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PH.D. THESIS

COLORED COMBINATORIAL STRUCTURES:
HOMOMORPHISMS AND COUNTING

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Introduction

The Four Color Theorem
is the tip of the iceberg,
the thin end of the wedge,
and the fist cuckoo of spring.
W.T. TUTTE

Starting with the four-color problem, the theory of graph coloring has existed for more than 150 years. It deals with the fundamental problem of partitioning a set of objects into classes according to certain rules. Historically, graph coloring involved finding the minimum number of colors to be assigned to the vertices of a graph so that adjacent vertices would have different colors. From this modest beginning, the theory has become central in discrete mathematics, with many contemporary generalizations and applications.

Even if many deep and interesting results in graph coloring theory have been obtained during the last century, there are many easily formulated interesting problems left. A very good illustration is the book devoted to unsolved graph coloring problems by T.R. Jensen and B. Toft [31]. For a general exposition of the subject, including proofs of several classical theorems, a good reference is the book on extremal graph theory by Bollobás [9].

In this thesis, our particular interest is in two very active areas of research which have emerged from coloring problems: *Graph Homomorphism Theory* and *Arithmetic Ramsey Theory*:

- The study of graph homomorphisms began in the early sixties in the framework of algebra and category theory; it was pioneered by G. Sabidussi [55], and by Z. Hedrlin and A. Pultr [26]. Homomorphism theory concerns the study of classes of combinatorial structures under natural morphisms. These include classes of graphs, oriented graphs, posets, digraphs or general relational structures. The chromatic number of a simple graph G can be stated, in this context, as the smallest complete graph to which G admits a homomorphism. Thus, graph homomorphism theory has been extensively studied as a generalization of colorings. An excellent reference on the subject is the book by Hell and Nešetřil [28].

- Ramsey theory can be described as the study of the preservation of properties under set partitions; in other words, Ramsey theory studies the existence of particular color patterns in colored structures. Starting with the Theorems of Ramsey, Hilbert, Schur and van der Waerden, the theory has been developed as a wide and beautiful area of combinatorics, in which a great variety of techniques are used from many branches of mathematics. Many of the classical results in the area are arithmetic versions of the theory and we are interested in this particular branch of Ramsey theory. Good references in the area are the books of Langman and Robertson [37] and Adhikari [1].

This thesis is organized in two parts. The first part deals with the study of homomorphisms in the class of colored mixed graphs, and in the second part we study some Ramsey and anti-Ramsey results in finite groups.

Part I: In this part of the thesis, our particular aim is to study homomorphisms of *colored mixed graphs*, which are graphs with vertices linked by both colored arcs and colored edges. The chromatic number of such a graph G is defined as the smallest order of a colored mixed graph H such that there exists a (color preserving) homomorphism from G to H. The *colored mixed chromatic number* of a simple graph G (*respect.* of a family of simple graphs $\mathcal{F}$) is then defined as the maximum of the chromatic numbers taken over all the possible colored mixed graphs having G (*respect.* a graph in $\mathcal{F}$) as underlying graph; we give some formal definitions in Section 2. These notions were introduced by Nešetřil and Raspaud [43] as a common generalization of the notion of homomorphisms of edge-colored graphs, and the notion of oriented colorings (see e.g. [4] and [60] respectively).

Most of the work related to the study of homomorphisms as a generalization of colorings has been done in the context of oriented graphs (which is a particular case of colored mixed graphs). The notion of oriented coloring was first introduced by Courcelle [16] in 1994. Since then, the oriented chromatic number of several families of graphs has been extensively studied. We specially recommend a wonderful survey by Sopena [60] on this topic. We mention here some of the most important families of graphs that have been studied: graphs with bounded degree [36, 59], graphs with bounded tree–width [48, 59], graphs with bounded acyclic chromatic number [36, 47, 53], planar graphs [11, 12, 13, 14, 44, 46, 50, 53, 61], outerplanar graphs [44, 52], grids [22, 63], Halin graphs [29], graphs subdivisions [68], and graphs with given excess [62].

Concerning homomorphisms of colored mixed graphs, as far as we know, the only published work was the one by Nešetřil and Raspaud in 2000 [43]. In this paper, the authors gave an upper bound of the colored mixed chromatic number of graphs with bounded acyclic chromatic number. By means of this result, they obtain an upper bound for the colored mixed chromatic number of planar graphs. Both results are a common generalization of the corresponding ones of Raspaud and Sopena [53] (for oriented graphs) and, Alon and Marshall [4] (for edge–colored graphs). They also gave the exact colored mixed chromatic number of the family of trees. In this thesis we present all this results with its proofs for the benefit of the reader. We also present our contributions in the area, which are the following:

1. We prove that the colored mixed chromatic number of trees given in [43], is reached by the much simpler family of paths (Theorem 3.2). By means of this result we give new lower bounds for the colored mixed chromatic number of outerplanar graphs and planar graphs (Theorem 6.2).

2. We show that the upper bound of the colored mixed chromatic number of graphs with acyclic chromatic number at most k, given in [43], is tight for every $k \geq 3$ (Theorem 4.2).

3. By means of the upper bound of the colored mixed chromatic number of graphs with bounded acyclic chromatic number given in [43], we provide an upper bound of the colored mixed chromatic number of partial k–trees (Theorem 5.1). Then we show that this upper bound has the correct order of magnitude on $2n + m$, where n is the number of colors in the arcs and m the number of colors in the edges (Theorem 5.2).

4. In particular, for 2–edge colored graphs, we obtain the precise colored mixed chromatic number of partial k–trees for $k = 2$ and 3; additionally, we improve the general upper bound for the colored mixed chromatic number of partial 2–trees in the cases of mixed graphs and 2–arc colored oriented graphs (Theorem 5.3).

5. For planar graphs, the main term in the lower bound of the colored mixed chromatic number is $(2n+m)^3$, while the main term in the upper bound is $(2n+m)^4$ (Corollary 6.2); for partial 2–trees and outerplanar graphs (which are special subclasses of planar graphs), the main term in both lower and upper bounds of the colored mixed chromatic number is $(2n+m)^2$ (Corollary 6.1). We prove that, if the girth is sufficiently large, the colored mixed chromatic number of these three classes of graphs becomes linear on $2n + m$ (Theorem 7.5).

6. In fact, we prove a kind of $(4n + 2m + 1)$–*color theorem* for sparse colored mixed planar graphs (Theorem 7.5–1); that is, we give the precise colored mixed chromatic number of planar graphs for suitable sufficiently large girths.

7. Concerning the particular case of 2–edge colored graphs, we give the complete classification on the colored mixed chromatic number of outerplanar graphs and partial 2–trees with a given girth (Theorems 8.1 and 8.2); we also obtain upper bounds for the colored mixed chromatic number of planar graphs with given girth (Theorem 8.3).

8. The class of bipartite 2–edge colored graphs is closely related to the one of bipartite oriented graphs. We make this statement precise and discuss the relationship between cores and dualities in the two categories.

Most of these results have been published (or accepted for publication) in:

- R. Fabila-Monroy, D. Flores, C. Huemer, A. Montejano. Lower bounds for the colored mixed chromatic number of some clases of graphs, *Comment. Math. Univ. Carolin.* vol. **49(4)** (2008) 637–645.
- A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, E. Sopena. Homomorphisms of 2–edge colored graphs, *Discrete Applied Mathematics.* to appear (2009).
- A. Montejano, A. Pinlou, E. Sopena. Chromatic number of sparse colored mixed planar graphs, *European Conference on Combinatorics, Graph Theory and Applications* (EuroComb 2009), submitted.

Part II: This part of the thesis is related to arithmetic Ramsey theory, which concerns the study of the existence of color patterns in every coloring of sets of integers. In Chapter 10, we present three classical theorems that became the starting point of an area that is still very active today: Schur's theorem, which states that for every k, if n is sufficiently large, then every k–coloring of $\{1,...,n\}$ contains a monochromatic solution of the equation $x+y=z$; van der Waerden's theorem, which states that for every k and t, if n is sufficiently large, then every k–coloring of $\{1,...,n\}$ contains a t–term arithmetic progression; and Rado's theorem, which is a generalization of Schur's theorem, concerning monochromatic solutions of systems of linear equations with integers coefficients.

Our interest lies not only on monochromatic structures, but also on the existence of *rainbow* (hetero–chromatic) structures in colored universes. The study of the existence of rainbow structures falls into the *anti–Ramsey theory* initiated by Erdős, Simonovits and Sós [21]. Arithmetic versions of this theory where initiated by Jungić et al. [32], where the authors study the existence of rainbow 3–term arithmetic progressions in colorings of integer intervals and cyclic groups; those results are thought as the first rainbow counterparts of clasical theorems in Ramsey theory. In Chapter 10, we give a historical overview of the so called *rainbow Ramsey theory*, which studies the the existence of rainbow structures, under certain density conditions on the colorings.

Our particular interest in this work is on the study of the existence and enumeration of colored structures (mainly monochromatic or rainbow structures) in colorings of finite groups. The structures under consideration can be described as solutions of systems of equations in the group, the main examples being arithmetic progressions and Schur triples. Our contributions in the area, are the following:

1. We describe the structure of those 3–colorings of abelian groups of odd order, which has no 3–term arithmetic progressions with its members having pairwise distinct colors (Theorem 11.1).

2. By means of the structural description in Theorem 11.1, we confirm a conjecture of Jungic et al. on the size of the smallest chromatic class of such colorings in cyclic groups (Corollary 11.1).

3. We present a counting lemma which gives a relationship between the number of vectors with some specific color patterns and the cardinalities of the color classes, for r–colorings in orthogonal arrays $OA(d, k)$ (Lemma 12.1). This result is a general formulation of the basic combinatorial counting argument used in [15].

4. For 3–colorings in orthogonal arrays $OA(d, d-1)$, we obtain a slight generalization of a result by Balandrau [8] (Corollary 12.1).

5. The set of Schur triples in a finite group form an orthogonal array $OA(3, 2)$. The same is true for 3–term arithmetic progressions if the order of the group is relatively prime with 6. For r–colorings in orthogonal arrays $O(3, 2)$, we get a nice relationship between monochromatic and rainbow vectors depending only on the cardinality of the color classes (Theorem 12.3).

6. For 2–colorings in orthogonal arrays $OA(3, 2)$, we do not have rainbow triples, so that Theorem 12.3 gives a formula for the total number of monochromatic triples in terms of the cardinalities of the color classes; by minimizing that formula we get the minimum number of monochromatic triples in an orthogonal array $OA(3, 2)$ for any 2-coloring (Corollary 12.2).

7. In the case of 3–colorings in orthogonal arrays $OA(3, 2)$, Theorem 12.3 has a nice interpretation in terms of the variance (Corollary 12.3).

8. In Sections 12.3.1 and 12.3.2 we collect some specific applications of our results, including the study of monochromatic and rainbow Schur triples and arithmetic progressions in finite groups.

Most of these results have been published (or accepted for publication) in:

- A. Montejano, O. Serra, Rainbow–free three colorings in abelian groups, *European Conference on Combinatorics, Graph Theory and Applications* (EuroComb 2009), submitted.

- A. Montejano, O. Serra, Color patterns in orthogonal arrays, *The 2009 British Combinatorial Conference*, submitted.

Contents

Part I

Homomorphisms of Colored Mixed Graphs

Chapter 1

Preliminaries

In this section we give the concepts and terminology of *Graph Theory* that we will use to develop our work. We proceed with the tedious but essential sequence of definitions.

1.1 Graphs

A *graph* is a pair of disjoint finite sets (V, E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of distinct elements of V (we have neither multiple edges nor loops). The set V is the set of *vertices* and E is the set of *edges*; if G is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set of G. Usually, when there is any danger of confusion, graphs are called *simple graphs*.

Let $G = (V, E)$ be a (simple) graph. An edge $\{u, v\}$ in $E(G)$ is said to *join* the vertices u and v and is denoted by uv; thus uv and vu mean exactly the same edge. If $uv \in E(G)$ then we say that u and v are *adjacent* vertices. We say that two edges are *adjacent* if they have exactly one common vertex. A set of vertices in G is called *independent* if not two elements of it are adjacent. Analogously, a set of mutually not adjacent edges is called *independent*, and if it contains all vertices of G is called a *perfect matching*. A graph G is called a *bipartite* graph, with *bipartition* $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, if every edge in G joints a vertex in V_1 with a vertex in V_2; thus V_1 and V_2 are independent sets. The *complement* of G is the graph $\overline{G} = (V, V^{(2)} - E)$; hence, two vertices are adjacent in $\overline{G}$, if and only if they are not adjacent in G. Two graphs are *isomorphic* if there is a one-to-one correspondence between their vertex sets that preserves adjacency.

The set of vertices adjacent to a vertex u in G, the *neighbourhood* of u, is denoted by $N(u)$. The *degree* of u is $d(u) = |N(u)|$. The minimal degree of the vertices of G is denoted by $\delta(G)$ and the maximal degree by $\Delta(G)$. The *order* of G is the number of vertices $|V(G)|$. A graph of order n and $\binom{n}{2}$ edges is called a *complete* graph and is denoted by K_n. We say that $G' = (V', E')$ is a *subgraph* of G, if $V' \subseteq V$ and $E' \subseteq E$; if G' contains all edges of G that join two vertices in V', then G' is said to be subgraph *induced* by V' and is denoted by $G[V']$. A maximal complete subgraph of G is called a *clique* of G; a clique of order k is called a k*–clique*.

A *path* is a graph P, with $V(P) = \{u_0, u_1, ..., u_k\}$ and $E(P) = \{u_0u_1, u_1u_2, ..., u_{k-1}u_k\}$. The vertices u_0 and u_k are called the *end-vertices* and $k = |E(P)|$ is the *length* of P. A *walk* W in a graph is an alternating sequence of vertices and edges say $u_0, e_1, u_1, e_2, ..., e_k, u_k$ where $e_i = u_{i-1}u_i$, $0 \leq i \leq k$. If a walk is such that $u_0 = u_k$ then is called a *closed walk*. A closed walk with $k \geq 3$, and all vertices u_i, $1 \leq i \leq k-1$, distinct from each other and u_0, then the walk is said to be a *cycle* and is usually denoted by C_k. If a cycle in G contains all vertices of G, then the cycle is said to be a *Hamiltonian cycle*. In accordance with the terminology above, the *length* of a cycle is the number of its edges. The *girth* of a graph is the length of it shortest cycle. A graph is said to be *sparse* if its girth is hight enough according to the context. A graph is *connected* if for every pair of distinct vertices u and v, there is a path from u to v. Unless it is explicit states otherwise, in this work we consider only connected graphs.

Let us now introduce some well-know graph classes, which are the ones that we mainly study in this work.

Trees:- A *tree* is a connected graph without any cycle. All trees are bipartite graphs. It is not difficult to argue that a tree of order at least 2, contains at least two vertices of degree 1. A vertex of degree 1 is usually called a *leaf*. Paths are the trees with exactly two leafs. A tree in which all the vertices except one are leafs is called a *star*.

Planar graphs:- A graph is *planar* if it can be drawn in the plane without edge crossing. A *plane graph* is a particular planar embedding of a planar graph. We will refer to the regions defined by a plane graph as it *faces*, and the unbounded region will be call the *outerface*. The following is a contribution of Leonhard Euler to graph theory, namely *Euler's formula*: if a connected plane graph has n vertices, m edges, and f faces, then:

$$n - m + f = 2$$

Outerplanar graphs:- A graph is outerplanar if it has a planar embedding such that every vertex belongs to the outerface.

Partial k–trees:- The notion of k*-tree* can be defined as follows: the complete graph K_k is a k–tree; if G is a k–tree then the graph G' obtained by G by adding a new vertex linked with every vertex of a k–clique subgraph of G is a k–tree and there are no further k–trees. By construction, every k–tree distinct to K_k has a vertex v of degree k whose neighbourhood is a k–clique subgraph of G, and whose deletion leads to another k–tree. The notion of a 1–tree corresponds to the usual notion of a tree. A *partial k-tree* is a subgraph of some k-tree. The class of outerplanar graphs is strictly contained in the class of partial 2–trees.

1.2 Colorings

A *partition* of a set X, is a set of subsets $X_1, ..., X_k$ such that $\bigcup_{i=1}^{k} X_i = X$ and $X_i \cap X_j = \emptyset$ for every $i \neq j$. A *coloring* corresponds to a partition of some objects in a graph (such as vertices, edges, etc.) into color classes according to certain rules.

For instance, a *proper k-coloring* of a simple graph G is a partition of $V(G)$ into k subsets, $V_1, V_2, ...V_k$, called *color classes*, such that no two adjacent vertices belong to the same color class; in other words, the color classes are independent sets of vertices. If G admits a proper k–coloring, then we say that G is *k–colorable*. The *chromatic number* $\chi(G)$, is then defined as the smallest integer k such that G is k–colorable.

For a simple graph G, $\chi(G) \geq 2$ if and only if G has an edge; and $\chi(G) = 2$ if and only if G is a bipartite graph with at least one edge. The most famous results in graph colorings is the *Four color theorem* [5, 6] which states that all planar graphs are 4–colorable.

In this work we will consider another variant of coloring named *acyclic colorings*. An *acyclic k–coloring* of a simple graph G is a proper k–coloring of G in which every cycle uses at least three colors. The *acyclic chromatic number*, $\chi_a(G)$, is then the minimum integer k such that G admits an acyclic k coloring. In analogy with the Four color theorem, it has been proved [10] that every planar graph admits an acyclic 5–coloring, and that there are planar graphs which cannot be acyclically 4–colored.

1.3 Homomorphisms

Homomorphisms are mappings of the vertices that preserve adjacency. Formally, let G and H be two simple graphs, a *homomorphism* of G to H, writing as $h : G \to H$, is a mapping $h : V(G) \to V(H)$ such that $h(u)h(v) \in E(H)$ whenever $uv \in E(G)$. If h is a homomorphism of G to H, then the graph with vertices $h(v)$, $v \in V(G)$, and edges $h(u)h(v)$, $uv \in E(G)$, is called the *homomorphic image* of G under h, and denoted by $h(G)$; note that $h(G)$ is a subgraph of H. Let us see some simple facts which are direct consequence of these definitions. Recall that $P_k = u_0, ..., u_k$ is the path of length k, and C_k denotes a cycle of length k.

- A mapping $h : V(P_k) \to V(G)$ is a homomorphism of P_k to G if and only if the sequence $h(u_0), h(u_2), ..., h(u_k)$ is a walk in G.
- A mapping $h : V(C_k) \to V(G)$ is a homomorphism of C_k to G if and only if the sequence $h(u_0), h(u_2), ..., h(u_k)$ is a closed walk in G.

Thus homomorphisms maps paths into walks, and cycles into closed walks. In particular the homomorphic image of an odd cycle contains a shorter odd cycle.

There is a close relation between homomorphisms and graph colorings. A proper k–coloring of G can be equivalently regarded as a homomorphism h of G to the complete graph K_k on k vertices. To see that, let $V(K_k) = \{v_1, v_2, ..., v_k\}$, then we can associate $V_i = h^{-1}(v_i)$. Thus, homomorphisms $h : G \to K_k$ are precisely the proper k–colorings of G.

If a graph G admits a proper k–coloring we say that G is *k–colorable*; in the same way, if there is a homomorphism of G to H we say that G is *H–colorable*. The *chromatic number* $\chi(G)$, defined as the smallest integer k such that G is k–colorable, corresponds then to the smallest $|H|$ such that G is H–colorable; note that such a graph H must be complete, else we could find a smaller graph H' such that $H \to H'$ and hence $G \to H'$.

Chapter 2

Basic definitions and notation

Now we focus on the concepts and terminology concerning *colored mixed graphs* which is our main object to study. We begin with some formal definitions and useful notation to handle colored mixed graphs. Then we define *colored homomorphisms* which are the natural morphisms in this class of objects. Finally we define the *colored mixed chromatic number* of a simple graph.

2.1 Colored mixed graphs

An *oriented graph* is an orientation of a simple graph, obtained by assigning to every edge one of the two possible orientations. An oriented edge is called an *arc*, and is denoted by (u, v), meaning that the orientation goes from u to v. The set of arcs of an oriented graph G is denoted by $A(G)$. A *mixed graph* is obtained from a simple graph, by assigning to some edges one of the two possible orientations; hence, a *mixed graph* G is usually denoted by an ordered triple $G = (V, A, E)$, where $V = V(G)$ (resp. $A = A(G)$, $E = E(G)$) is the set of vertices (resp. arcs, edges) of G. Simple graphs and oriented graphs are special cases of mixed graphs.

Definition 2.1 *An (n, m)**-colored mixed graph** is a mixed graph $G = (V, A, E)$, together with partitions $A = A_1 \cup \cdots \cup A_n$ and $E = E_1 \cup \cdots \cup E_m$ where A_i (resp. E_i) consists of the set of arcs (resp. edges) colored by color i.*

In Figures 2.2 and 2.3 you can see examples of $(3, 2)$–colored mixed graphs.

Observe that a $(1, 1)$–colored mixed graph is just a mixed graph. When $n = 0$ we adopt the convention that there are no arcs, and when $m = 0$ we adopt the convention

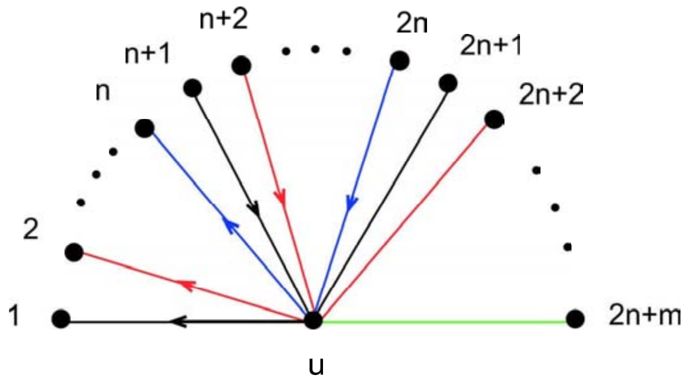


Figure 2.1: The $2n + m$ different types.

that there are no edges. In particular, a $(0, 1)$-colored mixed graph is a simple graph, and a $(1, 0)$-colored mixed graph is an oriented graph. Sometimes we will refer to a $(0, m)$-colored mixed graph as an *m–edge colored graph*, and similarly, we will refer to an $(n, 0)$-colored mixed graph as an *n–arc colored oriented graph*.

The *underlying graph* of an (n, m)-colored mixed graph G, is the simple graph obtaining by forgetting the colors and orientations of the arcs and edges in G. We will use all standard notions (order of a graph, independent set of vertices, planar graph, etc.) in colored mixed graphs as for its underlaying graph; for instance, the girth of a colored mixed graph is the girt of its underlying graph, or a colored mixed path is a colored mixed graph for which its underlying graph is a path.

Let G be a colored mixed graph. We say that a pair of vertices $u, v \in V(G)$ are *adjacent* vertices in G, if either there is an arc or an edge (of any color) between them. In the context of colored mixed graphs, many times we have to be more precise and specify the *type* adjacency between two vertices, thus we give a formal definition below.

For any vertex u of an (n, m)–colored mixed graph G, let $N_i^+(u)$ (resp. $N_i^-(u)$) be the set of all vertices in G adjacent from (resp. adjacent to) u by an arc of color i; similarly let $N_i^0(u)$ be the set of all vertices in G connected with u by an edge of color i. Note that the total number of possible edges and arcs, incident to u, of a particular color and orientation, is $2n+m$; label these possibilities from 1 to $2n+m$ as it is shown in Figure 2.1. According to this we use the following notation:

$$N_i(u) = \begin{cases} N_i^+(u) & \text{for } 1 \leq i \leq n \\ N_{(i-n)}^-(u) & \text{for } n+1 \leq i \leq 2n \\ N_{(i-2n)}^0(u) & \text{for } 2n+1 \leq i \leq 2n+m \end{cases}$$

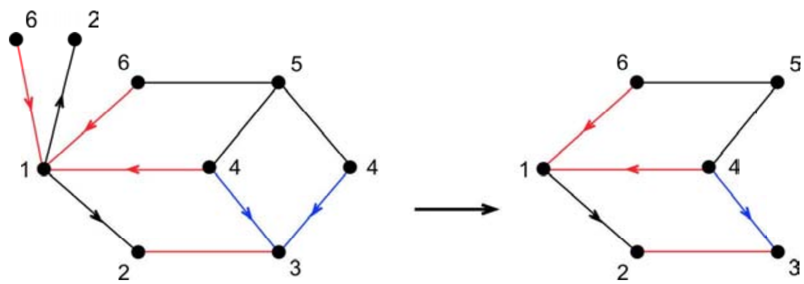


Figure 2.2: A colored homomorphism.

Definition 2.2 *Let (u, v) be an ordered pair of adjacent vertices in an (n, m)–colored mixed graph G. We say that (u, v) has* ***type*** *$i \in \{1, \dots 2n + m\}$, writing $t(u, v) = i$, if $v \in N_i(u)$.*

For $X \in V(G)$, we will use $N_i(X) := \bigcup_{u \in X} N_i(u)$.

2.2 Colored homomorphisms

Colored homomorphisms are mappings between vertex sets of colored mixed graphs that preserve the adjacency type. Thus a colored homomorphism maps edges to edges and arcs to arcs preserving directions and colors. More precisely:

Definition 2.3 *Let G and H be two (n, m)-colored mixed graphs. A* ***colored homomorphism*** *of G to H is a mapping $h\colon V(G) \to V(H)$ satisfying:*

(i) $(u, v) \in A_i(G)$ implies $(h(u), h(v)) \in A_i(H)$ for every $i \in \{1, \dots n\}$, and

(ii) $uv \in E_i(G)$ implies $h(u)h(v) \in E_i(H)$ for every $i \in \{1, \dots m\}$.

In other words, a colored homomorphism h of G to H satisfies $t(u, v) = t(h(u), h(v))$ for every ordered pair (u, v) of adjacent vertices in G. In Figure 2.2 you can see an example of a colored homomorphism.

The existence of a colored homomorphism from G to H is denoted by $G \to H$, and $G \nrightarrow H$ means there is no such homomorphism. The class of graphs G and H will be usually clear from the context.

By following the classical context of homomorphisms as a generalization of colorings, for G and H colored mixed graphs, we say that G is *H-colorable* if $G \to H$, and the vertices of H are called the *colors*. According to this view, an *admissible k–coloring* of a colored mixed graph G is defined as follows.

Definition 2.4 *An **admissible** k–**coloring** of a colored mixed graph G is a partition of $V(G)$ into k independent sets (called the **color classes**) such that no two adjacent vertices belong to the same color class and, there are only edges of the same color, or only arcs with the same orientation and the same color, between any pair of color classes.*

An admissible k–coloring of an (n, m)-colored mixed graphs G can be also stated as a mapping f from $V(G)$ to a set of k colors such that:

(i) $f(u) \neq f(v)$ whenever (u, v) is an adjacent pair,and

(ii) $f(u) \neq f(x)$ whenever (u, v) and (x, y) are both adjacent pairs with $t(u, v) \neq t(x, y)$, and $f(v) = f(y)$.

To finish this section we shall make the following:

Remark 2.1 *If the colored mixed star S depicted in Figure 2.1, is contained in some colored mixed graph G, then every vertex of S must be assigned distinct colors in any admissible coloring of G; the same is true for any colored mixed star subgraph of S.*

Hence, colored homomorphisms of colored mixed graphs "preserves" stars in which every leaf has a different type; homomorphisms preserving other configurations have been considered in [45].

2.3 The colored mixed chromatic number

Given a colored mixed graph G, our purpose is to find the smallest number of vertices of a colored mixed graph H such that G is H-colorable. The *colored chromatic number* of a colored mixed graph G, is the smallest k such that G admits an admissible k–coloring.

For instance, the colored chromatic number of the $(3, 2)$–colored mixed graph showed in Figure 2.3, is four (one can easily find a colored homomorphism from that graph to a $(3, 2)$–colored mixed graph of order 4 and it is clear that there is impossible to do it so for a $(3, 2)$–colored mixed graph of order 3).

