  are depicted by squares. Edges  $\{3, 6\}$ ,  $\{4, 10\}$ ,  $\{8, 10\}$ ,  $\{6, 13\}$ ,  $\{7, 13\}$ ,  $\{10, 13\}$  and  $\{12, 13\}$  are in  $\mathcal{H}$  but not in  $\mathcal{F}$ .

We say that a simple polygonal chain  $C$  is *strictly  $x$ -monotone* if, for every vertical line  $\ell$ ,  $|C \cap \ell| \leq 1$ . For two distinct points  $p$  and  $q$  in the plane, let  $\overline{pq}$  denote the *open* line-segment with endpoints  $p$  and  $q$ . A simple polygon  $C$  is *star-shaped* (from  $p$ ) if there is a point  $p$  such that for every point  $q \in C$ ,  $\overline{pq} \cap C = \emptyset$ . The following lemma is the main ingredient in the proof of Lemma 4.

**Lemma 6.** *The geometric planar graph  $G[V(\mathcal{H})]$  can be untangled such that each vertex of  $R$  is on the  $y$ -axis and it has the same  $y$ -coordinate in the untangling as in  $G[V(\mathcal{H})]$ . Moreover, all the internal faces of the untangling are star-shaped and the path on its outer face from  $x$  to  $y$  not containing  $xy$  is strictly  $x$ -monotone.*

We delay the proof of Lemma 6 until the end of the section. We first show how it implies our desired result when coupled with the following theorem by Hong and Nagamochi [7].

**Theorem 4.** [7] *Consider a 3-connected embedded planar graph  $\mathcal{E}$ , with outer facial cycle  $C$ . Given any geometric cycle  $\overline{C}$  that is star-shaped, and given any isomorphic mapping from  $V(C)$  to  $V(\overline{C})$ , there is a crossing-free geometric graph  $\overline{\mathcal{E}}$  isomorphic to  $\mathcal{E}$  with  $\overline{C}$  as its outer face and respecting the vertex mapping.*

*Proof of Lemma 4.* Since  $\prec_{\mathcal{F}}$  has an antichain of size  $t$ ,  $\prec_{\mathcal{F}}$  has a maximal antichain  $S$  of size  $t' \geq t$ . Then the subset  $R \subseteq S$ , on which  $\mathcal{H}$  is defined, has size  $|R| \geq \sqrt{t}$ . Thus by Lemma 6,  $G[V(\mathcal{H})]$  can be untangled such that the vertices of  $R$  are all on the  $y$ -axis and their  $y$ -coordinates are preserved. If  $z \notin R$ , then assign  $x$ - and  $y$ -coordinates to  $z$ , and connect  $z$  to its neighbours in  $\mathcal{H}$ , such that the resulting geometric graph  $H$  is crossing-free and all the internal faces of  $H$  are star-shaped. This is always possible since the path from  $x$  to  $y$  on the outer face of the above untangled graph is strictly  $x$ -monotone.  $H$  is an untangling of  $G[V(\mathcal{H}) \cup \{z\}]$ .

It remains to determine a placement of the remaining free vertices of  $G$ , that is vertices in  $V(G) \setminus V(H)$ . Vertices of  $V(G) \setminus V(H)$  can be partitioned into sets  $I_j$ ,  $1 \leq j \leq |E(H)| - |V(H)| + 1$ , where each vertex in  $I_j$  is inside the cycle in  $\mathcal{E}$  determined by the internal face  $f_j$  of  $H$ . For each internal face  $f_j$  of  $H$ , let  $G^j$  be the following subgraph of  $\mathcal{E}$ . The vertex set  $V(G^j)$  is the union of  $V(f_j)$  and  $I_j$ . The edge set  $E(G^j)$  is comprised of the edges of the cycle  $f_j$ , the edges in  $\mathcal{E}[I_j]$ , and the edges between  $V(f_j)$  and  $I_j$ . Each  $f_j$  is star-shaped in  $H$ , by Lemma 6. Therefore, to apply Theorem 4, it remains to show that  $G^j$  is 3-connected.

