MESH-FREE METHODS FOR DYNAMIC PROBLEMS
INCOMPRESSIBILITY AND LARGE STRAINS

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Doctoral Thesis
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Als meus pares,
a la meva família,
al Toni.
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Chapter 1

Introduction

Computational methods that use a finite grid to represent the domain, such as finite elements, finite volumes or finite difference methods, have been developed for the past four decades. These methods have become the industrial conventional tools for solving a wide range of problems in continuum mechanics. However, there are many industrial and academic problems which cannot be easily treated with these classical mesh based methods. For instance, the simulation of manufacturing processes such as extrusion and molding where it is necessary to deal with extremely large deformations, or fracture dynamic problems, where the simulation of the propagation of cracks with arbitrary and complex paths is needed.

Mesh based methods perform a partition of the domain into certain non overlapping elements, which are explicitly representative of a piece of the domain. In domains undergoing very high deformations, these individual elements may suffer from severe distortions, causing mesh deformation and/or entanglement leading to a loss of accuracy in the computations. These difficulties arise due to the rigid connectivity structure defined by elements. Meshless methods eliminate part of this mesh dependent structure by constructing the approximation entirely in terms of nodes (usually called particles in the context of mesh-free methods). In
meshless methods large deformations can be handled more robustly because the interpolation is not based on elements whose distortion may lead to a dramatically loss of accuracy. When problems involving large deformations and distortions are solved, both, mesh based and mesh-free methods require a remesh (which consists in a new neighbor search for mesh-free methods). Nevertheless, it is much more computationally expensive to calculate a new mesh and project quantities from the old to the new mesh rather than finding new neighbors in mesh-free methods.

Meshfree methods also present significant advantages in adaptivity. There are a priori error estimates for most of the meshfree methods. This allows the definition of adaptive refinement processes as in finite element computations, see Babuška and Melenk (1995); Melenk and Babuška (1996); Duarte and Oden (1996a); Babuška, Banerjee and Osborn (2002b). Strategies based on $h$-adaptivity in a mesh-based context consist on building a new mesh and adapt the element size to the requirements of the solution. In mesh-free methods no mesh generation is needed and $h$-adaptivity consists in adding particles where needed.

The range of problems that can be addressed by meshfree methods is much wider than mesh-based methods and the key issue is the elimination of the mesh. Finally just observe that the advantages of meshfree methods for 3D computations is evident as mesh generation is not needed anymore.

1.1 Scope of the Thesis

Although mesh-free methods were originated over twenty-five years ago, the popularization of these methods is still in an early state. The aim of this thesis is to advance in the development and understanding of these methods through some contributions in this novel research line which is experimenting an intense development nowadays. There are commercial softwares as LS-Dyna and Plexis dealing with fast transient dynamic problems which are incorporating mesh-free
methods.

Concretely, the feasibility of using meshless methods for large strain dynamic problems is investigated: the mathematical models and robust implementation. In this class of problems, when dealing with metals, it is assumed that the behavior of the material is incompressible. In finite elements, locking in the incompressible limit has been a major concern since its early developments and has been extensively studied. However, locking in mesh-free methods is still an open topic. The first developments in the framework of the thesis concern the study of locking in the incompressible limit and include a contribution in this area: the pseudo-divergence-free element free Galerkin method. This method asymptotically suppresses volumetric locking as the distribution of particles is refined.

Next, meshless formulations for solving large strain dynamic problems are studied. In finite deformation analysis two possible coordinate systems can be chosen to describe the continuum under consideration (Bonet and Wood 1997):

- a certain reference configuration (usually an “initial configuration”). This is called a Lagrangian or material description, and all relevant quantities are referred to an initial problem domain;

- the current continuum configuration. This is called an Eulerian or spatial description; relevant quantities are referred to the current problem domain.

The former is most frequent in solid mechanics and history-independent materials, whereas the latter is typical in fluid mechanics. These two descriptions will lead, in general, to non-equivalent discretizations in particle methods (Bonet and Kulasegaram 2001). On one hand Eulerian formulations present the tensile instability but on the other hand Lagrangian formulations are not suitable for problems involving large distortions. The contribution in this area is an updated Lagrangian formulation which does not present tensile instability conserving all advantages
of an Eulerian formulation.

1.2 Layout of the Thesis

The thesis is organized in the following way.

Chapter 2 contains a state of the art in mesh-free methods since nowadays. The aim of this chapter is to provide an overview of meshfree particle methods highlighting the main differences among them and also their similarities.

In Chapter 3 contribution done in the area of locking due to incompressibility is presented in detail by the papers:

“Locking in the incompressible limit: pseudo-divergence-free element free Galerkin”, published in the Communications in Numerical Methods in Engineering (Vidal, Villon and Huerta 2003) showing the solids applicability


Chapters 4 and 5 are related to large strain dynamic problems. The first one contains a review on existing Eulerian and Lagrangian CSPH formulations. The second one contains the contribution done in this area: an updated Lagrangian formulation which solves one of the main weak points of SPH method, namely the presence of zero energy modes.

Finally a summary (Chapter 6) of the major and noteworthy results is performed and a chapter is dedicated to future developments (Chapter 7).
Chapter 2

State of the art in mesh-free methods

2.1 Introduction

Mesh-free methods were originated over twenty-five years ago, but the research effort devoted to them until the nineties compared to other methods as finite elements was small. Smooth Particle Hydrodynamics (SPH) was initially developed by Lucy (1977) and Gingold and Monaghan (1977), see Section 2.2.1 for solving astrophysical hydrodynamics phenomena such as exploding stars and dust clouds. As has been said compared to other methods the rate of publications was small for many years and it is mainly represented in the papers of Monaghan (1982, 1988). In these papers, the method was explained as a kernel estimate to provide a more rational basis to the approximation method, however there was little effort in the estimation of the accuracy.

A parallel path to construct meshless approximations is the use of moving least-squares approximations, see Section 2.2.2. Moving least-squares (MLS) approximations originated in data fitting. Some of the first studies of their behavior were performed by Barnhill (1977), Gordon and Wixom (1978), McLain
In an overview, Lancaster and Salkauskas (1981) point out two special cases of the MLS approximation. If the polynomial basis consists of only a constant, the method reduces to a weighted averaging technique; if the weight functions are singular at the nodes with which they are associated, the approximation becomes an interpolation, i.e., the approximation passes through the nodal data. Shepard’s interpolant (Shepard 1968), is shown to be the combination of these two cases.