Now we are ready to define the *colored mixed chromatic number* of a simple graph.

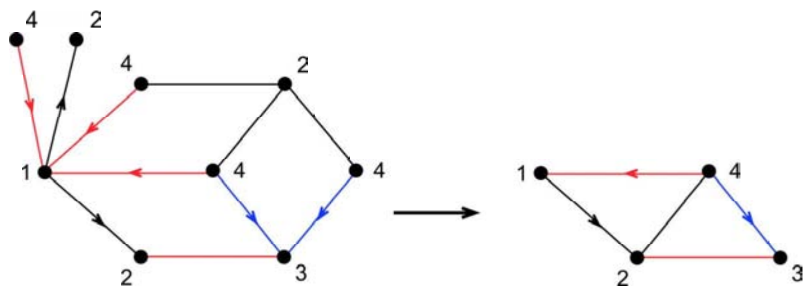


Figure 2.3: A $(3, 2)$–colored mixed graph with chromatic number 4.

Definition 2.5 *For a simple graph G, the (n, m)–**colored mixed chromatic number** of G, denoted by $\chi_{(n,m)}(G)$, is the maximum of the colored chromatic numbers taken over all the possible (n, m)-colored mixed graphs having as underlying graph G.*

We shall note that $\chi_{(0,1)}$ is the ordinary chromatic number, and $\chi_{(1,0)}$ is the oriented chromatic number. It is clear that for every simple graph G, $\chi_{(0,1)}(G) \leq \chi_{(1,0)}(G)$. In general, for the (n, m)–colored mixed chromatic number we have the following relation.

Proposition 2.1 *For every graph G, $\chi_{(n,m+1)}(G) \leq \chi_{(n+1,m)}(G)$...((ARREGLAR))*

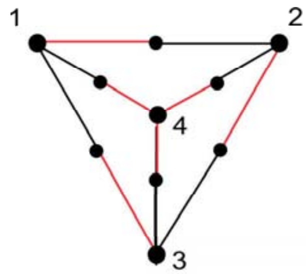
Proof:- For every $(n+1, m)$–colored mixed graph G, lets denote by G' the $(n, m+1)$–colored mixed graph obtained from G by replacing all arcs of color $n + 1$ with edges of color $m+1$. Observe that, a homomorphism from a $(n+1, m)$–colored mixed graph G to another $(n + 1, m)$–colored mixed graph H, is always a homomorphism from G' to H'. □

As a consequence of the above Proposition we can observe that, for a fixed number of colors in arcs and edges, the relations are:

$$\chi_{(0,k)}(G) \leq \chi_{(1,k-1)}(G) \leq \chi_{(2,k-2)}(G)... \leq \chi_{(k,0)}(G)$$

Now, we will define the *colored mixed chromatic number* of a family of simple graphs, which is our main object to study.

Definition 2.6 *For a finite or infinite family $\mathcal{F}$ of simple graphs, the (n, m)–**colored mixed chromatic number** of $\mathcal{F}$, denoted by $\chi_{(n,m)}(\mathcal{F})$, is the (possible infinite) maximum of the (n, m)–colored mixed chromatic numbers of its members.*



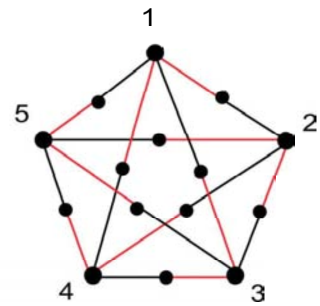


Figure 2.4: Bipartite $(0, 2)$–colored mixed graphs with chromatic number 4 and 5 respectively.

The most natural question to consider in this framework is whether or not a given family of graphs has a finite colored mixed chromatic number. When the answer is affirmative, we say that the family is *colorable*. For a colorable family of graphs, we are interested in determining or bounding its colored mixed chromatic number. Next we see an example of a family of simple graphs for which the ordinary chromatic number is bounded, while the (n, m)–colored mixed chromatic number is unbounded for every $(n, m) \neq (0, 1)$.

Example 2.1 *Let $\mathcal{B}$ be the class of simple bipartite graphs. Certainly $\chi_{(0,1)}(\mathcal{B}) = 2$, but $\chi_{(n,m)}(\mathcal{B}) = \infty$ for every $(n, m) \neq (0, 1)$; that is, for every $k > 0$ there is a bipartite graph B_k with $\chi_{(n,m)}(B_k) > k$ when $(n, m) \neq (0, 1)$. For instance, consider the bipartite graph B_k obtained from the complete graph K_k by replacing every edge by a 2-path; if $(n, m) \neq (0, 1)$, we can color and/or orient the edges of B_k in such a way that the original vertices get distinct colors in any admissible coloring.*

To illustrate this example, in Figure 2.4 you can see a bipartite graph with $(0, 2)$–colored mixed chromatic number 4, and a bipartite graph with $(0, 2)$–colored mixed chromatic number 5. In this way, we can construct for every $k > 0$ a bipartite graph B_k with $\chi_{(0,2)}(B_k) > k$.

At this point we should leave some things clear. For a family of simple graphs $\mathcal{F}$:

- $\chi_{(n,m)}(\mathcal{F}) = k$ means that every colored mixed graph for which its underlying graph is in $\mathcal{F}$, admits an admissible k'–coloring for a $k' \leq k$; and there exist at least one member of $\mathcal{F}$ with colored mixed chromatic number precisely k.
- $\chi_{(n,m)}(\mathcal{F}) = k$ does not means that there exist an (n, m)–colored mixed graph H of order k, such that every (n, m)–colored mixed graph with its underlying graph in $\mathcal{F}$, is H–colorable.

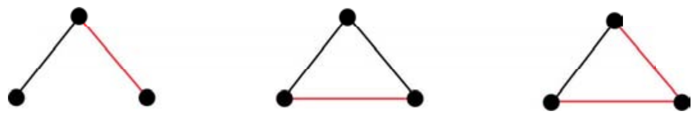


Figure 2.5: All non isomorphic $(0,2)$–colored mixed graphs for which their underlying graphs are in $\mathcal{G}_3$.

Definition 2.7 *An (n,m)–colored mixed graph U is said to be $\mathcal{F}$–**universal**, if every (n,m)–colored mixed graph for which its underlying graph is in $\mathcal{F}$, admits a colored homomorphism to U.*

Thus, the smallest colored mixed graph H which is $\mathcal{F}$–universal (if it exist) may be of order strictly greater than $\chi_{(n,m)}(\mathcal{F})$. Let us see an example to illustrate this situation.

Example 2.2 *Let k be an integer smaller than $2n+m+1$ and let $\mathcal{G}_k$ be the family of connected (simple) graphs of order k. Obviously $\chi_{(n,m)}(\mathcal{G}_k) \leq k$, and since there are graphs in $\mathcal{G}_k$ with colored mixed chromatic number precisely k (recall Remark 2.1) then we have $\chi_{(n,m)}(\mathcal{G}_k) = k$. However, if $(n,m) \neq (0,1)$, clearly there is non (n,m)–colored mixed graph of order k which is $\mathcal{G}_k$–universal (such a colored mixed graph should contain all complete colored mixed graphs of order k as subgraphs).*

To illustrate this example, fix $(n,m) = (0,2)$ and $k = 3$. In Figure 2.5 you can see all non isomorphic $(0,2)$–colored mixed graphs for which their underlying graphs are in $\mathcal{G}_3$. Clearly there is non $(0,2)$–colored mixed graph of order 3 which is $\mathcal{G}_3$–universal.

Nevertheless, to find universal colored mixed graphs is a good strategy to bound the colored mixed chromatic number. In fact, most of the results concerning upper bounds on the oriented chromatic number of some special families of graphs, have been obtained by exhibiting universal graphs.

A family $\mathcal{F}$ is say to be *optimally colorable* if there exist a graph H of order $\chi_{(n,m)}(\mathcal{F})$ which is $\mathcal{F}$–universal. The following Proposition provides a sufficient condition to a colorable family of graphs to be optimal. We say that a family of graphs $\mathcal{F}$ is *complete* if for every two graphs G_1 and G_2 in $\mathcal{F}$, there exist a graph $G_3 \in \mathcal{F}$ containing G_1 and G_2 as subgraphs.

Proposition 2.2 *Every complete family of graphs $\mathcal{F}$ which is colorable its optimal colorable.*

Proof. Since $\mathcal{F}$ is colorable, then $\chi_{(n,m)}(\mathcal{F})$ is finite, and thus every graph $G \in \mathcal{F}$ is H_i–colorable where H_i is a color graph of order $\chi_{(n,m)}(\mathcal{F})$. Suppose $\mathcal{F}$ is not optimally colorable. Then, let $\{H_1, ...H_l\}$, $l \geq 2$, be a minimal set of color graphs such that every graph $G \in \mathcal{F}$ is H_i–colorable for some $i \in \{1, ..., l\}$ (such a finite set exist since the number of color graphs of order $\chi_{(n,m)}(\mathcal{F})$ is finite). By the minimality of l, for every H_i, $1 \leq i \leq l$ there exist $G_i \in \mathcal{F}$ such that $G \nrightarrow H$ for every $H \in \{H_1, ...H_l\} \setminus \{H_i\}$. Since $\mathcal{F}$ is complete there exist a graphs G^* in $\mathcal{F}$ which contains all G_i's, $1 \leq i \leq l$. The graph G^* is H_i–colorable for some $i \in \{1, ..., l\}$, a contradiction. □

Chapter 3

Paths and trees

3.1 Introduction

Probably $\mathcal{T}$, the class of trees, is the most simple (but interesting) family to study its chromatic number. In the case of simple graphs, it is well-known that $\chi_{(0,1)}(\mathcal{T}) = 2$; since an edge require 2 colors, and every tree admits a homomorphism to K_2 (i.e. K_2 is $\mathcal{T}$–universal). In the context of oriented graphs, we have $\chi_{(1,0)}(\mathcal{T}) = 3$; since the direct path on 3 vertices require 3 colors, and it is not difficult to argue that $\overrightarrow{C}_3$, the circuit on three vertices, is $\mathcal{T}$–universal.

In the general context of colored mixed graphs, Nešetřil and Raspaud [43] provided, for every value of (n, m), the (n, m)–colored mixed chromatic number of $\mathcal{T}$. Instead of the original statement we present here the following which is equivalent.

Theorem 3.1 (Nešetřil, Raspaud, [43]) *$\chi_{(n,m)}(\mathcal{T}) = 2n + m + \epsilon$, where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$.*

To prove the upper bound, the authors exhibited an (n, m)–colored mixed graph of order $2n + m + 1$ which is $\mathcal{T}$–universal when m odd or $m = 0$, and an (n, m)–colored mixed graph of order $2n + m + 2$ which is $\mathcal{T}$–universal when m is positive and even. We will present these graphs in Section 3.2 and prove that they are universal.

On the other hand, to prove the lower bound of Theorem 3.1, the authors constructed, for m odd or $m = 0$, an (n, m)–colored mixed tree with chromatic number $2n+m+1$, and, for m positive and even, an (n, m)–colored mixed tree with chromatic number $2n+m+2$. As we will see in Section 3.2, both examples have maximum degree $2n+m$. This suggests the question of whether this upper bound for the colored mixed chromatic number of trees, can be improved within a simpler class of trees.

We succeeded to prove that the colored mixed chromatic number of trees given in Theorem 3.1, is reached by the much simpler family of paths. Thus, the colored mixed chromatic number of $\mathcal{L}$, the class of paths, is the same as for the class of trees. In other words, there are paths with colored mixed chromatic number as large as the highest colored mixed chromatic number of a tree.

Theorem 3.2 *$\chi_{(n,m)}(\mathcal{L}) = 2n + m + \epsilon$, where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.*

In Section 3.3 we will prove Theorem 3.2 by constructing an (n, m)–colored mixed path with chromatic number $2n + m + 1$ if m is odd or $m = 0$, and an (n, m)–colored mixed path with chromatic number $2n + m + 2$ if m is positive and even.

3.2 The colored mixed chromatic number of trees

In this section we prove Theorem 3.1. We first prove the upper bound. Next we describe an (n, m)–colored mixed graph of order $2n + m + \epsilon$ which is $\mathcal{T}$–universal (recall $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even).

- We will define the target graph $\mathcal{K}_{2n+m+\epsilon}$. It is a complete (n, m)–colored mixed graph, whose underlying graph is the complete graph $K_{2n+m+\epsilon}$ and his colored edges and colored arcs are given as follows. It is well known that the set of edges of K_{2n+m+1} (resp. K_{2n+m+2}) can be decomposed into $n+\frac{m-1}{2}$ (resp. $n+\frac{m}{2}$) Hamiltonian cycles and one perfect matching. Thus, we orient and color the edges in order to obtain, a monochromatic Hamiltonian circuit for each color $i \in \{1, \dots n\}$ and one perfect matching for each color $i \in \{1, \dots m\}$.

In Figure 3.1 you can see these target graphs for some values of (n, m). We shall note that, in the cases when $\epsilon = 1$ we use all edges, while in the cases when $\epsilon = 2$ there is a matching which is not used.

Proposition 3.1 *The (n, m)–colored mixed graph $\mathcal{K}_{2n+m+\epsilon}$ is $\mathcal{T}$–universal.*

Proof. Note that the complete (n, m)–colored mixed graph $\mathcal{K}_{2n+m+\epsilon}$ is such that each vertex is incident to one arc of each type (orientation and color) and with one edge of each color. Hence, for any colored mixed tree T, there is a colored homomorphism from T to $\mathcal{K}_{2n+m+\epsilon}$: we use a depth first search of T from any root of T; map the root into any

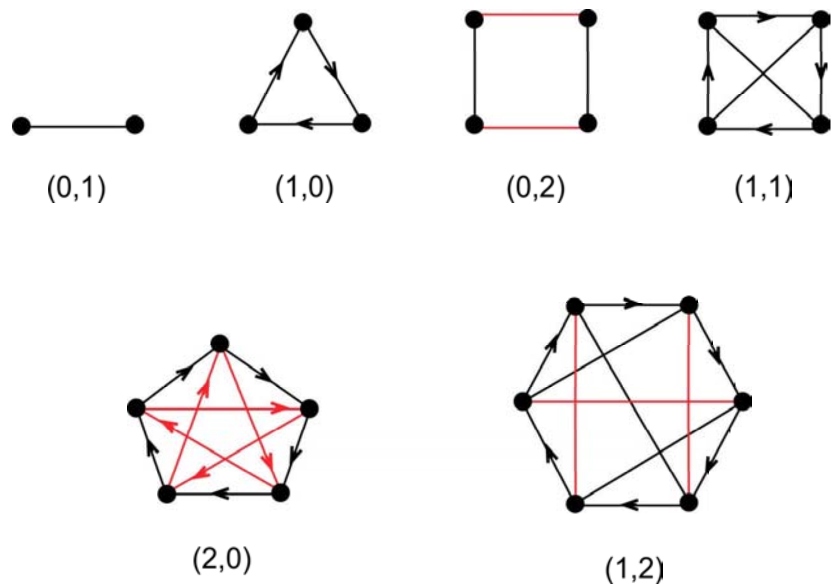


Figure 3.1: The target graph $\mathcal{K}_{2n+m+\epsilon}$ which is universal for the family of trees.

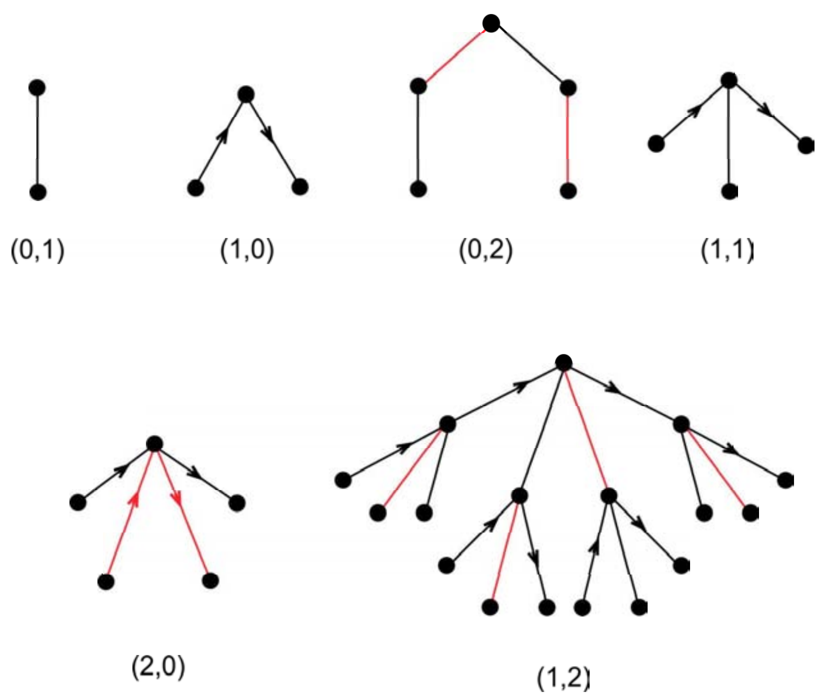


Figure 3.2: A colored mixed tree with highest chromatic number.

vertex of $\mathcal{K}_{2n+m+\epsilon}$, then each new vertex v can be mapped according to the type of the adjacency between v and another already visited vertex. □

Now we prove the lower bound. We construct an (n, m)–colored mixed tree with chromatic number at least $2n + m + \epsilon$.

- When m is odd or $m = 0$, consider the (n, m)–colored mixed star S with $2n+m+1$ vertices in which every leaf has a different type (see Figure 2.1). When m is positive and even, consider the tree S^* constructed from the star S as follows. Take $2n + m$ copies of S named $S_1, ..., S_{2n+m}$ and, for each $i \in \{1, ..., 2n+m\}$, let v_i be the vertex of S_i such that the adjacency from the central vertex of S_i to v_i is of type i; we identify these $2n + m$ vertices v_i's in order to obtain S^*.

In Figure 3.2 you can see the trees S and S^* for some values of (n, m). We shall note that, in all cases the maximum degree is $2n + m$.

Proposition 3.2 *The (n, m)–colored mixed tree S (resp. S^*) has chromatic number at least $2n + m + 1$ (resp. $2n + m + 2$).*

Proof. By Remark 2.1, every vertex of S must be assigned distinct colors in any admissible coloring, hence S has chromatic number $2n + m + 1$. To see that S^* has chromatic number at least $2n+m+2$ suppose first that $S^* \to H$ for some (n, m)–colored mixed graph H of order $2n+m+1$. By Remark 2.1, every vertex of S^* which has degree at leas two, must be assigned distinct colors, hence all vertices of H necessarily are such that there is exactly one edge/arc of each type incident to it. Now consider the subgraph of H induced by the edges of one color (recall we are in case $m > 0$), this graph should be a perfect matching which is impossible since H has odd order. □

This conclude the prove of Theorem 3.1.

3.3 The colored mixed chromatic number of paths

In this section we prove Theorem 3.2. We will construct an (n, m)–colored mixed path with chromatic number $2n + m + \epsilon$, where $\epsilon = 1$ if m is odd or $m = 0$, and $\epsilon = 2$ if m is positive and even. This colored mixed path is going to be the concatenation of many other colored mixed paths, one for each (n, m)-colored mixed complete graph on $(2n+m+\epsilon)-1$ vertices.

Consider a fixed (n, m)-colored mixed complete graph H on $(2n + m + \epsilon) - 1$ vertices. We will construct an (n, m)-colored mixed path L such that $L \nrightarrow H$. Since the number of (n, m)-colored mixed complete graphs of fixed order is finite, the concatenation of all such paths cannot be mapped onto any (n, m)-colored mixed complete graph of that size. Thus we get an (n, m)-colored mixed path with chromatic number $2n + m + \epsilon$.

In order to construct the path L such that $L \nrightarrow H$, where H is a fixed complete (n, m)-colored mixed graph on $(2n + m + \epsilon) - 1$ vertices, the key idea is to find:

1. A sequence of types: $t_1, \dots, t_r$ where $t_i \in \{1, \dots, 2n + m\}$ and,
2. A sequence of subsets $X_0, X_1, \dots, X_r$ of $V(H)$, with the following properties:
   i) $X_0 = V(H)$,
   ii) $X_i = N_{t_i}(X_{i-1})$ for every $i \in \{1, ..., r\}$ and,
   iii) $X_r = \emptyset$.

This allows us to define L. Indeed, define $L := v_0, v_1, ..., v_r$ where $t(v_{i-1}, v_i) = t_i$. Now, for every homomorphism from L to H, the first vertex of L must be mapped onto a vertex of $X_0 = V(H)$, the second vertex onto a vertex of X_1 and so on. Since X_r is the empty set, no such homomorphism can exist. To find the sequence of types and subsets with the properties defined above, we split the proof into two cases according to the value of ϵ. Before that, we prove a simple but useful counting lemma.

Lemma 3.1 *Let X be a subset of vertices of a complete (n, m)-colored mixed graph H. Then,*

$$\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(|V(H)| - 1) \ .$$

Proof. Consider the bipartite (n, m)-colored mixed graph B_X defined as follows. The set of vertices is the disjoint union of a copy of X and a copy of $V(H)$. We add every edge or arc $(x, v) \in X \times V(H)$, with the same color and orientation as (x, v) in H (thus the only edges we don't have in B_X are the ones for which x and v correspond to the same vertex in H). Denote by $E_i(B_X)$ the set of arcs or edges from X to $V(H)$ of type i. Observe that the total number of edges in B_X is $|X|(|V(H)|-1)$. Then $\sum_{i=1}^{2n+m} |E_i(B_X)| = |X|(|V(H)|-1)$. The result follows since $|N_i(X)| \leq |E_i(B_X)|$ for every $i \in \{1, \dots, 2n + m\}$. □

Case 1. $\epsilon = 1$ (m odd or $m = 0$)

Let H be a complete (n, m)-colored mixed graph on $2n + m$ vertices. We start with $X_0 = V(H)$. By means of Lemma 3.2, we are able to find a strictly decreasing sequence of subsets $|X_0| > |X_1| > \dots > |X_r|$ and a sequence of types $t_1, \dots t_r$ such that $X_i = N_{t_i}(X_{i-1})$. Since in every step the size of the subset decreases, eventually we get $X_r = \emptyset$.

Lemma 3.2 *For any subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m$ vertices, there exists $i \in \{1, \ldots, 2n + m\}$ such that $|N_i(X)| < |X|$.*

Proof. By Lemma 3.1, we have $\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(2n + m - 1)$, and the result follows. □

Case 2. $\epsilon = 2$ ($m > 0$ even)

Let H be a complete (n, m)-colored mixed graph on $2n + m + 1$ vertices. In this case we can not apply Lemma 3.2 to construct a strictly decreasing sequence of subsets as in Case 1. Instead, by means of Lemma 3.3 below, we can guarantee that if we can not decrease, then all neighborhoods have the same size.

Lemma 3.3 *For any subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m + 1$ vertices, either there exists $i \in \{1, \ldots, 2n + m\}$ such that $|N_i(X)| < |X|$ or $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$.*

Proof. By Lemma 3.1 we obtain $\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(2n+m)$ and the result follows. □

Now more work is required. Suppose $X \subset V(H)$ is such that $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$. In Lemma 3.5 we show that in at most three steps we can reduce the size of the subset. In order to prove it, we need the next Lemma.

Lemma 3.4 *In any (n, m)-colored mixed complete graph on $2n + m + 1$ vertices with $m > 0$ even, there exists a vertex incident to at least 2 edges of the same type.*

Proof. Any vertex v of H has degree $2n + m$. If v is not incident to an edge of a particular type, then v would be the desired vertex (being $2n+m$ types in total). Assume that every vertex is incident to exactly one edge of every type. Then any color class of edges would induce a perfect matching of H. This is a contradiction since H has an odd number of vertices. □

Lemma 3.5 *If a subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m + 1$ vertices with $m > 0$ even is such that $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$, then there exist $j, k, l \in \{1, \ldots, 2n + m\}$ such that $|N_l(N_k(N_j(X)))| < |X|$.*

Proof. By Lemma 3.4, there exists a vertex $u \in V(H)$ incident to at least two edges of type $k \in \{1, \ldots, 2n + m\}$. Since H is a complete (n, m)-colored mixed graph, $u \in N_j(X)$

for some $j \in \{1, \ldots, 2n+m\}$. By hypothesis, $|N_j(X)| = |X|$. We may assume that $N_j(X)$ is such that $|N_i(N_j(X))| = |N_j(X)|$ for every $i \in \{1, \ldots, 2n+m\}$, otherwise by Lemma 3.3 we are done. Then we have $|N_k(N_j(X))| = |N_j(X)| = |X|$. Name $Y := N_k(N_j(X))$. We will use the bipartite graph B_Y, defined as in the proof of Lemma 3.1. By construction, there are two vertices $v, w \in Y$ with a common k-neighbor (u). Therefore $|N_k(Y)| < |E_k(B_Y)|$. We suppose $|N_k(Y)| = |Y|$ (otherwise by Lemma 3.3 we are done). Thus we have $|Y| < |E_k(B_Y)|$, and the result follows since $\sum_{i=1}^{2n+m} |E_i(B_Y)| = |Y|(2n+m)$. □

This conclude the prove of Theorem 3.2.

Chapter 4

Graphs with bounded acyclic chromatic number

4.1 Introduction

As we said before, most of the work related to the study of homomorphisms as a generalization of colorings has been done in the context of oriented graphs. One of the first problems considered in this framework was to characterize the families of graphs having bounded oriented chromatic number. In 1994, Raspaud and Sopena [53] proved that families of graphs with bounded acyclic chromatic number have also bounded oriented chromatic number. A few years later, in 1996, Alon and Marshall [4] provide the analogous result for edge–colored graphs. In 2000, Nešetřil and Raspaud [43] came with the notion of colored mixed graphs and proved the result which unify and generalize these two former results (Theorem 4.1 below).

Recall that the *acyclic chromatic number* of a simple graph G, is the smallest number of colors needed in an *acyclic coloring*, which is a proper vertex coloring satisfying that every cycle receives at least three colors. We denote by $\mathcal{A}_k$ the family of graphs with acyclic chromatic number at most k.

Theorem 4.1 (Nešetřil, Raspaud, [43]) $\chi_{(n,m)}(\mathcal{A}_k) \leq k(2n+m)^{k-1}$.

We found interesting to include in this thesis the proof of Theorem 4.1, since it is a refinement of the technique used in [53] for oriented graphs, and the technique used in [4] for edge–colored graphs; thus we prove Theorem 4.1 in Section 4.2.

In the same paper [43] the authors proved that the main term in the bound of Theorem 4.1 is, in a sense, best possible. They show, for all $(n, m) \neq (0, 1)$, a graph with acyclic chromatic number at most k, and colored mixed chromatic number greater than $(2n+m)^{k-1}+k-1$. It is an interest problem then, to ask if the bound given in Theorem 4.1 is tight in particular cases; for instance one can consider particular values of k or particular values of n and m.

Example 4.1 *In the case of simple graphs $(n,m)=(0,1)$ the bound of Theorem 4.1 is tight. Recall that the colored mixed chromatic number in this case is just the ordinary chromatic number. Thus the statement says that, if a simple graph has acyclic chromatic number at most k, then it has chromatic number at most k (triviality). Since there are simple graphs with acyclic chromatic number at most k and chromatic number k, then the bound is tight; for instance consider the complete graph of order k for which $\chi_a(K_k)=\chi(K_k)=k$.*

Example 4.2 *Consider now the case when $k=2$. Then Theorem 4.1 states $\chi_{(n,m)}(\mathcal{A}_2)\leq 4n+2m$. Note that the family of graphs with acyclic chromatic number at most 2, is in fact the family of trees, for which we know that the colored mixed chromatic number is $2n+m+\epsilon$ (Theorem 3.1). Then is not difficult to see that the upper bound given in Theorem 4.1 for $k=2$ is tight in the cases of simple graphs $(n,m)=(0,1)$, and 2–edge colored graphs $(n,m)=(0,2)$.*

The techniques used to prove Theorem 4.1 suggested that may be for $k\geq 3$, the bound is not tight. However, Ochem [47] surprisingly proved that the bound in Theorem 4.1 is tight for every $k\geq 3$ in the class of oriented graphs: $(n,m)=(1,0)$. He constructed an oriented graph for which the acyclic chromatic number of the underlying graph is at most k and its oriented chromatic number is $k2^{k-1}$, which is the corresponding value of the bound in this case.