Assume, for the sake of contradiction, that  $G^j$  is not 3-connected. All the faces of  $G^j$  are triangles except possibly the outer face  $C^j$ . Therefore,  $G^j$  is internally 3-connected, that is, removing any two interior vertices of  $G^j$  does not disconnect it. Thus each cut-set of size 2 of  $G^j$  has a vertex, say  $v$ , that is in  $C^j$ .



Removing  $v$  from  $G^j$  results in a graph that is not 2-connected. The outer face  $C^j$  has no chords, since  $f_j$  is a face of  $H$ . Therefore, removing  $v$  from  $G^j$  results in graph whose outer face is a cycle and all internal faces are triangles. Thus that graph is a 2-connected graph, which provides the contradiction.

Applying Theorem 4 to embed each subgraph  $G^j$  yields an untangling of  $G$  in which the vertices of  $R$  are all on the  $y$ -axis and have their  $y$ -coordinates preserved. Applying Lemma 1 to this untangling completes the proof of the theorem.  $\square$

All that remains is to prove Lemma 6.

*Proof of Lemma 6.* The proof is by induction on the number of vertices in  $R$ . We start by considering some useful properties of the roofs of two vertices in  $R$ .

Consider two incomparable vertices,  $u$  and  $v$  in  $R$  (that is, two incomparable vertices in  $\langle \mathcal{F} \rangle$ ), where  $u <_\sigma v$ . Let  $x'$  be a vertex of  $\mathcal{F}$  such that  $x' \in \text{Lroof}(u)$  and  $x' \in \text{Lroof}(v)$ , and the vertex following  $x'$  in  $\text{Lroof}(u)$  is not the same as the vertex following  $x'$  in  $\text{Lroof}(v)$ , as illustrated in Figure 4. Similarly, let  $y'$  be a vertex of  $\mathcal{F}$  such that  $y' \in \text{Rroof}(u)$  and  $y' \in \text{Rroof}(v)$ , and the vertex before  $y'$  in  $\text{Rroof}(u)$  is not the same as the vertex before  $y'$  in  $\text{Rroof}(v)$ . Such vertices,  $x'$  and  $y'$ , exist since  $u$  and  $v$  are incomparable in  $\mathcal{F}$ . Then  $\text{roof}(v)$  and  $\text{roof}(u)$  have the following properties. The paths between  $x$  and  $x'$  in  $\text{roof}(u)$  and in  $\text{roof}(v)$  coincide in  $\mathcal{F}$ , that is, the two paths are both equal to  $\text{Lroof}(x')$ . Similarly, the paths between  $y'$  and  $y$  in  $\text{roof}(u)$  and in  $\text{roof}(v)$  coincide in  $\mathcal{F}$ , that is, they are both equal to  $\text{Rroof}(y')$ . The path between  $x'$  and  $y'$  in  $\text{roof}(u)$  contains  $u$ , the path between  $x'$  and  $y'$  in  $\text{roof}(v)$  contains  $v$ , and the two paths have only  $x'$  and  $y'$  in common. Finally,  $u$  is inside the cycle determined by  $\text{roof}(v)$  and edge  $xy$  in  $\mathcal{F}$ . To summarise, for all  $u, v \in R$ , if  $u <_\sigma v$  then each vertex of  $\text{roof}(u)$  is either on or inside the cycle determined by  $\text{roof}(v)$  and edge  $xy$  in  $\mathcal{F}$ .

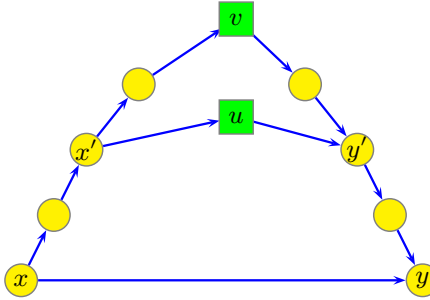


Figure 4: Roofs of two incomparable vertices  $u$  and  $v$  of  $\langle \mathcal{F} \rangle$ .

