

Recoloring Directed Graphs

Stefan Felsner*

Clemens Huemer†

Maria Saumell‡

Abstract

Let G be a directed graph and k a positive integer. We define the k -color graph of G ($D_k(G)$ for short) as the directed graph having all k -colorings of G as node set, and where two k -colorings β and φ are joined by a directed edge $\beta \rightarrow \varphi$ if φ is obtained from β by choosing a vertex v and recoloring v so that its color is different from the colors of all its out-neighbors. We investigate reachability questions in $D_k(G)$. In particular we want to know whether a fixed legal k' -coloring ψ of G with $k' \leq k$ is reachable in $D_k(G)$ from every possible initial k -coloring β . Interesting instances of this problem arise when G is planar and the orientation is an arbitrary α -orientation for fixed α . Our main result is that reachability can be guaranteed if the orientation has maximal out-degree $\leq k - 1$ and an accessible pseudo-sink.

1 Introduction

Given a k -colored directed graph G , we study the problem of recoloring the vertices of G , one at a time and maintaining a certain recoloring rule, in order to obtain a legal k -coloring of G . This kind of question has been investigated recently for the case of undirected graphs [1, 2, 3], where only legal k -colorings are considered and one wants to recolor vertices to reach any legal k -coloring from a given legal k -coloring of the graph.

A main object of interest are maximal bipartite planar graphs, or quadrangulations. It is well known that they admit a 2-orientation, that is, the edges of a quadrangulation G can be directed so that each vertex (but two special ones) has out-degree 2 (precise definitions are given below). Such a 2-orientation is not unique, and it is possible to modify a given 2-orientation by inverting the direction of the edges of a directed cycle. This local modification defines a graph on all 2-orientations of G , and some work has been done to identify the properties of this graph. For example, it is known that it has the structure of a distributive lattice [4] and that its diameter is $O(n^2)$ [9], where n is the number of vertices of G .

Here we consider an arbitrary fixed 2-orientation of a quadrangulation G , and look at the graph $D_3(G)$, which is defined on all 3-colorings of G . The edges of this graph correspond to local modifications obtained by recoloring a vertex: we allow to recolor a vertex if its new color is different from the colors of the two vertices it points to. We study reachability questions in $D_3(G)$. In particular, given any 3-coloring of the vertices of G , can a legal 3-coloring be reached? And how many recoloring steps are needed? We answer the first question in the positive -in fact we can even reach a legal two-coloring of G - and show an upper bound of $O(n^2)$ on the diameter of $D_3(G)$.

These 2-orientations are a particular instance of α -orientations. We generalize our result to a special class of α -orientations. This includes maximal planar graphs, or triangulations, which are known to admit a 3-orientation. We obtain that, given any 4-coloring of a 3-oriented triangulation, there is a sequence of $O(n^2)$ recoloring steps that transforms the given 4-coloring into a legal 4-coloring.

In the following we formally state the problem and some definitions.

*Institut für Mathematik, Technische Universität Berlin, felsner@math.tu-berlin.de.

†Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Barcelona, Spain, clemens.huemer@upc.edu.

‡Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain, maria.saumell@upc.edu.

Let G be a directed graph and k a positive integer. A k -coloring of G is an arbitrary assignment of colors from the set $\{1, 2, \dots, k\}$ to the vertices of G , that is, a mapping $\gamma : V \rightarrow \{1, 2, \dots, k\}$. A *legal k -coloring* is a k -coloring with the property that the end-vertices of every edge receive different colors.

Given G and k , we define the k -color graph of G , denoted by $D_k(G)$, as follows: its node set consists of all k -colorings of G , and two k -colorings β and φ are joined by a directed edge $\beta \rightarrow \varphi$ if they differ in color on precisely one vertex v of G and $\beta(v) \neq \varphi(w)$ for all out-neighbors w of v , in other words, if φ is obtained from β by choosing a vertex v and recoloring v so that its color is different from the colors of all its out-neighbors. A recoloring of a vertex in this way will be called a *valid recoloring step*.

If β and φ are two k -colorings of G , we may ask whether there is a path from β to φ in $D_k(G)$. In this case, we say that φ is *reachable* from β .