Nayroles, Touzot and Villon (1992) were the first to use MLS approximation in a Galerkin formulation framework. They called the resulting formulation the diffuse element method (DEM). Several key features of the method are recognized. First, the regularity of the shape functions is shown to be equivalent to that of the weight functions. Thus, it is simple to obtain shape functions with continuous first derivatives (and therefore continuous stresses), and the post-processing projection schemes typically applied to $C^0$ finite elements are not needed. Second, MLS approximations exactly reproduce functions appearing in its polynomial basis. Third, shape functions do not, in general, take a value of unity at the nodes with which they are associated nor vanish at all other nodes. This latter feature introduces an important difficulty for the imposition of essential boundary conditions. Finally, the relative ease with which meshes are generated and modified is suggested with reference to adaptive schemes.

Belytschko, Lu and Gu (1994) made three refinements to DEM in order to increase the accuracy of the method: Lagrange multipliers are introduced in the potential energy functional to enforce the essential boundary conditions, the derivatives of the shape functions are evaluated exactly, and the spatial integration is improved by introducing a regular cell structure which is independent of the nodes. The resulting formulation is called the element free Galerkin method (EFG). This class of methods is consistent and, in the forms proposed, quite
stable, although substantially more expensive than SPH. Recently, the work in (Duarte and Oden 1996b, Babuška and Melenk 1995) recognizes that the methods based on moving least-squares are specific instances of partitions of unity. These references and Liu, Li and Belytschko (1997) were among the first to prove convergence of this class of methods.

At the same time that moving least-squares methods were developed, SPH formulation was improved in several manners. For instance, Dyka (1994) and Swegle, Hicks and Attaway (1995) study its instabilities; Johnson and Beissel (1996) proposed a method for improving strain calculations; and Liu, Jun and Zhang (1995) present a correction function for kernels in both the discrete and continuous case. Reference (Vila 1999) has introduced a different mesh-free approximation: the renormalized Meshless Derivative (RMD) which turns out to give accurate approximation of derivatives in the framework of collocation approaches. Bonet and coworkers have developed a Corrected SPH method called CSPH. Reference (Bonet and Lok 1999) combines a correction of the window function and a gradient correction with applications to fluid dynamics. In (Bonet and Kulasegaram 2000) CSPH is further improved by introducing an integration correction which enables the method to pass the patch tests. The effectiveness of the method in simulating metal-forming problems is showed.

In recent papers the possibilities of mesh-free methods become more and more apparent. The special issue of CMAME (1996) shows the ability of mesh-free methods to handle complex situations, such as impact problems, crack simulations or fluid dynamics. Moreover, several authors have proposed to use mixed interpolations combining finite elements and mesh-free methods, in order to take profit of the advantages of each method (Belytschko, Organ and Kröngauz 1995, Hegen 1996, Liu, Uras and Chen 1997, Huerta and Fernández-Méndez 2000). Several review papers and books have been published on meshfree methods; Liu,

This chapter is devoted to the description of the most common mesh-free methods and to the analysis of the differences and similarities among the existing formulations. Firstly, the most popular mesh-free approximations are presented (SPH and MLS). Secondly, their application to the resolution of PDE boundary value problems its presented (collocation techniques and Galerkin formulations are used).

2.2 Mesh-free approximations

In this section, the most common approximations used in mesh-free methods are described. The name “approximants” will be used because these approximants usually do not pass through the data, so they are not interpolants. Meshfree approximants can be classified in those based on smooth particle hydrodynamics (SPH) and those based on moving least-squares (MLS).

2.2.1 Smooth Particle Hydrodynamic

The original SPH

The earliest mesh-free method is the SPH method (Lucy 1977). The method is based on a simple property of the Dirac delta function $\delta(x)$,

$$u(x) = \int \delta(x - y)u(y)dy.$$
where \( u(\mathbf{x}) \) is the function to be approximated. The key idea is to replace the Dirac delta function by a kernel or weight function \( C_\rho \phi \left( \frac{\mathbf{x} - \mathbf{y}}{\rho} \right) \) positive, even and compact supported (Figure 2.1),

\[
\phi((x_i - x)/\rho)
\]

Figure 2.1: Reproducing kernel.

\[
u(\mathbf{x}) \simeq \tilde{u}^\rho(\mathbf{x}) := \int C_\rho \phi \left( \frac{\mathbf{x} - \mathbf{y}}{\rho} \right) u(\mathbf{y}) d\mathbf{y},
\]

(2.2.1)

where \( \rho \) is called the dilation parameter and is usually the support radius of the kernel function. \( C_\rho \) is a normalization constant such that

\[
\int C_\rho \phi \left( \frac{\mathbf{x} - \mathbf{y}}{\rho} \right) d\mathbf{y} = 1,
\]

i.e. constant functions are exactly interpolated. Therefore, as \( \rho \) tends to zero the kernel function approaches the Dirac delta function, and consequently,

\[
\lim_{\rho \to 0} \tilde{u}^\rho(\mathbf{x}) = u(\mathbf{x}).
\]

Most commonly used kernel functions are,
Cubic Spline:

\[
\phi(\xi) = \begin{cases} 
\frac{2}{3} + 4(\xi - 1)\xi^2 & \text{if } \xi \leq 0.5 \\
\frac{4}{3}(1 - \xi)^3 & \text{if } 0.5 \leq \xi \leq 1, \\
0 & \text{if } 1 \leq \xi 
\end{cases}, \quad \xi = ||x||
\]

Gaussian:

\[
\phi(\xi) = \begin{cases} 
\frac{e^{-9\xi^2} - e^{-9}}{1 - e^{-9}} & \text{if } \xi \leq 1, \\
0 & \text{if } \xi \geq 1 
\end{cases}, \quad \xi = ||x||
\]

In Figure 2.2 most common 1D kernel functions are represented.

\[\text{Figure 2.2: Most common 1D kernel functions.}\]

In order to develop a computational technique, it is necessary to evaluate the integration in equation (2.2.1) in a discrete manner to give

\[
u(x) \approx \tilde{u}^\rho(x) \approx u^\rho(x) := \sum_i V_i C_{\rho} \phi \left( \frac{x - x_i}{\rho} \right) u(x_i),
\]

(2.2.2)
where \( \mathbf{x}_i \) and \( V_i \) are the points and weights of the numerical quadrature. Usually the quadrature points are called particles and the weights are called volumes. Gingold and Monaghan (1977) propose a typical expression, for the tributary or statistical volume associated with particle \( \mathbf{x}_i \), derived from Montecarlo integration theory namely

\[
V_i^{-1} = \sum_j C_{\rho} \phi \left( \frac{x - x_j}{\rho} \right).
\]

It is possible to re-write equation (2.2.2) in terms of standard shape function as

\[
u(x) \simeq u^\rho(x) := \sum_i N_i(x) u(x_i), \quad N_i(x) = V_i C_{\rho} \phi \left( \frac{x - x_i}{\rho} \right).
\]

