CHARACTERISTIC CYCLES OF LOCALIZATIONS: ALGORITHMIC APPROACH

JOSEP ÀLVAREZ MONTANER AND ANTON LEYKIN

Abstract. For a polynomial ring $R = k[x_1, ..., x_n]$, we present an algorithm for computing the characteristic cycle of the localization R_f for any nonzero polynomial $f \in R$. The approach is useful to answer certain questions regarding vanishing of local cohomology modules $H_I^k(R)$, which is illustrated by our examples.

1. Introduction

Let k be a field of characteristic zero and $R = k[x_1, \ldots, x_n]$ the ring of polynomials in n variables. For any ideal $I \subseteq R$, the local cohomology modules $H_I^r(R)$ have a natural finitely generated module structure over the Weyl algebra A_n. Recently, there has been an effort made towards effective computation of these modules by using the theory of Gröbner bases over rings of differential operators. Algorithms given by U. Walther [22] and T. Oaku and N. Takayama [20] provide a utility for such computation and are both implemented in the package `D-modules` [17] for `Macaulay 2` [10].

Walther's algorithm is based on the construction of the Čech complex in the category of A_n-modules so, it is necessary to give a description of the localization R_f at a polynomial $f \in R$. An algorithm to compute these modules was given by T. Oaku in [19]. The main ingredient of the algorithm is the computation of the Bernstein-Sato polynomial of f which turns out to be a major bottleneck due to its complexity.

To give a presentation of R_f as a A_n-module is out of the scope of this work. Our aim is to provide an algorithm to compute an invariant that can be associated to a finitely generated A_n-module, the characteristic cycle. This invariant gives a description of the support of the A_n-module as an R-module, so it is a useful tool to prove the vanishing of local cohomology modules. Moreover, the characteristic cycles of these modules also give us some extra information since their multiplicities are a set of numerical invariants of the quotient ring R/I (see [2]). Among these invariants we may find Lyubeznik numbers that were introduced in [18].

Our method is based on a geometric formula given by V. Ginsburg in [9] and reinterpreted by J. Briançon, P. Maisonobe and M. Merle in [6] to compute the characteristic

---

2000 *Mathematics Subject Classification.* Primary 13D45, 13N10.

Key words and phrases. Characteristic cycle, Local cohomology, $\mathcal{D}$-modules.

Research of the first author partially supported by a Fulbright grant and the Secretaría de Estado de Educación y Universidades of Spain and the European Social Funding.

cycle of localizations. The advantage of this approach is that we will not have to compute the Bernstein-Sato polynomial of f as well as that we will be operating in a commutative graded ring in $2n$ variables instead of operating in the Weyl algebra A_n. The algorithm we will present is an elaboration of [6, Thm. 3.4.2] but we have to point out that this result is stated in the complex analytic context. For our computational purposes we are interested in the algebraic context since we will need (absolute) primary decomposition so we have to make sure that, at least for the examples we will develop, this result is also true in the algebraic counterpart. The complex algebraic case may already be found in [9] but the approach we will use in this work is through flat base change. It allow us to work over any field of characteristic zero for a large sample of examples since localization modules and local cohomology modules have a good behavior with respect to this operation. For effective computations we will use the field of rational numbers but, even though absolute primary decomposition is not implemented in `Macaulay 2`, this is not an issue for the examples we present.

The scripts of the source codes we will use in this work as well as the output in full detail of the examples are available at the web page http://www2.math.uic.edu/∼leykin/CC.

2. Basics on the theory of $\mathcal{D}$-modules

Let $X = \mathbb{C}^n$ be the complex analytic space with coordinate system $x_1, \dots, x_n$. Given the convergent series ring $R = \mathbb{C}\{x_1, \dots, x_n\}$ consider the associated ring of differential operators $D_n := R\langle \partial_1, \dots, \partial_n \rangle$, i.e. the ring extension generated by the partial derivatives $\partial_i = \frac{d}{dx_i}$, with the relations given by $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i r - r\partial_i = \frac{dr}{dx_i}$, where $r \in R$. For any unexplained terminology concerning the theory of rings of differential operators we shall use [4], [5].