In Section 4.3 we extend Ochem's construction to show an (n,m)–colored mixed graph for which the acyclic chromatic number of the underlying graph is at most k and its chromatic number is $k(2n+m)^{k-1}$. By means of this result we prove that the upper bound given in Theorem 4.1 is still tight for every $k\geq 3$ in the more general class of $(n.m)$–colored mixed graphs for every $(n,m)\neq(0,1)$; since the same is true for $(n,m)=(0,1)$ (Example 4.1) we get the following result.

Theorem 4.2 *For every $k\geq 3$ and every $m\geq 0$ and $n\geq 0$, $\chi_{(n,m)}(\mathcal{A}_k)=k(2n+m)^{k-1}$.*

4.2 A universal colored mixed graph for $\mathcal{A}_k$

In this section we prove Theorem 4.1. We will construct a target graph H on $k(2n+m)^{k-1}$ vertices which is $\mathcal{A}_k$–universal.

Figure 4.1: The colored mixed graph $\mathcal{K}_{2n+m,2n+m}$, which is $\mathcal{T}$–universal, with the extra property that preserves the bipartition of trees.

We first define a colored mixed graph $\mathcal{K}_{2n+m,2n+m}$, which is $\mathcal{T}$–universal, with the extra property that preserves the bipartition of trees (according to the fact that a tree is a bipartite graph).

- We will define the target graph $\mathcal{K}_{2n+m,2n+m}$. It is a (n,m)–colored mixed complete bipartite graph, whose underlying graph is the complete bipartite graph $K_{2n+m,2n+m}$ and his colored edges and colored arcs are given as follows. It is well known that the set of edges of $K_{2n+m,2n+m}$ can be decomposed into $\frac{2n+m-1}{2}$ Hamiltonian cycles and one perfect matching if m is odd, or $\frac{2n+m}{2}$ Hamiltonian cycles if m is even. Thus, we orient and color the edges in order to obtain a monochromatic Hamiltonian circuit for each color $i \in \{1, \dots n\}$ and one perfect matching for each color $i \in \{1, \dots m\}$.

In Figure 4.1 you can see these target graphs for some values of (n,m). We will denote $V(\mathcal{K}_{2n+m,2n+m}) = U \cup W$ with $U = \{u_1, ..., u_{2n+m}\}$ and $W = \{w_1, ..., w_{2n+m}\}$.

Proposition 4.1 *Let T be any (n,m)–colored mixed tree and V_1, V_2 be any bipartition of its set of vertices into stable sets. Then there exist a colored homomorphism from T to $\mathcal{K}_{2n+m,2n+m}$ which maps V_1 into U and V_2 into W.*

Proof. Notice that $\mathcal{K}_{2n+m,2n+m}$ is such that each vertex is incident to one arc of each type (orientation and color) and with one edge of each color. Hence, we can use a similar argument as in the proof of Proposition 3.1. We can use a depth first search, by mapping the root into any vertex of the correct stable set. □

By means of $\mathcal{K}_{2n+m,2n+m}$ we define the target graph H on $k(2n+m)^{k-1}$ vertices which is $\mathcal{A}_k$–universal.

- The target graph H has the following set of vertices:

  $V(H) = \{(i;, a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_k)\}$ where $i \in \{1, ..., k\}$ and

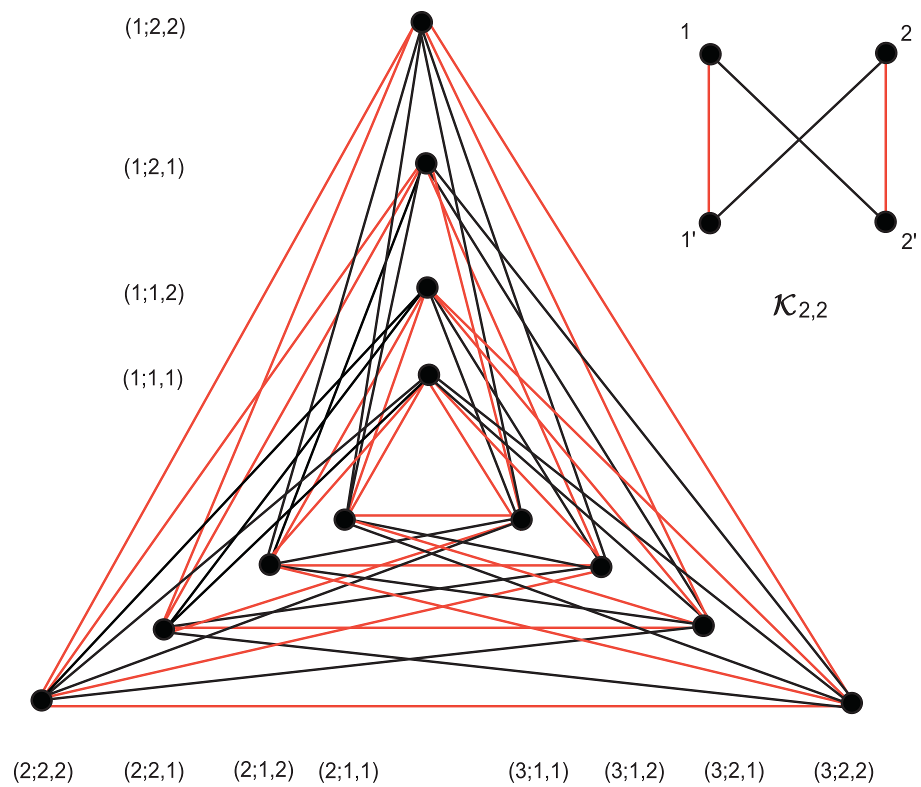


Figure 4.2: A $(0, 2)$–colored mixed graph on 12 vertices which is $\mathcal{A}_3$–universal.

$a_l \in \{1, ..., 2n+m\}$, $l \in \{1, ..., k\} \setminus \{i\}$.

For $1 \leq i < j \leq k$, two vertices of H, $(i; , a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_k)$ and $(j; , b_1, b_2, ..., b_{i-1}, b_{i+1}, ..., b_k)$ are linked by an edge or an arc with the direction and the color according to the one linking $u \in U$ and $w \in W$ where $u = a_j$ and $w = b_i$.

In Figure 4.2 you can see this target graph for $(n, m) = (0, 2)$ and $k = 3$.

Proposition 4.2 *H is $\mathcal{A}_k$–universal.*

Proof. Let G be an (n, m)–colored mixed graph for which its underlying graph is in $\mathcal{A}_k$. We define a colored homomorphism from G to H. Let $V_1, ..., V_k$ be the color classes of an acyclic coloring of G. For any i and j with $1 \leq i < j \leq k$, $G[V_i \cup V_j]$, the induced graph by $V_i \cup V_j$ is a forest. Hence, by Proposition 4.1 there exist a colored homomorphism $c_{i,j}$ which maps V_i (resp. V_j) into U (resp. W). We define the colored homomorphism $f : G \rightarrow H$ as follows. Let v be a vertex of G; then $v \in V_i$ for some $i \in \{1, ..., k\}$. The image of v will be $f(v) = (i; c_{1,i}(v), ..., c_{i,k}(v))$. The constraints of coloration and orientation is satisfied. Indeed, if $v \in V_i$, $w \in V_j$, and the type $t(u, w) = t$ in G; then, by definition of H, the type $t(f(u), f(w)) = t$ in H. □

Since the number of vertices of H is exactly $k(2n+m)^{k-1}$, we conclude the proof of Theorem 4.1.

4.3 Extending Ochem's construction

In this Section prove Theorem 4.2. We construct, for every $k \geq 3$, a simple graph $G(k)$ with acyclic chromatic number at most k and (n, m)–colored mixed chromatic number $k(2n+m)^{k-1}$ when $(n, m) \neq (0, 1)$.

- We now define the graph $G(k)$. Consider B the complete bipartite graph with independent sets $U = \{u_1, ..., u_{(2m+n)^{k-1}}\}$ and $W = \{w_1, ..., w_{k-1}\}$. Take k disjoint copies $B_1, B_2, ...B_k$ of B with their respective stable sets labeled $U_1, U_2, ...U_k$ and $W_1, W_2, ...W_k$. For each pair of subscripts $1 \leq i < j \leq k$ and each pair of vertices $(x, y) \in U_i \times U_j$, we add an extra vertex $z = z_{ij}(x, y)$, connected to x and y (see Figure 4.3).

Proposition 4.3 *For every $k \geq 3$, $G(k) \in \mathcal{A}_k$.*

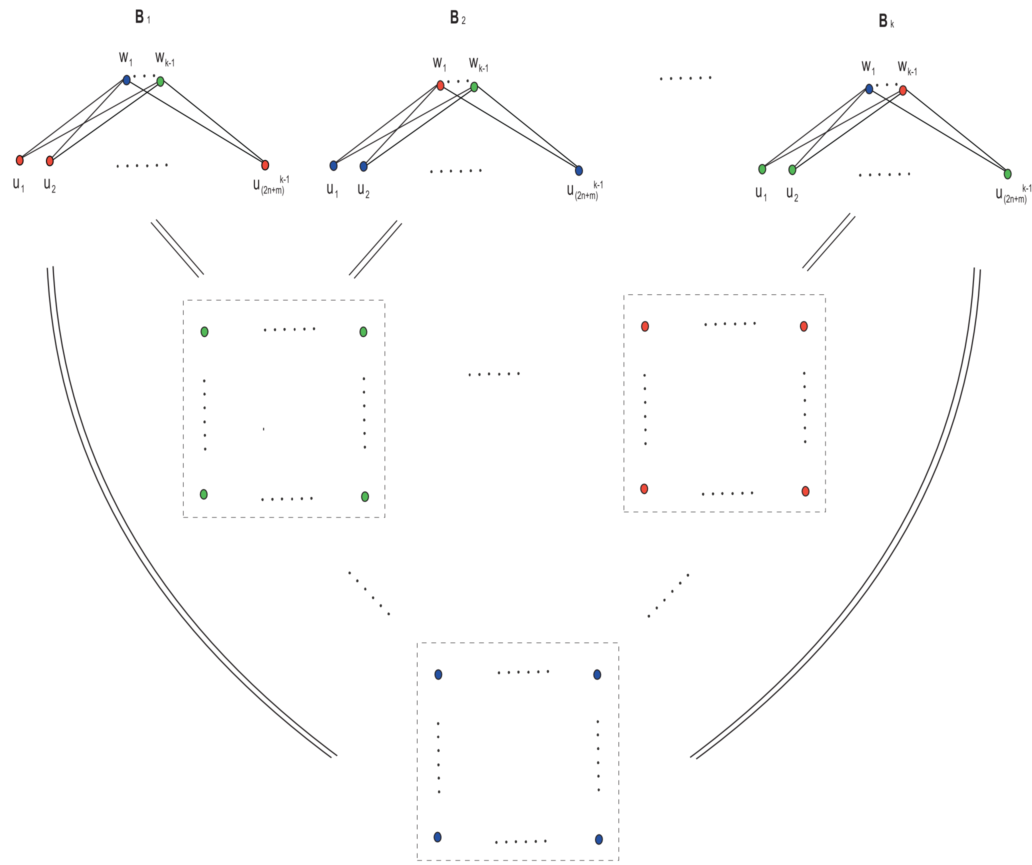


Figure 4.3: The simple graph $G(k)$ with acyclic chromatic number at most k and (n, m)–colored mixed chromatic number $k(2n + m)^{k-1}$.

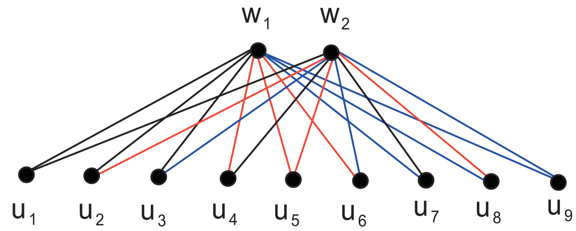


Figure 4.4: How to color the bipartite graph B, example for $k = 3$ and $(n, m) = (0, 3)$.

Proof. We acyclically color $G(k)$ as follows. Every vertex in U_i gets color i and all vertices in W_i get pairwise distinct colors in $\{1, 2...k\}\backslash\{i\}$. Thus every cycle in each copy of B gets at least three different colors. For each pair $(x, y) \in U_i \times U_j$ we color the extra vertex $z_{ij}(x, y)$ by any color in $\{1, 2...k\}\backslash\{i, j\}$, so that every cycle involving extra vertices has at least three colors and the resulting coloring is proper (see Figure 4.3). □

Proposition 4.4 *For every $k \geq 3$, $\chi_{(n,m)}(G(k)) = k(2n + m)^{k-1}$, when $(n, m) \neq (0, 1)$.*

Proof. We color and orient the edges of each copy of B in such a way that the vertices of U necessarily get distinct colors in every admissible coloring. This can be done since there are $(2m+n)^{k-1}$ different vectors of length $k-1$ with entries in $\{1, ...2n+m\}$. Thus we can color and orient the edges of B in order to have the sequences of types of vertices in U pairwise distinct, that is: for every pair of different vertices $u_i, u_j \in U$, we have:

$$(t(u_i, w_1), t(u_i, w_2)...t(u_i, w_{k-1})) \neq (t(u_j, w_1), t(u_j, w_2)...t(u_j, w_{k-1})).$$

In Figure 4.4 you can see an example with $k = 3$ and $(n, m) = (0, 3)$.

Now we color and orient the edges connecting the extra vertices, in such a way that $t(x, z) \neq t(y, z)$ (recall that $(n, m) \neq (0, 1)$.) By construction, the vertices in $U_1 \cup U_2 ... \cup U_k$ get pairwise distinct images. Since $|\cup_{i=1}^{k} U_i| = k(2m + n)^{k-1}$ then $\chi_{(n,m)}(G(k)) \geq k(2n + m)^{k-1}$. □

Hence, for every $k \geq 3$, we get a graph $G(k) \in \mathcal{A}_k$ with $\chi_{(n,m)}(G(k)) = k(2n + m)^{k-1}$ when $(n, m) \neq (0, 1)$. This conclude the proof of Theorem 4.2, since Example 4.1 provides the result for $(n, m) = (0, 1)$.

Chapter 5

Partial k–trees

5.1 Introduction

Up to now we have studied the colored mixed chromatic number of paths, trees and graphs with bounded acyclic chromatic number. In this Section we study the colored mixed chromatic number of the class of graphs with bounded treewidth, or partial k–trees. This is a very important class of graphs from the algorithmic point of view, since many problems that are NP–hard for general graphs become polynomial or linear time solvable when restricted to graphs with bounded treewidth.

Recall that a *k-tree* is a simple graph obtained from the complete graph K_k by repeatedly inserting new vertices linked to all vertices of an existing clique of order k. A *partial k-tree* is a subgraph of some k-tree. It is not difficult to see that every partial k-tree has acyclic chromatic number at most $(k+1)$: starting with a proper k–coloring of the complete graph K_k, every newly inserted vertex has exactly k neighbors and can be thus colored using a $(k+1)$–th color; moreover, this coloring is clearly acyclic since all the neighbors of a newly inserted vertex have pairwise distinct colors. Therefore, by Theorem 4.1 we get the following upper bound for the class $\mathcal{T}^k$ of partial k–trees.

Theorem 5.1 $\chi_{(n,m)}(\mathcal{T}^k) \leq (k+1)(2n+m)^k$.

Concerning the class of partial k–trees we are not able to get the precise colored mixed chromatic number as for the classes of paths (Theorem 3.2), trees (Theorem 3.1) or graphs with bounded acyclic chromatic number (Theorem 4.2). Instead we prove that the upper bound given in Theorem 5.1 has the correct order of magnitude on $(2n+m)$, that is:

Theorem 5.2 $\chi_{(n,m)}(\mathcal{T}^k) = \Theta((2n+m)^k)$

In order to prove Theorem 5.2, we will exhibit in Section 5.2 partial k–trees with colored mixed chromatic number greater than $(2n+m)^k$ (see Propositions 5.1 and 5.2 for precisely statements).

It is natural to try to improve the bound given in Theorem 5.1, by exhibiting universal graphs in particular cases: particular values of n and m, or particular values of k.

To prove that a target graph is universal for some family we need "useful" properties. Concerning the family of partial k–trees, in the context of oriented graphs, Sopena [59] described a property named *property* P_k. In Section 5.3 we generalize this property to the more general context of colored mixed graphs, and prove several results.

In particular we exhibit a 2–edge colored graph on 9 vertices with property P_2, and a 2–edge colored graph on 20 vertices with property P_3; both of these target graphs are optimal by the lower bounds given in Section 5.2. Additionally, we construct a mixed graph on 21 vertices and a 2–arc colored oriented graph on 28 vertices, both with with property P_2; these target graphs improve the upper bound given in Theorem 5.1 when $k = 2$, in the cases $(n, m) = (1, 1)$ and $(n, m) = (2, 0)$. With these target graphs we obtain the following result.

Theorem 5.3 *Let $\mathcal{T}^2$ (resp. $\mathcal{T}^3$) be the class of partial 2–trees (resp. partial 3–trees), then*

1. $\chi_{(0,2)}(\mathcal{T}^2) = 9$.
2. $\chi_{(0,2)}(\mathcal{T}^3) = 20$.
3. $\chi_{(1,1)}(\mathcal{T}^2) \leq 21$.
4. $\chi_{(2,0)}(\mathcal{T}^2) \leq 28$.

5.2 Constructions for lower bounds

In this Section we will construct partial k–trees with colored mixed chromatic number greater than $(2n + m)^k$. The key idea to do it, is to generalize a construction proposed by Sopena [59]; we shall note that if we take $(n, m) = (1, 0)$ in Proposition 5.1 below, we obtain $2^k + 2^{k-1} + \frac{2^{k-1}-1}{2-1}$ which is precisely $2^{k+1} - 1$ the bound given in Theorem 3.9 of [59].

Proposition 5.1 *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even. For every $k \geq 1$, $m \geq 0$ and $n \geq 0$, there are partial k–trees with (n, m)–colored mixed chromatic number at least:*

$$(2n+m)^k + \epsilon(2n+m)^{k-1} + \tfrac{(2n+m)^{k-1}-1}{(2n+m)-1}.$$

When m is positive and even, and $k \geq 3$, we can improve this lower bound by one. This will allow us to get a tight bound for the $(0,2)$–colored mixed chromatic number of partial 3–trees in Theorem 8.

Proposition 5.2 *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even. For every $k \geq 3$ and every $n \geq 0$ and $m > 0$ even, there are partial k–trees with (n,m)–colored mixed chromatic number at least:*

$$(2n+m)^k + 2(2n+m)^{k-1} + \tfrac{(2n+m)^{k-1}-1}{(2n+m)-1} + 1.$$

To make the writing easier, let:

$$a(x) = (2n+m)^x + \epsilon(2n+m)^{x-1} + \tfrac{(2n+m)^{x-1}-1}{(2n+m)-1}.$$

Thus Proposition 5.1 states that there are partial k–trees with colored mixed chromatic number at least $a(k)$ for every $k \geq 1$, $m \geq 0$ and $n \geq 0$; and Proposition 5.2 states that for $k \geq 3$ and $m > 0$ even, there are partial k–trees with (n,m)–colored mixed chromatic number at least $a(k)+1$.

In order to prove Proposition 5.1, we will use the following construction and the exact colored mixed chromatic number of trees.

- Let G be an (n,m)–colored mixed graph.

  Define G' as the (n,m)–colored mixed graph obtained by taking $\{G_1, G_2, ...G_{2n+m}\}$, $2n+m$ disjoint copies of G, and adding a new vertex u adjacent to all other vertices in such a way that, for every $v \in G_i$, the type $t(u,v) = i$ (see Figure 5.1).

  Let G^k be defined inductively by $G^0 = G$ and for every $k \geq 1$, $G^k = (G^{k-1})'$.

The vertex u is called the *central vertex* of G'. By construction, in every H–coloring of G^k, all vertices in G_i^{k-1} get distinct colors from those assigned to the vertices in G_j^{k-1} when $i \neq j$. Moreover the central vertex u most be assigned a distinct color from those assigned to all other vertices. Thus we have the next:

Remark 5.1 *For every (n,m)–colored mixed graph G and every $k \geq 1$, $\chi(G^k) \geq (2n+m)\chi(G^{k-1}) + 1$.*

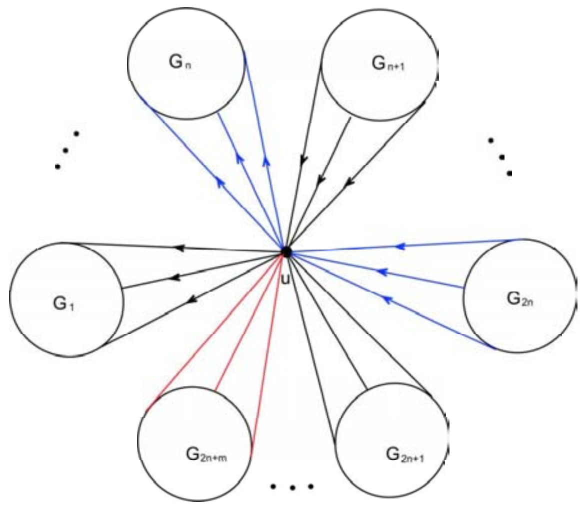


Figure 5.1: Given G we construct G' with higher chromatic number.

Proof of Proposition 5.1:- To prove our result we proceed by induction on k. For $k = 1$, Theorem 3.1 provided us an (n, m)–colored mixed tree with chromatic number $a(1) = 2n + m + \epsilon$. Suppose now that the result holds up to $(k - 1)$ and let $T^{(k-1)}$ be an (n, m)–colored mixed partial $(k - 1)$–tree with chromatic number at least $a(k - 1)$. By Remark 5.1, $T^k = (T^{(k-1)})'$ has chromatic number at least $(2n + m)a(k - 1) + 1$, which is precisely $a(k)$. It remains to show that T^k is a partial k–tree; we can verify this by taking $k' = k - 1$ in the next Proposition proved in [59]. □

Proposition 5.3 (Proposition 3.2 in [59]) *Let T_1 be a partial k–tree, $X = (x_1, x_2, ...x_k)$ a k–clique subgraph of T_1 and T_2 a partial k'–tree ($k' \leq k$). The graph T_3 obtained from T_1 and T_2 by adding edges from every vertex of T_2 to vertices $x_1, x_2, ...x_k$ is also a partial k–tree.*

Now let us prove Proposition 5.2; the basic fact that we use to do it, is the following:

Remark 5.2 *For m even and $k \geq 3$, both $a(k - 1)$ and $a(k)$ are odd numbers.*

We ask m to be positive in the sentence of Proposition 5.2, since in our proof we use the subgraph induced by the edges of one color. For $m > 0$ even and $k = 2$, our proof fails since $a(1)$ is an even number (in fact, the statement of Proposition 5.2 is not true for $k = 2$; in Section 5.3 we will show a $(0, 2)$–colored mixed graph on $a(2) = 9$ vertices which is universal for the family of partial 2–trees). For m odd, our proof fails since $a(k)$ is even when k is odd, while $a(k)$ is odd when k is even.

The construction that we will use to prove Proposition 5.2 is the following.

- Let T^k be the (n,m)–colored mixed partial k–tree constructed in the proof of Proposition 5.1.

  Define $(T^k)''$ as the (n,m)–colored mixed graph obtained from a copy of T^k, namely T^k_0, by gluing on each vertex $v \in V(T^k_0)$ a new copy of T^k by its central vertex.

Observe $(T^k)''$ is a partial k–tree.

Proof of Proposition 5.2:- To prove our result, we will show that there are no homomorphisms from $(T^k)''$ to any (n,m)–colored mixed complete graph of order $a(k)$. Suppose to the contrary that $(T^k)''$ is C–colorable for some (n,m)–colored mixed complete graph C of order $a(k)$. Recall that T^k is such that $\chi_{(n,m)}(T^k) = a(k)$ and since every vertex in T^k_0 is a central vertex of a copy of T^k, necessarily C is such that for every vertex $v \in C$ and every type $i \in \{1, ..., 2n+m\}$ it happens that $|N_i(v)| \geq a(k-1)$. Since C has $a(k)$ vertices of degree $a(k) - 1 = (2n+m)a(k)$, then actually $|N_i(v)| = a(k-1)$ for every $i \in \{1, ..., 2n+m\}$. In particular, look at the subgraph induced by the edges of one color (recall $m > 0$); this graph is $a(k-1)$–regular and has $a(k)$ vertices, which is a contradiction, since in any graph the number of vertices of odd degree is even, and by Remark 5.2 both $a(k-1)$ and $a(k)$ are odd numbers. □

5.3 The property P_k

In this Section we will prove Theorem 5.3.

As we said before, to get upper bound for the colored mixed chromatic number, one often tries to find a universal graph. To prove that a target graph is universal we need "useful" properties. Concerning the class of partial k–trees, in the context of oriented graphs, Sopena [59] described a property named *property* P_k. We generalize this definition to colored mixed graphs.

A *pattern* $\mathcal{Q} = q_1q_2...q_k$ of length k is a (non-empty) word in $\{1, ..., 2n+m\}^k$. Let $S = \{v_1, ..., v_k\}$ be a sequence of k vertices in a colored mixed graph G. A vertex $u \in V(G)$ is said to be a *$\mathcal{Q}$–neighbor of S* if $t(u, v_i) = q_i$ for every i.

Definition 5.1 *We say that a colored mixed graph G satisfies* ***property*** *P_k for some $k > 0$, if for every sequence $S = \{v_1, ..., v_n\}$ which induce an n–clique subgraph of G, with $1 \leq n \leq k$, and every pattern $\mathcal{Q} = q_1q_2...q_n$ of length n, there exist a $\mathcal{Q}$–neighbor of S.*

The next Proposition is a natural generalization of Theorem 3.4 in [59].

Proposition 5.4 *Every colored mixed graph G satisfying property P_k is universal for $\mathcal{T}^k$.*

Proof. Let G be a colored mixed graph satisfying property P_k. We will prove that every colored mixed k–tree T_k admits a homomorphism to G (observe that it suffices to consider the case of k–trees, since partial k–trees are subgraphs of k–trees). We proceed by induction on p, the number of vertices of T_k. If $p = k$ then T_k is a complete colored mixed graph. By definition, if G satisfies property P_k then G contains as a subgraph all complete colored mixed graphs. Let T_k be a colored mixed k–tree with $p+1$ vertices. Since there exist a vertex v in T_k with degree k whose deletion leads to another k–tree, the induction hypothesis ensures that $T_k - v$ admits a homomorphism to G. Since G has property P_k we can extend that homomorphism to a homomorphism from T_k to G. □

We shall notice that for $k = 1$, a k–tree is a usual tree, and a colored mixed graph with property P_1, is a colored mixed graph in which each vertex is incident with at least one arc of each type (color and orientation) and at least one edge of each color. In Section 3 we gave the colored mixed chromatic number of trees (Theorem 3.1); the upper bound was proved by exhibiting an (n, m)–colored mixed graph with property P_1 on $2n + m + \epsilon$ vertices; by the lower bound, it follows that this target graph is actually the smallest colored mixed graph with property P_1.