**Remark 2.2.1.** By careful selection of the parameter \( C_{\rho} \), it is possible to ensure that polynomials up to a given degree \( m \) are exactly reproduced by (2.2.1). To demonstrate this consider a simple one-dimensional case where the function \( u(y) \) inside the integral has been approximated as a Taylor series expansion around \( x \) to give,

\[
u(x) \simeq u^\rho(x) := \int \left( u(x) + \cdots + \frac{1}{m!} u^m(x)(y - x)^m \right) C_{\rho} \phi \left( \frac{x - y}{\rho} \right) dy
\]

\[= u(x) \int C_{\rho} \phi \left( \frac{x - y}{\rho} \right) dy + \cdots + \frac{1}{m!} u^m(x) \int (x - y)^m C_{\rho} \phi \left( \frac{x - y}{\rho} \right) dy\]

Hence if the following conditions are satisfied:

\[
\int C_{\rho} \phi \left( \frac{x - y}{\rho} \right) dy = 1, \quad \int (x - y)^k C_{\rho} \phi \left( \frac{x - y}{\rho} \right) dy = 0, \quad 0 < k \leq m
\]

the approximation is said to have \( m \)-order consistency.

**Remark 2.2.2.** Note that, unlike standard finite element approximations, the shape functions do not satisfy the Kronecker delta property that is, \( N_i(x_j) \neq \delta_{ij} \), and consequently \( u^\rho(x_i) \neq u(x_i) \), see Figure 2.3. This is common for all mesh-free methods and will lead to difficulties in enforcing the essential boundary conditions (see Section 2.4).
Figure 2.3: SPH shape functions $N$ and approximation of $u(x) = 1 - x^2$ with cubic spline window function, distance between particles $h = 0.5$ and Montecarlo assigned volumes, for $\rho/h = 1, 2, 3, 4$ (from top to bottom).
Remark 2.2.3. The dilation parameter \( \rho \) characterizes the support of the approximants \( N_i(x) \), see Figure 2.3.

Remark 2.2.4. In contrast to mesh-based methods, the neighbor particles (particles belonging to a given support) have to be identified during the course of the computation. This is of special importance if the support changes in time and requires fast neighbor search algorithms, a crucial feature for the effectiveness of a meshfree method, see e.g. Schweitzer (2003).

Remark 2.2.5. There is an optimal ratio between the dilation parameter \( \rho \) and the distance between particles \( h \). On one hand, the dilation parameter must be large enough in order to avoid that high frequencies are present in the approximated solution. On the other hand, a too large \( \rho \) will lead to a bad approximation of the Dirac delta function so it will produce large errors, see Figure 2.3. Thus, in a refinement process it is usual to maintain the ratio between \( \rho \) and \( h \) constant.

Although by careful selection of the parameter \( C_\rho \), it is possible to ensure that polynomials up to a given degree \( m \) are exactly reproduced, the design of \( C_\rho \) is not trivial in the presence of boundaries or with nonuniform distributions of particles. For example, taking

\[
C_\rho = \frac{27}{17} - \frac{120}{17} x^2
\]

with the cubic spline window function the second degree polynomial basis \( \{1, x, x^2\} \) is reproduced. In Figure 2.4 is shown that linear monomial \( u(x) = x \) is exactly reproduced in an unbounded domain (observe that shape functions have changed due to the correction). However, in a bounded domain the approximation is not exact near the boundaries, see Figure 2.5. In Figure 2.6 is shown that when nonuniform distributions of particles are used the approximation is not exact even in an unbounded domain.

**Corrected SPH method**

As a result of point integration in equation (2.2.2), the consistency conditions are no longer satisfied exactly. Bonet and Kulasegaram (2000) present a corrected
Figure 2.4: Shape functions and SPH approximation for $u(x) = x$ with $\rho/h = 2$ in an unbounded domain.

Figure 2.5: Shape functions and SPH approximation for $u(x) = x$ with $\rho/h = 2$ in a bounded domain.

Figure 2.6: Shape functions and SPH approximation for $u(x) = x$ with $\rho = 2$ in an unbounded domain and a nonuniform distribution of particles.
SPH approximation (CSPH). The foregoing is a brief review of the three main corrections introduced by Bonet and coworkers.

Firstly, the aforementioned discrepancy is eliminated by a kernel correction. As proposed by Liu, Jun and Zhang (1995), \( C_\rho \) is selected by enforcing linear consistency conditions (see Remark 2.2.1), now given by a point wise integration as,

\[
\sum_i V_i C_\rho \phi \left( \frac{x - x_i}{\rho} \right) = 1, \quad \sum_i V_i (x - x_i) C_\rho \phi \left( \frac{x - x_i}{\rho} \right) = 0. \tag{2.2.3}
\]

These equations lead to,

\[
C_\rho = \alpha(x) [1 + \beta(x) \cdot (x - x_i)]
\]

where

\[
\alpha(x) = \frac{1}{\sum_i V_i \phi \left( \frac{x - x_i}{\rho} \right) [1 + \beta(x) \cdot (x - x_i)]}
\]

\[
\beta(x) = \left[ \sum_i V_i \phi \left( \frac{x - x_i}{\rho} \right) (x - x_i)(x - x_i)^T \right]^{-1} \sum_i V_i(x_i - x) \phi \left( \frac{x - x_i}{\rho} \right).
\]

The use of this type of correction ensures that linear functions are perfectly interpolated (even in bounded domains and nonuniform distributions of particles, see Figure 2.7) and their gradients are exactly obtained. A possible way of simplifying the calculation is by using constant, rather than linear, correction. This is equivalent to taking \( \beta(x) = 0 \) in equation (2.2.1). Nevertheless, gradient evaluation using the above expressions is expensive, both in computer memory and time consuming.

Secondly, the gradient functions are directly amended to ensure that the gradient of a general constant or linear function is correctly evaluated. The corrected
Figure 2.7: CSPH shape functions and approximation for $u(x) = x$ with $\rho = 2$ in a bounded domain and nonuniform distribution of particles.

The gradient is defined as

$$\vec{\nabla} u^\rho(x) = \sum_i V_i [u(x_i) - u(x)] \vec{\nabla} \phi \left( \frac{x - x_i}{\rho} \right),$$

where

$$\vec{\nabla} \phi \left( \frac{x - x_i}{\rho} \right) = L(x) \vec{\nabla} \phi \left( \frac{x - x_i}{\rho} \right). \tag{2.2.4}$$

It is clear that using equation (2.2.4) will ensure that the gradient of a constant function vanishes. The correction matrix $L(x)$ is obtained after imposing the linear consistency condition, namely

$$\sum_i V_i \vec{\nabla} \phi \left( \frac{x - x_i}{\rho} \right) x_i^T = I.$$

This equation enables the explicit evaluation of the correction term as,

$$L(x) = \left[ \sum_i V_i \vec{\nabla} \phi \left( \frac{x - x_i}{\rho} \right) (x_i - x)^T \right]^{-1}.$$
This corrected gradient, proposed by Bonet and coworkers, is similar to the Renormalized Meshless Derivative (RMD) proposed by Randles and Libersky (1996), Krongauz and Belytschko (1998) and Vila (1999).