The ring D_n has a natural increasing filtration given by the total order; the corresponding associated graded ring $gr(D_n)$ is isomorphic to the polynomial ring $R[a_1, \dots, a_n]$. A finitely generated D_n-module M has an increasing sequence of finitely generated R-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(D_n)$-module. The characteristic ideal of M is the ideal in $gr(D_n) = R[a_1, \dots, a_n]$ given by the radical ideal $J(M) := \mathrm{rad}\,(\mathrm{Ann}_{gr(D_n)}(gr(M)))$. The ideal $J(M)$ is independent of the good filtration on M. The characteristic variety of M is the closed algebraic set given by:

$$C(M) := V(J(M)) \subseteq \mathrm{Spec}\,(gr(D_n)) = \mathrm{Spec}\,(R[a_1, \dots, a_n]).$$

The characteristic variety describes the support of a finitely generated D_n-module as R-module. Let $\pi : \mathrm{Spec}\,(R[a_1, \dots, a_n]) \longrightarrow \mathrm{Spec}\,(R)$ be the map defined by $\pi(x, a) = x$. Then $\mathrm{Supp}_R(M) = \pi(C(M))$.

We single out the important class of regular holonomic D_n-modules. Namely, a finitely generated D_n-module M is holonomic if $M = 0$ or $\dim C(M) = n$. It is regular if there exists a good filtration on M such that $\mathrm{Ann}_{gr(D_n)}(gr(M))$ is a radical ideal.

The characteristic cycle of M is defined as:

$$CC(M) = \sum m_i \Lambda_i$$

where the sum is taken over all the irreducible components $\Lambda_i = V(\mathfrak{q}_i)$ of the characteristic variety $C(M)$, where $\mathfrak{q}_i \in \operatorname{Spec}(gr(D_n))$ and m_i is the multiplicity of $gr(M)$ at a generic point along each component Λ_i. These multiplicities can be computed via Hilbert functions (see [7],[16]). Notice that the contraction of $\mathfrak{q}_i$ to R is a prime ideal so the variety $\pi(\Lambda_i)$ is irreducible.

The irreducible varieties Λ_i that appear in the characteristic cycle of a holonomic D_n-module can be described in terms of conormal bundles to $X_i := \pi(\Lambda_i)$ in X, see [21, §10], [15, §7.5]. For completeness we recall how to compute the conormal bundles $T^*_{X_i}X$. Let X_i° be the smooth part of a subvariety $X_i \subseteq X$. Set:

$$Z = \{(x,a) \in T^*X \mid x \in X_i^\circ \text{ and } a \text{ annihilates } T_x X_i^\circ\}. \tag{2.1}$$

The conormal bundle $T^*_{X_i}X$ is the closure of Z in $T^*X|_{X_i}$. By using the results in [14], there exists a Whitney stratification $\{X_i\}_{i\in\Im}$ of X such that the characteristic cycle of a holonomic D_n-module M is in the form

$$CC(M) = \sum_{i\in\Im} m_i \; T^*_{X_i}X.$$

In particular, the support of M is $\operatorname{Supp}_R(M) = \bigcup X_i$.

2.1. **Characteristic cycle of a localization.** Let M be a regular holonomic D_n-module. Then the localization M_f at a polynomial $f \in R$ is a regular holonomic D_n-module as well. A geometric formula that provides the characteristic cycle of M_f in terms of the characteristic cycle of M is given by V. Ginsburg in [9] and became known to us through the interpretation of J. Briançon, P. Maisonobe and M. Merle in [6].

First we will recall how to compute the conormal bundle relative to f. Let Y° be the smooth part of a subvariety $Y \subseteq X$ where $f|_Y$ is a submersion. Set:

$$W = \{(x,a) \in T^*X \mid x \in Y^\circ \text{ and } a \text{ annihilates } T_x(f|_Y)^{-1}(f(x))\}.$$

The conormal bundle relative to f, denoted by $T^*_{f|_Y}$, is then the closure of W in $T^*X|_Y$.

Theorem 2.1. ([6, Thm. 3.4.2]) *Let M be a regular holonomic D_n-module with characteristic cycle $CC(M) = \sum_i m_i \; T^*_{X_i}X$ and let $f \in R$ be a polynomial. Then*

$$CC(M_f) = \sum_{f(X_i)\neq 0} m_i(\Gamma_i + T^*_{X_i}X)$$

with $\Gamma_i = \sum_j m_{ij}\Gamma_{ij}$, where Γ_{ij} are the irreducible components of the divisor defined by f in $T^*_{f|_{X_i}}$ and m_{ij} are the corresponding multiplicities.