For $k = 2, 3, 4$ and 5, in the context of oriented graphs, some target graphs with property P_k have been found which improves the upper bound given in Theorem 5.1:

- The circulant tournament on 7 vertices QR_7, constructed by the non–zero quadratic residues, has property P_2 and is optimal since there are oriented partial 2–trees with oriented chromatic number 7 [59].
- The *Tromp graph* $Tr(QR_7)$ on 16 vertices, has property P_3 and is optimal since there are oriented partial 3–trees with oriented chromatic number 16 [59] (see the *Tromp construction* in Section 5.3.2).
- There exist oriented graphs with property P_k for $k = 4$ and 5, having respectively 40 and 96 vertices (these graphs are not known to be optimal) [2].

In order to prove our Theorem 5.3, we will next exhibit a 2–edge colored graph on 9 (resp. 20) vertices with property P_2 (resp. P_3); these target graphs are optimal by the lower bounds given in Proposition 5.1 and Proposition 5.2 respectively. We also will exhibit a mixed graph on 21 vertices and a 2–arc colored oriented graph on 28 vertices, both with property P_2; we can verify that these graphs improve the general upper bound given in Theorem 5.1, however we were not able to prove (or disprove) that they are

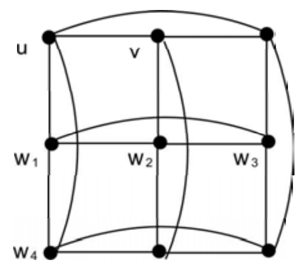


Figure 5.2: The graph $C_3\square C_3$ induced from one color in T_9.

optimal. To fine a colored mixed graph with property P_3 in the cases $(n,m) = (1,1)$ and $(n,m) = (2,0)$ which improves the general upper bound given in Theorem 5.1 is a challenging problem.

Now we proceed to define our target graphs.

5.3.1 A 2–edge colored graph on 9 vertices with property P_2

We define such a target graph T_9, by means of the graph $C_3\square C_3$ depicted in Figure 5.2. Observe this graph is self complementary (i.e. isomorphic to its complement graph). Thus, the complete graph of 9 vertices can be decomposed into two edge disjoint copies of $C_3\square C_3$.

- The 2–edge colored graph T_9, is the complete 2–edge colored graph on 9 vertices, where the edges of each color induce an isomorphic copy of $C_3\square C_3$.

Proposition 5.5 *The 2–edge colored graph T_9 has property P_2.*

Proof. First observe that each vertex in T_9 has a neighbor of each color. Now, we must show that for every edge (u,v) in T_9, and every pattern $\{t_1,t_2\}$ with $t_i \in \{1,2\}$, there exist a vertex $w \in V(T_9)$ such that $t(w,u) = t_1$ and $t(w,v) = t_2$. By the symmetry of T_9 is enough to verify this condition for the edge (u,v) in Figure 5.2. Observe that $\{t(w_i,u), t(w_i,v)\}$ where $1 \le i \le 4$ correspond to the 4 patterns $\{t_1,t_2\}$ with $t_i \in \{1,2\}$.
$\square$

5.3.2 A 2–edge colored graph on 20 vertices with property P_3

We define such a target graph by means of the so called *Tromp construction* proposed by Tromp [65]; this construction was already used to bound the oriented chromatic number

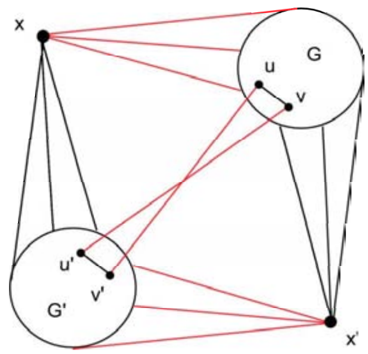


Figure 5.3: The $(0, 2)$–colored mixed Tromp graph $tr(G)$.

of partial 3–trees [59].

Let G be a 2–edge colored graph $G = (V(G), E_1(G), E_2(G))$, and G' be an isomorphic copy of G. The *Tromp 2–edge colored graph* $Tr(G) = (V(Tr(G)), E_1(Tr(G)), E_2(Tr(G)))$, has $2|V(G)| + 2$ vertices, and is defined as follows:

$V(Tr(G)) = V(G) \cup V(G') \cup \{x, x'\}$;

for every $u \in V(G)$, $ux, u'x' \in E_1(Tr(G))$ and $u'x, ux' \in E_2(Tr(G))$;

for every $uv \in E_i(G)$, $uv, u'v' \in E_i(Tr(G))$ and $u'v, uv' \in E_{3-i}(Tr(G))$.

Figure 5.3 illustrates the construction of $Tr(G)$.

- The 2–edge colored graph we consider, is the 2–edge colored graph on 20 vertices $T_{20} = Tr(T_9)$.

Proposition 5.6 *The 2–edge colored graph T_{20} has property P_3.*

Proof. We must show that for every triangle $[u_1, u_2, u_3]$ in T_{20}, and every pattern (t_1, t_2, t_3) with $t_i \in \{1, 2\}$, there exist a vertex $v \in V(T_{20})$ such that $t(v, u_i) = t_i$ for each $i \in \{1, 2, 3\}$. Let the vertex set $V(T_{20}) = V(T_9) \cup V(T_9') \cup \{x, x'\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \cup \{1', 2', 3', 4', 5', 6', 7', 8', 9'\} \cup \{x, x'\}$.

By the symmetry of T_{20} it is not difficult to verify that is sufficient to consider the following four cases:

(1) Suppose that $u_1, u_2, u_3 \in V(T_9)$. Is enough to consider the triangles $[1, 2, 3]$, $[1, 2, 4]$, $[1, 2, 5]$ and $[1, 2, 6]$.

(2) Suppose $u_1, u_2 \in V(T_9)$ and $u_3 \in V(T_9')$. Is enough to consider the triangles $[1, 2, 3']$, $[1, 2, 4']$, $[1, 2, 5']$ and $[1, 2, 6']$.

(3) Suppose $u_1, u_2 \in V(T_9)$ and $u_3 \in \{x, x'\}$. Is enough to consider the triangles $[1, 2, x]$ and $[1, 2, x']$.

(4) Finally suppose $u_1 \in V(T_9)$ $u_2 \in V(T_9')$ and $u_3 \in \{x, x'\}$. Is enough to consider the triangles $[1, 2', x]$ and $[1, 2', x']$.

The table below gives, for each case mentioned below and each pattern (t_1, t_2, t_3), the required vertex v.

	[123]	[124]	[125]	[126]	[123′]	[124′]	[125′]	[126′]	[12x]	[12x']	[12′x]	[12′x']
(1,1,1)	x'	x'	x'	x'	6′	3	3	9′	3	6′	4	5′
(1,1,2)	6′	3	3	9′	x'	x'	x'	x'	6′	3	5′	4
(1,2,1)	5′	7	5	4	4	5′	7	5′	4	5′	3	6′
(1,2,2)	4	5′	7	5′	5′	7	5	4	5′	4	6′	3
(2,1,1)	4′	5	7′	5	5	7′	4′	4′	5	4′	6	3′
(2,1,2)	5	7′	4′	4′	4′	5	7′	5	4′	5	3′	6
(2,2,1)	6	3′	3′	9	x	x	x	x	6	3′	5	4′
(2,2,2)	x	x	x	x	6	3′	3′	9	3′	6	4′	5

□

5.3.3 A mixed graph on 21 vertices with property P_2

A *circulant graph* $G = G(n; c_1, ..., c_d)$ has $V(G) = \{0, 1, ..., n-1\}$ and $(x, y) \in A(G)$ if and only if $y = x + c_i \pmod{n}$ for some i, $1 \le i \le d$. If n is a prime number of the form $4k+3$ and the c_i's are the non–zero quadratic residues of n, then $d = \lfloor (n-1)/2 \rfloor$ and $G = QR(n; c_1, ..., c_d)$ is a tournament.

We define our mixed graph on 21 vertices T_{21}, by means of the circulant tournament $QR(7; 1, 2, 4)$. It is not difficult to prove that this graph is arc transitive and has property P_2.

- The mixed graph T_{21}, is obtained by taking three copies of $QR(7; 1, 2, 4)$, named Q_1, Q_2 and Q_4. For each $i \in \{1, 2, 4\}$, we replace in Q_i the arcs of the form (x, y), $y = x + i \pmod{7}$, with simple edges (see Figure 5.4). Then, for distinct $i, j, k \in \{1, 2, 4\}$, we put an arc or an edge from $x \in QR_i$ to $y \in Q_j$ if the corresponding arc or edge is in Q_k.

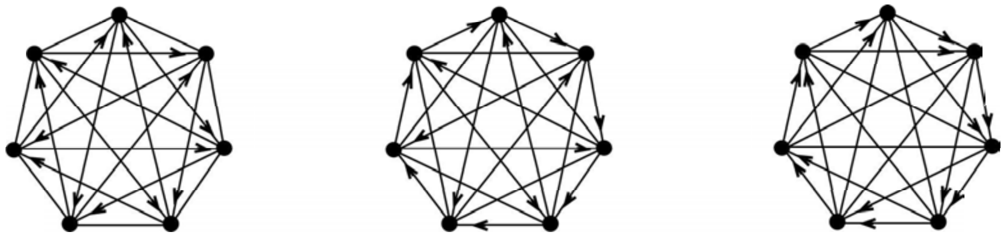


Figure 5.4: The mixed graphs Q_1, Q_2 and Q_4.

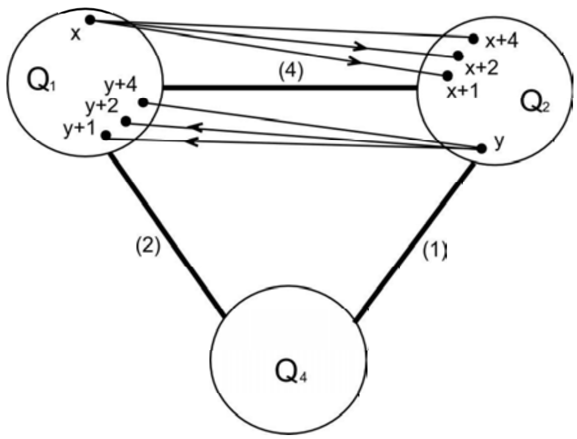


Figure 5.5: Construction for the mixed graph T_{21}.

Figure 5.5 illustrates this construction.

Proposition 5.7 *The mixed graph T_{21} has property P_2.*

Proof. We use a program to confirm that for every edge (u, v) in T_{21}, and every pattern $\{t_1, t_2\}$ with $t_i \in \{1, 2\}$, there exist a vertex $w \in V(T_{21})$ such that $t(w, u) = t_1$ and $t(w, v) = t_2$. □

5.3.4 A 2–arc colored oriented graph on 28 vertices with property P_2

We define such a target graph T_{28}, by means of the circulant tournament $QR(7; 1, 2, 4)$. First consider the edge–coloring of K_4 (with vertices a, b, c and d) given by:

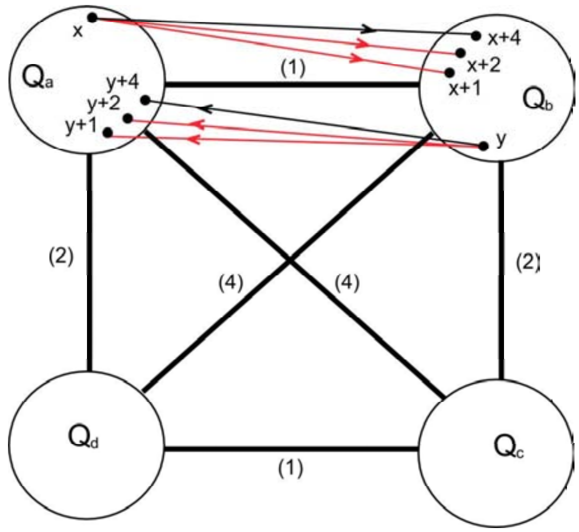


Figure 5.6: Construction for the 2–arc colored oriented graph T_{21}.

$$c(ab) = c(cd) = 1, c(ad) = c(bc) = 2, c(ac) = c(bd) = 4$$

- The 2–arc colored oriented graph T_{28}, is obtained by taking four copies of $QR(7; 1, 2, 4)$, named Q_a, Q_b, Q_c and Q_d; all arcs in this four copies are of color 1. Then, for distinct $j, k \in \{a, b, c, d\}$, we put an arc from $x \in Q_j$ to $y \in Q_k$ if the corresponding arc is in $QR(7; 1, 2, 4)$; if $y = x + c_i$ (c_i is the generator 1, 2 or 4) and $c(jk) = c_i$, then the arc is of color 1, otherwise is of color 2.

Figure 5.6 illustrates this construction.

Proposition 5.8 *The 2–arc colored oriented graph T_{28} has property P_2.*

Proof. We use a program to confirm that for every edge (u, v) in T_{28}, and every pattern $\{t_1, t_2\}$ with $t_i \in \{1, 2\}$, there exist a vertex $w \in V(T_{28})$ such that $t(w, u) = t_1$ and $t(w, v) = t_2$. □

Chapter 6

Planar and outerplanar graphs

6.1 Introduction

Certainly the most important family to study its chromatic number is the family of planar graphs. In the context of simple graphs, it took more than a hundred years to prove that every planar graph has chromatic number at most four. There is no doubt that *the four color conjecture* was the most famous problem in graph theory until it was solved in 1976 by Appel and Haken [5, 6].

In 1979, Borodin [10] proved a remarkable result showing that every planar graph has acyclic chromatic number at most five. This result together with Theorem 4.1 (the upper bound of the colored mixed chromatic number in terms of the acyclic chromatic number) give us an upper bound of the colored mixed chromatic number of $\mathcal{P}$, the family of planar graphs.

Theorem 6.1 (Nešetřil, Raspaud, [43]) $\chi_{(n,m)}(\mathcal{P}) \leq 5(2n+m)^4$.

This is the best known upper bound even for oriented graphs where the corresponding value is 80. In 1997 an oriented planar graph with oriented chromatic number at least 16 has been constructed [61]; this gap between the upper and the lower bound in the class of oriented graphs has not been reduced up to now. Therefore, to improve any of these bounds is an important challenge in the framework.

In the context of edge–colored graphs, Alon and Marshall [4] define a class of planar graphs, named *triangular graphs*, for which the $(0, m)$–colored mixed chromatic number is at least $m^3 + 3$. It is not difficult to generalize the argument they used, to establish the following lower bound in the more general context of colored mixed graphs:

$$\chi_{(n,m)}(\mathcal{P}) \geq (2n+m)^3 + 3. \tag{6.1}$$

In Section 6.2 we present the class of triangular graphs, and prove the lower bound given in (6.1). This was the best known lower bound for the colored mixed chromatic number of $\mathcal{P}$ [43]. However, we improve it (see Theorem 6.2 below) by means of the colored mixed chromatic number of the class of paths (Theorem 3.2); we also provide (by means of Theorem 3.2) a lower bound for the colored mixed chromatic number of $\mathcal{O}$, the class of outerplanar graphs. We will prove the following Theorem in section Section 6.3.

Theorem 6.2 *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even, then:*

1. $\chi_{(n,m)}(\mathcal{O}) \geq (2n+m)^2 + \epsilon(2n+m) + 1.$
2. $\chi_{(n,m)}(\mathcal{P}) \geq (2n+m)^3 + \epsilon(2n+m)^2 + (2n+m) + \epsilon$

Since the class of outerplanar graphs is strictly included in the class of partial 2–trees, by Theorem 5.1 we get: $\chi_{(n,m)}(\mathcal{O}) \leq 3(2n+m)^2$, thus by Theorem 6.2 we obtain the following:

Corollary 6.1 $\chi_{(n,m)}(\mathcal{O}) = \Theta((2n+m)^2)$

Concerning the class of planar graphs, it is widely believed that the real value of the oriented chromatic number $\chi_{(1,0)}(\mathcal{P})$, is closer to the known lower bound than the upper one. In the context of colored mixed graphs, by means of Theorems 6.1 and Theorem 6.2, we have:

Corollary 6.2 $\Omega((2n+m)^3) \leq \chi_{(n,m)}(\mathcal{P}) \leq \Theta((2n+m)^4)$

If we are not able to improve the known bonds for the colored mixed chromatic number of the class of planar graphs, at least to determine the asymptotic behavior is an interesting problem.

6.2 The class of triangular graphs

We define the class of *triangular graphs* Δ, which is a class of simple planar graphs, inductively defined as follows:

- The simple graph C_3 (a triangle) is in Δ.

  If G is in Δ, then the graph obtained by adding a new vertex adjacent to the three vertex of an existing face of G, is also in Δ.

By a simple counting argument we will show the next:

Proposition 6.1 $\chi_{(n,m)}(\Delta) \geq (2n+m)^3 + 3$.

Proof. Note that since $\Delta \subset \mathcal{P}$, by Theorem 6.1 Δ is colorable. Since Δ is a complete family of graphs, then by Proposition 2.2 in Section 2, Δ is optimally colorable. That is, there exist a Δ–universal graph of order $\chi_{(n,m)}(\Delta)$.

Now, let $\Delta_{(n,m)}$ be the class of colored mixed graphs having as underlying graph a graph in Δ. Let H be a colored mixed graph which is universal for $\Delta_{(n,m)}$. We suppose for contradiction that $|H| < (2n+m)^3 + 3$. Let H_0 be the underlying simple graph of H. For each $G \in \Delta$ let $h(G)$ be the set of homomorphisms from G to H_0, and $c(H)$ be the set of (n,m)–colored mixed graphs for which the underlying graphs is G. We shall note that each $h \in h(G)$ induces a unique $G^* \in c(G)$ for which h is also a colored homomorphism (from G^* to H). This gave a mapping $h(G) \rightarrow c(H)$ which, by assumption, is onto. We thus have:

$$|c(G)| \leq |h(G)|. \tag{6.2}$$

Now construct the graph $G' \in \Delta$ by subdividing a face of G. A homomorphism in $h(G)$ can be extended to G' in at most $(2n+m)^3 - 1$ ways (the image of the new vertex must differ from that of its three neighbors and $|H| < (2n+m)^3 + 3$) so that, $|h(G')| \leq (2n+m)^3 - 1|h(G)|$. On the other hand, each of the three new edges in G' can be oriented and/or colored in $2n+m$ ways, so that $|c(G')| \leq (2n+m)^3|c(G)|$. Hence, by repeatedly subdividing, we obtain a graph $G'' \in \Delta$ for which $|c(G'')| > |h(G'')|$, contrary to (6.2).

□

6.3 The lower bound giving by means of colored mixed paths

In this Section we will prove Theorem 6.2. We will use the constructions described in Section 5.2 in order to exhibit outerplanar and planar graphs with the require colored mixed chromatic number. Recall $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.

Proof of Theorem 6.2(1):- Observe that if G is a path then G' is an outerplanar graph. Thus, starting with an (n,m)–colored mixed path with chromatic number at least $2n+m+\epsilon$ (provided by Theorem 3.2), we get (according to Remark 5.1) an (n,m)–colored mixed outerplanar graph with chromatic number at least $(2n+m)(2n+m+\epsilon)+1$ which is precisely $(2n+m)^2+\epsilon(2n+m)+1$. □

Proof of Theorem 6.2(2):- Observe that if G is a outerplanar graph then G' is a planar graph. Thus, starting with an (n,m)–colored mixed outerplanar graph with chromatic number at least $(2n+m)^2+\epsilon(2n+m)+1$ (provided by Theorem 6.2(1)), we get (according to Remark 5.1) an (n,m)–colored mixed planar graph with chromatic number at least $(2n+m)((2n+m)^2+\epsilon(2n+m)+1)+1$ which is:

$$(2n+m)^3+\epsilon(2n+m)^2+(2n+m)+1. \tag{6.3}$$

Now look that, for $k=3$ and $m>0$ even, the partial 3–tree $(T^3)''$ constructed in the proof of Proposition 5.2 is also a planar graph; thus we improve by one the bound in (6.3) when $m>0$ even, and obtain the require bound. □

Chapter 7

Sparse planar graphs

7.1 Introduction

Before the 4–*color theorem* (every simple planar graph is 4–colorable) was proved in 1976 by Appel and Haken [5, 6], Grötzsch [24] proves in 1959 his celebrated 3–*color theorem*: every simple triangle–free planar graph is 3–colorable. Since the complete graph K_4 is a planar graph with chromatic number 4, and odd cycles have chromatic number 3, we have the following classification.

Theorem 7.1 *Let $\mathcal{P}_g$ be the class of planar graphs with girth at least g, then:*

1. $\chi_{(0,1)}(\mathcal{P}_3) = 4$ *(Appel, Haken [5, 6], 1976).*
2. $\chi_{(0,1)}(\mathcal{P}_g) = 3$ *for every* $g \geq 4$ *(Grötzsch [24], 1959).*

Motivated by this result, Nešetřil, Raspaud and Sopena [44] propose the so-called *Girth Problem* for oriented planar graphs. In the general context of colored mixed graphs, the problem is the following.

The Girth Problem: Determine the quantity $\chi_{(n,m)}(\mathcal{P}_g)$ for every $g > 2$.

The *Girth Problem* is then completely solved for simple graphs. In the oriented case, it appears that *The Girth Problem* is much harder and presently seemingly hard to solve. Certainly, it is shown that the oriented chromatic number of planar graphs can be significantly lowered under a high girth assumption, but a lot of work still remains to be done, in order to completely solve the problem. Next we summarize what is known.

Theorem 7.2 *Let $\mathcal{P}_g$ be the class of planar graphs with girth at least g, then:*

1. $\chi_{(1,0)}(\mathcal{P}_3) \leq 80$ *(Raspau, Sopena [53] 1994).*
2. $\chi_{(1,0)}(\mathcal{P}_4) \leq 47$ *(Borodin, Ivanova [12] 2005).*
3. $\chi_{(1,0)}(\mathcal{P}_5) \leq 16$ *(Pinlou [50] 2008).*
4. $\chi_{(1,0)}(\mathcal{P}_6) \leq 11$ *(Borodin, Kostochka, Nešetřil, Raspaud, Sopena [14] 1999).*
5. $\chi_{(1,0)}(\mathcal{P}_7) \leq 7$ *(Borodin, Ivanova [11] 2005).*
6. $\chi_{(1,0)}(\mathcal{P}_{11}) \leq 6$ *(Pinlou [51] 2006).*
7. $\chi_{(1,0)}(\mathcal{P}_g) = 5$ *for every* $g \geq 14$ *(Nešetřil, Raspaud, Sopena [44] 1997, Borodin, Kostochka, Nešetřil, Raspaud, Sopena [14] 1999).*

Restricted to outerplanar graphs (still in the oriented case), Pinlou and Sopena gave the complete classification of the oriented chromatic number of outerplanar graphs with a given girth.

Theorem 7.3 *Let $\mathcal{O}_g$ be the class of outerplanar graphs with girth at least g, then:*

1. $\chi_{(1,0)}(\mathcal{O}_3) = 7$ *(Sopena [59] 1997).*
2. $\chi_{(1,0)}(\mathcal{O}_4) = 6$ *(Pinlou, Sopena [52] 2006).*
3. $\chi_{(1,0)}(\mathcal{O}_g) = 5$ *for every* $g \geq 5$ *(Pinlou, Sopena [52] 2006).*

Concerning partial 2–trees (which is also a subclass of planar graphs which strictly contains outerplanar graphs), Ochem and Pinlou [48] recently gave the analogous classification.

Theorem 7.4 *Let $\mathcal{T}^2_g$ be the class of partial 2–trees with girth at least g, then:*

1. $\chi_{(1,0)}(\mathcal{T}^2_g) = 7$ *for* $3 \leq g \leq 4$ *(Sopena [59] 1997, Pinlou, Sopena [52] 2006).*
2. $\chi_{(1,0)}(\mathcal{T}^2_g) = 6$ *for* $5 \leq g \leq 6$ *(Ochem, Pinlou [48] 2007).*
3. $\chi_{(1,0)}(\mathcal{T}^2_g) = 5$ *for every* $g \geq 7$ *(Ochem, Pinlou [48] 2007).*

To summarize, *The Girth Problem* for planar graphs in the oriented case, is solved when restricted to certain subclasses (outerplanar graphs and partial 2–trees) and for planar graphs with girth $g \geq 14$ (Theorem 7.2–7).

We shall note that the condition $g \geq 14$ in Theorem 7.2–7, may not be the best possible, since no planar graph with girth 13 and oriented chromatic number strictly greater than 5 is known. Nevertheless, this has not been improved up to now. Just, concerning lower bounds, it is shown a planar graph with girth 7 and oriented chromatic number strictly grater that 5 [44]; thus the gap between girths 8 and 13 provides a challenge.

Even so, the authors in [44] viewed Theorem 7.2.7 as yet another 5–*color Theorem* for high girth (sparse) oriented planar graphs. In this paper, following this view, we prove a $(4n+2m+1)$–*color Theorem* for high girth (n,m)–colored mixed planar graphs. In fact we give the colored mixed chromatic number of $\mathcal{P}_g$, $\mathcal{T}_g^2$ and $\mathcal{O}_g$, for a suitable sufficiently large girth in each case. The following is our main result.

Theorem 7.5 *Let $\mathcal{P}_g$, $\mathcal{T}_g^2$ and $\mathcal{O}_g$ be the classes of planar graphs, partial 2–trees, and outerplanar graphs with girth at least g respectively. Then:*

1. $\chi_{(n,m)}(\mathcal{P}_g) = 2(2n+m)+1$ *for every* $g \geq 20n+10m-4$.
2. $\chi_{(n,m)}(\mathcal{T}_g^2) = 2(2n+m)+1$ *for every* $g \geq 8n+4m-1$.
3. $\chi_{(n,m)}(\mathcal{O}_g) = 2(2n+m)+1$ *for every* $g \geq 4n+2m+\beta$, *where* $\beta = 0$ *if* $n = 0$, *and* $\beta = 1$ *otherwise.*

In the previous Section we give general upper and lower bounds for the colored mixed chromatic number of planar graphs, outerplanar graphs and partial 2–trees (see Theorems 5.1, 6.1 and 6.2). We shall note that, for the class of planar graphs, the main term in the lower bound is $(2n+m)^3$ while the main term in the upper bound is $(2n+m)^4$; concerning partial 2–trees and outerplanar graphs, the main term in both lower and upper bounds is $(2n+m)^2$. However, Theorem 7.5 states that, if the girth is sufficiently large, the colored mixed chromatic number of these three classes of graphs becomes linear on $2n+m$.

Most of the techniques used to prove the upper bounds in Theorem 7.2, substantially use the property that every planar graph with large girth, contains either a vertex with degree one or a long path whose internal vertices have degree two. Motivated by this fact, Hell, Kostochka, Raspaud and Sopena [27], introduced the concept of *nice* digraph and *nice* edge colored multigraph. We adapt this concept to the context of colored mixed graphs. In Section 7.2 we give the appropriate definitions and describe the target graphs that we will use to show the upper bounds, in Section 7.3. Finally, in Section 7.4 we prove the lower bounds by constructing, for every $g > 2$, a colored mixed outerplanar graph with girth g and chromatic number $2(2n+m)+1$.

7.2 Nice colored mixed graphs

In this Section we give the definition of a *k–nice* and *k–quasi–nice* colored mixed graph and, after seen some examples, we will define the target graph which allows us to prove Theorem 7.5.

We will use the following notation. The degree of a vertex u is denoted by $d(u)$ and the minimum degree of a graph is denoted by $\delta(G)$. A *k-vertex* is a vertex of degree k, and a *k-path* is a path $P = u_0, u_1 \dots u_k$ of length k (i.e. formed by k edges), with 2 *end-vertices* (u_0 and u_k) and $k-1$ *internal vertices* (all u_i's with $1 \leq i \leq k-1$). A $(k,2)$*-path* in a graph G, is a k–path in which all internal vertices have degree 2. A *pattern* $\mathcal{Q} = q_1 q_2 ... q_k$ of length k is a (non-empty) word in $\{1, ..., 2n+m\}^k$. A $\mathcal{Q}$*–walk* is a walk $W = u_0, u_1, ..., u_k$ such that for every $1 \leq i \leq k$ the type $t(u_{i-1}, u_i) = q_i$. For $X \subset V(G)$ we denote by $N_{\mathcal{Q}}(X)$ the set of all vertices $v \in V(G)$ such that there is a $\mathcal{Q}$–walk going from some vertex $u \in X$ to v. If $\mathcal{Q} = q_1$ is a pattern of length one, we write $N_{q_1}(X)$; we also use $N_{\mathcal{Q}}(u)$ instead of $N_{\mathcal{Q}}(\{u\})$.