Finally, as a procedure to validate the accuracy of the method, a patch test (see Bazeley, Cheung, Irons and Zienkiewicz 1965, Taylor, Simo, Zienkiewicz and Chan 1986, Razzaque 1986) is performed. However, the test fails due to the error introduced by the point integration procedure and new correction methods have to be added to the CSPH method. In order to obtain a consistent approximation the gradient is modified by the introduction of an integration corrector vector $\gamma$ as,

$$\hat{\nabla} u^p(x_k) = \nabla u^p(x_k) + \gamma_k [u]_k,$$

where $[u]_k = u(x_k) - u^p(x_k)$. After imposing the patch test a linear system of equations, with dimension equal to the number of particles, must be solved to obtain the correction vector and define the derivatives of the approximation (Bonet and Kulasegaram 2000).

2.2.2 Moving Least Squares

Continuous Moving Least Squares

Let us consider a bounded, or unbounded, domain $\Omega$. The moving least-squares approach is based on the local (i.e., at any point $z$ in the neighborhood of $x$) approximation of the unknown scalar function $u(z)$ by $u^p$ as

$$u(z) \simeq u^p(x, z) = P^T(z) a(x) \quad \text{for } z \text{ near } x,$$

(2.2.5)

where the coefficients $a(x) = \{a_0(x), a_1(x), \ldots, a_l(x)\}^T$ are not constant, they depend on point $x$, and $P(z) = \{p_0(z), p_1(z), \ldots, p_l(z)\}^T$ includes a complete basis of the subspace of polynomials of degree $m$. The coefficients $a(x)$ are
obtained through least-squares fitting with the scalar product
\[ <u, v> = \int_\Omega \phi \left( \frac{x - y}{\rho} \right) u(y) \, v(y) \, dy, \]
where \( \phi \left( \frac{x - y}{\rho} \right) \) is the weighting function as seen in SPH methods. Thus, vector \( a(x) \) is the solution of the standard normal equations in a weighted least-squares problem,
\[ M(x) \, a(x) = <P, u> \quad (2.2.6) \]
where, as usual, the Gram matrix \( M(x) \) is the scalar product of the interpolation polynomials
\[ M(x) = \int_\Omega \phi \left( \frac{x - y}{\rho} \right) P(y) \, P^T(y) \, dy. \quad (2.2.7) \]
Substituting \( a(x) \) as obtained from (2.2.6) into (2.2.5), yields,
\[ u(z) \approx u^0(x, z) = P^T(z) \, M^{-1}(x) \int_\Omega \phi \left( \frac{x - y}{\rho} \right) u(y) \, P(y) \, dy, \quad \text{for } z \text{ near } x. \]
Since the weighting function \( \phi \) favors the central point \( x \), it seems reasonable to assume that such an approximation is more accurate precisely at \( z = x \) and thus the approximation (2.2.5) is particularized at \( x \), that is,
\[ u(x) \approx u^0(x) = P^T(x) \, a(x) = \int_\Omega P^T(x) \, M^{-1}(x) \, P(y) \phi \left( \frac{x - y}{\rho} \right) u(y) \, dy. \quad (2.2.8) \]
This formula (2.2.8) is equal to SPH approximation (2.2.1) taking,
\[ C_\rho(x, y) := P^T(x) \, M^{-1}(x) \, P(y). \]
In fact, Liu, Jun and Zhang (1995) have called this term a correction function: a correction of the kernel which allows the MLS approximation to exactly reproduce all the polynomials in \( P \). To prove this, take into account definition (2.2.7) and rearrange terms or see Belytschko et al. (1994).
Remark 2.2.6. (Polynomial space) In one dimension, we can let $p_l(x)$ be the monomials $x^l$, and, in this particular case, $l = m$. For larger spatial dimensions two types of polynomial spaces are usually chosen: the set of polynomials, $P_m$, of total degree $\leq m$, and the set of polynomials, $Q_m$, of degree $\leq m$ in each variable. Both include a complete basis of the subspace of polynomials of degree $m$. This, in fact, characterizes the a priori convergence rate, see Liu, Li and Belytschko (1997) or Fernández-Méndez, Díez and Huerta (2003).

Reproducing Kernel Particle Method

In order to develop a computational technique, it is necessary to apply a numerical quadrature in equation (2.2.8). This leads to the reproducing kernel particle method (RKPM) approximation,

$$u(x) \approx u^\rho(x) = P^T(x) a(x) = P^T(x) M^{-1}(x) \sum_i P(x_i) V_i \phi \left( \frac{x-x_i}{\rho} \right) u(x_i).$$

This expression can also be written in a standard form

$$u^\rho(x) = \sum_i N^\rho_i(x) u(x_i) = \sum_i V_i \left[ \phi \left( \frac{x-x_i}{\rho} \right) P^T(x) M^{-1}(x) P(x_i) \right] u(x_i),$$

where $x_i$ and $V_i$ are the points (particles) and weights of the numerical quadrature.

Remark 2.2.7. As a result of point integration in equation (2.2.2), the consistency conditions are no longer satisfied exactly in the SPH method. In the RKPM approximation the only requirement to preserve consistency is that matrix $M$ must be computed with the same quadrature used for the discretization of (2.2.8), see Chen, Pan, Wu and Liu (1996) for details. That is, matrix $M$ must be computed as,

$$M(x) = \sum_i V_i \phi(x, x_i) P(x_i)^T P(x_i).$$

Remark 2.2.8. Liu, Li and Belytschko (1997) and Huerta and Fernández-Méndez (2000) discuss the necessary conditions for matrix $M$ to be regular at every point $x$ in the domain.
**Discrete MLS: Element Free Galerkin approximation**

An analogous development, as the one performed in the continuous MLS, can be done with the discrete scalar product

\[ < u, v > = \sum_i \phi \left( \frac{x - x_i}{\rho} \right) u(x_i) v(x_i), \]

leading to the discrete MLS approximation (usually called Element Free Galerkin approximation),

\[ u^0(x) = \sum \phi \left( \frac{x - x_i}{\rho} \right) P(x_i) M^{-1}(x) P(x_i) u(x_i). \]

**Remark 2.2.9.** Note that EFG approximation coincides with the RKPM one with weights \( V_i = 1 \) for all particles.

**Remark 2.2.10.** Liu, Li and Belytschko (1997) proved convergence of the RKPM. The \textit{a priori} error bound is very similar to the bound in finite elements. The dilation parameter plays the role of the element size in finite elements, and the order of consistency plays the role of the degree of the interpolation polynomials in the finite element mesh.