Remark 2.2. Assume for simplicity that M is a regular holonomic D_n-module such that $CC(M) = T^*_Y X$ and let $f \in R$ be a polynomial such that $f(Y) \neq 0$. By the formula above we have $CC(M_f) = T^*_Y X + \Gamma$. It is worthwhile to point out that the reduced variety associated to Γ is the characteristic variety of the local cohomology module $H^1_{(f)}(M)$.

The multiplicities m_{ij} appearing in the formula are the multiplicities of a generic point x along each component Γ_{ij} of Γ_i and can be computed via Hilbert functions as in [16]. The main inconvenience is how to choose a generic point but we can avoid this step using the following

Lemma 2.3. *Let $e(\Gamma, x)$ be the multiplicity of the variety $\Gamma \subseteq T^*X$ defined by the ideal $I \subseteq R[a_1, \dots, a_n]$ at a point x. Then, the multiplicity m of a generic point x along Γ is*

$$m = e(\Gamma, x)/e(\Gamma^{red}, x),$$

where Γ^{red} is the variety defined by $\mathrm{rad}\,(I)$.

Proof. A reformulation of [11, Prop. 3.11] for the particular case of x being a generic point gives us the desired result, i.e. $e(\Gamma, x) = e(\Gamma^{red}, x) \cdot m$. □

2.2. **Algebraic $\mathcal{D}$-modules.** Let $X = \mathbb{C}^n$ be the complex affine space with coordinate system $x_1, \dots, x_n$. Given the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ consider the associated ring of differential operators $A_n := R\langle \partial_1, \dots, \partial_n \rangle$, i.e. the Weyl algebra. The theory of algebraic $\mathcal{D}$-modules and analytic $\mathcal{D}$-modules are very closely related. If one mimics the constructions given for the ring D_n, one can check that the results we have considered before, conveniently reformulated, remain true for A_n. In particular we may construct an algebraic characteristic cycle as a counterpart to the analytic characteristic cycle described before. Our aim is to explain how both cycles are related.

Set $\mathbb{C}\{x\} := \mathbb{C}\{x_1, \dots, x_n\}$ and $\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]$. Let M be a regular holonomic A_n-module. The D_n-module $M^{an} := \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} M$ is also regular holonomic. For a good filtration $\{M_i\}_{i \geq 0}$ on M the filtration $\{M_i^{an} := \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} M_i\}_{i \geq 0}$ is also good due to the fact that $\mathbb{C}\{x\}$ is flat over $\mathbb{C}[x]$. Therefore $gr(M^{an}) \simeq \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} gr(M)$ so the characteristic variety of M^{an} is the extension of the characteristic variety of M, i.e. $C(M^{an}) = C(M)^{an}$. However, we should notice that the components of the characteristic variety may differ depending on the ring we are considering. In particular we may have algebraically irreducible components that are analytically reducible.

The regular holonomic A_n-modules we will consider in this work, i.e the polynomial ring $R = \mathbb{C}[x]$, the localization R_f for a polynomial $f \in R$ and the local cohomology modules $H^r_I(R)$ have a good behavior with respect to flat base change. We state that, roughly speaking, the formulas of the algebraic and analytical characteristic variety of these modules are the same but the components and multiplicities of the corresponding characteristic cycle may differ.

Remark 2.4. The results of this section can be stated in general for X being any smooth algebraic variety over $\mathbb{C}$. It is worth to point out that $M \to M^{an}$ gives an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules and the category of regular holonomic $\mathcal{D}_X^{an}$-modules when X is projective.

3. Algorithmic approach to Briançon-Maisonobe-Merle's Theorem

From now on we will assume that $R = \mathbb{C}[x_1, \dots, x_n]$ is the polynomial ring so we will be working in the algebraic context. Let M be a regular holonomic A_n-module with algebraic characteristic cycle $CC(M) = \sum_i m_i \; T^*_{X_i}X$ and let $f \in R$ be a polynomial. Our aim is to compute the characteristic cycle of the localization M_f operating in the commutative graded ring $gr(A_n) = R[a_1, \dots, a_n]$. We are going to provide two algorithms that are an elaboration of Theorem 2.1. The first one is devoted to compute the part Γ_i of the formula in Theorem 2.1 corresponding to each irreducible component $T^*_{X_i}X$ in the characteristic cycle of M. The second one will compute the components and the corresponding multiplicities of the varieties Γ_i.

