INVERTIBLE CONTRACTIONS AND ASYMPTOTICALLY STABLE ODE'S THAT ARE NOT $\mathcal{C}^1$-LINEARIZABLE.

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ABSTRACT. We present an example of a contraction diffeomorphism in infinite dimensions that is not $\mathcal{C}^1$-linearizable, and we construct a regular ordinary differential equation in a Hilbert space whose time-one map is that diffeomorphism. With this we have an example of an asymptotically stable ODE that is not $\mathcal{C}^1$-conjugate to its linear part.

Keywords: linearization, conjugacy, contraction.

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1. INTRODUCTION AND MAIN RESULT.

In two previous papers, we obtained positive results for smooth linearization in infinite dimensional systems. Under some sufficient conditions, that include a nonresonance condition, in Rodrigues & Sola-Morales [10] we proved that $\mathcal{C}^1$-linearization is possible for contractions. The main theorem of that paper extends a classical result for finite dimensional systems by P. Hartman [6] and a result for infinite dimensions by Mora & Sola-Morales [8]. In the works ElBialy [3], Bin Tan [13] and Abbaci [1] results are obtained in the same direction.

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Also in Rodrigues & Sola-Morales [11], we proved a similar result for a more particular saddle case. It extends a result proved by P. Hartman (see [6]) for two-dimensional systems (see also Aroson, Belitskii and Zhuzhoma, [2]).

In particular in Rodrigues & Sola-Morales [10] we obtained the following theorem:

Theorem 1. (The Linearization Theorem For Contractions.) *Let $\mathcal{Z}$ be a Banach space with the property that there exists function ρ such that*

$$\rho \in \mathcal{C}^{1,1}(\mathcal{Z}, \mathbb{R}), \quad \text{with } \rho(z) = 1, \text{ when } |z| \leq 1/2 \quad \text{and } \rho(z) = 0, \text{ when } |z| \geq 1. \tag{1.1}$$

Suppose that $L,\ L^{-1} \in \mathcal{L}(\mathcal{Z})$. We assume that there exist real numbers $\nu_i^-,\ \nu_i^+$, $i = 1, \cdots, n$ such that:

$$\left.\begin{array}{l} 0 < \nu_n^- < \nu_n^+ < \nu_{n-1}^- < \nu_{n-1}^+ < \cdots \nu_1^- < \nu_1^+ < 1 \\ \nu_1^+ \nu_i^+ < \nu_i^-,\ i = 1, \cdots, n \quad \text{(nonresonance condition)} \\ |\sigma(L)| \subset (\nu_n^-, \nu_n^+) \cup (\nu_{n-1}^-, \nu_{n-1}^+) \cup \cdots \cup (\nu_1^-, \nu_1^+). \end{array}\right\} \tag{1.2}$$

Let $F = F(z)$ be a $\mathcal{C}^{1,1}$-function in a neighborhood of the origin with values in $\mathcal{Z}$, such that $F = 0,\ \partial_z F = 0$, at $z = 0$.

Then, for the map $T : z \mapsto z',\ z' = Lz + F(z)$, there exists a $\mathcal{C}^1$-map $R : z \mapsto u$, $u = z + \psi(z)$, satisfying $\psi = 0,\ \partial_z \psi = 0$, at $z = 0$, such that $RTR^{-1} : u \mapsto u'$ has the form $u' = Lu$ in a sufficiently small neighborhood of the origin.

In a short Note, Rodrigues & Sola-Morales [12], we presented a first example of an analytic invertible contraction that is not $\mathcal{C}^1$-linearizable. This result is interesting because it is in contrast with the finite dimensional case, since as proved by P. Hartman (see [6]), for finite dimensional systems, every $\mathcal{C}^{1,1}$- contraction can be linearized in the class $\mathcal{C}^1$. Before the existence of this example, the possibility or not of extending Hartman's result to all infinite dimensional contractions was an open question in this field, as it was said for example in [1].

Of course, the nonresonance condition of (1.2) is not satisfied in the example, but (1.1) is.

In the present paper we improve the result of Rodrigues & Sola-Morales [12], by presenting a different example, that is simpler in some respects, and a more complete analysis of the problem. This is stated in Theorem 2 below, that will be proved in Section 2.

Our first example contained in [12] was constructed using a sequence of Jordan blocks of increasing order. The example presented now is somehow simpler because instead, we use only one infinite-dimensional Jordan block.