**Remark 2.2.11.** One important feature of mesh-free approximations is that if the weight function \( \phi \) is \( C^k \) then the shape functions are \( C^k \). See Liu, Li and Belytschko (1997) where a study of this feature, with the EFG method, is performed.

**Remark 2.2.12.** For computational purposes, it is usual to use a centered and scaled approach of the method. This means to center in \( x_j \) and scale with \( \rho \) also the polynomials involved in the definition of the approximation functions, see Liu, Li and Belytschko (1997) or Fernández-Méndez (2001).

**The diffuse derivative**

The approximation of the derivative of \( u \) is the derivative of \( u^0 \). This requires to derive (2.2.8), that is

\[ \frac{\partial u}{\partial x_i} \approx \frac{\partial u^0}{\partial x_i} = \frac{\partial \mathbf{P}^T}{\partial x_i} a(x) + \mathbf{P}^T \frac{\partial a}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, n_{sd}. \]
Note that the derivative of the polynomials in $\mathbf{P}$ is trivial but the derivative of the coefficients $\mathbf{a}$ requires the resolution of a linear system of equations with the same matrix $\mathbf{M}$ (see Belytschko, Krongauz, Fleming, Organ and Liu 1996).

Thus the concept of diffuse derivative proposed by Villon (1991) and Nayroles et al. (1992) and defined as

$$
\frac{\partial u^0}{\partial x_i} = \frac{\partial u^0}{\partial z_i} \bigg|_{z=\mathbf{x}} = \frac{\partial \mathbf{P}^T}{\partial z_i} \mathbf{a}(\mathbf{x}) = \frac{\partial \mathbf{P}^T}{\partial x_i} \mathbf{a}(\mathbf{x}) \quad \text{for } i = 1, \ldots, n_{sd}
$$

is, from a computational cost point of view, an interesting alternative to (2.2.10). Moreover, Villon (1991) shows for 1D that the diffuse derivative converges at optimal rate to the derivative of $u$. In Section 3.2 the convergence proof for higher spatial dimensions can be found.

In fact, this strategy is the basis of the diffuse element method: the diffuse derivative is used in the weak form of the problem, see Nayroles et al. (1992); Breitkopf, Rassineux and Villon (2001). Moreover, the generalized finite difference interpolation or meshless finite difference method, see Orkisz (1998), coincides also with this MLS development.

In Chapter 3 the concept of diffuse derivative will be of crucial importance to develop a new formulation which will preclude locking in the incompressible limit.

**Partition of unity methods**

The methods based on moving least-squares are specific instances of partitions of unity (Babuška and Melenk 1995, Duarte and Oden 1996b). In the partition of unity methods a domain is covered by overlapping patches, or subdomains $\Omega$, each of which is associated with a function $\Phi_i(\mathbf{x})$ which is nonzero only in $\Omega_i$ and has the property that

$$
\sum \Phi_i(\mathbf{x}) = 1 \quad \text{in } \Omega. \quad (2.2.11)
$$
Note that the essential condition of a partition of unity method is identical to the zeroth order consistency condition, i.e.,

\[ \sum_i N_i = 1 \text{ in } \Omega. \]

This viewpoint has led to several new approximations for meshless methods. Duarte and Oden (1996b) use the concept of partition of unity in a general manner by constructing it from an MLS shape function of order \( k \); they call the method \( hp \) clouds. The proposed approximation is

\[ u(\mathbf{x}) \simeq \Psi(\mathbf{x}) = \sum_i N_i^p(\mathbf{x}) u_i + \sum_{I=1}^{n_i} b_{ij} [N_i^p(\mathbf{x}) q_{ij}(\mathbf{x})], \]

where \( N_i(\mathbf{x}) \) are the MLS approximation functions, \( q_{ij} \) are \( n_i \) polynomials of any order greater than \( k \) associated to each particle \( \mathbf{x}_i \), and \( u_i, b_{ij} \) are coefficients to determine.

**Remark 2.2.13.** The major advantage of the Duarte-Oden formulation is that it enables the basis \( \Psi(\mathbf{x}) \) (often called the extrinsic base) to vary from node to node, thus facilitating \( hp \)-adaptivity. In fact, the concept of an extrinsic base is essential for obtaining \( p \)-adaptivity. In MLS approximations, the base \( \Psi(\mathbf{x}) \) cannot vary from particle to particle without introducing a discontinuity.

### 2.2.3 Equivalent approximations: CSPH versus RKPM

In this section it will be shown that the correction function obtained from the RKPM is the same as the obtained from the CSPH method.

In table 2.1 the equivalence between CSPH and centered and scaled RKPM is shown. It's obvious from the table that:

\[ \alpha(\mathbf{x}) = C_0(\mathbf{x}) \quad \alpha(\mathbf{x}) \beta^T(\mathbf{x}) = -\frac{C_0^T(\mathbf{x})}{\rho} \]
Table 2.1: Equivalence: CSPH versus RKPM

As an example the correction term $C_\rho$ here is derived for linear polynomials in two dimensional problems. The polynomial basis is $P(\mathbf{x}) = \left(1, x_1, x_2\right)^T$ and the scalar product matrix reads,

$$
M(\mathbf{x}) = \sum_i V_i \phi\left(\frac{\mathbf{x} - \mathbf{x}_i}{\rho}\right) \begin{pmatrix} 1 \\ x_1^{(i)} \\ x_2^{(i)} \end{pmatrix} = \begin{pmatrix} m_{00} & m_{10} & m_{01} \\ m_{10} & m_{20} & m_{11} \\ m_{01} & m_{11} & m_{02} \end{pmatrix} ; \quad m_{jk} = \sum_i V_i \phi\left(\frac{\mathbf{x} - \mathbf{x}_i}{\rho}\right) (x_1^{(i)})^j (x_2^{(i)})^k
$$