An α -orientation of a graph $G = (V, A)$ is an orientation of its edges such that the out-degree of every vertex $v \in V$ is given by a function $\alpha : V \rightarrow \mathbb{N}$ (see, for example, [7, 4, 8]). We will say that a vertex v is *accessible* if G contains a directed path from every other vertex to v . We deal with α -orientations with an special vertex accessible from every other vertex and in which the out-degree of each vertex is bounded from above. More precisely, in Section 2 we look at the particular case of 2-orientations of planar quadrangulations (of which we describe the 3-color graph). In Section 3 we generalize our result and we highlight its consequences on 3-orientations of planar triangulations.

2 2-orientation and 3-colorings of planar quadrangulations

2.1 Definitions and results

A *plane quadrangulation* is a plane graph $Q = (V \cup \{s, t\}, E)$ such that Q is a maximal bipartite graph with $|V| = n$, and s, t are two non-consecutive vertices on the outer face. A *2-orientation* of Q is an orientation of its edges such that every vertex different from s and t has out-degree two. Given a 2-orientation of Q , it is easy to see that s and t are sinks, i.e., vertices with out-degree zero. It has been proved that every quadrangulation admits a 2-orientation [7, 9].

Since a quadrangulation is a bipartite graph, its vertices can be divided into two groups, the white ones and the black ones, in such a way that every edge connects a white and a black vertex. Let us assume that s and t are black vertices. The following definition is useful for our purposes:

Definition 2.1. A *separating decomposition* is an orientation and coloring of the edges of Q with colors red and blue satisfying that:

- (1) The edges incident to s are colored red and the edges incident to t are colored blue.
- (2) Every vertex $v \neq s, t$ is incident to an interval of red edges and an interval of blue edges. Among these edges, only one red edge and one blue edge are outgoing. If v is white, the two outgoing edges are the first ones in clockwise order in the interval of edges of their color. If v is black, the outgoing edges are the clockwise last in their color.

It can be shown that separating decompositions fulfill the additional property that the red edges form a directed tree rooted in s (T_r for short) and the blue edges form a tree rooted in t (T_b for short) [5]. See Figure 1 for an example. This ensures that there exist paths in Q from every vertex $v \neq s, t$ to s and t .

For each non-special vertex $v \neq s, t$ we denote the directed paths in T_r and T_b from v to the corresponding sink by $P_r(v)$ and $P_b(v)$, respectively. Then one can prove that the paths $P_r(v)$ and $P_b(v)$ only share vertex v , so they split the quadrangulation into two regions [6]. We define $R_r(v)$ (respectively, $R_b(v)$) to be the region to the right of $P_r(v)$ (respectively, $P_b(v)$) and including both paths. A nice observation here is that, for each vertex of the path, all incoming edges belonging to a fixed region have the same color.

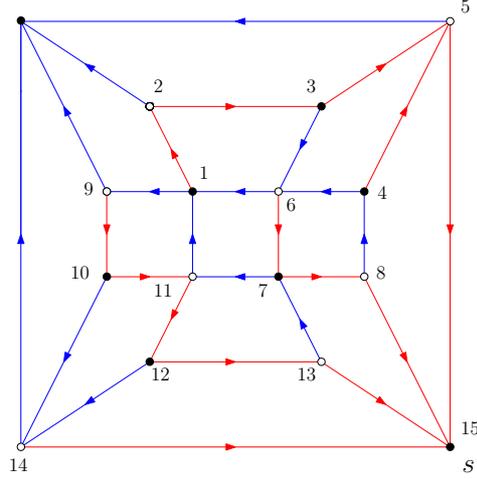


Figure 1: A quadrangulation with a separating decomposition.

Separating decompositions and 2-orientations are closely related. Indeed let $Q = (V \cup \{s, t\}, E)$ be a plane quadrangulation; then it has been proved that there is a bijection between separating decompositions and 2-orientations of Q [5, 7]. To be precise, a separating decomposition yields a 2-orientation, while the edges of a 2-orientation can be colored so as to obtain a separating decomposition.

Theorem 2.2. *Let G be a 2-orientation of a planar quadrangulation. Then, given any 3-coloring β of G , it is always possible to reach a legal 2-coloring in $D_3(G)$.*

2.2 Proof of Theorem 2.2

Our proof is constructive, that is, we provide an algorithm to obtain the path from β to a legal 2-coloring in $D_3(G)$, and we show its correctness.