We prove that the conjugation map does not exist not only in the class of $\mathcal{C}^1$ diffeomorphisms but even in the wider class of local homeomorphisms that are differentiable at the origin, together with their inverses. This possibility was only slightly mentioned in [12].

So, our example shows also the impossibility to extend to infinite dimensions the result of Guysinsky, Hasselblatt and Rayskin, [5]. These authors prove that if the map of the classical Hartman-Grobman theorem is $\mathcal{C}^\infty$, then the linearizing homeomorphism can be taken to be differentiable at the origin and its derivative at zero being the identity.

It will also be shown in Remark 2 below, that our example fits in the hypotheses of Cabré, Fontich and de la Llave [4] as a sharp example of both existence and non-existence of some invariant non-hyperbolic manifolds.

Besides, as a meaningful contribution of the present paper, we prove that the stated contraction is the time one map of an asymptotically stable ordinary differential equation defined in the space ℓ_2, of the square summable sequences, with the usual inner product. This property is based on a careful spectral analysis of some operators involved. This property was not considered in [12], and we believe that it is quite significant. However, with the new ideas developed in the present paper it would be possible to prove that the first example of [12] is also a time-one map of an ODE. To embed a diffeomorphism into the flow of an autonomous ODE is not always possible, even in finite dimensions, but we succeeded in proving that for our example thanks to its special form. See the works of Palis [9] and Li, Llibre and Zhang [7] as some references concerning this problem.

Another meaningful contribution of the present paper will be the construction of a family of examples that approach the nonresonance condition (1.2). They can be seen as modifications of the example of Theorem 2, and will be constructed in Section 3. The meaning and the interest of these families of examples will be explained in the Remark 1, at the end of the present Section.

The next result will play an important role in the construction of our example.

Proposition 1. *Let a, ε, δ be positive real numbers, with $a < 1$. Consider the map, $(x, y_1, y_2, \cdots, y_n, \cdots)^T \in \ell_2 \mapsto (x', y_1', y_2', \cdots, y_n', \cdots)^T \in \ell_2$, defined as:*

$$\begin{aligned} x' &= ax \\ y_1' &= ay_1 + \varepsilon x^2 \\ y_2' &= ay_2 + \delta y_1 \\ &\vdots \\ y_{n+1}' &= ay_{n+1} + \delta y_n \\ &\vdots \end{aligned} \tag{1.3}$$

If $\vec{y} = (y_1, y_2, \cdots, y_n, \cdots)^T = \Phi(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots)^T$ defines a local invariant manifold for the above map, differentiable at $x = 0$, such that $\Phi(0) = 0$, $\partial_x \Phi(0) = 0$ then necessarily,

$$\phi_1(x) = \frac{1}{a - a^2}\varepsilon x^2,\ \phi_2(x) = \frac{\delta}{(a - a^2)^2}\varepsilon x^2,\ \cdots,\ \phi_{n+1}(x) = \frac{\delta^n}{(a - a^2)^{n+1}}\varepsilon x^2,\ \cdots\ .$$

Let us now introduce some notations. We define the infinite matrix J and the nonlinear function $\vec{f}(x)$ as:

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \vec{f}(x) := \begin{pmatrix} x^2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{1.4}$$

For the scalars a, δ, ε, we consider the following infinite matrix L and the nonlinear function F, acting in the Hilbert space ℓ_2.

$$L := \begin{pmatrix} a & 0 \\ 0 & \delta J + aI \end{pmatrix}, \quad F(z) := \begin{pmatrix} 0 \\ \varepsilon \vec{f}(x) \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix} \tag{1.5}$$

In the next theorem, that is the main result of the present paper, we use the above notations.

Theorem 2. *Let $\varepsilon \neq 0$ and $0 < a < 1$. Under the hypothesis,*

$$a - a^2 \leq \delta < \min\{1 - a, a\} \tag{1.6}$$

the operator L is an invertible contraction on ℓ_2, its spectrum $\sigma(L)$ is the closed disk of center a and radius δ and the local analytic diffeomorphism defined in ℓ_2 by,

$$z' = Tz := Lz + F(z)$$

does not conjugate, even locally, with its linear part L, through a conjugation R, with R and R^{-1} differentiable at $z = 0$.