In centered and scaled RKPM we know that:

$$
C_\rho = P^T(0) M^{-1}(\mathbf{x}) P\left(\frac{\mathbf{x}_i - \mathbf{x}}{\rho}\right)
$$
substituting the polynomial basis into the previous expression:

\[
C_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{00} & \mu_{10} & \mu_{01} \\ \mu_{10} & \mu_{20} & \mu_{11} \\ \mu_{01} & \mu_{11} & \mu_{02} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{x_1^{(i)} - x_1}{\rho} \\ \frac{x_2^{(i)} - x_2}{\rho} \end{pmatrix}
\]

where \((\mu_{jk})\) is the inverse of \(\mathbf{M}(\mathbf{x})\). After simple algebra we find that:

\[
C_\rho = \mu_{00} + \mu_{10} \left( \frac{x_1^{(i)} - x_1}{\rho} \right) + \mu_{01} \left( \frac{x_2^{(i)} - x_2}{\rho} \right)
\]

and therefore it can be written:

\[
C_\rho = C_0(\mathbf{x}) + C_1^T(\mathbf{x}) \left( \frac{\mathbf{x} - \mathbf{x}^{(i)}}{\rho} \right)
\]

where

\[
C_0(\mathbf{x}) = \mu_{00} ; \quad C_1^T(\mathbf{x}) = \begin{pmatrix} \mu_{10} \\ \mu_{01} \end{pmatrix}
\]

This is equivalent to CSPH taking:

\[
\alpha(\mathbf{x}) = C_0(\mathbf{x}) = \mu_{00} ; \quad \alpha(\mathbf{x}) \beta(\mathbf{x}) = -\frac{C_1^T(\mathbf{x})}{\rho} = -\frac{1}{\rho} \begin{pmatrix} \mu_{10} \\ \mu_{01} \end{pmatrix}
\]

### 2.3 Partial Differential Equations (PDE)

In Section 2.2 specific approximation techniques based on different meshless formulations were described. However, meshless methods in computational mechanics are not simply different approximation schemes but constitute, indeed,
a powerful and ambitious attempt to solve the equations of continuum mechanics without the computational limitations associated to the explicit partition of the domain into certain non-overlapping cells.

Usually SPH methods are combined with a collocation or point integration techniques, while the approximations based on MLS are usually combined with a Galerkin discretization and Gauss integration techniques. However, a lot of variety can be found in this area.

Probably, the best known, among those using MLS approximation, are: the meshless (generalized) finite difference method (MFDM) developed by Liszka and Orkisz (1980), the diffuse element method (DEM) by Nayroles et al. (1992), the element-free Galerkin (EFG) method by Belytschko et al. (1994), the reproducing kernel particle method (RKPM) by Liu, Jun and Zhang (1995), the nodal integration of EFG (NIEFG) by Beissel and Belytschko (1996), the Petrov-Galerkin DEM (PGDEM) by Krongauz and Belytschko (1997), the meshless local Petrov-Galerkin (MLPG) by Zhu and Atluri (1998) and the finite point method (FPM) by Oñate and Idelsohn (1998).

Among those using SPH approximation the best known are: corrected smooth particle hydrodynamics (CSPH) by Bonet and Kulasegaram (2000), the renormalized meshless derivative (RMD) by Krongauz and Belytschko (1998) and Vila (1999) and the normalized SPH (NSPH) by Randles and Libersky (1996).

But, as has been said, a lot of variety can be found, for example, Dilts (1999) use SPH with a collocation-Galerkin approximation (which is Petrov-Galerkin with weight functions equal to the Dirac delta) and call their method Moving Least Squares Particle Hydrodynamics (MLSPH); Zhang, Liu, Song and Lu (2001) use MLS with collocation methods and call their method Least-Squares Collocation Method (LSCM); Chen, Wu, Yoon and You (2001) use Galerkin meshfree methods with a strain smoothing stabilization for nodal integration and call the method
stabilized conforming nodal integration (SCNI)... Perhaps a good question is: how many (really different) meshless methods there exist? Tables 2.2 and 2.3 classify all aforementioned methods depending on three factors:

▷ Which kind of approximation is used: SPH or MLS.
▷ How do they evaluate the derivatives: non-corrected or corrected for SPH and consistent or diffuse for MLS.
▷ How the PDE is solved: Galerkin, Petrov-Galerkin or point collocation.

<table>
<thead>
<tr>
<th>Non-corrected Deriv.</th>
<th>Corrected Deriv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss quad.</td>
<td></td>
</tr>
<tr>
<td>Galerkin</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Particle quad.</td>
<td>CSPH</td>
</tr>
<tr>
<td>Petrov-Galerkin</td>
<td>MLSPH</td>
</tr>
<tr>
<td></td>
<td>RMD</td>
</tr>
<tr>
<td>Point Collocation</td>
<td>SPH</td>
</tr>
<tr>
<td></td>
<td>NSPH</td>
</tr>
</tbody>
</table>

Table 2.2: Classification of SPH based meshfree methods for PDE’s

<table>
<thead>
<tr>
<th>Consistent Deriv.</th>
<th>Diffuse Deriv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss quad.</td>
<td>EFG</td>
</tr>
<tr>
<td>Galerkin</td>
<td>MFDM</td>
</tr>
<tr>
<td></td>
<td>RKPM</td>
</tr>
<tr>
<td></td>
<td>DEM</td>
</tr>
<tr>
<td>Particle quad.</td>
<td>SCNI</td>
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<tr>
<td></td>
<td>NIEFG</td>
</tr>
<tr>
<td>Petrov-Galerkin</td>
<td>MLPG</td>
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<td></td>
<td>PGDEM</td>
</tr>
<tr>
<td>Point Collocation</td>
<td>LSCM</td>
</tr>
<tr>
<td></td>
<td>FPM</td>
</tr>
</tbody>
</table>

Table 2.3: Classification of MLS based meshfree methods for PDE’s
Although the most important and best known meshfree approximations are based in SPH or MLS interpolants, there are two other important kind of methods to comment: the natural neighbor interpolants based and the Radial basis functions (RBF).

The natural neighbor interpolants are based on a Voronoi tessellation of the set of particles. The interpolants are smooth ($C^\infty$) everywhere, except at the particles where they are $C^0$. The Natural Element Method (NEM) by Sukumar, Moran and Belytschko (1998) was among the firsts of these nature. In the Natural Element Method, the discrete model of the domain consists of a set of distinct particles, and a polygonal description of the boundary. Among the last contributions in natural neighbor interpolants based methods are the meshless finite element method by Idelsohn, Oñate, Calvo and Pin (2003) and the stabilized conforming nodal integration in the natural-element method by Yoo, Moran and Chen (2004). The main important advantage of the NEM is that unlike most of meshless methods, the NEM shape functions are strictly interpolant: the NEM interpolant passes through the nodal values. This implies that essential boundary conditions in the NEM can be directly imposed on the nodes, in accordance with the finite element method. Nevertheless, it must be noted that the above inference is rigorously true for convex domains; however, for non-convex domains is not straight forward: numerical simulations need to be carried out on a per problem basis to determine the appropriate nodal discretization of the part of the boundary which renders the domain to be non-convex. Last, Sukumar et al. (1998) have also reported that NEM does not pass the patch test within machine precision and found error norms of about $10^{-5}$, in the worst case. The discrepancies are due to the numerical integration of the Galerkin equations which is still a matter of research.