Definition 7.1 *An (n,m)–colored mixed graph G is k–**nice** (resp. k–**quasi–nice**) if for every pair of vertices $v, w \in V(G)$ allowing (resp. not allowing) $v = w$, and for each (n,m)–colored mixed path $P_k = u_0, u_1, ..., u_k$ of length k, there exist a homomorphism h from P_k to G such that $h(u_0) = v$ and $h(u_k) = w$.*

In other words, an (n,m)–colored mixed graph G is k–nice (resp. k–quasi–nice), if for every pattern $\mathcal{Q}$ of length k and every vertex $u \in V(G)$ we have $N_{\mathcal{Q}}(u) = V(G)$. (resp. $V(G) \setminus \{u\} \subseteq N_{\mathcal{Q}}(u)$).

Since colored mixed graphs have neither loops nor multiple edges or arcs, then there are neither 1–nice nor 2–nice colored mixed graphs. Also, the only possibility to have a 1–quasi–nice colored mixed graph is when $(n,m) = (0,1)$ (actually, just complete simple graphs are 1–quasi–nice).

Notice that, for every $k \geq 2$, a k–quasi–nice colored mixed graph is also $(k+1)$–nice; and, for every $k \geq 3$, a k–nice colored mixed graph is also k–quasi–nice. Hence, if a colored mixed graph is 2–quasi–nice it is also 3–nice and, for every $k \geq 3$ we have:

$$k\text{–nice} \Rightarrow k\text{–quasi–nice} \Rightarrow (k+1)\text{–nice}$$

Thus, the strongest assumption in this context, is to have a 2–quasi–nice colored mixed graph. Let us see some examples of nice and quasi–nice colored mixed graphs:

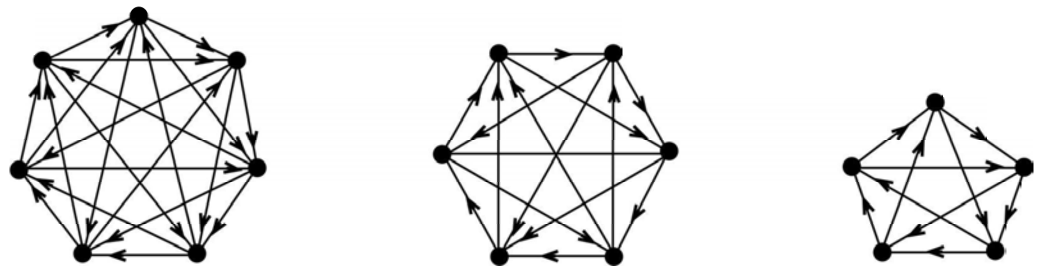


Figure 7.1: The oriented target graphs $T_7(1,0)$, $T_6(1,0)$ and $T_5(1,0)$ which are 2–quasi–nice, 3–nice and 4–nice respectively.

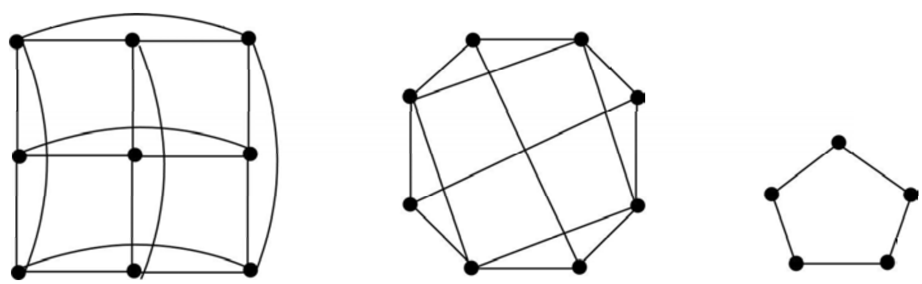


Figure 7.2: The graphs induced by one color of the target graphs $T_9(0,2)$, $T_8(0,2)$ and $T_5(0,2)$, which are 2–quasi–nice, 3–nice, and 3–quasi–nice respectively.

- In the oriented case, the target graphs depicted in Figure 7.1 were used in the literature to bound the oriented chromatic number of the families of outerplanar graphs and partial 2–trees with given girth ([59, 52, 48]); it was proved that $T_7(1,0)$ (resp. $T_6(1,0)$, $T_5(1,0)$) is 2–quasi–nice (resp. 3–nice, 4–nice).
- In the case of 2–edge colored graphs, the analogous target graphs are the following. Observe that the graphs depicted in Figure 8.1 are all self complementary (i.e. isomorphic to their complements). Thus, let $T_9(0,2)$ (resp. $T_8(0,2)$, $T_5(0,2)$) be the complete $(0,2)$–colored mixed graph on 9 (resp. 8, 5) vertices where the edges of each color induces an isomorphic copy of the first (resp. second, third) graph in Figure 8.1. It appears that $T_9(0,2)$ (resp. $T_8(0,2)$, $T_5(0,2)$) is 2–quasi–nice (resp. 3–nice, 3–quasi–nice); we will prove these claims in Propositions 7.1, 8.2 and 7.2 respectively. Then, we will use these target graphs in Section 8, to bound the oriented chromatic number of the families of outerplanar graphs and partial 2–trees with given girth.

Observe that in both cases the colored mixed graphs which are 2–quasi–nice have also property P_2; this is always the case since:

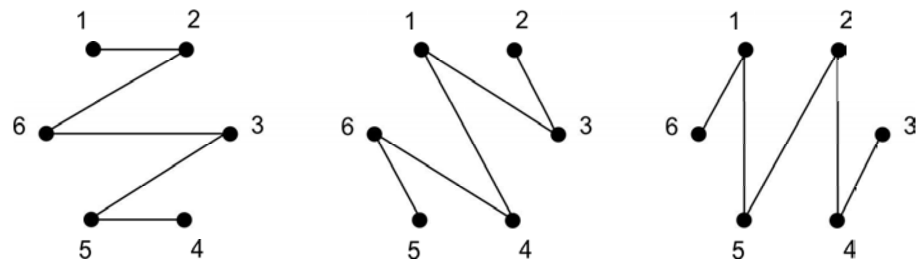


Figure 7.3: Standard path decomposition of K_6.

Proposition 7.1 *A colored mixed graph is 2–quasi–nice if and only if satisfies property P_2.*

Proof. Recall that, by definition, a colored mixed graph G satisfies property P_2 if both of the following conditions holds:

(i) for every vertex $v \in V(G)$ and every type $t \in \{1, ..., 2n+m\}$, there exist a vertex $u \in V(G)$ such that $t(u, v) = t$.

(ii) for every ordered pair of adjacent vertices (v_1, v_2) in G, and every sequence of types $\{t_1, t_2\}$, there exist $u \in V(G)$ such that u is adjacent to both v_1 and v_2 with $t(u, v_1) = t_1$ and $t(u, v_2) = t_2$.

Thus clearly, a colored mixed graph G satisfying property P_2, is such that for every pair of different vertices $v_1, v_2 \in V(G)$, and every colored mixed path $P = u_0u_1u_2$ of length 2, there is a homomorphism h from P to G such that $h(u_0) = v_1$ and $h(u_2) = v_2$. On the other hand, if G is 2–quasi–nice then clearly satisfies conditions (i) and (ii). □

Next we describe an (n, m)–colored mixed graph $T_{4n+2m+1} := T_{4n+2m+1}(n, m)$ of order $4n + 2m + 1$ which is $(4n + 2m)$–nice and $(2m - 1)$–quasi–nice when $n = 0$.

We will use the following (standard) decomposition of the complete graph K_{2k+1} into k edge disjoint Hamiltonian cycles. Place $2k$ vertices on a regular $2k$-gon numbered cyclically $v_1, v_2, \ldots, v_{2k}$ and a vertex v_0 in its center. The set of edges parallel to v_1v_2 together with the set of edges parallel to v_2, v_{2k} form a hamiltonian path on the vertices of the polygon (see Figure 7.3) and, together with the edges v_0v_1 and $v_{k+1}v_0$, a hamiltonian cycle, call it H_1. By rotating this hamiltonian cycle by an angle of $j\pi/k$ we obtain a hamiltonian cycle H_j and $H_1, H_2, \ldots, H_k$ form an edge decomposition of K_{2k+1}. Denote by π the permutation with v_0 fixed and shifting cyclically the vertices $v_1, \ldots, v_{2k}$, so that π^j is an automorphism which sends H_1 to H_j.

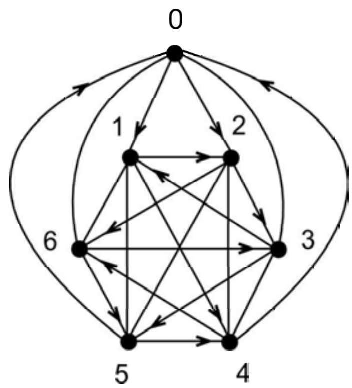


Figure 7.4: The target graph $T_{4n+2m+1}$ for $(n, m) = (1, 1)$ which is universal for the family of sparse planar graphs.

We now define the target graph $T_{4n+2m+1}$. It is a complete (n, m)–colored mixed graph, whose underlying graph is the complete graph $K_{4n+2m+1}$, and its colored edges and colored arcs are given as follows.

- Let $H_1, H_2, \ldots, H_{2n+m}$ be the decomposition of $K_{4n+2m+1}$ into hamiltonian cycles as described above. For $i = 1, \ldots 2n$, the arcs of H_i are colored by color $\lceil i/2 \rceil$, and orient the cycles $H_1, H_2, \ldots, H_{2n}$ as to obtain hamiltonian oriented cycles with the edge $v_0 v_i$ oriented from v_0 to v_i. For $i = 2n + 1, \ldots, 2n + m$ the edges of H_i are colored $i - 2n$.

Thus we obtain 2 monochromatic Hamiltonian circuits for each color $i \in \{1, \ldots n\}$ and one monochromatic Hamiltonian cycle for each color $i \in \{1, \ldots m\}$. In Figure 7.4 you can see this target graph for $(n, m) = (1, 1)$. Note that each vertex in $T_{4n+2m+1}$ is incident with two arcs of each type (orientation and color) and with two edges of each color.

In order to prove that $T_{4n+2m+1}$ is $(4n + 2m)$–nice, we will use the following Lemma.

Lemma 7.1 *Let X be a subset of $V(T_{4n+2m+1})$. For every type $t \in \{1, ..., 2n + m\}$ we have*

$$|N_t(X)| \geq \min\{|X| + 1, 4n + 2m + 1\}.$$

Proof. Let $M = 4n + 2m + 1$. If $|X| = M$ then clearly $|N_t(X)| = M$ for every $t \in \{1, ..., 2n + m\}$. If $|X| < M$ then, consider first the case when $t \in \{2n + 1, ..., 2n + m\}$ (i.e. t corresponds to a colored edge), then $N_t(X)$ is the neighborhood of X in the subgraph spanned by the edges of color $t - 2n$ in T_M. Since this subgraph is a cycle of length M, then we have $|N_t(X)| \geq |X|+1$. Suppose now that t corresponds to an oriented type. The subgraph spanned by the arcs colored by t is isomorphic to $H_1 \cup H_2$. Since each of H_1 and H_2 is a hamiltonian directed cycle, $|N_{H_i}(X)| = |X|$, $i = 1, 2$. By the construction of the decomposition, H_2 is obtained from H_1 by cyclically rotating the vertices $v_1, v_2, \ldots, v_{2k}$ and leaving v_0 fixed. Therefore we could only have $N_{H_1}(X) = N_{H_2}(X)$ if both sets equal $\{v_0\}$, which is not the case for any subset X. Therefore $|N_t(X)| = |N_{H_1}(X) \cup N_{H_2}(X)| = |N_{H_1}(X)| + |N_{H_2}(X) \setminus N_{H_1}(X)| \geq |X| + 1$. □

Now we are ready to prove that $T_{4n+2m+1}$ is $(4n+2m)$–nice. In fact, we will prove a stronger statement.

Lemma 7.2 *For every positive integer k, every pattern $\mathcal{Q}$ of length k, and every vertex v in $V(T_{4n+2m+1})$, we have*

$$|N_{\mathcal{Q}}(v)| \geq \min\{k+1, 4n+2m+1\}.$$

Proof. The proof proceeds by induction on k. Let $M = 4n+2m+1$. If $k = 1$ the result follows from Lemma 7.1. Now let $\mathcal{Q} = q_1q_2...q_{k-1}q_k$ be a pattern of length $k < M$, and let $\mathcal{Q}' = q_1q_2...q_{k-1}$. By Lemma 7.1 and the induction hypothesis,

$$|N_{\mathcal{Q}}(v)| = |N_{q_k}(N_{\mathcal{Q}'}(v))| \geq \min\{|N_{\mathcal{Q}'}(v)|+1, M\} \geq \min\{|\mathcal{Q}|+1, M\}$$

Similarly, if $k = M$ then $|N_{\mathcal{Q}}(v)| \geq |N_{\mathcal{Q}'}(v)| = M$. □

As a consequence of Lemma 7.2 we obtain the following,

Proposition 7.2 *The (n,m)–colored mixed graph $T_{4n+2m+1}$ is $(4n+2m)$–nice, and moreover, is $(2m-1)$–quasi–nice when $n = 0$.*

Proof. We can see that $T_{4n+2m+1}$ is $(4n+2m)$–nice by taking $k = (4n+2m)$ in Lemma 7.2. To see that $T_{4n+2m+1}$ is $(2m-1)$–quasi–nice for $n = 0$, observe that for every pattern $\mathcal{Q}$ of length $2m-1$, and every vertex v in $V(T_{2m+1})$, $|N_{\mathcal{Q}}(v)| \geq 2m$ by Lemma 7.2. Moreover, $v \notin N_{\mathcal{Q}}(v)$ since Q has odd length and thus $N_{\mathcal{Q}}(v) = V(T_{2m+1}) \setminus \{v\}$. □

7.3 The upper bounds

Now we prove the upper bounds of Theorem 11.1. More precisely, we prove that every (n,m)–colored mixed planar graph, outerplanar graph, or partial 2–tree, with the required girth, admits a homomorphism to $T_{4n+2m+1}$. We will use the following propositions which have already been used to bound the oriented chromatic number of planar graphs, outerplanar graphs and partial 2–trees with given girth.

Proposition 7.3 ([44]) *Every planar graph with girth g, contains either a 1–vertex or a $(\frac{g+4}{5}, 2)$-path.*

Proposition 7.4 ([48]) *Every partial 2–tree with girth g, contains either a 1–vertex or a $(\lceil \frac{g}{2} \rceil, 2)$-path.*

Proposition 7.5 ([52]) *Every outerplanar graph with girth g, contains either a 1–vertex or a $(\ell - 1, 2)$–path, for some $\ell \geq g$, in which the end-vertices is an adjacent pair.*

Our claim is a direct consequence of Proposition 7.2 and Lemma 7.3 below. An (n, m)–colored mixed graph U is said to be *$\mathcal{F}$–universal*, if every (n, m)–colored mixed graph whose underlying graph is in $\mathcal{F}$, admits a colored homomorphism to U.

Lemma 7.3 .

1. *If H is a k–nice colored mixed graph, then H is $\mathcal{P}_{5k-4}$–universal (resp. $\mathcal{T}^2_{2k-1}$–universal).*
2. *If H is a k–quasi–nice colored mixed graph, then H is $\mathcal{O}_{k+1}$–universal*

Proof.

1. Let H be a k–nice (n, m)–colored mixed graph. We prove that every (n, m)–colored mixed planar graph (resp. partial 2–tree) with girth at least $5k - 4$ (resp. $2k - 1$) admits a homomorphism to H. Note that is sufficient to consider the case $g = 5k-4$ (resp. $g = 2k - 1$), since $P_{g+1} \subset P_g$ for every g. Let G be a minimal (with respect to the number of vertices) counter–example. We show that G contains neither a 1–vertex, nor a $(k, 2)$–path.
   - Suppose that G contains a 1–vertex u. By the minimality of G, the graph $G' = G \setminus \{u\}$, admits a homomorphism h to H. Since H is nice, every vertex in H has at least one neighbor of every type. Thus h can be extended to G.
   - Suppose that G contains a $(k, 2)$–path $P = u, v_1 \dots v_{k-1}, w$. By the minimality of G, the graph $G' = G \setminus \{v_1, v_2 ... v_{k-1}\}$ admits a homomorphism to H. Since H is k–nice, this homomorphism can be extended to G.

We thus get a contradiction by Proposition 7.3 (resp. Proposition 7.4).

2. Let H be a k–quasi–nice (n, m)–colored mixed graph. We prove that every (n, m)–colored mixed outerplanar graph with girth at least $k+1$ admits a homomorphism to H. Note that it is sufficient to consider the case $g = k+1$, since $O_{g+1} \subset O_g$ for every g. Let G be a minimal (with respect to the number of vertices) counter–example. We show that G contains neither a 1–vertex, nor a $(k, 2)$–path whose end-vertices are adjacent.

   - Suppose that G contains a 1–vertex u. By the minimality of G, the graph $G' = G \setminus \{u\}$, admits a homomorphism h to H. Since H is quasi–nice, every vertex in H has at least one neighbor of every type. Thus h can be extended to G.
   - Suppose that G contains a $(k, 2)$–path $P = u, v_1 \dots v_{k-1}, w$ such that u and w are adjacent. By the minimality of G, the graph $G' = G \setminus \{v_1, v_2 ... v_{k-1}\}$ admits a homomorphism h to H. Since u and w are adjacent, then $h(u) \neq h(w)$. Hence, since H is k–quasi–nice, h can be extended to G.

   We thus get a contradiction by Proposition 7.5.

□

7.4 The lower bounds

Now we prove the lower bounds of Theorem 11.1. We will construct, for every $g \geq 3$, an (n, m)–colored mixed outerplanar graph $O_g(n, m)$ with girth g which has no homomorphism to any (n, m)–colored mixed graph on $4n + 2m$ vertices. Since $\mathcal{O} \subset \mathcal{T}^2 \subset \mathcal{P}$, this will complete the proof.

For each pattern $\mathcal{Q} = q_1 q_2 ... q_k$ of length k we consider the (n, m)–colored mixed path $P_k(\mathcal{Q}) = u_0, u_1, ..., u_k$ of length k such that for every $1 \leq i \leq k$ the type $t(u_{i-1}, u_i) = q_i$. We now define the colored mixed outerplanar graph $O_g(n, m)$.

- For each of the $(2n+m)^{\lfloor \frac{g}{2} \rfloor}$ distinct patterns of length $\lfloor \frac{g}{2} \rfloor$, we take two copies P, P' of $P_{\lfloor \frac{g}{2} \rfloor}(\mathcal{Q}_i)$, $1 \leq i \leq (2n+m)^{\lfloor \frac{g}{2} \rfloor}$. We identify all the $2(2n+m)^{\lfloor \frac{g}{2} \rfloor}$ initial u_0–vertices, and connect (with an edge or an arc of an arbitrary color) the two end $u_{\lfloor \frac{g}{2} \rfloor}$–vertices of each pair of $P_{\lfloor \frac{g}{2} \rfloor}(\mathcal{Q}_i)$. The graph obtained in this way is $O_g(n, m)$ if g is odd. For even g we glue a cycle of length g to an arbitrary vertex.

Figure 7.5 shows this outerplanar colored mixed graph for $(n, m) = (1, 1)$ and $g = 4$ and 5.

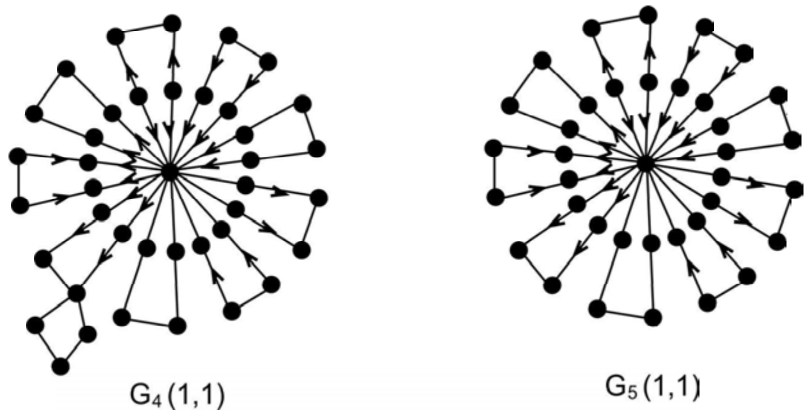


Figure 7.5: Examples of the outerplanar graph $O_g(n,m)$ with girth g and colored mixed chromatic number $4n+2m+1$.

Now we prove that $O_g(n,m)$ admits no homomorphism to any (n,m)–colored mixed graph H on $4n+2m$ vertices. Let v be the vertex of $O_g(n,m)$ with maximum degree (the central vertex). Suppose that there exists a homomorphism $h: O_g(n,m) \to H$ and $h(v) = u$. Observe that every vertex in H has degree at most $4n+2m-1$; hence, for every vertex $x \in V(H)$ there exists a type $t(x) \in \{1, ..., 2n+m\}$ such that $|N_{t(x)}(x)| \leq 1$. Let Q be the walk in H starting in $u_0 = u = h(v)$ obtained by selecting the unique $t(u_i)$–neighbour u_{i+1} of u_i. Either Q is a finite path (if for some u_i, $t(u_i)$ is such that $|N_{t(u_i)}(u_i)| = 0$), or Q contains a cycle (if for some u_i, $N_{t(u_i)}(u_i) = u_j$ for some $j < i$). If Q is a finite path of length $< \lfloor \frac{g}{2} \rfloor$, then there is a pair of paths P, P' in $O_g(n,m)$ which cannot be mapped into H. If Q is a finite path of length $> \lfloor \frac{g}{2} \rfloor$ or Q contains a cycle, then there is a pair of paths P, P' in $O_g(n,m)$, so that h must map each of P and P' into Q, but then the adjacent end-vertices of P and P' must be mapped to the same vertex, a contradiction.

Chapter 8

The case of 2–edge colored graphs

8.1 Introduction

In this Section we focus on the class of 2–edge colored graphs which are $(0, 2)$–colored mixed graphs. We mainly study three particular classes of graphs: planar graphs, outerplanar graphs and partial 2–trees. Indeed, we give the complete classification on the $(0, 2)$–colored mixed chromatic number of outerplanar graphs and partial 2–trees with a given girth; we also obtain upper bounds for the colored mixed chromatic number of planar graphs with given girth.

We shall note that a complexity result of Edwards and McDiarmid [20] on the harmonious chromatic number implies that to find the $(0, 2)$–colored mixed chromatic number of a graph is, in general, an NP–complete problem.

Our results in this Section are the following.

Theorem 8.1 *Let $\mathcal{O}_g$ be the class of outerplanar graphs with girth at least g, then:*

1. $\chi_{(0,2)}(\mathcal{O}_3) = 9$.
2. $\chi_{(0,2)}(\mathcal{O}_g) = 5$ *for every* $g \geq 4$.

Theorem 8.2 *Let $\mathcal{T}_g^2$ be the class of partial 2–trees with girth at least g, then:*

1. $\chi_{(0,2)}(\mathcal{T}_3^2) = 9$.
2. $\chi_{(0,2)}(\mathcal{T}_g^2) = 8$ *for* $4 \leq g \leq 5$.

3. $\chi_{(0,2)}(\mathcal{T}_g^2) = 5$ *for every* $g \geq 6$.

Theorem 8.3 *Let* $\mathcal{P}_g$ *be the class of planar graphs with girth at least* g*, then:*

1. $\chi_{(0,2)}(\mathcal{P}_5) \leq 20$.
2. $\chi_{(0,2)}(\mathcal{P}_6) \leq 12$.
3. $\chi_{(0,2)}(\mathcal{P}_8) \leq 8$.
4. $\chi_{(0,2)}(\mathcal{P}_g) = 5$ *for every* $g \geq 14$.

In Section 8.3 we describe the target graphs that we will use to prove our results. We will use the so called method of *reducible configurations* and *discharging procedure* used in particular by Appel an Haken [5] in their proof of the four color theorem, and proposed by Borodin et al. [14] to bound the oriented chromatic number of sparse planar graphs. In Section 8.2 we will explain in general this method and Sections 8.4, 8.5 and 8.6 are devoted to the proofs of Theorems 8.1, 8.2 and 8.3 respectively.

8.2 Reducible configuration and discharging procedure

In order to prove Theorems 8.2 and 8.3 we will use the so called method of *reducible configurations* and *discharging procedure*. Next we will explain this method in general.

Suppose that we want to prove: $G \in \mathcal{C} \Rightarrow G \rightarrow H$, where $\mathcal{C}$ is a graph class and H is a suitable target graph with "useful" properties. The method of *reducible configurations* is the following:

1. Define a partial order $\prec$ that extends the subgraph partial order.
2. Consider a potential counter–example $G \in \mathcal{C}$ with $G \nrightarrow H$ which is minimal with this property according to $\prec$. Then, every $G' \prec G$ satisfies $G' \rightarrow H$.
3. Provide a "small" set S of *forbidden configurations* (i.e. a set of configurations that G can not contain due to its minimality property). To show that $C \in S$, we suppose that G contains C, and find a suitable proper subgraph G' of G such that, by the "useful" properties of the target graph H, every H–coloring of G' can be extended to an H–coloring of G. This is a contradiction since G' suppose to be H–colorable but not G.

4. Show that no counter–example exist since every graph in $\mathcal{C}$ contains at least one configuration in S.

There are several ways to do this last step. We now present the *discharging procedure* proposed by Borodin et al. in [14] and used in some other papers [46, 50] to bound the oriented colored chromatic number of planar graphs with given girth. It involve the following auxiliary graph parameter.

The *average degree* of a simple graph G, denoted by $\mathrm{ad}(G)$, is defined as twice the number of edges over the number of vertices ($\mathrm{ad}(G) = \frac{2|E(G)|}{|V(G)|}$). The *maximum average degree* of a simple graph G, denoted by $\mathrm{mad}(G)$, is then defined as the maximum of the averages degrees taken over all subgraphs of G:

$$\mathrm{mad}(G) = \max_{H \subseteq G}\{ad(H)\}$$

Assume that the graph class $\mathcal{C}$ is such that $G \in \mathcal{C}$ if and only if $\mathrm{mad}(G) < q$. We assign to every vertex v of G an initial charge equal to its degree $d(v)$, and define a *discharging procedure* which specifies some transfer of values among the vertices in G keeping the sum of all the values constant. Then we show that if the discharging procedure is applied to a graph G avoiding the configurations in S, then the final charge $d^*(v)$ of every vertex in G is greater than q. This shows that every graph in $\mathcal{C}$ contains at least one configuration in S, and thus no potential counter–example exist.

In Sections 8.5 and 8.6 we will use the following:

Drawing conventions: In all the figures depicting forbidden configurations, all neighbors of "white" vertices are drawn, while "black" vertices may have other neighbors in the graph. Also, two or more "black" vertices may coincide in a single vertex, provided that they do not share a common "white" neighbor.

Languages conventions: Let H be any target graph, recall that the vertices of H will be called the *colors*. Suppose that we want to construct a homomorphism h of a given graph G to H and let u, v be two vertices of G to be colored. We will say that *y allows k colors for x* if for any choice of the color of y we have at least k choices for coloring x. Similarly, we will say that *y forbids k colors for x* if for any choice of the color of y we have at least $n - k$ choices for coloring x.