Also, this map T is the time-one map of an ordinary differential equation in ℓ_2 of the form

$$\dot{z} = Az + G(z) \tag{1.7}$$

where $A := \log L$ is a bounded linear operator and $G : \ell_2 \to \ell_2$ is defined by

$$G(z) := \begin{pmatrix} 0 \\ x^2\vec{\beta} \end{pmatrix}$$

for some $\vec{\beta} \in \ell_2$.

Remark 1. *Observe that $|\sigma(L)| = [a - \delta, a + \delta]$ and that (1.6) implies that*

$$(a + \delta)^2 > a^2 \geq a - \delta,$$

so the nonresonance condition (1.2) is not satisfied.

Observe also that the spectrum of L consists of a single block (that is, n is equal to 1 in the notation of Theorem 1). It would have been very satisfactory to find examples of this kind but with the set of moduli of the spectrum of the linear part being equal to $[b^2, b]$, and to do that for all $b \in (0, 1)$. This kind of examples would fill up the complementary of the condition (1.2) for the case of a single block.

Unfortunately, we are not able to construct a family of examples of this kind. We can only construct an approximate version of them, and this will be the purpose of the last section below.

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2. Proof of Theorem 2

Lemma 1. *If* $0 < a < 1$ *and* $r \in \mathbb{R}$ *the functional equation,*

$$\phi(ax) = a\phi(x) + rx^2 \tag{2.1}$$

has a unique local solution ϕ *that is differentiable at* $x = 0$ *such that* $\phi(0) = 0,\ \partial_x\phi(0) = 0$. *This solution is given by* $\phi(x) = \frac{r}{a^2-a}x^2$.

Proof: (For the proof see [12]). The proof essentially follows from two facts. First, that the above function is indeed a solution of equation (2.1) and second, that the unique solution in the above class of the homogeneous equation, $\phi(ax) = a\phi(x)$, is the zero function.

Proof of Proposition 1

Suppose that $\vec{y} = (y_1, y_2, y_3, \cdots, y_n, \cdots)^T = \Phi(x) = (\phi_1(x), \phi_2(x), \cdots)^T$ is an invariant manifold for the map $T(z) = Lz + F(z)$, such that $\Phi(0) = 0$, $\partial_x\Phi(0) = 0$.

Then $\phi_1, \phi_2, \phi_3, \cdots, \phi_n, \cdots$ should satisfy the following system of equations:

$$\begin{aligned}
&\phi_1(ax) = a\phi_1(x) + \varepsilon x^2\\
&\phi_2(ax) = \delta\phi_1(x) + a\phi_2(x)\\
&\phi_3(ax) = \delta\phi_2(x) + a\phi_2(x)\\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\
&\phi_{n+1}(ax) = \delta\phi_n(x) + ay_{n+1}(x)\\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots
\end{aligned}$$

Using Lemma 1 recursively, we obtain

$$\phi_1(x) = \frac{1}{a-a^2}\ \varepsilon\ x^2,\ \phi_2(x) = \frac{\delta}{(a-a^2)^2}\ \varepsilon\ x^2,\ \cdots,\ \phi_{n+1}(x) = \frac{\delta^n}{(a-a^2)^{n+1}}\ \varepsilon\ x^2,\ \cdots\ .$$

■

Lemma 2. *The spectrum* $\sigma(J)$ *of the operator* J *defined above is the closed disk of radius* 1 *centered at zero. When* $a < 1$ *and* $0 < \delta < \min\{a, 1-a\}$ *the operator*

$$L = \begin{pmatrix} a & 0 \\ 0 & \delta J + aI \end{pmatrix}$$

is an invertible contraction and its spectrum is the disk of center a *and radius* δ *in the complex plane.*

Proof:

It is clear that $\|J\| = 1$. Suppose $|\lambda| > 1$. Then $\|\lambda^{-1}J\| = |\lambda|^{-1} < 1$, $I - \lambda^{-1}J$ is invertible and $\|I - \lambda^{-1}J\| \leq |\lambda|(|\lambda| - 1)^{-1}$. So, λ belongs to the resolvent set of J.