Radial basis functions (RBF) have been studied for the past 30 years and are closely related to meshless approximations. There are two main differences in
between the current approaches in radial basis functions and those described in previous sections: (1) most radial basis functions have non compact support and (2) completeness is provided by adding a global polynomial to the basis. The major applications of radial basis functions has been in data fitting. One of the first applications to the solution of PDE’s is given by Kansa (1990) who used multiquadrics for smooth problems in fluid dynamics. In Sharan, Kansa and Gupta (1997), the method was applied to elliptic PDE’s. In both cases, collocation was employed for the discretization. Exceptional accuracy was reported. Although a good understanding of this behavior is not yet available, evidently very smooth global approximants have intrinsic advantages over rough approximants for elliptic PDE’s and other smooth problems (any locally supported approximant will have some roughness at the edge of its support). The low cost of RBF evaluation is another advantage. Wendland (1999) has studied Galerkin discretization of PDE’s with radial basis functions. Compactly supported RBFs are also under development. Local error estimates for radial basis approximations of scattered data are given by Wu and Schaback (1993).

Finally observe that when Galerkin or Petrov-Galerkin based methods are used the final step to solve a PDE boundary value problem corresponds to the numerical integration of the weak form. Next an overview of the different numerical integration techniques used in mesh-free methods is performed.

### 2.3.1 Numerical Integration

Mesh based methods provide a natural spatial framework to integrate the globally defined Galerkin weak form, which is split into assembled elemental contributions. In meshless methods the spatial framework is no longer preserved turning the simple integration of the Galerkin weak form into an open question far from being closed. Numerical integration is an important issue in mesh-
less methods and is the source of well known inaccuracies and instabilities (see Bonet and Kulasegaram 2000, Belytschko, Guo, Liu and Xiao 2000, Beissel and Belytschko 1996, Dolbow and Belytschko 1999a).

Nodal integration has been used, at least implicitly, in all SPH formulations, and lies, indeed, in the basis of its early formulation. Obviously, this is the cheapest option and the resulting scheme does not need a grid. The nodes are used as quadrature points and the corresponding integration weights are their tributary volumes.

The most frequent approach in EFG and RKPM is the definition of a background integration grid (Belytschko, Krongauz, Organ, Fleming and Krysl 1996), composed by non overlapping cells covering the whole domain, where Gauss quadratures are defined (Figure 2.8). In general, these cells do not match integration domains; however, the spatial framework required by the Galerkin method is recovered (at the cost of the generation of an integration grid which do not need to be conforming). These techniques ensure an accurate approximation of the integrals.

However, domain integration by Gauss quadrature in the Galerkin mesh-free methods adds considerable complexity to solution procedures. Direct nodal integration, on the other hand, leads to a numerical instability due to under integration and vanishing derivatives of shape functions at the nodes. Chen et al. (2001) propose a strain smoothing stabilization for nodal integration to eliminate spatial instability in nodal integration.

In the context of SPH, it is important to highlight that the use of alternative numerical quadratures appeared implicitly within the concept of stress points. The concept of stress points was introduced by Dyka and Ingel (1995), as an attempt to eliminate tensile instabilities (Swegle et al. 1995) in SPH. The basic idea, which still remains in the stress-point SPH literature, is to calculate stresses away from
Figure 2.8: Background integration mesh.

Figure 2.9: Particles and stress points (double particle distribution).
the centroids (particles). This is equivalent to use a quadrature other than nodal integration in the Galerkin weak form. The integration points are called stress points, which are moving integration points spread among the cloud of particles, with no reference to any background mesh (Figure 2.7). Stress points are set up in certain positions and their movement is completely determined by the movement of the particles: this allows the integration “mesh” adapt the moving domain, avoiding the rigid background mesh.

2.4 Essential boundary conditions

One of the major difficulties in the implementation of mesh-free methods is the noninterpolatory character of the approximation, see Remark 2.2.2. As a consequence, the imposition of essential or Dirichlet boundary conditions is quite awkward. There has been work with several different approaches to this problem: (1) Lagrange multipliers approaches (Belytschko et al. 1994), (2) modified variational principles (Belytschko et al. 1994), (3) penalty methods (Zhu and Atluri 1998, Bonet and Kulasegaram 2000) (4) perturbed Lagrangian (Chu and Moran 1995), (5) coupling to finite elements (Belytschko et al. 1995, Huerta and Fernández-Méndez 2000, Wagner and Liu 2001), or (6) modified shape functions (Gosz and Liu 1996, Gunter and Liu 1998, Wagner and Liu 2001) among others.

The Lagrange multiplier method allows the imposition of essential boundary conditions in a simple and accurate way. However, it has several disadvantages: discrete equations for a linear self-adjoint PDE are no longer positive definite nor banded, it augments the dimension of the linear system and, possibly the worst, in some problems it is difficult to establish the number of multipliers. The approach based on the modified variational principle results in banded equations, but the boundary conditions are not imposed with as high a degree of accuracy. For practical purposes, mixed interpolations, that combine finite elements with meshfree
methods approach, appears to be most satisfactory. The idea is to discretize the domain using finite elements only in a neighborhood of the Dirichlet boundary and a mesh-free approximation in the rest of the domain. Thus, the essential boundary conditions can be imposed as in standard finite elements.

Another approach to impose boundary conditions is due to Nitsche. In the early 1970’s Nitsche developed a general approach for the treatment of essential boundary conditions where the trial and test functions do not have to fulfill the boundary conditions. In this sense, it can be interpreted as a consistent improvement of the penalty method. Nitsche’s method, however, seems to be quite unknown. Recently, Stenberg (1995) revived the interest in this non-standard method by showing its connection to some stabilization techniques for the Lagrange multiplier method. Griebel and Schweitzer (2002) present the implementation of essential boundary conditions by Nitsche’s method in the context of a partition of unity method. In Chapter 3 this method will be used to solve Stokes problem.