A key ingredient is the following Lemma 2.3. The idea behind it is as follows. Suppose that v is a non-special vertex such that v and all its out-neighbors have different colors in β (v is a *rainbow* in β). Although the two outgoing edges from v are legal, we might need to recolor v to reach a legal coloring in $D_3(G)$, which is not possible in one single valid recoloring step. Lemma 2.3 shows that v can nevertheless be recolored.

Suppose that ς, τ are two different colors in the set $\{1, 2, 3\}$. The expression $\overline{\{\varsigma, \tau\}}$ will denote the color in $\{1, 2, 3\}$ different from ς and τ .

Lemma 2.3. *Let $P = \{v_l, v_{l-1}, \dots, v_2, s\}$ be a directed path in G without any directed chord. Let $f(v_l)$ denote the color of the vertex pointed to by v_l different from v_{l-1} . If v_l is colored ς and $f(v_l) = \tau$, there exists a sequence of valid recoloring steps involving only $v_l, v_{l-1}, \dots, v_2, s$ such that, at the end, v_l has color $\overline{\{\varsigma, \tau\}}$.*

Proof. We prove the statement by induction on the length l of P . First observe that, in any situation, we can assume that s has the most convenient color, as all edges incident to this vertex are incoming.

Let us look at the case $l = 2$, i. e., $P = \{v_2, s\}$. Suppose s is colored ς . Then, in a valid recoloring step, v_2 can be assigned color $\overline{\{\varsigma, \tau\}}$. Now assume that the lemma is valid for all paths of length at most $l - 1$. If v_{l-1} has color different from $\overline{\{\varsigma, \tau\}}$, then v_l can be recolored with this color. Otherwise, by hypothesis of induction, v_{l-1} can be assigned a color different from $\overline{\{\varsigma, \tau\}}$ by applying a sequence of valid recoloring steps that only involves vertices $v_{l-1}, v_{l-2}, \dots, v_2, s$. Since v_l does not point to any of the vertices $v_{l-2}, v_{l-3}, \dots, v_2, s$, these recoloring steps do not modify the value of $f(v_l)$. Therefore, in the new coloring v_l points to a vertex with color τ and a vertex with color different from $\overline{\{\varsigma, \tau\}}$, so it can be assigned color $\overline{\{\varsigma, \tau\}}$. \square

Observe that the previous proof already provides an algorithm to recolor vertex v_l in $O(l)$ valid recoloring steps. It consists of visiting the vertices of the path (starting from v_l) until a non-rainbow vertex is found, changing the color of this vertex and then recoloring backwards. Figure 2 shows an example of this process.