We proceed by induction on the number of vertices in  $R$ , but require a somewhat stronger inductive hypothesis than the statement of the lemma. Let  $C$  be a simple strictly  $x$ -monotone polygonal chain. We say that  $C$  is  $\varepsilon$ -ray-monotone from a point  $p = (x_p, y_p)$  if for every point  $r = (x_p, y_p + t)$  with  $t \geq \varepsilon$ , and every point  $q \in C$ ,  $\overrightarrow{rq} \cap C = \emptyset$ . Informally,  $C$  is  $\varepsilon$ -ray-monotone from  $p$  if every point sufficiently far above  $p$  sees all of  $C$ . Note that, under this definition, if  $C$  is  $\varepsilon$ -ray-monotone from  $p$  then  $C$  is  $\varepsilon$ -ray-monotone from any point  $q = (x_p, y_p + t)$ ,  $t > 0$ , above  $p$ . Furthermore, there exists a value  $\delta = \delta(p, C, \varepsilon)$  such that  $C$  is  $\varepsilon$ -ray-monotone from any point  $p'$  whose distance from  $p$  is at most  $\delta$ . (This follows from the fact that the set of points  $p$  from which  $C$  is  $\varepsilon$ -ray-monotone is an open set.)

Let  $\varepsilon'$  be the minimum difference between the  $y$ -coordinates of any two vertices in  $R$ . We will construct a crossing-free geometric graph  $\overline{\mathcal{H}}$  that is an untangling of  $G[V(\mathcal{H})]$ . In addition to the conditions of the lemma,  $\overline{\mathcal{H}}$  will have the following property: If  $|R| > 0$  then the outer face of  $\overline{\mathcal{H}}$  is bounded by the edge

$xy$  and a path  $C$  from  $x$  to  $y$  such that  $C \cap R = \{v\}$ , for some vertex  $v \in R$ , and  $C$  is  $\varepsilon$ -ray-monotone from  $v$  for some  $\varepsilon < \varepsilon'$ .

The base case occurs when  $|R| = 0$ . Then  $\mathcal{H}$  consists of the single edge  $xy$ , which can be untangled by placing  $x$  at  $(-1, t)$  and  $y$  at  $(1, t)$ , where  $t$  is smaller than any  $y$ -coordinate in  $G$ . Clearly this crossing-free geometric graph satisfies the conditions of the lemma as well as the inductive hypothesis. Next, suppose  $|R| \geq 1$  and let  $v$  be the largest vertex of  $R$  in the total order  $\sigma$ . If  $|R| = 1$ , let  $\mathcal{H}'$  be the subgraph of  $\mathcal{H}$  induced by  $\{x, y\}$ , otherwise,  $|R| > 1$  and let  $\mathcal{H}'$  be the subgraph of  $\mathcal{H}$  induced by the vertices in  $\cup\{\text{roof}(u) : u \in R \setminus v\}$ . By induction, we can untangle  $G[V(\mathcal{H}')$  to obtain a crossing-free geometric graph  $\overline{\mathcal{H}'}$  that satisfies the inductive hypothesis and the conditions of the lemma. It remains to place  $v$  and the vertices of  $\text{roof}(v)$  that are not yet placed. As described above, these vertices form a path  $P$  that goes from some vertex  $x'$  of  $\mathcal{H}'$  to  $v$  to some vertex  $y'$  of  $\mathcal{H}'$ .

The conditions of the lemma specify the location of  $v$ . In particular,  $v$  is on the  $y$ -axis, with its  $x$ -coordinate equal to its  $y$ -coordinate in  $G$ . The inductive hypothesis guarantees that the vertex  $v$  and any point sufficiently close to  $v$  can see<sup>3</sup> all vertices of the outer face of  $\overline{\mathcal{H}'}$ . Finally, we note that, if  $|R| > 1$ , then directly below  $v$ , on the  $y$ -axis, is a vertex  $u \in R$ . The fact that  $u$  is on the  $y$ -axis and that the outer face of  $\overline{\mathcal{H}'}$  is strictly  $x$ -monotone implies that the  $x$ -coordinate of  $x'$  is less than 0 and that the  $x$ -coordinate of  $y'$  is greater than 0. (For the special case when  $x' = x$  and/or  $y' = y$ , the above statement is still true.)