Theorem 2.1 is a geometric reformulation of a result given by V. Ginsburg [9, Thm. 3.3]. Even though it is stated in the analytic context me may find in Ginsburg's paper the algebraic counterpart to the same result, see [9, Thm. 3.2]. We may interpret it as in Section 2.2. through flat base change.

Algorithm 3.1. *(Divisor defined by f in $T^*_{f|_Y}$, the conormal relative to f)*

Input: Generators $g_1, ..., g_d$ of an ideal $I \subset R$ defining the algebraic variety $Y = V(I) \subseteq X$ and a polynomial $f \in R$.
Output: Divisor defined by f in the conormal $T^*_{f|_Y}$ relative to f.

Compute the smooth part Y° of Y where $f|_Y$ is a submersion:

(0a) Compute $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$
(0b) Compute the ideal $I^\circ \subset R$ such that $Y^\circ = \{x \in Y \mid \nabla f(x) \notin T_x Y\}$ is described as $Y^\circ = Y \setminus V(I^\circ)$.

Compute the conormal relative to f

(1a) Compute $K = \ker \phi$, where the $\phi : R^n \to R^{d+1}/I$ sends

$$s \mapsto (\nabla f, \nabla g_1, ..., \nabla g_d) \cdot s \in R^{d+1}/I.$$

(1b) Let $J \subset \operatorname{gr} A_n = R[a_1, ..., a_n]$ be the ideal generated by $\{(a_1, ..., a_n) \cdot b \mid b \in K\}$.
(1c) Compute $J_{sat} = J : ((\operatorname{gr} A_n) I^\circ)^\infty$; then $I(T^*_{f|_Y}) = \sqrt{J_{sat}}$. .

Compute the divisor defined by f in $T^*_{f|_Y}$

(2a) Compute $K_f = \ker \phi_f$, where the map $\phi_f : R^n \to R^{d+1}/(I + (f))$ sends

$$s \mapsto (\nabla f, \nabla g_1, ..., \nabla g_d) \cdot s \in R^{d+1}/(I + (f)).$$

(2b) Let $J_f \subset \operatorname{gr} A_n = R[a_1, ..., a_n]$ be the ideal generated by $\{(a_1, ..., a_n) \cdot b \mid b \in K_f\}$.
(2c) $C = J_{sat} + (f) + J_f \subset \operatorname{gr} A_n$.

Return: The ideal C that defines the divisor f in $T^*_{f|_Y}$

Proof. (Correctness of the algorithm) The steps (0a), (0b) follow from the definition of $f|_Y$ being a submersion. The relative conormal $T^*_{f|_Y}$ is the closure of

$$W = \{(x, a) \in T^*X \mid x \in Y^\circ, \forall s \in K,\ a(s) = 0\}$$

according to (2.1). For every point $x \in Y^\circ$, the tangent space T_xY° is a specialization of $V(K)$, where K is computed in step (1a). A defining ideal of W is produced in (1b) and, finally, taking the closure amounts to the saturation in (1c). In order to restrict to $f = 0$, it is not enough to compute $J_{sat} + (f)$. However, step (2a) and (2b) that follow closely the idea of (1a) and (1b) provide the necessary correction term in (2c).

Recall that the analytic extension of the ideal C we obtain with the algorithm is what we would obtain applying Theorem 2.1 in order to compute the analytic characteristic cycle of the localization module (see Section 2.2). In our case, the ideal C will give us the components of the algebraic characteristic variety.

□

Algorithm 3.2. *(Components and multiplicities of the characteristic cycle)*

INPUT: The characteristic cycle $CC(M) = \sum_i m_i\ T^*_{X_i}X$ of a regular holonomic A_n-module M and a polynomial $f \in R$.
OUTPUT: The characteristic cycle $CC(M_f) = \sum_{f(X_i)\neq 0} m_i(\Gamma_i + T^*_{X_i}X)$.

For every component $Y = X_i$ we have to compute the ideal C_i corresponding to the divisor defined by f in $T^*_{f|_Y}$ using Algorithm 3.1. Then:

Compute the components of C_i

- **(1a)** Compute the associated primes C_{ij} of C_i.
- **(1b)** Compute $I_{ij} = C_{ij} \cap R$ (if you need to know the defining ideal of $X_{ij} = \pi(\Gamma_{ij})$ in Theorem 2.1).