Suppose now $|\lambda| \leq 1$. Let $\vec{e}_1 = (1, 0, 0, \cdots)^\top$, and let us write the infinite system $(J - \lambda I)\vec{y} = \vec{e}_1$:

$$\begin{aligned}
&-\lambda\ y_1 = 1 \\
&y_1 - \lambda\ y_2 = 0 \\
&y_2 - \lambda\ y_3 = 0 \\
&\cdots\cdots\cdots\cdots \\
&y_n - \lambda\ y_{n+1} = 0 \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

If $\lambda = 0$, the system is clearly incompatible. For $0 < |\lambda| \leq 1$ the components of the solution should be $y_n = -\lambda^{-n}$ and so $\vec{y}$ does not belong to ℓ_2. Thus $\lambda \in \sigma(L)$. This completes the proof of the first part of our lemma.

The second part of our lemma follows from the fact that

$$L = \begin{pmatrix} a & 0 \\ 0 & \delta J + aI \end{pmatrix} = \delta \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} + aI,$$

and so $\sigma(L) = \delta\sigma(J) + a$. Since $0 \notin \sigma(L)$ and $\|L\| \leq a + \delta < 1$ we conclude that L is an invertible contraction.

Proof of Theorem 2.

For $T = L + F$ we suppose that a local linearization map R exists such that $RTR^{-1} = L$. If both R and R^{-1} are differentiable at zero then from $RTR^{-1} = L$ one obtains that $\partial_x R(0)L = L\partial_x R(0)$ and so $(\partial_x R(0))^{-1}RT((\partial_x R(0))^{-1}R)^{-1} = L$. So we can suppose that $\partial_x R(0) = I$.

Now, the linear subspace $\{(x, 0)^T\} \subset \ell_2$ is invariant by L, so $R^{-1}\{(x, 0)^T\}$ is invariant by T. Let us write $R^{-1}(x, 0)^T = (x + \psi(x), \Phi(x))^T$. We have $\psi(0) = 0$, $\partial_x\psi(0) = 0$, $\Phi(0) = 0$, $\partial_x\Phi(0) = 0$. Let us see that $\psi \equiv 0$. From $TR^{-1}(x, 0)^T = R^{-1}L(x, 0)^T$ we take the first component and we see that $a\psi(x) = \psi(ax)$ for all x near zero. Then, by applying Lemma 1 with $r = 0$ we see that $\psi(x) = 0$ in a neighborhood of zero. So, the set $R^{-1}\{(x, 0)^T\}$ can be expressed in a neighborhood of zero as $\{(x, \Phi(x))^T\} \subset \ell_2$.

From Proposition 1 it follows that for any integer $n \geq 1$,

$$\|\Phi(x)\| \geq \frac{\delta^n}{(a-a^2)^{n+1}}\varepsilon x^2.$$

From the assumption (1.6), since $a - a^2 \leq \delta$, it follows that $\Phi(x)$ does not belong to ℓ_2. This completes the first part of the proof of our main theorem.

Let us consider now the differential equation:

$$\dot{z} = Az + G(z) \tag{2.2}$$

where $A := \log L$, and $G : \ell_2 \longrightarrow \ell_2$ is defined by

$$G(z) := \begin{pmatrix} 0 \\ x^2\vec{\beta} \end{pmatrix}$$

for some $\vec{\beta} \in \ell_2$ that will be found later.

It is easy to see that the operator $A = \log L$ is a bounded operator, but next we will obtain an estimate for its norm. Since

$$L = \begin{pmatrix} a & 0 \\ 0 & \delta J + aI \end{pmatrix} = a\left(I - \left(\frac{-\delta}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}\right),$$

if we let

$$D := \left(\frac{-\delta}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$$

we obtain $\log L = (\log a)I + \log(I - D) = (\log a)I - (D + \frac{D^2}{2} + \cdots \frac{D^n}{n} + \cdots)$.

Therefore,

$$\|\log L\| \leq -\log a + \frac{\delta}{a} + \frac{(\frac{\delta}{a})^2}{2} + \cdots \frac{(\frac{\delta}{a})^n}{n} + \cdots = -\log a - \log(1 - \frac{\delta}{a}) = -\log(a - \delta).$$

Using the variation of constants formula in the time $t = 1$, we obtain

$$z(1, z) = e^A z + \int_0^1 e^{A(1-s)} G(z(s, z))ds \tag{2.3}$$

where $z(t, z)$ indicates the solution such that $z(0, z) = z$.