Notation: The degree of a vertex u is denoted by $d(u)$ and the minimum degree of a graph is denoted by $\delta(G)$. A *k-vertex* (*respect.* a *$^{\geq}k$–vertex*) is a vertex of degree k, (*respect.* at least k). A *k-path* is a path $P = u_0, u_1 \dots u_k$ of length k (i.e. formed by k edges), with 2 *end-vertices* (u_0 and u_k) and $k-1$ *internal vertices* (all u_i's with $1 \leq i \leq k-1$). A *$(k, 2)$-path* in a graph G, is a k–path in which all internal vertices have

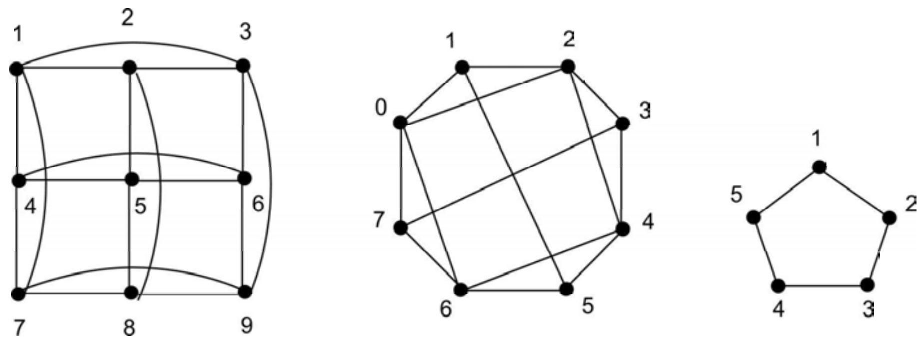


Figure 8.1: The graphs induced by one color of the $(0, 2)$–colored mixed target graphs T_9, T_8 and T_5.

degree 2. A *pattern* $\mathcal{Q} = q_1q_2...q_k$ of length k is a (non-empty) word in $\{1, 2\}^k$. If G is a 2–edge colored graph, a *$\mathcal{Q}$–neighbor of* a sequence $S = \{v_1, ..., v_k\}$ of vertices in G, is a vertex u in G such that $uv_i \in E_{q_i}(G)$ for every i.

8.3 The target graphs

The target graphs T_5, T_8, T_9, T_{12} and T_{20} that we will use to prove Theorems 8.1, 8.2 and 8.4 are the following. Observe that the graphs depicted in Figure 8.1 are all self complementary (i.e. isomorphic to their complements).

- Let T_9 (resp. T_8, T_5) be the complete $(0, 2)$–colored mixed graph on 9 (resp. 8, 5) vertices where the edges of each color induces an isomorphic copy of the first (resp. second, third) graph in Figure 8.1.
- Let $T_{12} = Tr(T_5)$ and $T_{20} = Tr(T_9)$ be the Tromp graphs obtained from T_5 and T_9 respectively (see Figure 8.2 to recall the Tromp construction described in Section 5.3.2).

Remark 8.1 *The target graphs T_5, T_9, T_{12} and T_{20} are vertex transitive.*

In previous sections, we shown that some of this target graphs have "useful" properties. Here we recall these properties and prove some others that we need to proceed with our proofs.

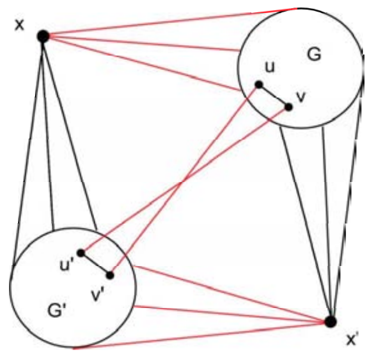


Figure 8.2: The $(0,2)$–colored mixed Tromp graph $tr(G)$.

Let G be a $(0,2)$–colored mixed graph and $P = u_0, u_1, ..., u_k$ be a $(0,2)$–mixed k–path. For $u \in V(G)$ we denote by:

$$N_P(G,u) = \{v \in G : \exists h : P \to G \text{ with } h(u_0) = u \text{ and } h(u_k) = v\}$$

Definition 8.1 *We say that G is k–**nice** (resp. k–**quasi–nice**) if for every k–path P, and every vertex $u \in V(G)$, we have $N_P(G,u) = V(G)$ (resp. $V(G) \setminus \{u\} \subseteq N_P(G,u)$).*

Note that this definition is equivalent to Definition 7.1 in Section 7. Hence, if a 2–edge colored graph is 2–quasi–nice it is also 3–nice and, for every $k \geq 3$ we have:

$$k\text{–nice} \Rightarrow k\text{–quasi–nice} \Rightarrow (k+1)\text{–nice}$$

A *pattern* $\mathcal{Q} = q_1q_2...q_k$ of length k is a (non-empty) word in $\{1,2\}^k$. Let $S = \{v_1, ..., v_k\}$ be a sequence of k vertices in a $(0,2)$–colored mixed graph G. A vertex $u \in V(G)$ is said to be a *$\mathcal{Q}$–neighbor of* S if $(u, v_i) \in E_{q_i}(G)$ for every i.

Definition 8.2 *We say that a colored mixed graph G satisfies **property** $P_{k,n}$ if for every sequence $S = \{v_1, ..., v_k\}$ which induce a k–clique subgraph of G, and every pattern $\mathcal{Q} = q_1q_2...q_k$ of length k, there exist n $\mathcal{Q}$–neighbors of S.*

Note that property $P_{k,n}$ differs from property P_k (given in Section 5.3) by requiring n $\mathcal{Q}$–neighbors instead of one. Hence, property $P_{k,1}$ is precisely property P_k, and property $P_{2,n}$ implies that the graph is 2–quasi–nice.

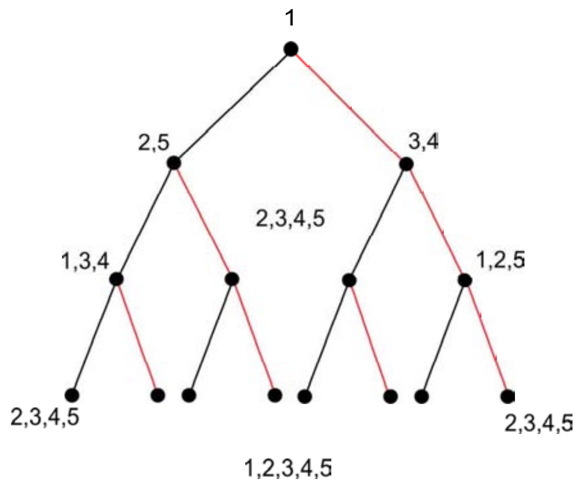


Figure 8.3: Vertices of T_5 reachable from vertex 1 by each k–path with $1 \leq k \leq 3$.

Proposition 8.1 *The target graph T_5 satisfies the following properties:*

1. *It has property $P_{1,2}$.*
2. *For every vertex u and for every $(2,2)$–path P, $|N_P(T_5, u)| \geq 3$.*
3. *It is 3–quasi–nice.*

Proof. Since T_5 is vertex transitive is sufficient to prove the statement for vertex 1. In Figure 8.3 are shown the vertices in T_5 reachable from 1 by each $(0,2)$–colored mixed k–path with $1 \leq k \leq 3$. □

Proposition 8.2 *The target graph T_8 satisfies the following properties:*

1. *It has property $P_{1,3}$.*
2. *For every vertex u and for every $(2,2)$–path P, $V(T_8) \backslash \{u, u+4 \pmod{8}\} \subseteq N_P(T_8, u)$.*
3. *It is 3–nice.*

Proof. Observe that there are two kinds of vertices in T_8, the odd vertices $\{1,3,5,7\}$, and the even vertices $\{2,4,6,8\}$. It is not difficult to check that for every two odd (*respect.* even) vertices u and v, there exist an automorphism of T_8 that maps u to v. Thus is sufficient to prove the statement for vertices 1 and 2. In Figure 8.4 (a) (*respect.* (b)) are shown the vertices in T_8 reachable from 1 (*respect.* 2) by each $(0,2)$–colored mixed k–path with $1 \leq k \leq 3$. □

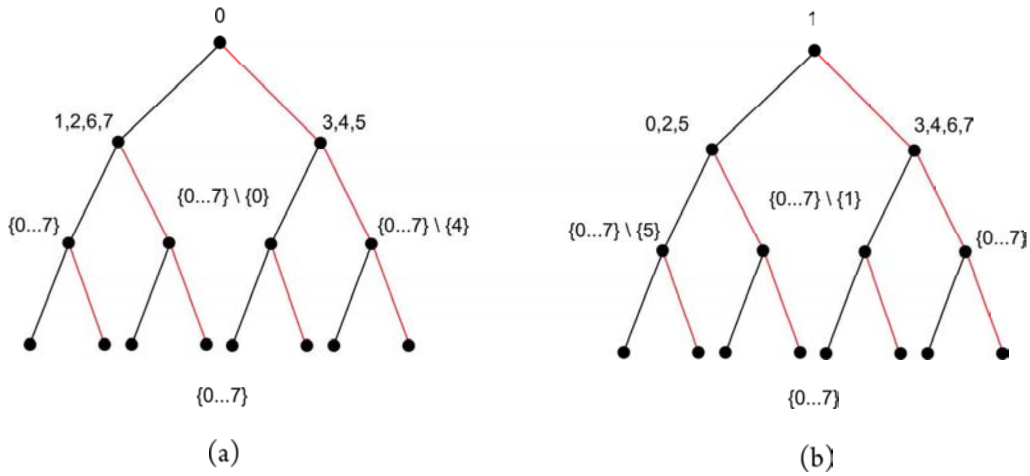


Figure 8.4: Vertices of T_8 reachable from vertices 0 and 1 by each k–path with $1 \leq k \leq 3$.

Proposition 8.3 *The target graph T_9 satisfies the following properties:*

1. *It has property $P_{1,4}$.*
2. *It has property $P_{2,1}$ (it is 2–quasi–nice).*

Proof. To check property $P_{1,4}$ is trivial. Property $P_{2,1}$ is equivalent to property P_2 described in Section 5.3. In Proposition 5.5 it was proved that T_9 has property P_2. □

Proposition 8.4 *The target graph T_{12} satisfies the following properties:*

1. *It has property $P_{1,5}$.*
2. *It has property $P_{2,2}$ (in particular it is 2–quasi–nice).*

Proof. To check property $P_{1,5}$ is trivial. Since T_{12} is vertex transitive, to show that it has property $P_{2,2}$, is enough to check that for every sequence $S = (x, i)$, $i \in V(T_5) \cup V(T_5')$, and every pattern $\mathcal{Q} = (q_1, q_2)$, there exist at least two $\mathcal{Q}$–neighbors of S. However, there exist an obvious automorphism $h : V(T_{12}) \to V(T_{12})$ that fixes x and x' (i.e. $h(x) = x$ and $h(x') = x'$) with orbits $(1, 2, 3, 4, 5)$ and $(1', 2', 3', 4', 5')$. Therefore, we only need to consider the sequences $(x, 1)$ and $(x, 1')$. The table below gives, for each the two $\mathcal{Q}$–neighbors. □

	$(x,1)$	$(x,1')$
(1,1)	$2;5$	$1;3$
(1,2)	$3;4$	$4;5$
(2,1)	$3';4'$	$4';5'$
(2,2)	$2';5'$	$1';3'$

□

Proposition 8.5 *The target graph T_{20} satisfies the following properties:*

1. *It has property $P_{1,9}$.*
2. *It has property $P_{2,4}$ (in particular it is 2–quasi–nice).*
3. *It has property $P_{3,1}$.*

Proof. To check property $P_{1,5}$ is trivial. Property $P_{3,1}$ is equivalent to property P_3 described in Section 5.3. In Proposition 5.6 it was proved that T_{20} has property P_3. Since T_{20} is vertex transitive, to show that it has property $P_{2,4}$, is enough to check that for every sequence $S = (x,i)$, $i \in V(T_9) \cup V(T_9')$, and every pattern $\mathcal{Q} = (q_1, q_2)$, there exist at least four $\mathcal{Q}$–neighbors of S. The table below gives, for each sequence $S = (x,i)$ and each pattern $\mathcal{Q}$, the four $\mathcal{Q}$–neighbors of S. □

	$(x,1)$	$(x,2)$	$(x,3)$	$(x,4)$	$(x,5)$	$(x,6)$	$(x,7)$	$(x,8)$	$(x,9)$
(1,1)	$2;3;4;7$	$1;3;5;8$	$1;2;6;9$	$1;5;6;7$	$2;4;6;8$	$3;4;5;9$	$1;4;8;9$	$2;5;7;9$	$3;6;7;8$
(1,2)	$5;6;8;9$	$4;6;7;9$	$4;5;7;8$	$2;3;8;9$	$1;3;7;9$	$1;2;7;8$	$2;3;5;6$	$1;3;4;6$	$1;2;4;5$
(2,1)	$5;6;8;9$	$4;6;7;9$	$4;5;7;8$	$2;3;8;9$	$1;3;7;9$	$1;2;7;8$	$2;3;5;6$	$1;3;4;6$	$1;2;4;5$
(2,2)	$2;3;4;7$	$1;3;5;8$	$1;2;6;9$	$1;5;6;7$	$2;4;6;8$	$3;4;5;9$	$1;4;8;9$	$2;5;7;9$	$3;6;7;8$

□

8.4 Outerplanar graphs

Proof of Theorem 8.1–1. Since T_9 has property P_2 (Proposition 8.3–2), Proposition 5.4 in Section 5.3, ensures that every 2–edge colored partial 2–tree is T_9–colorable (Theorem 5.3–1). Then we get the upper bound, since outerplanar graphs form a strict subclass of partial 2–trees. The lower bound is obtained by exhibiting an outerplanar 2–edge colored graph with colored mixed chromatic number 9. Such a graph is obtained by taking $(n,m) = (0,2)$ in Proposition 5.1 (Section 5.3); in Figure 8.5 you can see this outerplanar 2–edge colored graph. □

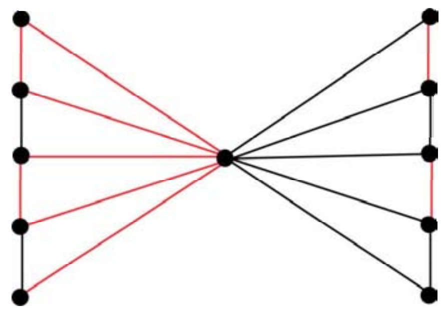


Figure 8.5: An outerplanar 2–edge colored graph with colored mixed chromatic number 9.

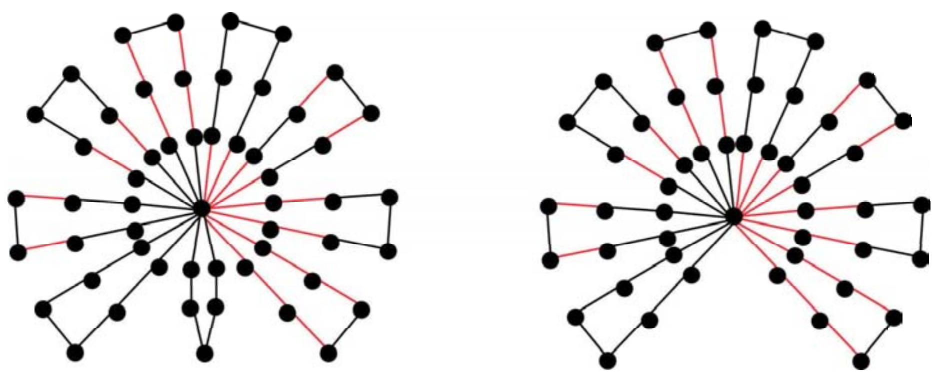


Figure 8.6: Outerplanar 2–edge colored graph with girth 6 and 7 and colored mixed chromatic number 5.

Proof of Theorem 8.1–2. It can be proved that every 2–edge colored outerplanar graph with girth at least 4 is T_5–colorable (upper bound of Theorem 7.5–3 in Section 7 by taking $(n, m) = (0, 2)$). Also, for every $g \geq 4$, there is a 2–edge colored outerplanar graph with girth g and colored mixed chromatic number at least 5 (lower bound of Theorem 7.5–3 in Section 7 by taking $(n, m) = (0, 2)$); in Figure 8.6 you can see this outerplanar 2–edge colored graph for $g = 6$ and 7. □

8.5 Partial 2–trees

Proof of Theorem 8.3–1. This is precisely Theorem 5.3–1 in Section 5. It was proved since T_9 has property P_2, thus every 2–edge colored partial 2–tree is T_9–colorable; the lower bound is obtained since an outerplanar graph is a partial 2–tree (see Figure 8.5). □

In order to prove Theorem 8.3–2 and Theorem 8.3–3 we will use the following structural Lemma recently given by Ochem an Pinlou [48] as a generalization of a previous result proposed by Lih, Wang, and Zhu [39]. For a graph G with girth at least g and a vertex $v \in V(G)$, we denote:

$$S_g^G(v) = \{u \in V(G), d(u) \geq 3, \text{ such that}$$
$$\text{there exist a unique } (k,2)\text{–path linking } u \text{ and } v,$$
$$\text{or } u \text{ and } v \text{ are the end points of at least a } (\lceil \tfrac{g}{2} \rceil, 2)\text{–path}\}.$$

Then, we denote $D_g^G(v) = |S_g^G(v)|$

Lemma 8.1 ([48]) *Let G be a partial 2–tree with girth g such that $\delta(G) \geq 2$. Then, one of the following holds:*

1. *there exist a $(\lceil \frac{g}{2} \rceil + 1, 2)$–path;*
2. *there exist a $^{\geq}3$–vertex v such that $D_g^G(v) \leq 2$.*

Proof of Theorem 8.3–2. We show that every 2–edge colored partial 2–tree with girth at leas 4 admits a T_8–coloring. Note that is sufficient to consider the case $g = 4$. Let G be a minimal (with respect to order) counter–example. In order to get a contradiction by Lemma 8.1, we show that G contains neither a $^{\leq}1$–vertex, nor a $(3,2)$–path, nor a $^{\geq}3$–vertex v such that $D_4^G(v) \leq 2$.

(1) Suppose that G contains a $^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_8–coloring h. By Property $P_{1,3}$ (Proposition 8.2–1), h can be extended to a T_8–coloring of G.

(2) Suppose that G contains a $(3,2)$–path $P_4 = u, v_1, v_2, w$. Due to the minimality of G the graph $G' = G \setminus \{v_1, v_2\}$ admits a T_8–coloring h. Since T_8 is 3–nice (Proposition 8.2–3), then h can be extended to a T_8–coloring of G.

(3) Suppose that G contains $^{\geq}3$–vertex v such that $D_4^G(v) = 1$. Then $S_4^G(v) = \{w\}$ and since G does not contain a $(3,2)$–path, there exist at least three $(2,2)$–paths linking v and w (see Figure 8.7). Due to the minimality of G the graph $G' = G \setminus \{v, u_1, ..., u_k\}$ admits a T_8–coloring h. We then set $h(v) \neq h(w)$ and $h(v) \neq h(w) + 4 \pmod 8$, hence Proposition 8.2–2 ensures that h can be extended to a T_8–coloring of G.

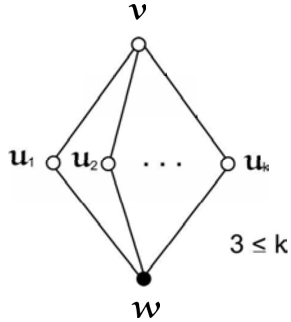


Figure 8.7: A ${}^{\geq}3$–vertex v such that $D_4^G(v) = 1$.

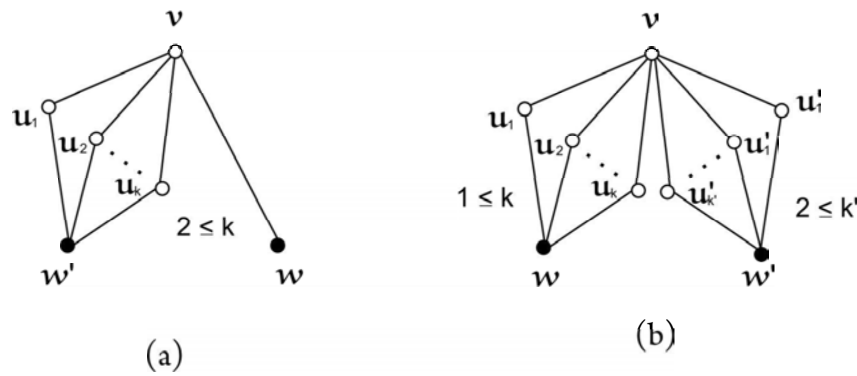


Figure 8.8: A ${}^{\geq}3$–vertex v such that $D_4^G(v) = 2$.

(4) Suppose that G contains ${}^{\geq}3$–vertex v such that $D_4^G(v) = 2$. Then $S_4^G(v) = \{w, w'\}$.

Suppose first that for w or w', say w, there is a unique $(k, 2)$–path linking v and w. In this case since G does not contain a $(3, 2)$–path, the edge vw is the only path linking v and w; then since $d(v) \geq 3$, there are at least two $(2, 2)$–paths linking v and w' (see Figure 8.8–(a)). Due to the minimality of G the graph $G' = G \backslash \{v, u_1, ..., u_k\}$ admits a T_8–coloring h. By Property $P_{1,3}$ (Proposition 8.2–1), w allows three colors for v, while w' forbids only two color for v, named $h(w')$ and $h(w') + 4$ (mod) 8 (by Proposition 8.2–2). Then h can be extended to a T_8–coloring of G.

Suppose now that there exist at least a $(2, 2)$–paths linking v and w (resp. w'), see Figure 8.8–(b). Due to the minimality of G the graph $G' = G \backslash \{v, u_1, ..., u_k, u'_1, ..., u'_{k'}\}$ admits a T_8–coloring h. By Proposition 8.2–2, w and w' each forbids two color for v. We thus have four available colors for v and thus h can be extended to a T_8–coloring of G.

To complete the proof of Theorem 8.3–2, we construct a 2–edge colored partial 2–tree

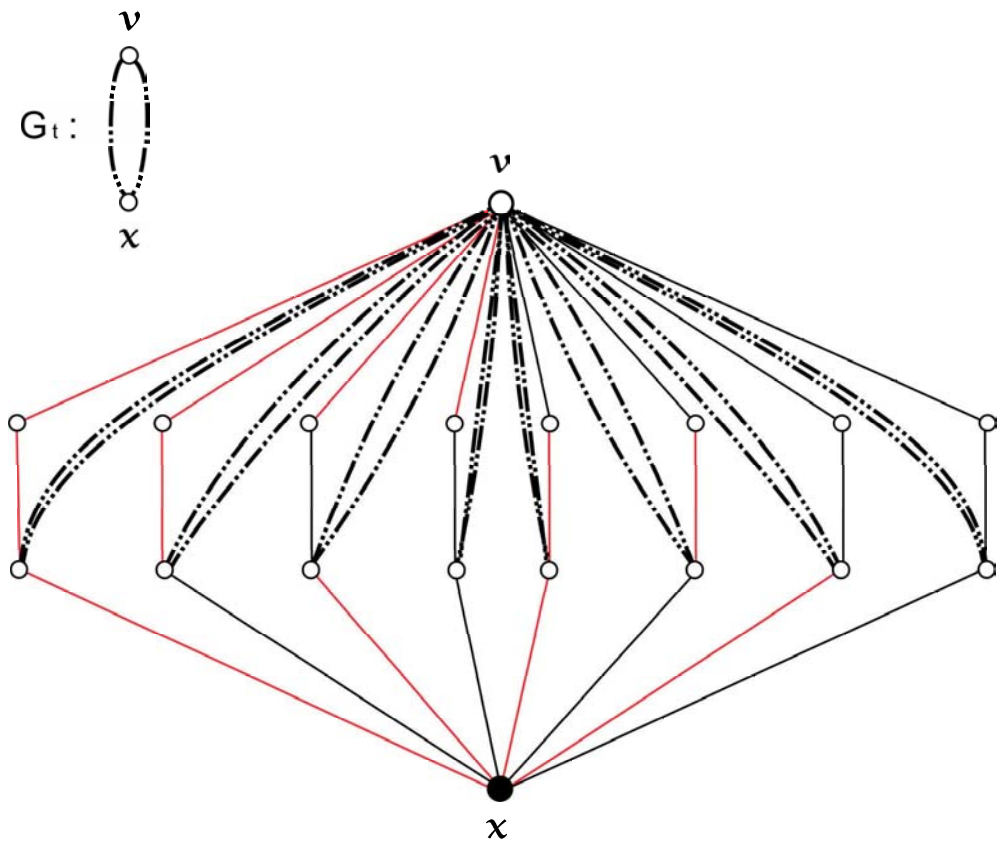


Figure 8.9: The family G_t, $t \geq 0$, of series–parallel graphs of girth 5.

with girth 5 and colored mixed chromatic number 8. Let us construct the family G_t, $t \geq 0$, of series–parallel graphs of girth 5, inductively as follows:

- G_0 consist in two non–adjacent vertices u and x.
- G_{t+1} consist in two non–adjacent vertices u and x joined by the eight possible $(3, 2)$–paths of the form $[u, v_i, w_i, x]$, $1 \leq i \leq 8$; and eight copies of G_t, named $G_t^1, ..., G_t^8$, such that vertex u of G_{t+1} is identify with the vertex u of every G_t^i, $1 \leq i \leq 8$, and each vertex w_i of G_{t+1} is identify with the vertex x of G_t^i, $1 \leq i \leq 8$.

Figure 8.9 illustrates this construction.

Consider a complete 2–edge colored graph T_n of order n. We define a family M_t, $t \geq 0$, of boolean square matrices of order n as follows:

- For $1 \leq i < j \leq n$, $M_t[i,j]$ is 1 (resp. 0) if there exist (resp. not exist) a T_n–coloring of G_t such that its vertex u gets color i and its vertex x gets color j.

If there exist $t \geq 0$ and $1 \leq i \leq n$ such that $M_t[i,j]$ is 0 for every $1 \leq j \leq n$, then T_n is not universal. We use a program to rule out target graphs on 5, 6 and 7 vertices, in that order (we can start with 5 by Theorem 8.1–(2)). Since every potential target graph less that n has been previously rule out, there exist a partial 2–tree W, with girth 5 and $\chi_{0,2}(W) = n$. Now, if we identify each vertex of W with the vertex u of a copy of G_t, the vertex x of some copy of G_t attached to a vertex of T colored by i can not be colored.

Our program runs over every complete 2–edge colored graphs by increasing the number of vertices. All entries of the matrix M_0 are 1, and then, depending on T_n, we compute iteratively M_{t+1} from M_t until every entry of some column of some M_t is all 0.

□

Before continue, we shall note that, by taking $(n,m) = (0,2)$ in Theorem 7.5–2, we obtain $\chi_{0,2}(\mathcal{T}_g^2) = 5$ for every $g \geq 7$. Here we improve that result by showing that $\chi_{0,2}(\mathcal{T}_g^2) = 5$ for every $g \geq 6$.

Proof of Theorem 8.3–3. We show that every 2–edge colored partial 2–tree with girth at leas 6 admits a T_5–coloring. Note that is sufficient to consider the case $g = 6$. Let G be a minimal (with respect to order) counter–example. In order to get a contradiction by Lemma 8.1, we show that G contains neither a ${}^{\leq}1$–vertex, nor a $(4,2)$–path, nor a ${}^{\geq}3$–vertex v such that $D_6^G(v) \leq 2$.

(1) Suppose that G contains a ${}^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_5–coloring h. By Property $P_{1,2}$ (Proposition 8.1–1), h can be extended to a T_5–coloring of G.

(2) Suppose that G contains a $(4,2)$–path $P_4 = u, v_1, v_2, v_3, w$. Due to the minimality of G the graph $G' = G \setminus \{v_1, v_2, v_3\}$ admits a T_5–coloring h. Since T_5 is 3–quasi–nice (Proposition 8.1–3) it is also 4–nice, then h can be extended to a T_8–coloring of G.