Compute the multiplicities

- **(2)** Compute the multiplicity m_{ij} in Theorem 2.1 as the multiplicity of a generic point x along each component C_{ij} of C_i as in Lemma 2.3 via Hilbert functions.

RETURN: The components of $CC(M_f)$ and their corresponding multiplicities.

Proof. The correctness of the algorithm is straightforward and follows from Lemma 2.3. □

3.1. **Implementation of the algorithm.** The algorithm we propose requires the computation of the associated primes of an ideal; primary decomposition is also needed in the implementation if we want to avoid choosing generic points when computing the multiplicities (see Lemma 2.3). Therefore, we have to restrict ourselves to computations in the polynomial ring $R = \mathbb{Q}[x_1, \dots, x_n]$ as we implemented the algorithm in the computer system `Macaulay 2`. What we are going to construct is the characteristic cycle of a regular holonomic A_n-module where now A_n stands for the Weyl algebra with rational coefficients. By flat base change we can extend the ideal C we obtain with Algorithm 3.1 to any ring of polynomials over a field of characteristic zero or to the convergent series ring over $\mathbb{C}$. As we stated in Section 2.2, the primary components may differ depending on the ring we are considering.

In order to construct the algebraic characteristic cycle over $\mathbb{Q}$ we would need to find the absolute primary decomposition of the ideal C we obtain with Algorithm 3.1. Even though the `Macaulay 2` command for primary decomposition is not implemented over the algebraic closure of $\mathbb{Q}$, it suffices for the examples we will treat in the next section.

Another fine point in the implementation is the treatment of embedded components of the ideal C outputted in the Algorithm 3.1. The ideal C contains complete information about all of the components of the divisor f on $T^*_{f|_Y}$ and the multiplicities of the maximal ones. However, the primary ideal in the decomposition of C that corresponds to an embedded component may not contain the correct information about its multiplicity due to the global nature of our computations. Therefore, extra care is needed: we restrict the divisor to the embedded component in order to get the needed multiplicity.

The scripts of the source codes we will use for the examples in the next section as well as the outputs are available at http://www2.math.uic.edu/∼leykin/CC.

4. Examples

We want to study localizations R_f of the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_n]$ at a polynomial $f \in R$. To compute its characteristic cycle directly one needs to:

· Construct a presentation of the A_n-module R_f,
· Compute the characteristic ideal $J(R_f)$,
· Compute the primary decomposition of $J(R_f)$ and its corresponding multiplicities.

The first two steps require expensive computations in the Weyl algebra A_n since we have to compute the Bernstein-Sato polynomial of f. For some short examples we can do the job just using the `Macaulay 2` commands `Dlocalize` and `charIdeal`.

Following the approach of this work we have developed some scripts written in `Macaulay 2` that compute and print out the list of components and the corresponding multiplicities showing up in the characteristic cycles of the localizations R_f in the examples we present in this section. In fact we develop two different strategies that we may use depending on the examples we want to treat.

· *Single localization:* Since the characteristic cycle of R is $CC(R) = T^*_X X$, the characteristic cycle of R_f is $CC(R_f) = T^*_X X + \Gamma$, where Γ is computed according to Theorem 2.1 so we may compute it in one step. Notice that the defining ideal of Γ may be quite large so computing its primary decomposition can be expensive.

· *Iterative localization:* We can apply Theorem 2.1 in an iterative way on the components of the polynomial f. This strategy is useful to treat large examples since we reduce the computation of primary decompositions but we must be careful with the embedded components we may find at each step.

For the examples we present in this work both strategies can be applied.

4.1. **Localization of arrangements of hyperplanes.** Arrangements of hyperplanes are good test examples in the computation of Bernstein-Sato polynomials. Consider a seemingly simple arrangement $\mathcal{A} \subseteq \mathbb{A}_k^4$ defined by the polynomial $f = x_1x_2x_3x_4(x_1+x_2+x_3+x_4)$ in $R = k[x_1, x_2, x_3, x_4]$. `Macaulay 2` runs out of memory before computing the Bernstein-Sato polynomial associated to f so we are not able to compute the characteristic cycle of the localization R_f using the direct computation. Our approach allow us to do so. The output of the script we have developed gives us the following list of components and their corresponding multiplicities:

- Let $y_i = x_i$ $(i = 1, ..., 4)$ and $y_5 = x_1+x_2+x_3+x_4$. The components corresponding to $V(y_i)$, $V(y_i, y_j)$ and $V(y_i, y_j, y_k)$ have multiplicity 1 for any i, j, k.
- The component corresponding to $V(x_1, x_2, x_3, x_4)$ has multiplicity 4.