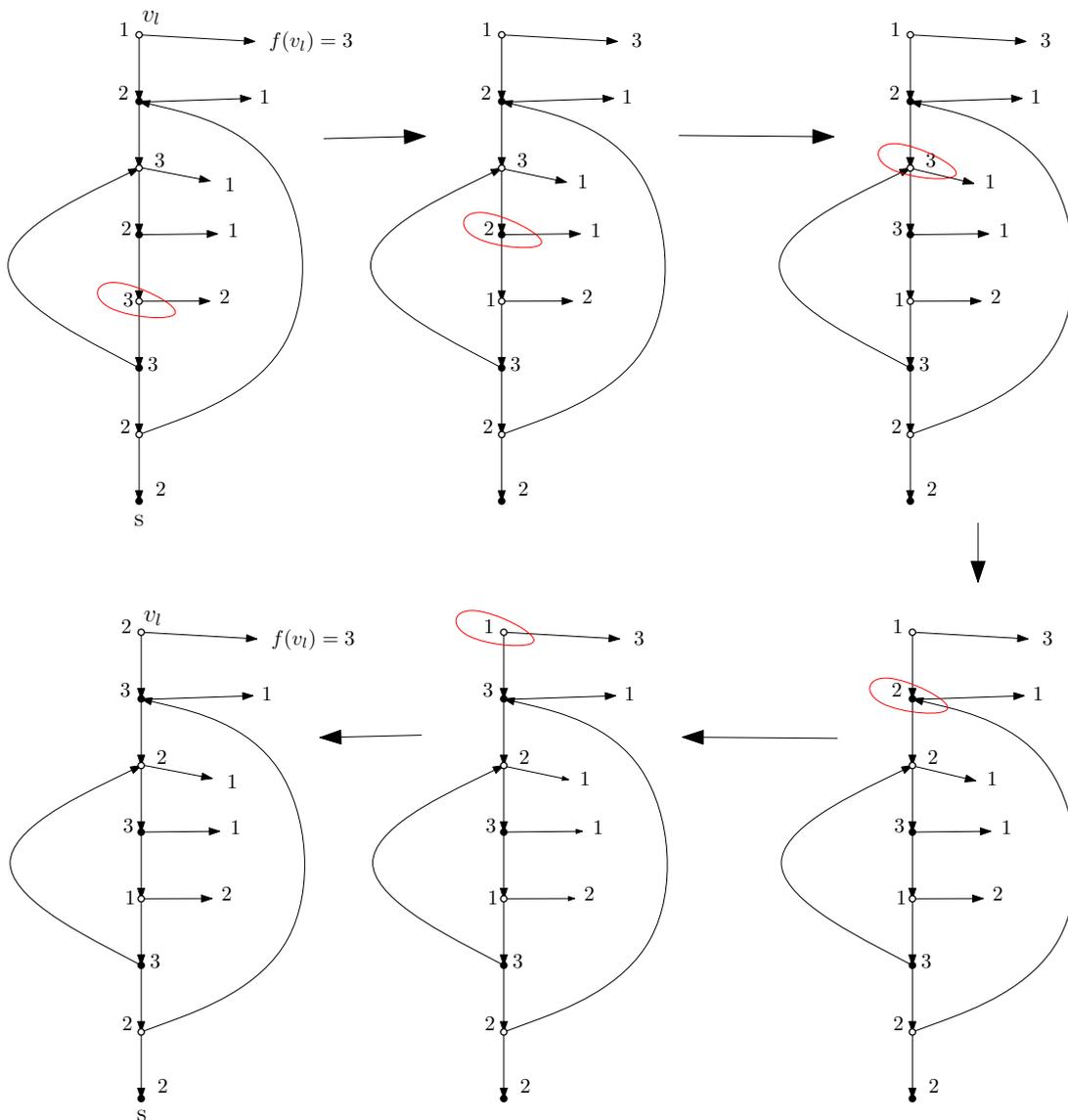


Figure 2: An example for Lemma 2.3. It is shown how v_l can be recolor.

Let ψ be the (legal) coloring of G where the black vertices have color 1 and the white vertices have color 2. Algorithm 1 gives a path in $D_3(G)$ from the coloring β to the coloring ψ .

Step 2 of Algorithm 1 involves postorder traversal in the red tree. In a binary tree, postorder traversal performs the following operations recursively at each node: (1) traverse the left subtree; (2) traverse the right subtree; (3) visit the root. This process can be generalized in the natural way to plane m -ary trees, as the notions of left and right are well-defined. The vertices in Figure 1 are labelled according to postorder in the red tree.

Let us introduce some notation. If $v \neq s, t$ is a vertex in G , then the red (respectively, blue) edge outgoing from v points to a vertex that will be denoted by v_r (respectively, v_b). If the path $P_r(v)$ is of the form $P_r(v) = \{v, v_{l-1}, \dots, v_2, s\}$, we define $P'_r(v)$ as one of the paths from v to s on a subset of $\{v, v_{l-1}, \dots, v_2, s\}$ which cannot be shortened by directed chords.

Lemma 2.4. *Given β , Algorithm 1 gives a sequence of $O(n^2)$ valid recoloring steps to reach ψ .*