(3) Suppose that G contains ${}^{\geq}3$–vertex v such that $D_6^G(v) = 1$. Then $S_6^G(v) = \{w\}$ and since G does not contain a $(4,2)$–path, there exist at least three $(3,2)$–paths linking v and w (see Figure 8.10). Due to the minimality of G the graph $G' = G \setminus \{v, u_1, ..., u_k, u'_1, ..., u'_k\}$ admits a T_5–coloring h. We then set $h(v) \neq h(w)$, and since T_5 is 3–quasi–nice (Proposition 8.1–3) then h can be extended to a T_5–coloring of G.

(4) Suppose that G contains ${}^{\geq}3$–vertex v such that $D_6^G(v) = 2$. Then $S_6^G(v) = \{w, w'\}$.

Suppose first that for w or w', say w, there exist a unique $(k,2)$–path linking u and v (since G does not contain a $(3,2)$–path, then $k = 1$ or $k = 2$). Then since

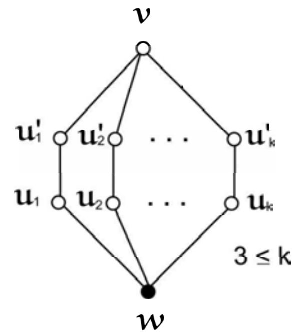


Figure 8.10: A $^{\geq}3$–vertex v such that $D_6^G(v) = 1$.

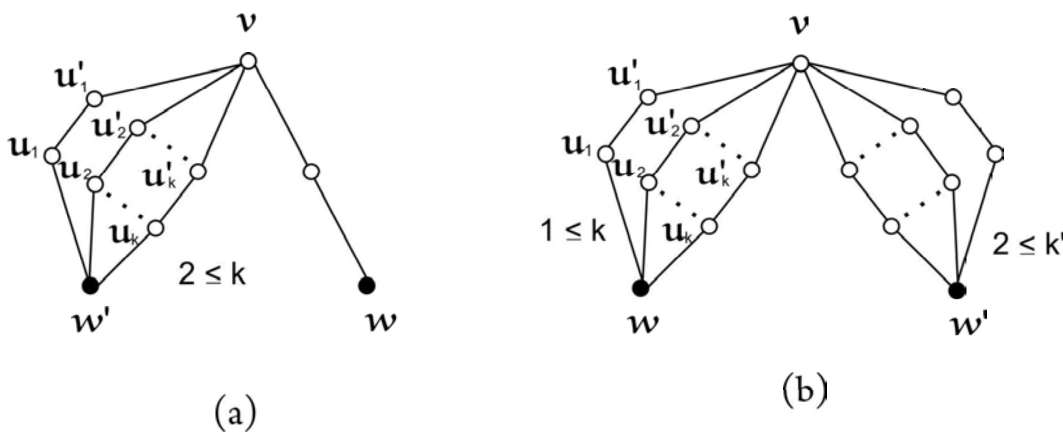


Figure 8.11: A $^{\geq}3$–vertex v such that $D_6^G(v) = 2$.

$d(v) \geq 3$, there are at least two $(3, 2)$–paths linking v and w' (see Figure 8.11–(a)). Due to the minimality of G the graph $G' = G \setminus \{v, u_1, ..., u_k, u'_1, ..., u'_k\}$ admits a T_5–coloring h. By Proposition 8.1–1 and 2, w allows at leat two colors for v, while w' forbids only one color for v, named $h(w')$ (by Proposition 8.1–3). Then h can be extended to a T_5–coloring of G.

Suppose now that there exist at least a $(3, 2)$–path linking v and w (resp. w'), see Figure 8.11–(b). Due to the minimality of G the graph $G' = G\setminus\{v, u_1, ..., u_k, u'_1, ..., u'_{k'}\}$ admits a T_5–coloring h. By Proposition 8.1–3, w and w' each forbids one color for v. We thus have three available colors for v and thus h can be extended to a T_5–coloring of G.

8.6 Planar graphs

The proof of Theorem 8.3 involve an auxiliary graph parameter named the maximum average degree. The *average degree* of a simple graph G, denoted by $\mathrm{ad}(G)$, is defined as twice the number of edges over the number of vertices ($\mathrm{ad}(G) = \frac{2|E(G)|}{|V(G)|}$). The *maximum average degree* of a simple graph G, denoted by $\mathrm{mad}(G)$, is then defined as the maximum of the averages degrees taken over all subgraphs of G: $\mathrm{mad}(G) = \max_{H \subseteq G}\{ad(H)\}$. The maximum average degree and the girth of a planar graph are linked by the following relation.

Proposition 8.6 *Let G be a planar graph with girth g. Then $mad(G) < \frac{2g}{g-2}$*

Proof. Observe that, if G has girth at least g then the number of faces of G is at most $\frac{2|E(G)|}{g}$. By Euler's formula we then get $\frac{2|E(G)}{|V(G)|} \leq \frac{2g|E(G)|}{2g+(g-2)|E(G)}$ and thus $\mathrm{mad}(G) < \frac{2g}{g-2}$ □

Hence Theorem 8.3 can be deduced from the previous Proposition and the following:

Theorem 8.4 *Let G be a $(0,2)$–colored mixed graph.*

1. *If $mad(G) < \frac{10}{3}$, then $\chi_{(0,2)}(G) \leq 20$.*
2. *If $mad(G) < 3$, then $\chi_{(0,2)}(G) \leq 12$.*
3. *If $mad(G) < \frac{8}{3}$, then $\chi_{(0,2)}(G) \leq 8$.*
4. *If $mad(G) < \frac{7}{3}$ and G does not contain a triangle whose edges are of the same type, then $\chi_{(0,2)}(G) \leq 5$ and moreover this bound is tight.*

Next we prove Theorem 8.4 by using the method of reducible configuration and discharging procedure explained in Section 8.2. For simplicity, we start with maximum average degree less than $\frac{7}{3}$ until maximum average degree less than $\frac{10}{3}$.

8.6.1 Graphs with maximum average degree less than $\frac{7}{3}$

We will prove that every triangle–free 2–edge colored graph with maximum average degree less than $\frac{7}{3}$ admits a T_5–coloring.

We shall note that the statement is not true if we permit triangles whose edges are of the same type. Indeed, for every fixed odd $n \geq 7$, we provide a 2–edge colored graph G (containing triangles whose edges are of the same type) with $\mathrm{mad}(G)$ which tends to 2 as n tends to infinity, and colored mixed chromatic number 6 (see Figure 8.12)

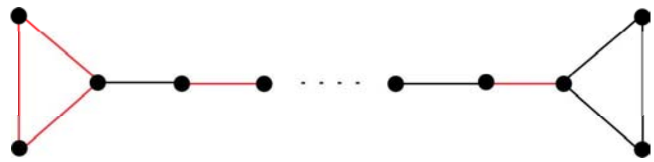


Figure 8.12: A 2–edge colored graph G with $\mathrm{mad}(G)$ which tends to 2 as n tends to infinity, containing triangles whose edges are of the same type and $\chi_{(0,2)}(G) = 6$.

Lemma 8.2 *A minimal (with respect to order) counter–example to Theorem 8.4–4 does not contain the following configurations.*

1. *A* $^{\leq}1$*–vertex.*
2. *A* $(4, 2)$*–path.*
3. *The configuration in Figure 8.13.*

Proof. Let G be a minimal (with respect to order) 2–edge colored graph with $\mathrm{mad}(G) < \frac{7}{3}$ and $\mathrm{girt}(G) > 3$ which does not admit a T_5–coloring.

1. Suppose that G contains a $^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_5–coloring h. By Property $P_{1,2}$ (Proposition 8.1–1), h can be extended to a T_5–coloring of G.
2. Suppose that G contains a $(4, 2)$–path $P_4 = u, v_1, v_2, v_3, w$. Due to the minimality of G the graph $G' = G \setminus \{v_1, v_2, v_3\}$ admits a T_5–coloring h. Since T_5 is 3–quasi–nice (Proposition 8.1–3), it is also 4–nice. Hence, h can be extended to a T_5–coloring of G.
3. Suppose that G contains the configuration in Figure 8.13. Due to the minimality of G the graph $G' = G \setminus \{x, u_1, v_1, v_2, w_1, w_2\}$ admits a T_5–coloring h. Since G is 3–quasi–nice (Proposition 8.1–3), each v and w forbids one color for x. By Proposition 8.1–2, u forbids two colors for x. Then we have at least one color for x and thus h can be extended to a T_5–coloring of G.

□

Proof of Theorem 8.4–4. Let G be an hypothetical minimal (with respect to order) 2–edge colored graph with $\mathrm{mad}(G) < \frac{7}{3}$ and and $\mathrm{girt}(G) > 3$ which does not admit a T_5–coloring. A *weak* 2–vertex is a 2–vertex adjacent to a 2–vertex, while a *strong* 2–vertex is a 2–vertex not adjacent to a 2–vertex. The discharging rules (R1) and (R2) are defined as follows.

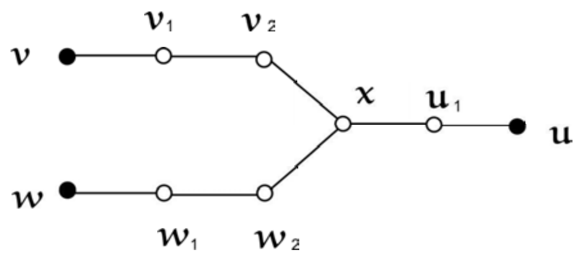


Figure 8.13: Unavoidable configuration for $\text{mad}(G) < \frac{7}{3}$.

(R1) Each $^{\geq}3$–vertex gives $\frac{1}{3}$ to each adjacent weak 2–vertex.

(R2) Each $^{\geq}3$–vertex gives $\frac{1}{6}$ to each adjacent strong 2–vertex.

Let v be a k–vertex. By Lemma 8.2–1, $k \geq 2$.

- If $k = 2$ and v is strong, then it receives $\frac{1}{6}$ from each of its two neighbors of degree at least three. If v is weak, Lemma8.2–2 ensures that v is adjacent to a $^{\geq}3$–vertex, thus it receives $\frac{1}{3}$. In both cases we have $d^*(v) = \frac{7}{3}$.
- If $k = 3$, then by Lemma 8.2–3 (and since $\text{girt}(G) > 3$), v gives at most $\max\{2 \times \frac{1}{3}, 2 \times \frac{1}{6} + \frac{1}{3}, 3 \times \frac{1}{6}\} = \frac{2}{3}$. Hence, $d^*(v) \geq 3 - \frac{2}{3} = \frac{7}{3}$.
- If $k \geq 4$, then v gives at most $\frac{k}{3}$ and hence $d^*(v) \geq k - \frac{k}{3} > \frac{7}{3}$.

Then, for all $v \in V(G)$, $d^*(v) \geq \frac{7}{3}$ ones the discharging is completed, that shows that $\text{mad}(G) \geq \frac{7}{3}$, a contradiction. □

8.6.2 Graphs with maximum average degree less than $\frac{8}{3}$

Here we prove that every 2–edge colored graph with maximum average degree less than $\frac{8}{3}$ admits a T_8–coloring.

Lemma 8.3 *A minimal (with respect to order) counter–example to Theorem 8.4–3 does not contain the following configurations.*

1. *A $^{\leq}1$–vertex.*
2. *A $(3, 2)$–path.*

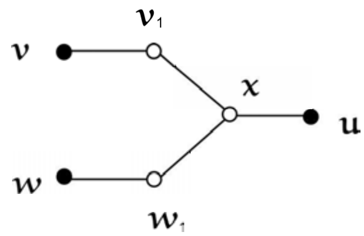


Figure 8.14: Unavoidable configuration for $\mathrm{mad}(G) < \frac{8}{3}$.

3. The configuration in Figure 8.14.

Proof. Let G be a minimal (with respect to order) 2–edge colored graph with $\mathrm{mad}(G) < \frac{8}{3}$ which does not admit a T_8–coloring.

1. Suppose that G contains a $^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_8–coloring h. Property $P_{1,3}$ (Proposition 8.2–1) ensures that h can be extended to a T_8–coloring of G.
2. Suppose that G contains a $(3, 2)$–path $P_4 = u, v_1, v_2, w$. Due to the minimality of G the graph $G' = G \setminus \{v_1, v_2\}$ admits a T_8–coloring h. Since T_8 is 3–nice (Proposition 8.2–3), then h can be extended to a T_8–coloring of G.
3. Suppose that G contains the configuration in Figure 8.14. Due to the minimality of G the graph $G' = G \setminus \{x, v_1, w_1\}$ admits a T_8–coloring h. By Property $P_{1,3}$, u allows three colors for x, while each of v and w forbids only one color for x (by Proposition 8.2–2). Then h can be extended to a T_8–coloring of G.

□

Proof of Theorem 8.4–3. Let G be an hypothetical minimal (with respect to order) 2–edge colored graph with $\mathrm{mad}(G) < \frac{8}{3}$ which does not admit a T_8–coloring. The discharging rule is the following

(R) Each $^{\geq}3$–vertex gives $\frac{1}{3}$ to each adjacent 2–vertex.

Let v be a k–vertex. By Lemma 8.3–1, $k \geq 2$.

- If $k = 2$, then by Lemma8.3–2 v has two neighbors of degree at leas 3, thus it receives $2 \times \frac{1}{3}$, hence $d^*(v) = \frac{8}{3}$.

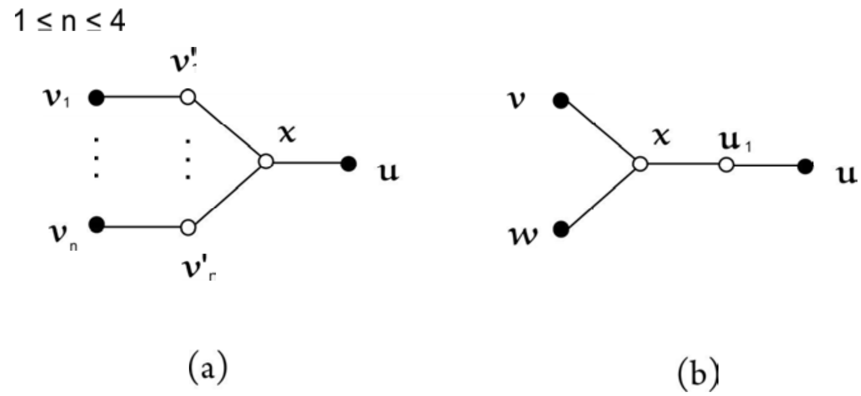


Figure 8.15: Unavoidable configurations for $\text{mad}(G) < 3$.

- If $k = 3$, then by Lemma 8.2–2 and 3, v has at most one neighbor of degree 2. Therefore, v gives at most $\frac{1}{3}$ and hence $d^*(v) \geq \frac{8}{3}$.
- If $k \geq 4$, then v gives at most $\frac{k}{3}$ and hence $d^*(v) \geq k - \frac{k}{3} > \frac{8}{3}$.

Then, for all $v \in V(G)$, $d^*(v) \geq \frac{8}{3}$ ones the discharging is completed, that shows that $\text{mad}(G) \geq \frac{8}{3}$, a contradiction. □

8.6.3 Graphs with maximum average degree less than 3

Now we prove that every 2–edge colored graph with maximum average degree less than 3 admits a T_{12}–coloring.

Lemma 8.4 *A minimal (with respect to order) counter–example to Theorem 8.4–2 does not contain the following configurations.*

1. *A* $^{\leq}1$*–vertex.*
2. *The configuration in Figure 8.15(a).*
3. *The configuration in Figure 8.15(b).*

Proof. Let G be a minimal (with respect to order) 2–edge colored graph with $\text{mad}(G) < 3$ which does not admit a T_{12}–coloring.

1. Suppose that G contains a ${}^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_{12}–coloring h. Property $P_{1,5}$ (Proposition 8.4–1) ensures that h can be extended to a T_{12}–coloring of G.

2. Suppose that G contains the configuration in Figure 8.15 (a). Due to the minimality of G the graph $G' = G \setminus \{x, v'_1, ..., v'_n\}$ admits a T_{12}–coloring h. By Property $P_{1,5}$, u allows five colors for x, while each of $v_1, ..., v_n$ forbids only one color for x (since G is 2–quasi–nice, Proposition 8.4–2). Then h can be extended to a T_{12}–coloring of G.

3. Suppose that G contains the configuration in Figure 8.15 (b). Due to the minimality of G the graph $G' = G \setminus \{x, u_1\}$ admits a T_{12}–coloring h. By property $P_{2,2}$ (Proposition 8.4–2), u and v allows one color from x which is distinct from $h(u)$. Then, since T_{12} is 2–quasi–nice, h can be extended to a T_{12}–coloring of G.

□

Proof of Theorem 8.4–2. Let G be an hypothetical minimal (with respect to order) 2–edge colored graph with $\text{mad}(G) < 3$ which does not admit a T_{12}–coloring. The discharging rule is the following

(R) Each ${}^{\geq}3$–vertex gives $\frac{1}{2}$ to each adjacent 2–vertex.

Let v be a k–vertex. By Lemma 8.4–1, $k \geq 2$.

- If $k = 2$, then by Lemma8.4–2 with $n = 1$, v has two neighbors of degree at leas 3, thus it receives $2 \times \frac{1}{2}$, hence $d^*(v) = 3$.
- If $k = 3$, then by Lemma 8.4–2 with $n = 1$ and Lemma 8.4–3, v has not neighbors of degree 2. Hence, $d^*(v) = d(v) = 3$.
- If $4 \leq k \leq 5$, then by Lemma 8.4–2 for $n = 1, 3$ and 4, v has at most $k - 2$ neighbors of degree 2. Hence, $d^*(v) \geq k - \frac{k-2}{2} \geq 3$.
- If $k \geq 6$, then v gives at most $\frac{k}{2}$ and hence $d^*(v) \geq k - \frac{k}{2} > 3$.

Then, for all $v \in V(G)$, $d^*(v) \geq 3$ ones the discharging is completed, that shows that $\text{mad}(G) \geq 3$, a contradiction. □

8.6.4 Graphs with maximum average degree less than $\frac{10}{3}$

Next we prove that every 2–edge colored graph with maximum average degree less than $\frac{10}{3}$ admits a T_{20}–coloring. In this case we need ...

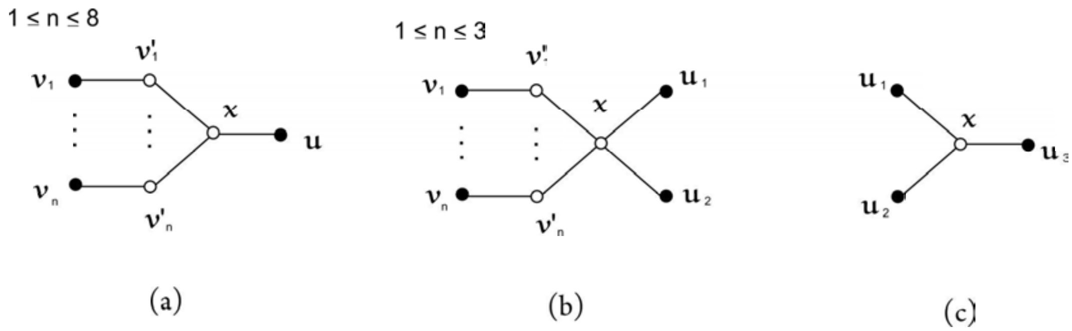


Figure 8.16: Unavoidable configurations for $\mathrm{mad}(G) < \frac{10}{3}$.

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $^{\leq}3$–vertices in G. for any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions holds:

- G_1 is a proper subgraph of G_2.
- $n_3(G_1) < n_3(G_2)$.

Note that this partial order is well–defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \leq n_3(G_2)$. So $\preceq$ is a partial linear extension of the subgraph poset.

Lemma 8.5 *A minimal counter–example (according to $\prec$) to to Theorem 8.4–1 does not contain the following configurations.*

1. *A $^{\leq}1$–vertex.*
2. *The configuration in Figure 8.16 (a).*
3. *The configuration in Figure 8.16 (b).*
4. *The configuration in Figure 8.16 (c).*

Proof. Let G be a minimal (with respect to $\prec$) 2–edge colored graph with $\mathrm{mad}(G) < \frac{10}{3}$ which does not admit a T_{20}–coloring.

1. Suppose that G contains a $^{\leq}1$–vertex v. The case $d(v) = 0$ is trivial. Suppose that $d(v) = 1$. Due to the minimality of G the graph $G' = G \setminus \{v\}$ admits a T_{20}–coloring h. Property $P_{1,9}$ (Proposition 8.5–1) ensures that h can be extended to a T_{20}–coloring of G.

2. Suppose that G contains the configuration in Figure 8.16 (a). Due to the minimality of G the graph $G' = G \setminus \{x, v'_1, ..., v'_n\}$ admits a T_{20}–coloring h. By Property $P_{1,9}$, u allows nine colors for x, while each of $v_1, ..., v_n$ forbids only one color for x (since G is 2–quasi–nice, Proposition 8.5–2). Then h can be extended to a T_{20}–coloring of G.

3. Suppose that G contains the configuration in Figure 8.16 (b). Due to the minimality of G the graph $G' = G \setminus \{x, v'_1, ..., v'_n\}$ admits a T_{20}–coloring h. By property $P_{2,4}$ (Proposition 8.5–2), u_1 and u_2 allows one color from x which is distinct from $h(v_n)$ for $1 \leq n \leq 3$. Then, since T_{20} is 2–quasi–nice, h can be extended to a T_{20}–coloring of G.

4. Suppose that G contains the configuration in Figure 8.16 (c). Since G contains neither a 1–vertex, neither the configuration in Figure 8.16 (b) for $n = 1$, u_1, u_2 and u_3 are $^{\geq}3$–vertices. Let G' be the graph obtained from $G \setminus \{x\}$ by adding, for every $1 \leq i < j \leq 3$, a $(2, 2)$–path joining u_i to u_j in such a way that its type is the same type of the path u_i, x, u_j in G. We have $G' \prec G$ since $n_3(G') = n_3(G) - 1$, and one can check that $\text{mad}(G') < \frac{10}{3}$. Any T_{20}–coloring of G' induces a coloring of $G \setminus \{x\}$ that can be extended to a T_{20}–coloring of G by property $P_{3,1}$ (Proposition 8.5–3).

□

Proof of Theorem 8.4–1. Let G be an hypothetical minimal (with respect to $\prec$) 2–edge colored graph with $\text{mad}(G) < \frac{10}{3}$ which does not admit a T_{20}–coloring. The discharging rule is the following

(R) Each $^{\geq}4$–vertex gives $\frac{2}{3}$ to each adjacent 2–vertex.

Let v be a k–vertex. By Lemma 8.5–1 and 4, $k \geq 2$ and $k = 3$.

- If $k = 2$, then by Lemma8.5–2 with $n = 1$, v has two neighbors of degree at leas 3, thus it receives $2 \times \frac{2}{3}$, hence $d^*(v) = \frac{10}{3}$.
- If $4 \leq k \leq 5$, then by Lemma 8.5–3, v has at most $k - 3$ neighbors of degree 2. Hence, $d^*(v) \geq k - \frac{2(k-3)}{3} \geq \frac{10}{3}$.
- If $6 \leq k \leq 9$, then by Lemma 8.5–4, v has at most $k - 2$ neighbors of degree 2. Hence, $d^*(v) \geq k - \frac{2(k-2)}{3} \geq \frac{10}{3}$.
- If $k \geq 10$, then v gives at most $\frac{2k}{3}$ and hence $d^*(v) \geq k - \frac{2k}{3} > \frac{10}{3}$.

Then, for all $v \in V(G)$, $d^*(v) \geq \frac{10}{3}$ ones the discharging is completed, that shows that $\text{mad}(G) \geq \frac{10}{3}$, a contradiction. □

Chapter 9

The class of bipartite 2–edge colored graphs

9.1 Motivation

The *constraint satisfaction problem* with template H (CSP–H) is the question whether, for a given structure G, there is an homomorphism from G to H. Here structure can mean anything we like: graphs, directed graphs, edge colored graphs or more general any relational system.

Definition 9.1 *A* ***general relational system*** *S is a finite set of vertices $V(S)$ together with a finite set of relations $R_i(S)$, $i \in I$, where $R_i(S)$ is a k_i–ary relation on $V(S)$. The* ***type*** *of S is the set of integers $\{k_i : i \in I\}$.*

A *binary relational systems* is a general relational systems in which $k_i = 2$ for every $i \in I$. A *digraph* G is a binary relational system with only one binary relation $E(G)$. The elements (u, v) of $E(G)$ are called the *arcs* of G. A *simple graph* G is a digraph in witch the binary relation $E(G)$ is symmetric and irreflexive (no loops allowed). Symmetric binary relations are more conveniently viewed as undirected edges. *Oriented graphs* are obtained from simple graphs by assigning to each (undirected) edge only one of the two possible orientations, and 2–*edge colored graphs* are obtained from simple graphs by assigning to each (undirected) edge only one of 2 possible colors. Thus a 2–edge colored graph G is a binary relational systems with two symmetric irreflexive relations (colors) $E_1(G)$ and $E_2(G)$.

Definition 9.2 *A* ***homomorphism*** *from a general relational system S to a general* *relational system T of the same type, is a mapping $h : V(S) \to V(T)$ such that, if $(v_1, v_2, ...v_{k_i}) \in R_i(S)$ then $(h(v_1), h(v_2), ...h(v_{k_i})) \in R_i(T)$ for all $i \in I$.*

We denote the class of digraphs by $\mathcal{D}$, the class of simple graphs by $\mathcal{G}$, the oriented one by $\overrightarrow{\mathcal{G}}$ and the 2–edge colored class by $\widetilde{\mathcal{G}}$. In each case the existence of a homomorphism from object G to object H is denoted by $G \to H$, and $G \nrightarrow H$ means there is no such homomorphism.

The *dichotomy conjecture* states that CSP–H is always either P or NP-complete. The dichotomy conjecture is true for undirected graphs and it is not known for digraphs. The role of digraphs is central for dichotomy because they form a *complete* class for this problem. That is, the general dichotomy conjecture can be reduced to the class of digraphs, which means that every CSP–T with T a general relational system, is polynomial equivalent to some CSP–H for a suitable digraph H. We can find proofs of these results in [28].

There are simple classes of digraphs which are complete for dichotomy, for example some special subclass of acyclic oriented graphs with 5 levels. We include in Section 9.1.1 a proof of this fact. By following the same technique, we will also show a subclass of bipartite 2–edge colored graphs which is complete for dichotomy. This result gave us a good reason to study the class of bipartite 2–edge colored graphs.

The class of bipartite 2–edge colored graphs is closely related to the one of bipartite oriented graphs. We make this statement precise in Section 9.2 and discuss the relationship between cores and dualities in the two categories.

9.1.1 Two classes complete for dichotomy

As we have said before, it is known [28] that the class of digraphs is complete for dichotomy, that is:

Theorem 9.1 *[28] Every CSP–T with T a general relational system, is polynomial equivalent to some CSP–H for a suitable digraph H.*

We can further sharpen Theorem 9.1 by means of the following construction.

Let J be a fixed graph with two specified vertices a, b. For any digraph H we denote by $H * (J, a, b)$ the graph obtained from H by replacing each arc $(x, y) \in E(H)$ by an isomorphic copy $J_{(x,y)}$ of J, identifying x with a and y with b; it is assumed that all copies $J_{(x,y)}$ are pairwise vertex disjoint. The graph $H * (J, a, b)$ is said to arise from H by the *replacement operation* with respect to the *replacement graph* (J, a, b).

We say that a replacement graph (J, a, b) is *strong*, if for any irreflexive digraph H and any homomorphism $h : J \to H * (J, a, b)$ the homomorphic image $h(J)$ is contained in some copy $J_{(x,y)}$.