We have to point out that in order to compute the characteristic cycle of the localization of any arrangement of hyperplanes one may also use the combinatorial indirect approach given in [3, Corollary 1.3].

4.2. **Local cohomology modules.** Consider the ideal $I \subset R = \mathbb{Q}[x_1, ..., x_6]$ generated by the minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

It is a nontrivial problem to show that the local cohomology module $H_I^3(R)$ is nonzero (see [12, Remark 3.13], [13]). For example, `Macaulay 2` runs out of memory before computing this module with the command `localCohom`. U. Walther [22, Example 6.1] gives a complete description of this module using a tailor-made implementation of his algorithm which is based on the construction of the Čech complex. The difference with the implementation of the `Macaulay 2` command is that he uses iterative localization to reduce the complexity in the computation of Bernstein-Sato polynomials.

Our method makes it possible to prove algorithmically that $H_I^3(R) \neq 0$ from the computation of the characteristic cycles of the localization modules in the Čech complex which for this particular example looks like

$$0 \to R \to R_{f_1} \oplus R_{f_2} \oplus R_{f_3} \to R_{f_1f_2} \oplus R_{f_1f_3} \oplus R_{f_2f_3} \to R_{f_1f_2f_3} \to 0, \tag{4.1}$$

where $f_1 = x_1x_5 - x_2x_4$, $f_2 = x_1x_6 - x_3x_4$ and $f_3 = x_2x_6 - x_3x_5$. It is enough to prove that a component of $CC(R_{f_1f_2f_3})$ does not belong to $CC(R_{f_if_j})$ for any $i \neq j$.

Remark 4.1. By flat base change we can also deduce the non-vanishing of the local cohomology module $H_I^3(R)$ where $R = k[x_1, ..., x_6]$ is the polynomial ring over any field k of characteristic zero.

The list of components and their corresponding multiplicities showing up in the characteristic cycles of the chains in the complex (4.1) and different from the whole space X that we get with our script contains 14 elements. A sample entry is as follows:

```
Component = V(ideal (x x  - x x , x x  - x x , x x  - x x ))
                      3 5   2 6   3 4   1 6   2 4   1 5
entries-> HashTable{{0, 1, 2} => 2}
                    {0, 1} => 1
                    {0, 2} => 1
                    {0} => 0
                    {1, 2} => 1
                    {1} => 0
                    {2} => 0
```

Namely, the component corresponding to the ideal I is present with multiplicity one in $R_{f_1 f_2}$, $R_{f_2 f_3}$, $R_{f_1 f_3}$ and with multiplicity two in $R_{f_1 f_2 f_3}$. The following is the complete list of 14 components:

$$\begin{array}{lll}
A_1 = V(f_1), & A_2 = V(f_2), & A_3 = V(f_3), \\
B_1 = V(x_3, x_6), & B_2 = V(x_2, x_5), & B_3 = V(x_1, x_4), \\
C_1 = V(x_3, x_6, f_3), & C_2 = V(x_2, x_5, f_2), & C_3 = V(x_1, x_4, f_1), \\
D_1 = V(x_1, x_2, x_4, x_5), & D_2 = V(x_1, x_3, x_4, x_6), & D_3 = V(x_2, x_3, x_5, x_6), \\
E = V(x_1, x_2, x_3, x_4, x_5, x_6), & F = V(I). &
\end{array}$$

Piecing the results of our computation together we can draw the diagram in Figure 1.

$$\emptyset \to \left[\begin{matrix} X \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right] \to \left[\begin{matrix} X[3] \\ \\ \\ D_1 \\ A_1 \\ \\ \\ D_2 \\ A_2 \\ \\ \\ D_3 \\ A_3 \\ \\ \\ \end{matrix}\right] \to \left[\begin{matrix} X[3] \\ B_1 \\ C_1 \\ D_1[2] \\ A_1[2] \\ B_2 \\ C_2 \\ D_2[2] \\ A_2[2] \\ B_3 \\ C_3 \\ D_3[2] \\ A_3[2] \\ \mathbf{F}[3] \\ \\ \end{matrix}\right] \to \left[\begin{matrix} X \\ B_1 \\ C_1 \\ D_1 \\ A_1 \\ B_2 \\ C_2 \\ D_2 \\ A_2 \\ B_3 \\ C_3 \\ D_3 \\ A_3 \\ \mathbf{F}[2] \\ \mathbf{E} \end{matrix}\right] \to \emptyset$$

FIGURE 1. Components of characteristic cycles for the Ĉech complex (4.1) (multiplicity > 1 is specified in square brackets).