Next we place the interior vertices of  $P$  to obtain the crossing-free geometric graph  $\overline{\mathcal{H}}$ . To do this, we draw a unit circle  $c$ , containing  $v$ , whose center is on the  $y$ -axis and below  $v$ . We place all interior vertices of  $P$  on  $c$  and sufficiently close to  $v$  so that:

- (1) the path on the outer face of  $\overline{\mathcal{H}}$  from  $x$  to  $y$  not containing  $xy$  is strictly  $x$ -monotone,
- (2) all interior vertices of  $P$  see all other vertices of  $P$  in  $\overline{\mathcal{H}}$ ,
- (3) all interior vertices of  $P$  see all vertices on the outer face of  $\overline{\mathcal{H}'}$  between  $x'$  and  $y'$ , and
- (4) the path on the outer face of  $\overline{\mathcal{H}}$  from  $x$  to  $y$  not containing  $xy$  is  $\varepsilon$ -ray-monotone from  $v$  for some  $\varepsilon < \varepsilon'$ .

That the first condition can be achieved follows from the fact that  $x'$  and  $y'$  are to the left and right, respectively, of the  $y$ -axis. That the second condition can be achieved follows from the fact that we are placing the interior vertices of  $P$  on a convex curve (a circle) as close to  $v$  as necessary. The third condition can be achieved since the upper chain of  $\overline{\mathcal{H}'}$  is  $\varepsilon$ -ray-monotone from  $u$  and hence also from  $v$ . That the fourth condition can be achieved follows from the definition of  $\varepsilon$ -ray-monotonicity and the first condition.

Consider the path in  $\overline{\mathcal{H}'}$  from  $x$  to  $y$  not containing  $xy$  along the outer face of  $\overline{\mathcal{H}'}$ . This path is comprised of the same vertices and edges as a directed path from  $x$  to  $y$  in  $\mathcal{F}$ . Thus, by Lemma 5, the outer face of  $\overline{\mathcal{H}'}$  has no outer chords in  $\overline{\mathcal{H}}$ . Therefore, an edge of  $\overline{\mathcal{H}}$  that is not an edge of  $\overline{\mathcal{H}'}$  is either an edge on  $P$ , or it is an edge accounted for in Conditions (2) or (3) above. Thus  $\overline{\mathcal{H}}$  is crossing-free. The vertices in  $R$  are all on the  $y$ -axis and all have the same  $y$ -coordinates in  $G$  as in  $\overline{\mathcal{H}}$ . Conditions (1) (and (4)) imply that the path between  $x$  and  $y$  on the outer face of  $\overline{\mathcal{H}}$  is strictly  $x$ -monotone. It remains to show that the internal faces of  $\overline{\mathcal{H}}$  are star-shaped. The only new faces in  $\overline{\mathcal{H}}$  not present in  $\overline{\mathcal{H}'}$  are the faces having interior vertices of  $P$  on their boundary. However, Conditions (2) and (3) above imply that each such face is star-shaped from some interior vertex of  $P$ . This completes the proof of the lemma.  $\square$

## 5 Trees – upper bound

In this section we prove the following theorem.

<sup>3</sup>Given a geometric graph, we say that a point  $p$  in the plane sees a point  $q$ , if  $\overline{pq}$  does not intersect the graph.

**Theorem 5.** For every positive number  $n$  such that  $\sqrt{n}$  is an integer, there exists a geometric forest (of stars)  $G$  on  $n$  vertices, such that  $\text{fix}(G) = 3(\sqrt{n} - 1)$ . That is,  $G$  cannot be untangled while keeping less than  $3(\sqrt{n} - 1)$  vertices fixed, and  $G$  can be untangled while keeping exactly that many vertices fixed.