Using the special form of G, we can show that the above equation is equivalent to the system:

$$\begin{aligned} &x(1,x) = e^{-\alpha}x \\ &\vec{y}(1,\vec{y}) = e^{A_1}\vec{y} + \int_0^1 e^{A_1(1-s)}\ \vec{\beta}\ (x(s,x))^2\ ds = e^{A_1}\vec{y} + \int_0^1 e^{A_1(1-s)}e^{-2\alpha s}\ ds\ \vec{\beta}\ x^2 \end{aligned} \tag{2.4}$$

where $-\alpha = \log a$ and $A_1 = \log(\delta J + aI)$.

Since $F(z) = (0, \varepsilon\vec{f}(x))^T$ and $\vec{f}(x) = (x^2, 0, 0, \cdots)^T$ we must show that there exists $\vec{\beta} \in \ell_2$ such that:

$$\int_0^1 e^{A_1(1-s)}e^{-2\alpha s}ds\vec{\beta} = (\varepsilon, 0, 0, \cdots)^T.$$

Now we let $\tau = 1 - s$. Then $d\tau = -ds$ and $2\alpha\tau - 2\alpha = -2\alpha s$ and so the previous equation is equivalent to:

$$\int_0^1 e^{(A_1+2\alpha I)\tau}\ d\tau\ \vec{\beta} = (\varepsilon\ e^{2\alpha}, 0, 0, \cdots)^T.$$

Let $B := A_1 + 2\alpha I$ and $h(B) := \int_0^1 e^{B\tau}d\tau$.

By taking the series expansion of $e^{B\tau}$ and integrating term by term, we obtain:

$$h(B) = \int_0^1 e^{B\tau}\ d\tau = I + \frac{B}{2!} + \frac{B^2}{3!} + \cdots \frac{B^n}{(n+1)!}\cdots$$

Next we are going to prove that $(h(B))^{-1}$ exists and is bounded, or in other words, that 0 does not belong to $\sigma(h(B))$.

Let $h : \mathbb{C} \to \mathbb{C}$ the analytic function:

$$h(\xi) := \int_0^1 e^{\xi\tau}d\tau = \begin{cases} 1, \text{ if } \xi = 0 \\ \xi^{-1}(e^{\xi} - 1), \text{ if } \xi \neq 0 \end{cases}$$

From the Spectral Mapping Theorem it follows that $h(\sigma(B)) = \sigma(h(B))$. But $h(\xi) = 0$ if and only if $\xi = 2n\pi i$ for n integer, $n \neq 0$.

Let us now estimate the spectrum of $B = \log(\delta J + aI) + 2\alpha I$. Here we are considering log as the principal branch of the logarithm function.

If $\xi \in \log(\sigma(\delta J + aI))$ then from Lemma 2 it follows that $\xi = \log(re^{i\theta}) = \log r + i\theta$, where $r \in [a - \delta, a + \delta]$ and $\theta \in (-\gamma, \gamma)$, for some $\gamma \in (0, \frac{\pi}{2})$. Since $\alpha = -\log a$, we can conclude that $\sigma(B) = \sigma(\log(\delta J + aI) + 2\alpha I)$ is contained in the rectangle of the complex plane,

$$[2\alpha + \log(a-\delta), 2\alpha + \log(a+\delta)] \times [-\frac{\pi}{2}, \frac{\pi}{2}] = [\log(\frac{a-\delta}{a^2}), \log(\frac{a+\delta}{a^2})] \times [-\frac{\pi}{2}, \frac{\pi}{2}].$$

This rectangle does not contain any complex number of the form $2n\pi i$ where n is a nonzero integer.

Therefore $0 \notin \sigma(h(B))$, $h(B)$ has a bounded inverse and so we can take

$$\vec{\beta} = (h(B))^{-1}(\frac{\varepsilon}{a^2}, 0, 0, \cdots)^T \in \ell_2.$$

This completes the proof of the second part of our theorem. ■

Remark 2. *From the preceding proof one concludes also that if* $0 < \delta < a - a^2$ *then the invariant manifold* $\vec{y} = \Phi(x)$*, with* $\Phi(0) = 0$ *and* $\partial_x \Phi(0) = 0$ *does exist.*

So the value $\delta = a - a^2$ *is the critical value separating existence and non-existence of this invariant manifold. This indicates the sharpness of the hypothesis (4) in Theorem 1.1 of* [4], *on existence of invariant manifolds, not necessarily hyperbolic. To verify this hypothesis (4) in our example, one has to compute previously a number that in their notation is called* L, *that jumps from* $L = 1$ *to* $L = 2$ *when* δ *jumps from* $\delta < a - a^2$ *to* $\delta \geq a - a^2$.