Algorithm 1 Reaching a legal 2-coloring

- 1: If necessary, recolor vertex t with 1.
 - 2: Sort all vertices different from s and t according to the postorder traversal sequence in the red tree. Let L be the ordered list containing the vertices.
 - 3: Let v be the first element in L . Remove v from L .
 - If v is white and v_r has color 2, apply the algorithm in Lemma 2.3 to recolor v_r using the path $P'_r(v_r)$. Afterwards, assign color 2 to v .
 - If v is black, apply (if necessary) the algorithm in Lemma 2.3 to make vertices v_r and v_b have color different from 1 by using the paths $P'_r(v_r)$ and $P'_r(v_b)$, respectively. Afterwards, assign color 1 to v .
 - Repeat until L is empty.
 - 4: If necessary, recolor vertex s with 1.
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Proof. Let us look at Step 3 of the algorithm. Let v be the first element in L at some point of the algorithm. If v is a white vertex, the red edge outgoing from v_b belongs to the region $R_b(v)$ (see Figure 3). This implies that v_b has already been processed and, as will be shown in a few lines, has its final color, which, in this case, is 1. Our aim is to assign to v its final color, namely, 2. If v_r is not colored 2, we can do it in one single valid recoloring step. Otherwise we first change the color of v_r by means of the algorithm in Lemma 2.3; we use the path $P_r(v_r) = \{v_r, v_{l-1}, \dots, v_2, s\}$, possibly shortened by directed chords. Since this algorithm only recolors vertices in the path and all of them are after v in L , we do not change the color of any vertex that has already been processed.

If v is black, the red edge outgoing from v_b belongs to $R_r(v)$ (see Figure 3). Since all red edges pointing to v are in $R_b(v)$, v_b is after v if T_r is traversed in postorder. So, in this case, neither v_r nor v_b have already been processed and we might change their color to be able to assign color 1 to v . Again, if we use the paths $P'_r(v_r)$ and $P'_r(v_b)$, we do not recolor any vertex that has already been treated.

To conclude, after an iteration of Step 3 the number of vertices that are assigned its final color and will not be recolored in future steps increases by one. Consequently, the algorithm is correct.

As for the number of valid recoloring steps performed by the algorithm, the assignment of the final color of each vertex v costs $O(l)$ recolorings, where l is the length of the path $P_r(v)$. Therefore, the total number of recoloring steps is $O(n^2)$. \square

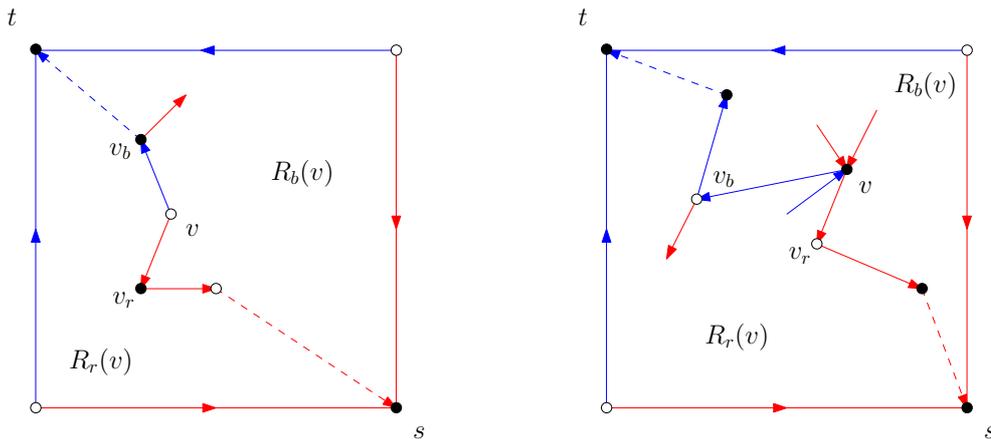


Figure 3: Postorder in the red tree of the vertices v, v_r, v_b .

3 A general result and an interesting application

In this section we generalize Theorem 2.2 to a special class of α -orientations that includes 2-orientations of planar quadrangulations and 3-orientations of planar triangulations.

Let G be a directed graph in which the maximum out-degree of the vertices is $k - 1$, and let β be a k -coloring of G . As in the case $k = 3$, we will say that a vertex v is a *rainbow in β* if v has out-degree $k - 1$, and v and all its out-neighbors have different colors in β . We will say that v is a *pseudo-sink* if it is guaranteed to be a non-rainbow in every k -coloring of G . Finally, recall that v is *accessible* if G contains a directed path from every other vertex to v .