*Proof.* We first define  $G$ . A  $k$ -star is a rooted tree on  $k + 1$  vertices one of which is the root and the rest of the vertices are leaves adjacent to that root.  $G$  is a forest on  $n$  vertices comprised of trees,  $T_i$ ,  $1 \leq i \leq \sqrt{n}$ , where each  $T_i$  is a  $(\sqrt{n} - 1)$ -star. All the vertices of  $G$  lie on the  $x$ -axis. For each  $i$ , the vertices of  $T_i$  have the following  $x$ -coordinates  $i, i + \sqrt{n}, \dots, i + \sqrt{n}(\sqrt{n} - 1)$  where the vertex with the maximum  $x$ -coordinate is the root of  $T_i$ . This completes the description of  $G$ .

*Upper bound:*

We first prove that  $\text{fix}(G) \leq 3\sqrt{n} - 3$ ; that is, we prove that  $G$  cannot be untangled while keeping more than  $3\sqrt{n} - 3$  vertices fixed. Let  $H$  be an untangling of  $G$  with  $\text{fix}(G)$  vertices fixed. Let  $\ell$  denote the number of fixed leaves and  $r$  the number of fixed roots. Let  $r'$  denote the number of fixed roots that are adjacent to a fixed leaf. Given the ordering of the vertices of  $G$  on the  $x$ -axis, it is clear that  $r' \leq 1$ .

Partition the set of free roots into two sets. Let  $A$  be the set containing the free roots that are on or above the  $x$ -axis in  $H$ . Let  $B$  be the set containing the free roots that are strictly below the  $x$ -axis in  $H$ . Our reason for this non-symmetric definition of  $A$  and  $B$  is to avoid double counting, and not because free roots on the  $x$ -axis have any special meaning. The total number of roots of  $G$  is  $|A| + |B| + r$ .

Suppose that the number of fixed leaves with a neighbour (i.e., a parent) in  $A$  is at most  $\sqrt{n} - 2 + |A|$ , and similarly for the number of fixed leaves with a neighbour in  $B$ . As noted above, at most one fixed leaf can be adjacent to a fixed root, thus  $\ell \leq 2\sqrt{n} - 4 + |A| + |B| + r'$ . Since  $\text{fix}(G) = \ell + r$ , we get  $\text{fix}(G) \leq 2\sqrt{n} - 4 + |A| + |B| + r' + r$ . Having  $|A| + |B| + r = \sqrt{n}$  further implies that  $\text{fix}(G) \leq 3\sqrt{n} - 4 + r'$ . Since  $r' \leq 1$ , we get the desired upper bound.

Thus to complete the proof of the upper bound it remains to prove that the number of fixed leaves with a neighbour in  $A$  is at most  $\sqrt{n} - 2 + |A|$ . The proof below has no special case for the free roots that are on the  $x$ -axis, so the proof for the number of fixed leaves with a neighbour in  $B$  is analogous.

Partition the leaves of  $G$  into a set of blocks  $\{P_j : 1 \leq j \leq \sqrt{n} - 1\}$ , such that  $P_1$  contains the first  $\sqrt{n}$  leaves on the  $x$ -axis,  $P_2$  the next  $\sqrt{n}$  leaves, and so on. More formally,  $P_j$  contains all the leaves with  $x$ -coordinate in the range  $[1 + (j - 1)\sqrt{n}, j\sqrt{n}]$ . Note that each block contains exactly one leaf from each star of  $G$ . There are  $\sqrt{n} - 1$  blocks, each containing  $\sqrt{n}$  vertices.

Define an auxiliary graph  $Q$  with vertex set  $V(Q) = A \cup \{p_j : 1 \leq j \leq \sqrt{n} - 1\}$ , where  $vp_j \in E(Q)$  precisely if  $v$  is a vertex of  $A$  and  $v$  has a fixed neighbour in block  $P_j$ . Thus  $Q$  is a bipartite graph, where one bipartition is precisely the set  $A$ . Note that  $|V(Q)| = |A| + \sqrt{n} - 1$ . Since each vertex of  $A$  has exactly one neighbour in each block, the number of fixed leaves whose parents are in  $A$  is precisely  $|E(Q)|$ . We now show that  $Q$  has no cycles. That will complete the proof of the upper bound since in that case  $|E(Q)| \leq |V(Q)| - 1 = |A| + \sqrt{n} - 2$ .