Due to the additivity of characteristic cycle one can prove that $CC(H^2_I(R)) = T^*_F X$ and $CC(H^3_I(R)) = T^*_E X$. It is not hard to show, for this particular example, that the other components vanish. Notice, that in general, it is not correct to claim that the alternating sum of multiplicities of a component in a complex leads its absence from the characteristic cycles of the homology. What we really do for this example is to compute first the characteristic cycles of the local cohomology modules $H^i_{(f_1)}(R)$, $H^i_{(f_1,f_2)}(R)$ $\forall i$.

Finally, it is worth to point out that the obtained result is coherent with the fact that the local cohomology module $H^3_I(R)$ is isomorphic to the injective hull of the residue field $E_R(R/(x_1, ..., x_6))$.

4.3. **Lyubeznik numbers.** Let $R = k[x_1, ..., x_n]$ be the polynomial ring over a field k of characteristic zero. Let $I \subseteq R$ be an ideal and $\mathfrak{m} = (x_1, ..., x_n)$ be the homogeneous maximal ideal. G. Lyubeznik [18] defines a new set of numerical invariants of the quotient ring R/I by means of the Bass numbers

$$\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H^{n-i}_I(R)) := \dim_k \operatorname{Ext}^p_R(k, H^{n-i}_I(R)).$$

These invariants can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$ (see [2]). Namely,

$$CC(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \lambda_{p,i}\, T^*_E X$$

Lyubeznik numbers carry interesting topological information of the quotient ring R/I as it is pointed in [18] and [8]. To compute them for a given ideal $I \subseteq R$ and arbitrary i, p we also have to refer to U. Walther's algorithm [22] even though it has not been implemented yet. When I is a squarefree monomial ideal, a description of these invariants is given in [1]. Some other particular computations may also be found in [8] and [13].

Let $I \subset R = \mathbb{Q}[x_1, ..., x_6]$ be the ideal generated by the minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

considered above, i.e. $I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)$. We want to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^i_I(R))$ for $i = 2, 3$ and $\forall p$ so we have to construct the Čech complex

$$0 \to M \to \bigoplus_{i=1}^{6} M_{x_i} \to \cdots \to M_{x_1 \cdots x_6} \to 0, \tag{4.2}$$

where M is either $H^2_I(R)$ or $H^3_I(R)$. Then we have to compute the characteristic cycles of the localization modules and use the additivity with respect to short exact sequences.

- For $M = H^3_I(R)$ we know that its characteristic cycle is $T^*_E X$ so, applying Theorem 2.1, the Čech complex 4.2 reduces to the first term. Then,

$$CC(H^0_{\mathfrak{m}}(H^3_I(R))) = T^*_E X$$

and the other local cohomology modules vanish.

- For $M = H^2_I(R)$ we obtain

$$CC(H^2_{\mathfrak{m}}(H^2_I(R))) = T^*_E X$$
$$CC(H^4_{\mathfrak{m}}(H^2_I(R))) = T^*_E X$$

and the other local cohomology modules vanish. We are not going to present the complete output with all the components as in Figure 1 for this case but at least we are going to show the multiplicities of the component $T^*_E X$ appearing in the Čech complex 4.2 in Figure 2.

$$\emptyset \to \big[\,\emptyset\,\big] \to \big[\,\emptyset\,\big] \to \big[\,E[12]\,\big] \to \big[\,E[34]\,\big] \to \big[\,E[39]\,\big] \to \big[\,E[18]\,\big] \to \big[\,E[3]\,\big] \to \emptyset$$

FIGURE 2. Component $T^*_E X$ appearing in the Čech complex (4.2) (multiplicity > 1 is specified in square brackets).