A graph J is *rigid* if the only homomorphism from J to J is the identity.

Proposition 9.1 *If J is both rigid and strong, then for any two irreflexive digraphs G, H without isolated vertices, $G \to H$ if and only if $G * (J, a, b) \to H * (J, a, b)$.*

Proof. Let h be a homomorphism from G to H. Since G has not isolated vertices, $V(G * (J, a, b)) = E(G) \times V(J)/_{\backsim}$ were $[(x, y), b] \backsim [(y, w), a]$; and $([(x, y), c], [(w, z), d]) \in E(G * (J, a, b))$ if and only if $(x, y) = (w, z)$ and $(c, d) \in E(J)$. Define h^* from $G * (J, a, b)$ to $H * (J, a, b)$ as follows, $h^*([(x, y), c]) = [(h(x), h(y)), c]$. It is easy to argue that h^* is a homomorphism. Now let g be a homomorphism from $G * (J, a, b)$ to $H * (J, a, b)$ we will define a homomorphism f from G to H such that $f^* = g$. Since J is strong, each copy $J_{(x,y)}$ of $G * (J, a, b)$ maps to a copy $J_{(u,v)}$ of $H * (J, a, b)$. Since J is rigid, this mapping is such that for every $c \in J$, $g([(x, y), c]) = [(u, v), c]$. Let f be the restriction of g to the vertices $[(x, y), c]$ where $c = a$ or $c = b$, then f is a homomorphism from G to H and $f^* = g$. $\square$

Define $\mathcal{D}(J, a, b) = \{H * (J, a, b) : H \in \mathcal{D}\}$. A consequence of Theorem 9.1 and Proposition 9.1 is the next.

Corollary 9.1 *If J is both rigid and strong, the class $\mathcal{D}(J, a, b)$ is complete for dichotomy.*

We say a digraph G is *acyclic* if it does not contain a direct cycle. The *algebraic length* of a digraph G is the minimal k such that there exist a homomorphism from G to $\overrightarrow{P}_k$ (the directed path of length k). The algebraic length is the number of *levels* in a "leveled" drawing of G.

By using Corollary 9.1 we next show a subclass of acyclic oriented graphs with five levels, complete for dichotomy. This is the class $\mathcal{D}(Z, a, b)$ for Z the graph in Figure 9.1.

Proposition 9.2 *The class $\mathcal{D}(Z, a, b)$ is complete for dichotomy.*

Proof. According to Corollary 9.1 we have to show that Z is both rigid and strong. We first prove Z is rigid. Let (Z, x, y) be the subpath of Z with initial vertex x and terminal vertex y. The zig-zag Z_k is defined as follows: $Z_1 = (Z, a, d)$, $Z_2 = (Z, b, f)$, $Z_3 = (Z, g, e)$, $Z_4 = (Z, c, e)$. We can deduce that Z is rigid from the next two facts. A homomorphism from Z to Z must preserve the levels and, $Z_k \to Z_{k'}$ if and only if $k' < k$.

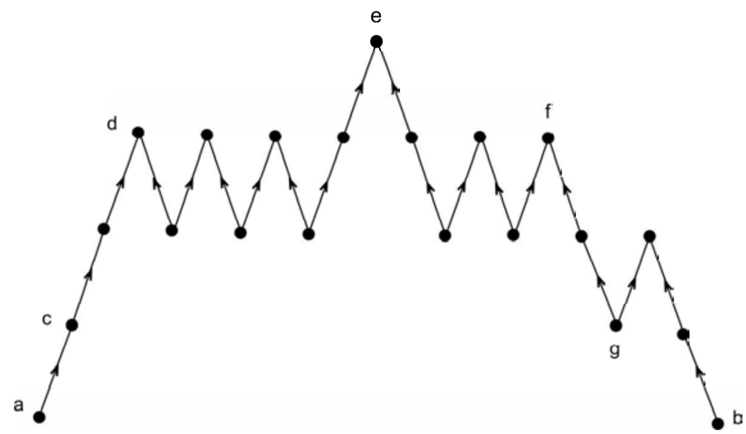


Figure 9.1: The strong and rigid acyclic oriented graph Z with five levels.

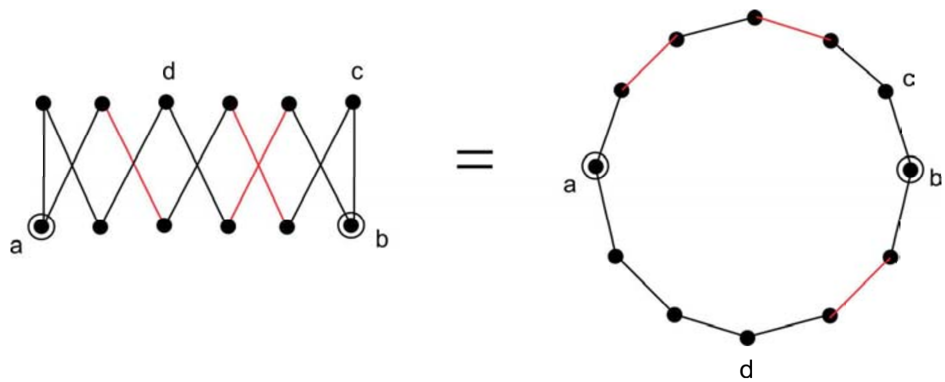


Figure 9.2: A strong and rigid bipartite 2-edge colored graph Z'.

Now we prove Z is strong. Let H be any digraph, and h any homomorphism from Z to $H * (Z, a, b)$. Is enough to observe that any subgraph of $H * (Z, a, b)$, isomorphic to $\overrightarrow{P}_3$, correspond to some (Z, a, d) in $Z_{(x,y)}$ for some $(x, y) \in E(H)$. □

Another example of complete class for dichotomy is given by the bipartite 2–edge colored graph Z' described in Figure 9.2.

Proposition 9.3 *The class $\mathcal{D}(Z', a, b)$ is complete for dichotomy.*

Proof. According to Corollary 9.1 we have to show that Z' is both rigid and strong. We first prove Z' is rigid. Let (Z', x, y) be the subpath of Z' with initial vertex x and terminal vertex y. Observe that for any homomorphism $h : Z' \to Z'$, the homomorphic

Figure 9.3:

image $h((Z', a, c))$ must be (Z', a, c). It is not difficult to argue that this force h to be the identity.

Now we prove Z' is strong. Let H be any digraph, and h any homomorphism from Z' to $H * (Z', a, b)$. Is enough to observe that any subgraph of $H * (Z, a, b)$, isomorphic to (Z', a, c), correspond to some (Z', a, c) in $Z_{(x,y)}$ for some $(x, y) \in E(H)$. □

Observe that the class $\mathcal{D}(Z', a, b)$ of bipartite 2-edge colored graphs is such that the edges of one color form a matching.

9.2 The class of bipartite 2–edge colored graphs

Let $\overrightarrow{\mathcal{B}}$ be the class of all connected bipartite oriented graphs and $\widetilde{\mathcal{B}}$ the class of all connected bipartite 2–edge colored graphs. There is a close relation between this classes that we want to understand.

The natural way to define a correspondence between $\overrightarrow{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ is the following:

- For every $\overrightarrow{G} \in \overrightarrow{\mathcal{B}}$ with bipartition $V(\overrightarrow{G}) = V_1(\overrightarrow{G}) \cup V_2(\overrightarrow{G})$, consider the 2–edge colored graph $\widetilde{G}$ with the same set of vertices, and replace all arcs from $V_1(G)$ to $V_2(G)$ with edges of color 1, and all arcs from $V_2(G)$ to $V_1(G)$ with edges of color 2.

There is a useful way to understand this correspondence by looking at the oriented paths. For any oriented path $\overrightarrow{P} = e_1, e_2...e_n$ where $e_i \in \{+, -\}$ (forward arcs and backward arcs), the corresponding 2–edge colored path $\widetilde{P} = c_1, c_2, ...c_n$ where $c_i \in \{1, 2\}$ is the following: choose c_i and then take $c_k = c_{k-1}$ if $e_k \neq e_{k-1}$ and $c_k \neq c_{k-1}$ if $e_k = e_{k-1}$.

Example 9.1 *Let $\overrightarrow{P}_n$ be the directed path of length n, then the corresponding $\widetilde{P}_n$ is the 2–edge colored path of length n with alternate colors, see Figure 9.3.*

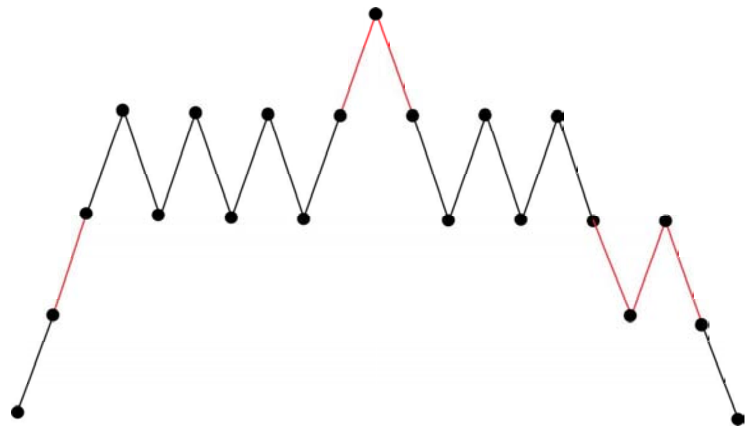


Figure 9.4:

Example 9.2 *Let Z be the directed path in Figure 9.1, then the corresponding path $\widetilde{Z}$ is the 2–edge colored path in Figure 9.4.*

There is an ambiguity in this correspondence and that is a problem if we want to preserve homomorphisms, as it is shown in Figure 9.5.

A homomorphism h from a bipartite graph G to a bipartite graph H, preserve the bipartition. That is, if the bipartitions are $V(G) = V_1(G) \cup V_2(G)$ and $V(H) = V_1(H) \cup V_2(H)$, then h has two possibilities, either $h(V_1(G)) \subseteq V_1(H)$ and $h(V_2(G)) \subseteq V_2(H)$, or $h(V_1(G)) \subseteq V_2(H)$ and $h(V_2(G)) \subseteq V_1(H)$. For the first case we say h is of *type 1*, and for the second case we say h is of *type 2*.

Consider the followings categories $\overrightarrow{\mathcal{B}}_2$ and $\widetilde{\mathcal{B}}_2$.

- An object in $\overrightarrow{\mathcal{B}}_2$ (respectively in $\widetilde{\mathcal{B}}_2$) is $(\overrightarrow{G}; V_1, V_2)$ (respectively $(\widetilde{G}; V_1, V_2)$) where $\overrightarrow{G} \in \overrightarrow{\mathcal{B}}$ (respectively $\widetilde{G} \in \widetilde{\mathcal{B}}$) and $V(G) = V_1(G) \cup V_2(G)$ is the bipartition. Homomorphisms in $\overrightarrow{\mathcal{B}}_2$ and $\widetilde{\mathcal{B}}_2$ are homomorphisms of type 1.

Now we can define $\phi : \overrightarrow{\mathcal{B}}_2 \to \widetilde{\mathcal{B}}_2$ as follows. Given $(\overrightarrow{G}; V_1, V_2) \in \overrightarrow{\mathcal{B}}_2$ define $\phi(\overrightarrow{G}; V_1, V_2) = (\widetilde{G}; V_1, V_2)$ as the 2–edge colored graph with $V_i(\overrightarrow{G}) = V_i(\widetilde{G})$, $i \in \{1, 2\}$, obtained from $\overrightarrow{G}$ by replacing all arcs from V_1 to V_2 with edges of color 1 and all arcs from V_2 to V_1 with edges of color 2.

Proposition 9.4 *Let $\phi : \overrightarrow{\mathcal{B}}_2 \to \widetilde{\mathcal{B}}_2$ defined as above, then $(\overrightarrow{G}; V_1, V_2) \to (\overrightarrow{H}; V_1, V_2)$ if and only if $\phi((\overrightarrow{G}; V_1, V_2)) \to \phi((\overrightarrow{H}; V_1, V_2))$.*

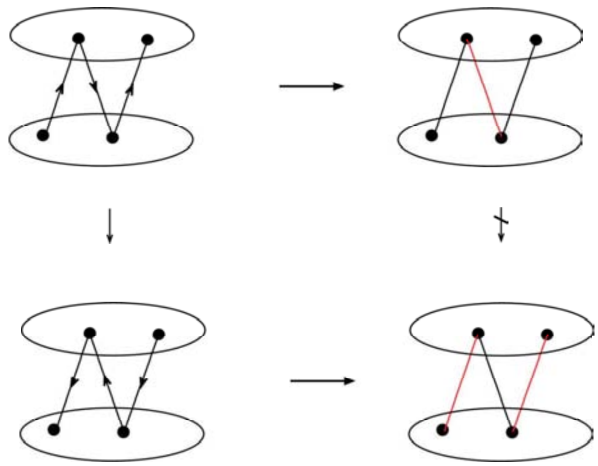


Figure 9.5:

Proof. Let h be a homomorphism from $(\overrightarrow{G}; V_1, V_2)$ to $(\overrightarrow{G}; V_1, V_2)$ in $\overrightarrow{\mathcal{B}}_2$. Since $h : V(G) \to V(H)$ is a homomorphism of type 1, all arcs from $V_1(G)$ to $V_2(G)$ are mapped into arcs from $V_1(H)$ to $V_2(H)$ and all arcs from $V_2(G)$ to $V_1(G)$ are mapped into arcs from $V_2(H)$ to $V_1(H)$. By definition $V(G) = V(\phi(G))$ and $V(H) = V(\phi(H))$. Consider the same mapping h from $V(\phi(G))$ to $V(\phi(H))$, clearly h is a homomorphism in $\widetilde{\mathcal{B}}_2$. The same argument shows that $\phi(G) \to \phi(H)$ implies $G \to H$. □

By means of this result we will prove in Section 9.2.1, that cores in $\widetilde{\mathcal{B}}$ are the same than cores in $\overrightarrow{\mathcal{B}}$.

We now define another two categories $\overrightarrow{\mathcal{B}}^*$ and $\widetilde{\mathcal{B}}^*$ and state the analogous of Proposition 9.4. For every $\overrightarrow{G} \in \overrightarrow{\mathcal{B}}$, denote by $\overleftarrow{G}$ the oriented graph obtained from $\overrightarrow{G}$ by reversing all his arrows. Let $\backsim$ denotes the relation $\overrightarrow{G} \backsim \overrightarrow{H}$ if and only if $\overrightarrow{H} \in \{\overrightarrow{G}, \overleftarrow{G}\}$.

- Let $\overrightarrow{\mathcal{B}}^* = \overrightarrow{\mathcal{B}}/_\backsim$. The objects in $\overrightarrow{\mathcal{B}}^*$ are the equivalence classes $\{\overrightarrow{G}, \overleftarrow{G}\}$. We will denote an object in $\overrightarrow{\mathcal{B}}^*$ by $[\overrightarrow{G}]$.

  We define a homomorphism from $[\overrightarrow{G}]$ to $[\overrightarrow{H}]$ in $\overrightarrow{\mathcal{B}}^*$ as a mapping $h : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that h is either an homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$, or h is a homomorphism from $\overrightarrow{G}$ to $\overleftarrow{H}$.

It is important to note that homomorphisms in $\overrightarrow{\mathcal{B}}^*$ compose: if $[\overrightarrow{G}] \to [\overrightarrow{H}]$ and $[\overrightarrow{H}] \to [\overrightarrow{K}]$ then $[\overrightarrow{G}] \to [\overrightarrow{K}]$. These happen because $\overrightarrow{G} \to \overrightarrow{H}$ implies $\overleftarrow{G} \to \overleftarrow{H}$ and $\overrightarrow{G} \to \overleftarrow{H}$ implies $\overleftarrow{G} \to \overrightarrow{H}$.

For every $\widetilde{G} \in \widetilde{\mathcal{B}}$, denote by $\widetilde{G}^*$ the 2-edge colored graph obtained from $\widetilde{G}$ by exchanching the colors. Let $\backsim$ denotes the relation $\widetilde{G} \backsim \widetilde{H}$ if and only if $\widetilde{H} \in \{\widetilde{G}, \widetilde{G}^*\}$.

- Consider $\widetilde{\mathcal{B}}^* = \widetilde{\mathcal{B}}/_{\backsim}$. The objects in $\widetilde{\mathcal{B}}^*$ are the equivalence classes $\{\widetilde{G}, \widetilde{G}^*\}$. We will denote an object in $\widetilde{\mathcal{B}}^*$ by $[\widetilde{G}]$.

  We define a homomorphism from $[\widetilde{G}]$ to $[\widetilde{H}]$ in $\widetilde{\mathcal{B}}^*$ as a mapping $h : V(\widetilde{G}) \to V(\widetilde{H})$ such that h is either a homomorphism from $\widetilde{G}$ to $\widetilde{H}$, or h is a homomorphism from $\widetilde{G}$ to $\widetilde{H}^*$.

Naturally homomorphisms in $\widetilde{\mathcal{B}}^*$ also compose.

Now we can define $T : \overrightarrow{\mathcal{B}}^* \to \widetilde{\mathcal{B}}^*$ as follows. Given $\overrightarrow{G} \in \overrightarrow{\mathcal{B}}$ take an arc $(u, v) \in E(\overrightarrow{G})$ and define $t_{(u,v)}(\overrightarrow{G})$ as the 2-edge colored graph obtained from $\overrightarrow{G}$ by replacing all arcs from V_u to V_v with red edges and all arcs from V_v to V_u with blue edges (here V_u is the stable set containing u and V_v is the stable set containing v). Observe that for any other arc $(u', v') \in E(\overrightarrow{G})$, we have $[t_{(u,v)}(\overrightarrow{G})] = [t_{(u',v')}(\overrightarrow{G})]$ since, if $u' \in V_u$ then $t_{(u,v)}(\overrightarrow{G}) = t_{(u',v')}(\overrightarrow{G})$, and otherwise $t_{(u,v)}(\overrightarrow{G}) = t_{(u',v')}(\overrightarrow{G})^*$. Observe also $[t_{(u,v)}(\overrightarrow{G})] = [t_{(v,u)}(\overleftarrow{G})]$. Define then $T([\overrightarrow{G}]) = [t_{(u,v)}(\overrightarrow{G})]$ for any $(u, v) \in E(\overrightarrow{G})$ and set $T([\overrightarrow{G}]) = [\widetilde{G}]$.

Proposition 9.5 *Let $T : \overrightarrow{\mathcal{B}}^* \to \widetilde{\mathcal{B}}^*$ defined as above, then $[\overrightarrow{G}] \to [\overrightarrow{H}]$ if and only if $T[\overrightarrow{G}] \to T[\overrightarrow{H}]$.*

Proof. By definition $[\overrightarrow{G}] \to [\overrightarrow{H}]$ implies $\overrightarrow{G} \to \overrightarrow{H}$ or $\overrightarrow{G} \to \overleftarrow{H}$, without lost of generality suppose the first and let h be a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. If h is of type 1, then $\widetilde{G} \to \widetilde{H}$ (also $\widetilde{G}^* \to \widetilde{H}^*$). If h is of type 2, then $\widetilde{G} \to \widetilde{H}^*$ (also $\widetilde{G}^* \to \widetilde{H}$). An argument similar shows that $T[\overrightarrow{G}] \to T[\overrightarrow{H}]$ implies $[\overrightarrow{G}] \to [\overrightarrow{H}]$. □

9.2.1 Cores of bipartite 2–edge colored graphs

Let H be a subgraph of G. A *retraction* of G to H is a homomorphism r such that $r(v) = v$ for all $v \in V(H)$. If there is a retraction of G to H we say that G *retracts* to H. A *core* is an object which does not retract to a proper subgraph.

An immediate consequence of Proposition 9.4 is that, cores in $\widetilde{\mathcal{B}}$ are the same than cores in $\overrightarrow{\mathcal{B}}$. More precisely we have the next proposition.

Proposition 9.6 *$\overrightarrow{G}$ is a core in $\overrightarrow{\mathcal{B}}$, if and only if $\phi(\overrightarrow{G})$ is a core in $\widetilde{\mathcal{B}}$.*

Proof. Suppose $\phi(\overrightarrow{G})$ is a core and $\overrightarrow{G}$ is not. Consider r the retraction from $\overrightarrow{G}$ to $c(\overrightarrow{G})$, the core of $\overrightarrow{G}$. Clearly r is of type 1. It follows from proposition 4 that r is a retraction from $\phi(\overrightarrow{G})$ to $\phi(c(\overrightarrow{G}))$ which is a contradiction. The argue is similar if we suppose $\overrightarrow{G}$ is a core and $\phi(\overrightarrow{G})$ is not. □

9.2.2 Duality results in the class of bipartite 2-edge colored graphs

We say that a pair of digraphs (F, H) is a *simple duality pair* in $\mathcal{D}$ (or H is a *dual* of F in $\mathcal{D}$) if every $G \in \mathcal{D}$ satisfies:

$$G \nrightarrow H \text{ if and only if } F \rightarrow G.$$

It follows from the definition that a dual is unique up to homomorphic equivalence, so we refer to the core of H as *the dual* of F.

A typical simple duality pair in $\mathcal{D}$ is the next.

Example 9.3 *Let $\overrightarrow{T}_n$ is the transitive tournament with n vertices and $\overrightarrow{P}_n$ is the directed path of length n, then $(\overrightarrow{P}_n, \overrightarrow{T}_n)$ is a simple duality pair in $\mathcal{D}$.*

We can find a proof of this in [28].

More generally, we say that $(\{F_1, F_2, ...F_t\}, H)$ have *finite duality* in $\mathcal{D}$, or H is dual of $\{F_1, F_2, ...F_t\}$ in $\mathcal{D}$, if for all $G \in \mathcal{D}$ the following holds:

$$G \nrightarrow H \text{ if and only if } F_i \rightarrow G \text{ for some } i \in \{1, ...t\}.$$

Are there simple duality pairs in $\widetilde{\mathcal{B}}$? Does $\widetilde{P}_n$ (the 2-edge colored path whit alternate colors) has a dual in $\widetilde{\mathcal{B}}$? Are there finite duality in $\widetilde{\mathcal{B}}$?

In order to answer this questions, consider $\overrightarrow{BT}_n = \overrightarrow{T}_n \times \overrightarrow{C}_2$ where $\overrightarrow{C}_2$ is the directed cycle of length 2. Observe $\overrightarrow{BT}_n$ is bipartite and, since $\overrightarrow{C}_2 = \overleftarrow{C}_2$ and $\overrightarrow{T}_n = \overleftarrow{T}_n$, also $\overrightarrow{BT}_n = \overleftarrow{BT}_n$. Recall $T[\overrightarrow{BT}_n] = [\widetilde{BT}_n]$.

Proposition 9.7 *For n even, $(\widetilde{P}_n, \widetilde{BT}_n)$ is a simple duality pair in $\widetilde{\mathcal{B}}$ and for n odd, $(\{\widetilde{P}_n, \widetilde{P}^*_n\}, \widetilde{BT}_n)$ has finite duality in $\widetilde{\mathcal{B}}$.*

Proof. The prove proceeds in three steps. First we prove that $(\overrightarrow{P}_n, \overrightarrow{BT}_n)$ is a simple duality pair in $\overrightarrow{\mathcal{B}}$. Second we establish that $([\overrightarrow{P}_n], [\overrightarrow{BT}_n])$ is a simple duality pair in $\overrightarrow{\mathcal{B}}^*$ and, using proposition 1, we translate these to $\widetilde{\mathcal{B}}^*$. Finally we develop the statement.

For the first step we use the followings facts. For any digraphs G and H:

1. $(G \times H) \to G$ also $(G \times H) \to H$.
2. $X \to G$ and $X \to H$ implies $X \to (G \times H)$.

Claim, for every $G \in \overrightarrow{\mathcal{B}}$ the following holds: $G \nrightarrow \overrightarrow{BT}_n$ if and only if $\overrightarrow{P}_n \to G$. If $G \nrightarrow \overrightarrow{BT}_n$ then, since $G \to \overrightarrow{C}_2$, we conclude from 2) that $G \nrightarrow \overrightarrow{T}_n$. We know $\overrightarrow{T}_n$ is the dual of $\overrightarrow{P}_n$ in $\mathcal{D}$, then $\overrightarrow{P}_n \to G$. Conversely, let $\overrightarrow{P}_n \to G$ and suppose $G \to \overrightarrow{BT}_n$, then $\overrightarrow{P}_n \to \overrightarrow{BT}_n$, by 1) these implies $\overrightarrow{P}_n \to \overrightarrow{T}_n$ which is a contradiction.

Now we know $(\overrightarrow{P}_n, \overrightarrow{BT}_n)$ is a simple duality pair in $\overrightarrow{\mathcal{B}}$, but since $\overrightarrow{P}_n = \overleftarrow{P}_n$ and $\overrightarrow{BT}_n = \overleftarrow{BT}_n$, we conclude $([\overrightarrow{P}_n], [\overrightarrow{BT}_n])$ is a simple duality pair in $\overrightarrow{\mathcal{B}}^*$. By proposition 1 we have $([\widetilde{P}_n], [\widetilde{BT}_n])$ is a simple duality pair in $\widetilde{\mathcal{B}}^*$.

We know $\widetilde{BT}_n = \widetilde{BT}_n^*$ and, for n even also $\widetilde{P}_n = \widetilde{P}_n^*$ but for n odd $\widetilde{P}_n \neq \widetilde{P}_n^*$. Then, for n even, $(\widetilde{P}_n, \widetilde{BT}_n)$ is a simple duality pair and for n odd $(\{\widetilde{P}_n, \widetilde{P}_n^*\}, \widetilde{BT}_n)$ has finite duality in $\widetilde{\mathcal{B}}$. □

9.3 Finite dualities in $\mathcal{D}_H$.

In the previous section (first step of the proof of Proposition 7) we found a simple duality pair in $\overrightarrow{\mathcal{B}}$, from a simple duality pair in $\mathcal{D}$. More precisely, from $(\overrightarrow{P}_n, \overrightarrow{T}_n)$ which is a simple duality pair in $\mathcal{D}$, we proved that $(\overrightarrow{P}_n, (\overrightarrow{T}_n \times \overrightarrow{C_2}))$ is a simple duality pair in $\overrightarrow{\mathcal{B}}$. Can we do the same for all duality pairs in $\mathcal{D}$?

We answer this question in a more general way.

The class $\overrightarrow{\mathcal{B}}$ is the class of all digraphs which are homomorphic to $\overrightarrow{C_2}$. That is, $\overrightarrow{\mathcal{B}} = \{G \in \mathcal{D} : G \to \overrightarrow{C_2}\}$. Define $\mathcal{D}_H = \{G \in \mathcal{D} : G \to H\}$.

Proposition 9.8 *If $(\{F_1, F_2, ...F_t\}, D)$ has finite duality in $\mathcal{D}$, then $(\{F_1, F_2, ...F_t\}, (D \times H))$ has finite duality in $\mathcal{D}_H$.*

Proof.

We use the followings facts. For any digraphs G and H:

1. $(G \times H) \to G$ also $(G \times H) \to H$.
2. $X \to G$ and $X \to H$ implies $X \to (G \times H)$.

Claim, for every $G \in \mathcal{D}_H$ the following holds:

$$G \nrightarrow (D \times H) \text{ if and only if } F_i \to G \text{ for some } i \in \{1, ...t\}.$$

If $G \nrightarrow (D \times H)$ then, since $G \to H$, we conclude from 2) that $G \nrightarrow D$. We know D is the dual of $\{F_1, F_2, ...F_t\}$ in $\mathcal{D}$, then $F_i \to G$ for some $i \in \{1, ...t\}$. Conversely, let $F_i \to G$ (for some $i \in \{1, ...t\}$) and suppose $G \to (D \times H)$, then $F_i \to (D \times H)$. By 1) these implies $F_i \to D$ which is a contradiction. □