Assume for the sake of contradiction that  $Q$  has a cycle. Let  $C$  be a shortest cycle in  $Q$ . Every second vertex of  $C$  is a vertex of  $A$ . The remaining vertices of  $C$  correspond to blocks of leaves. Let  $C_H$  be the subset of  $V(H)$  containing all the roots in  $V(C) \cap A$  and for each of those roots,  $C_H$  also contains all its fixed leaves contained in blocks  $P_j$  for which  $p_j$  is in  $C$ .

Consider the geometric graph  $H[C_H]$ . The fact that  $C$  is a (shortest) cycle and that each vertex in  $A$  has exactly one leaf in each block, implies that  $H[C_H]$  is a geometric forest of 2-stars, where the vertices in  $V(C) \cap A$  have degree 2 in  $H[C_H]$  and each block  $P_j$  such that  $p_j \in V(C) \setminus A$  has precisely two fixed leaves in  $H[C_H]$ .

Since  $H$  is crossing-free, so is  $H[C_H]$ . Furthermore, since all the roots in  $C_H$  are on or above the  $x$ -axis and all the leaves of  $C_H$  are on the  $x$ -axis,  $H[C_H]$  is fully contained in a closed half-plane determined by the  $x$ -axis. We now show that  $H[C_H]$  cannot be crossing-free, which will provide the desired contradiction.  $H[C_H]$  is a crossing-free geometric forest of 2-stars. We first expand  $H[C_H]$  into a crossing-free geometric cycle  $H[C_H]$  into a crossing-free geometric cycle by adding some segments to it, as follows. Consider blocks that contain a leaf of  $H[C_H]$ . Each such block  $P_j$  contains exactly two leaves of  $H[C_H]$ , denoted by  $j_1$  and  $j_2$  (see Figure 5). We claim that  $\overline{j_1 j_2} \cap H[C_H] = \emptyset$ . There is no edge of  $H[C_H]$  that properly crosses  $\overline{j_1 j_2}$ , since  $H[C_H]$  is fully contained in a closed half-plane determined by the  $x$ -axis. Therefore,  $\overline{j_1 j_2} \cap H[C_H]$  can be non-empty only if there is an edge of  $H[C_H]$  fully contained in  $\overline{j_1 j_2}$ . That implies that there is a root of  $H[C_H]$  that is located on the  $x$ -axis between  $j_1$  and  $j_2$ . That however is impossible, since one of the two edges of  $H[C_H]$  incident to that root would contain  $j_1$  or  $j_2$  in its interior. This observation implies that  $H[C_H]$  can be extended into a *crossing-free* geometric cycle  $R$  by adding the appropriate line segments into each block that contains a leaf of  $H[C_H]$ .

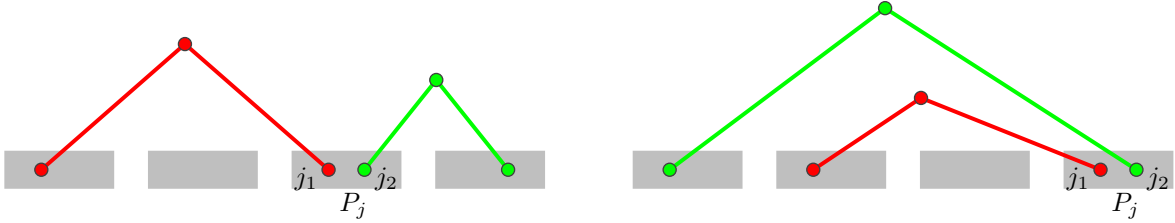


Figure 5: Two 2-stars (one depicted in green and the other in red) with leaves in a common block.