For this example, the alternating sum of multiplicities of the components in the complex does not lead us to the final result. What we have done to get it is to compute first the characteristic cycles of the local cohomology modules

$$H^p_{(x_1)}(H^2_I(R)) \quad ,\ldots, \quad H^p_{(x_1,\ldots,x_5)}(H^2_I(R)) \quad \forall p.$$

Notice that the Čech complexes we use to compute these modules are subcomplexes of 4.2. It gives us an iterative method that allow us to use the additivity of the characteristic cycle for complexes that are shorter than the initial one. This method does not depend on the order of the generators of the maximal ideal $\mathfrak{m}$ we choose to compute the intermediate local cohomology modules.

Using the properties that Lyubeznik numbers satisfy (see [18, Section 4]), we can collect the multiplicities in a triangular matrix as follows:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix}$$

This computation is consistent with the results in [8].

References

[1] J. Àlvarez Montaner, Characteristic cycles of local cohomology modules of monomial ideals, J. Pure Appl. Algebra **150** (2000), 1–25.

[2] J. Àlvarez Montaner, Some Numerical Invariants of Local Rings, Proc. Amer. Math. Soc. **132**, (2004), 861-865

[3] J. Àlvarez Montaner, R. García López and S. Zarzuela, *Local cohomology, arrangements of subspaces and monomial ideals,* Adv. Math. **174** (2003), no. 1, 35–56.

[4] J.– E. Björk, *Rings of differential operators,* North Holland Mathematics Library, 1979.

[5] J.– E. Björk, *Analytic $\mathcal{D}$-modules and applications,* Mathematics and its Applications **247**, Kluwer Academic Publishers, 1993.

[6] J. Briançon, P. Maisonobe and M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom,* Invent. Math. **117** (1994), 531–550.

[7] F. J. Castro-Jiménez, *Théorème de division pour les opérateurs différentiels et calcul des multiplicités,* These 3ème cycle, Université Paris VII (1984).

[8] R. Garcia and C. Sabbah, Topological computation of local cohomology multiplicities, Collect. Math. **49** (1998), 317–324.
[9] V. Ginsburg, *Characteristic varieties and vanishing cycles,* Invent. Math.**84** (1986), 327–402.
[10] D. Grayson and M. Stillman, *Macaulay 2,* (http://www.math.uiuc.edu/Macaulay2).
[11] M. Herrmann, S. Ikeda and U. Orbanz, *Equimultiplicity and blowing up,* Springer-Verlag, Berlin, 1988.
[12] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules,* Invent. Math. **102** (1990), 73–93.
[13] M. Kashiwara and N. Lauritzen, *Local cohomology and $\mathcal{D}$-affinity in positive characteristic,* arXiv math. AG/0210171.
[14] M. Kashiwara and P. Shapira, *Micro-hyperbolic systems,* Acta Math. **142** (1979), 1–55.
[15] F. Kirwan, *An introduction to intersection homology theory,* Pitman Research Notes in Mathematics, Longman Scientific and Technical, 1988.
[16] M. Lejeune, *Multiplicité d'un $\mathcal{D}$-module coherent,* in Séminaire sur les opérateurs différentiels et pseudo-différentiels, 1975-76, III. Uni. Sci. et Méd. de Grenoble (1976).
[17] A. Leykin, M. Stillman and H. Tsai, *D-modules for Macaulay 2,* (http://www2.math.uic.edu/ leykin/Research/Dmodules/Dmods.html).
[18] G. Lyubeznik, *Finiteness properties of local cohomology modules,* Invent. Math., 113 (1993) 41-55.
[19] T. Oaku, *Algorithms for b-functions, restrictions and local cohomology groups of D-modules,* Adv. in Appl. Math **19** (1997), 61–105.
[20] T. Oaku and N. Takayama, *Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups,* J. Pure Appl. Algebra **156** (2001), 267–308.
[21] F. Pham, *Singularités des systèmes différentiels de Gauss-Manin,* Progress in Mathematics **2**, Birkhäuser, 1979.
[22] U. Walther. *Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties.* J. Pure Appl. Algebra 139 (1999), no. 1-3, 303–321.
[13] U. Walther, On the Lyubeznik numbers of a local ring, Proc. Amer. Math. Soc. **129**(6) (2001), 1631–1634.

Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Avinguda Diagonal 647, Barcelona 08028, SPAIN
E-mail address: `Josep.Alvarez@upc.es`

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan (M/C 249), Chicago, IL 60607-7045, USA
E-mail address: `leykin@math.uic.edu`