Remark 3. *The result of non-existence of an invariant manifold tangent to the* x*-axis can be easily paraphrased for the case of the flow defined by (2.2) in the following way: it does not exist a solution* $(x(t), \vec{y}(t))^T$ *with* $x(0) \neq 0$ *of 2.2 such that* $\lim_{t\to\infty} \frac{\|\vec{y}(t)\|}{|x(t)|} = 0$.

3. Examples near resonance.

Following the ideas expressed in Remark 1 above, we are going to show that for all $b \in (0,1)$ and all sufficiently small $r > 0$ there exists a map $T_{b,r} := L + F : \ell_2 \to \ell_2$, with L the linear part and F a nonlinear (quadratic) polynomial, such that $[b^2, b] \subset |\sigma(L)| \subset [b^2, b + r]$ and that $T_{b,r}$ is not $\mathcal{C}^1$-conjugate to its linear part L in any neighborhood of $\vec{0} \in \ell_2$.

As in Theorem 2 above, one can also show that $T_{b,r}$ does not conjugate, even locally, with its linear part through a conjugation homeomorphism R with R and R^{-1} differentiable at the origin. And also, that $T_{b,r}$ is the time-one map of an ordinary differential equation in ℓ_2 of the form (2.2). The construction in these points is very similar to that of the Theorem 2, and we will only underline the points of difference.

First, we show the existence of a linear operator $M \in \mathcal{L}(\ell_2)$ that can be represented by a low-triangular infinite matrix of the form

$$M := \begin{pmatrix} b & 0 & 0 & 0\cdots & \cdots \\ b_1 & b & 0 & 0\cdots & \cdots \\ b_2 & b_1 & b & 0\cdots & \cdots \\ b_3 & b_2 & b_1 & b\cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{3.1}$$

and such that $[b^2, b] \subset |\sigma(M)| \subset [b^2, b+r]$. The way of defining this operator is very simple: if $g(\xi)$ is an analytic function in a neighborhood of $\{|\xi| \le 1\} \subset \mathbb{C}$, it is clear that the operator $M = g(J)$, where J was defined in (1.4), will have the form of a matrix like (3.1), since $g(\xi)$ admits a representation as a power series in ξ, namely $g(\xi) = b + b_1\xi + b_2\xi^2 + \cdots$, and these coefficients will appear in the matrix of M as in (3.1). In order to have all these coefficients to be real we need only to ask $g(\xi)$ to be real if $\xi \in \mathbb{R}$. Then, by the Spectral Mapping Theorem, $\sigma(M) = g(\{|\xi| \le 1\})$. If we want to have $[b^2, b] \subset |\sigma(M)| \subset [b^2, b+r]$, it is very easy to find such a function $g(\xi)$ (for the needs we will have below, we also ask $b^2 \in \sigma(M)$, not only $b^2 \in |\sigma(M)|$):

Take r small enough and take the Möbius map $N_{1-r}(\xi) = [(1-r) - \xi]/[1 - (1-r)\xi]$ that is an automorphism of the unit disk and maps $\xi = 0$ into $1 - r$. Then define

$$g(\xi) := \frac{b - b^2}{2 - r} N_{1-r}(\xi) + \frac{b + b^2 - rb^2}{2 - r}.$$

We have that $g(0) = b$, $g(1) = b^2$ and $g(-1) = b + r\frac{b-b^2}{2-r}$. So, with this particular function $g(\xi)$ we have

$$[b^2, b] \subset |\sigma(g(J))| = [b^2, b + r\frac{b - b^2}{2 - r}] \subset [b^2, b + r].$$

Second, we observe that the point $\lambda = b^2$ belongs to $\sigma(M)$ in such a way that the range $R(M - b^2 I)$ is not the whole space ℓ_2. This is also very easy to see: we first observe that $M - b^2 I$ is necessarily one-to-one, because $(M - b^2 I)\vec{y} = 0$ is equivalent to the infinite set of equations

$$\begin{aligned}&(b-b^2)y_1 = 0\\&b_1y_1 + (b-b^2)y_2 = 0\\&b_2y_1 + b_1y_2 + (b-b^2)y_3 = 0\\&\cdots\cdots\cdots\cdots\end{aligned}$$

that can only have the solution $y_1 = y_2 = y_3 \cdots = 0$. By the Open Mapping Theorem, if $M - b^2I$ is one-to-one but it is not boundedly invertible, necessarily $R(M - b^2I) \neq \ell_2$.