Let  $v$  be, among all the roots in  $C_H$ , the one with the smallest index; that is, there is no other root  $w \in C_H$  where  $v \in T_i$  and  $w \in T_j$  and  $j < i$ . Vertex  $v$  has two neighbours (fixed leaves) in  $H[C_H]$ ,  $s_1 \in P_s$  and  $t_1 \in P_t$  (see Figure 6). Vertex  $s_1$  has two neighbours in  $R$ . One is  $v$ , and the other is a vertex (fixed leaf)  $s_2 \in P_s$ . Similarly,  $t_1$  is adjacent in  $R$  to  $v$  and to a vertex (fixed leaf)  $t_2 \in P_t$ . Therefore,  $R$  contains two vertex disjoint paths:  $R_1$ , between  $s_1$  and  $t_1$ , and  $R_2$ , between  $s_2$  and  $t_2$ . Since  $v$  belongs to the smallest indexed tree, the ordering of their endpoints on the  $x$ -axis is  $s_1 < s_2 < t_1 < t_2$ . With such ordering of endpoints and since  $R$  is fully contained in the closed half-plane above the  $x$ -axis, it is impossible to draw  $R_1$  and  $R_2$  without crossings (since  $R_1$  separates the closed half-plane above the  $x$ -axis into two components, one containing  $s_2$  and one containing  $t_2$ ). That is the desired contradiction.

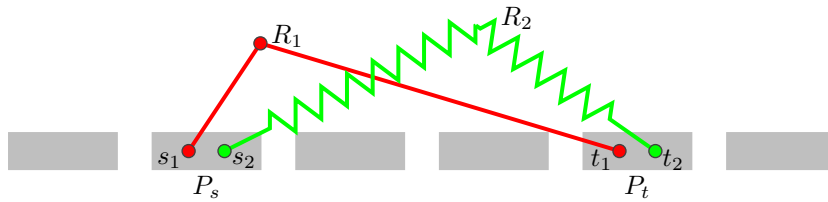


Figure 6: Illustration for the proof of the upper bound of Theorem 5.

*Lower bound:*

We now prove that  $\text{fix}(G) \geq 3\sqrt{n} - 3$ , that is, we prove that  $G$  can be untangled while keeping  $3\sqrt{n} - 3$  vertices fixed. Keep the followings vertices of  $G$  fixed:

- (1) all the leaves of  $T_1$  and  $T_2$ , and
- (2) all the vertices in the block  $P_{\sqrt{n}-1}$ , and
- (3) the root of  $T_{\sqrt{n}}$ .

Move the root of  $T_1$  to the half-plane above the  $x$ -axis and move the root of  $T_2$  to the half-plane

below the x-axis. For all  $3 \leq i \leq \sqrt{n} - 1$ , move all the free vertices of  $T_i$  to a very small disk centered at the fixed leaf of  $T_i$ . Move all the free leaves of  $T_{\sqrt{n}}$  to a small disk centered at the root of  $T_{\sqrt{n}}$ . Clearly, this can be done such that the resulting geometric forest  $H$  is crossing-free, as illustrated in Figure 7. The number of fixed vertices of  $H$  is  $2(\sqrt{n} - 1) + (\sqrt{n} - 2) + 1 = 3\sqrt{n} - 3$ , as claimed.  $\square$

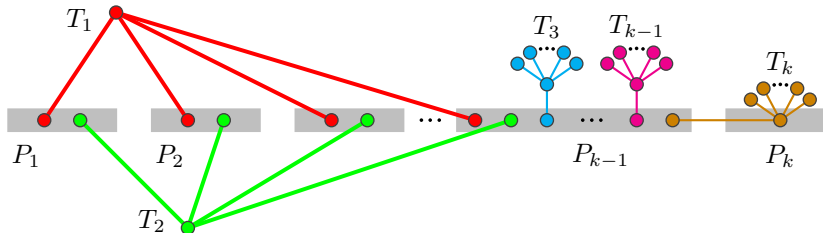


Figure 7: Untangled forest  $G$  with  $3\sqrt{n} - 3$  vertices fixed ( $k = \sqrt{n}$ ).

## 6 Conclusions

Polynomial bounds are now known for untangling all classes of planar graphs. Tight bounds (up to a constant) are known for untangling trees and outerplanar graphs. The gap remains open for untangling geometric cycles where the best known lower and upper bounds are  $\sqrt{n}$  and  $\mathcal{O}((n \log n)^{2/3})$ , and geometric planar graphs where the best known lower and upper bounds are  $\Omega(n^{1/4})$  and  $\mathcal{O}(\sqrt{n})$ .

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