Theorem 3.1. *Let G be a directed graph where the maximum out-degree of the vertices is $k - 1$. Assume that G has an accessible pseudo-sink s . Then, given any k -coloring β of G and any legal k' -coloring ψ of G with $k' \leq k$, ψ is reachable from β in $D_k(G)$.*

The method we propose to obtain a path from β to ψ in $D_k(G)$ is detailed in Algorithm 2. Here we also use an auxiliary lemma ensuring that each vertex in G can be recolored modifying only the color of some particular vertices. We omit its proof because its very similar to the one in Lemma 2.3:

Lemma 3.2. *Let $P = \{v_l, v_{l-1}, \dots, v_2, s\}$ be a directed path in G without any directed chord. If v_l is colored ς , there exists a sequence of $O(l)$ valid recoloring steps involving only $v_l, v_{l-1}, \dots, v_2, s$ such that, at the end, v_l has a color different from ς .*

In Algorithm 2 we talk about distances in the graph G . The *distance* between a vertex v and a vertex w in G is the length (i.e., the number of edges) of the shortest directed path from v to w in G ($d_G(v, w) = +\infty$ if there are no directed paths from v to w).

Algorithm 2 Reaching a legal k' -coloring

- 1: Sort all vertices by decreasing distance to s (choose any order if there are ties). Let L be the ordered list containing the vertices. For each $v \in L$, let $\delta_{\leq}(v)$ be the set of out-neighbors of v that are after v in L .
 - 2: Let v be the first element in L . Remove v from L .
Apply (if necessary) the algorithm in Lemma 3.2 to make the vertices in $\delta_{\leq}(v)$ have color different from $\psi(v)$ by using their shortest paths in G to s . Afterwards, assign color $\psi(v)$ to v .
Repeat until L is empty.
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Lemma 3.3. *Given β and ψ , Algorithm 2 gives a sequence of $O(n^2)$ valid recoloring steps to reach ψ from β .*

Proof. The vertices are processed in such a way that the recolorings necessary to assign the definite color to a vertex do not involve the vertices that have already been treated. Thus, after each iteration of Step 2, the number of vertices that are assigned its final color and will not be recolored increases.

The number of valid recoloring steps performed by the algorithm can be counted as in the proof of Lemma 2.4. □

An remarkable example of a family of directed graphs that fulfill the hypothesis of Theorem 3.1 are 3-orientations of planar triangulations.

A *plane triangulation* T is a maximal plane graph with n vertices and three special vertices a_1, a_2, a_3 in the outer face. A *3-orientation* of T is an orientation of its edges such that every vertex different from a_1, a_2, a_3 has out-degree three. It can be proved that every triangulation admits a 3-orientation [7].

Furthermore, it is known that the interior edges of a triangulation can be colored with 3 colors with the property that the edges of each color form a directed tree that spans all the inner vertices and is rooted at one of the vertices in the outer face [10]. These are so-called *Schnyder woods*. An example of a triangulation with a Schnyder wood is shown in Figure 4.

An important analogy with separating decompositions and 2-orientations is the fact that, if T is a plane triangulation with outer vertices a_1, a_2, a_3 , then Schnyder woods and 3-orientations of T are in bijection (see [5, 7]). Furthermore, the edges of a 3-orientation of T can be colored to obtain a Schnyder wood.

Corollary 3.4. *Let G be a 3-orientation of a planar triangulation. Then, given any 4-coloring β of G , it is always possible to reach any legal 4-coloring in $D_4(G)$.*

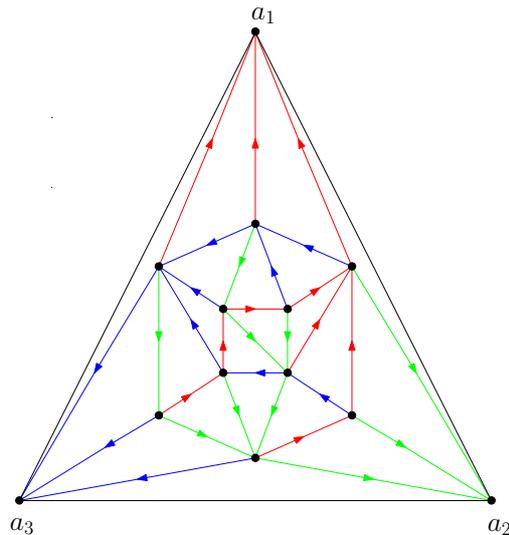


Figure 4: A triangulation with a Schnyder wood.

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