With this linear operator M we construct the linear operator L acting on $z \in \ell_2$ as in (1.5), that is

$$L := \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix}, \quad z := \begin{pmatrix} x \\ \vec{y} \end{pmatrix}.$$

It is clear that $\sigma(L) = \sigma(M)$.

To see that L is a contraction we compute the Taylor expansion of $g(\xi)$ around $\xi = 0$:

It is easy to see that

$$N_{1-r}(\xi) = (1-r) - r(2-r)[\xi + (1-r)\xi^2 + (1-r)^2\xi^3 + \cdots + (1-r)^{n-1}\xi^n + \cdots]$$

and therefore one can see that

$$g(\xi) = b - r(b-b^2)[\xi + (1-r)\xi^2 + (1-r)^2\xi^3 + \cdots + (1-r)^{n-1}\xi^n + \cdots].$$

So, we conclude that

$$b_1 = -r(b-b^2),\ b_2 = -r(b-b^2)(1-r), \cdots,\ b_n = -r(b-b^2)(1-r)^{n-1}, \cdots$$

Then

$$\|L\| \leq b + \sum_{k=1}^{\infty} |b_k| = b + r(b-b^2)\sum_{k=1}^{\infty}(1-r)^{(k-1)} =$$

$$= b + r(b-b^2)\frac{1}{1-(1-r)} = 2b - b^2 < 1.$$

The conclusion is that the operator L is in fact a contraction.

Now we define the nonlinear part $F(z) := (0, x^2\vec{\gamma})$, where $\vec{\gamma} = (\gamma_1, \gamma_2, \cdots)^T \in \ell_2$ has to be chosen outside the range of $M - b^2I$.

Now we proceed as in the proof of Theorem 2 to show that an invariant manifold of the form $\{(x, \Phi(x))^T\} \subset \ell_2$ such that $\partial\Phi(0) = 0$ cannot exist. Writing in components

$\Phi(x) = (\phi_1(x), \phi_2(x), \cdots)^T$ we obtain that the following infinite system of equations has to be satisfied:

$$\begin{aligned}
&\phi_1(bx) = b\phi_1(x) + \gamma_1 x^2 \\
&\phi_2(bx) = b\phi_2(x) + b_1\phi_1(x) + \gamma_2 x^2 \\
&\phi_3(bx) = b\phi_3(x) + b_1\phi_2(x) + b_2\phi_1(x) + \gamma_3 x^2 \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\end{aligned}$$

Applying recursively Lemma 1 to this system we obtain that the only possible solution must have the form $\phi_n(x) = \alpha_n x^2$, for all n, for some sequence of real numbers $(\alpha_n) \in \ell_2$.

But then, the previous infinite system of equations reads

$$b^2(\alpha_n) = M(\alpha_n) + (\gamma_n)$$

and this system has no solution $(\alpha_n) \in \ell_2$ if (γ_n) has been chosen outside the range of $M - b^2 I$.

The differential equation in the form (2.2) whose time-one map is the map $T_{b,r}$ can also be constructed as in the proof of Theorem 2:

To see that $\log L$ is a well defined, bounded and real operator we can observe that

$$L = \begin{pmatrix} b & 0 \\ 0 & M \end{pmatrix} = b\left[I - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix}\right] := b[I - D],$$

where

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b}M - I \end{pmatrix}$$

$$\|D\| = \frac{1}{b}\sum_{k=1}^{\infty}|b_k| = \frac{1}{b}(b - b^2) = 1 - b < 1.$$

We have that,

$$\log L = (\log b)I + \log(I - D) = (\log b)I - (D + \frac{D^2}{2} + \cdots + \frac{D^n}{n} + \cdots).$$

$$\|\log L\| \le -\log b + \sum_{k=1}^{\infty}\frac{\|D\|^n}{n} = -\log b - \log(1 - \|D\|) = -\log b^2$$

Now, following also the proof of Theorem 2, we define $B = \log M + 2\alpha I$ with $\alpha = -\log b$ and we have to show that $0 \notin \sigma(h(B))$. But this is easy to see, since the spectrum of M is the disc of center $\dfrac{b + b^2 - rb^2}{2 - r}$ and radius $\dfrac{b - b^2}{2 - r}$.

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