In the SPH literature, essential boundary conditions have been rather treated as fluid-structure contact and specific techniques such as mirror particles and boundary forces have been extensively used. In the classical boundary force approach, certain forces are applied to those particles that approach the boundary. The contact with solid boundaries is detected by checking the distance between fluid particles and certain boundary particles, which are fixed at the solid boundary and are not included in the general computations. Another way to impose in a simple manner essential boundary conditions consists in blending with the finite element formulation. Coupling finite elements near the Dirichlet boundaries and using mesh-free methods in the interior of the computational domain simplifies considerably the prescription of essential boundary conditions. In Huerta, Fernández-Méndez and Liu (2004) two recently proposed formulations to couple mesh-free and finite element methods are discussed and compared. Fernández-Méndez and
Huerta (2004) present a general overview on the existing techniques to enforce essential boundary conditions in Galerkin based mesh-free methods.
Locking in the incompressible limit: pseudo-divergence-free EFG
Chapter 3

Locking in the incompressible limit: pseudo-divergence-free element free Galerkin

3.1 Introduction

Locking of standard finite elements has been widely studied. It appears because poor numerical interpolation leads to an over-constrained system. It is acknowledged that in a displacement-based finite element method, linear approximations perform poorly for the modeling of incompressible materials. For incompressible, or nearly incompressible, materials an additional constraint appears in the field equations which requires the divergence of the displacement field to be zero in the domain. This constraint is difficult to fulfill for low order elements. Locking is attenuated and can be suppressed for increasing polynomial degrees, in the context of an $hp$ adaptive strategy, Babuška and Suri (1992) and Suri (1996) present a review on this issue. Moreover, several techniques are available to alleviate or
completely remove the locking phenomena in finite element approximations (see Hughes 2000).

However, locking in meshless methods is still an open topic. Even recently, Zhu and Atluri (1998) claimed that meshless methods do not exhibit volumetric locking. Now it is clear that this is not true. For instance, Dolbow and Belytschko (1999b) analyze the EFG method using the numerical inf-sup condition. Moreover, several authors claim that increasing the dilation parameter locking phenomena in mesh-free methods can be suppressed, or at least attenuated. Their argument is based on numerical examples (Askes, de Borst and Heeres 1999, Dolbow and Belytschko 1999b) or on the heuristic constraint ratio (Chen, Yoon, Wang and Liu 2000) proposed by Hughes (2000). In a recent paper by Huerta and Fernández-Méndez (2001) this issue is clarified and the influence of the dilation parameter on the locking behavior of EFG near the incompressible limit is established. This is done performing a modal analysis: studying the fundamental modes (base of the solution space) and their corresponding energy (eigenvalue). In particular EFG behavior is compared with standard finite elements, bilinear and biquadratic. It concludes that an increase of the dilation parameter attenuates, but never suppresses, volumetric locking and that, as in standard finite elements, an increase in the order of reproducibility reduces the relative number of locking modes.

Thus, large domains of influence alleviate locking but for small domains of influence, however, the direct application of the EFG approximation can result in volumetric locking. In dynamic problems and many non-linear problems, small domains of influence are preferred because they improve the local resolution and enhance the sparsity of the system of equations. Therefore, procedures which avoid locking, even for small domains of influence, are needed. Until now the remedies proposed in the literature are extensions of the methods developed for
finite elements.

As noticed before, there are several techniques available to alleviate or remove the locking phenomena in finite element approximations. For example, Suri (1996) shows that locking can be alleviated through the use of higher-order \( p \) elements. Alternatively, locking can be removed by mixed methods in which different approximations are implemented for the displacement and pressure fields (see, for instance, Hermann 1965, Hughes 2000). However, mixed methods are more expensive due to the need for additional unknowns. Alternatives which do not require additional degrees of freedom are selective reduced integration or strain projection methods. Extensions of these techniques to meshless methods can be found. For example, Dolbow and Belytschko (1999b) propose a new formulation of the EFG method using a selective reduced integration and Chen et al. (2000) suggest an improved Reproducing Kernel Particle Method (RKPM) using a pressure projection method.

Here a novel approach is explored. It consists in using approximation functions that verify approximately the divergence-free constraint. These approximating functions can be defined \textit{a priori} and are independent of the particle distribution. Moreover, as the density of particles is increased the divergence-free condition is better approximated. This method is based on diffuse derivatives (see Nayroles et al. 1992), which, as proven by Villon (1991), converge to the derivatives of the exact solution when the radius of the support goes to zero (for a fixed dilation parameter).

### 3.2 The diffuse derivatives

In Section [2.2.2](#) the concept of diffuse derivative was introduced and it was recalled that Villon (1991) showed for 1D that the diffuse derivative converges at optimal rate to the derivative of \( u \). Here a convergence proof for higher spatial
Proposition 3.2.1. If \( u^p \) is an approximation to \( u \) with an order of consistency \( m \) (i.e. \( P \) includes a complete basis of the subspace of polynomials of degree \( m \)) and \( \rho / h \) is constant, then

\[
\left\| \frac{\partial |k| u}{\partial x^k} - \frac{\partial |k| u^p}{\partial x^k} \right\|_\infty \leq C(\mathbf{x}) \frac{\rho^{m+1-|k|}}{(m+1)!} \quad \forall |k| = 0, \ldots, m. \tag{3.2.1}
\]

where \( k \) is a multi-index, \( k = (k_1, k_2, \ldots, k_n) \) and \( |k| = k_1 + k_2 + \cdots + k_n \).

Proof. Let's assume \( u \in C^{m+1}(\Omega) \) where \( C^{m+1} \) is the space of \( (m+1) \) times continuously differentiable functions. Recall that Taylor’s formula of order \( m \) can be written as:

\[
\begin{align*}
\begin{split}
\sum_{|\mathbf{\alpha}|=0}^{m} \frac{1}{\alpha!} h^\mathbf{\alpha} \frac{\partial |\alpha| u}{\partial x^\mathbf{\alpha}}(\mathbf{x}) + R_{m+1}(\mathbf{x} + \theta h),
\end{split}
\end{align*}
\]

\[
(3.2.2)
\]

where \( \theta \in [0, 1], R_{m+1}(\mathbf{x} + \theta h) \) is the error term and \( \alpha \) is a multi-index such that,

\[
h^\alpha := h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}, \quad \mathbf{\alpha}! := \alpha_1! \alpha_2! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.
\]

Equation (3.2.2) can be rewritten taking \( z = \mathbf{x} + h \)

\[
\begin{align*}
\begin{split}
\sum_{|\mathbf{\alpha}|=0}^{m} \frac{1}{\alpha!} \left( \frac{z - \mathbf{x}}{\rho} \right)^\alpha \frac{\partial |\alpha| u}{\partial x^\mathbf{\alpha}}(\mathbf{x}) + R_{m+1}(\mathbf{x}, z).
\end{split}
\end{align*}
\]

Thus, Taylor’s formula can also be written as:

\[
\begin{align*}
\begin{split}
\sum_{|\mathbf{\alpha}|=0}^{m} \frac{1}{\alpha!} \frac{\partial |\alpha| u}{\partial x^\mathbf{\alpha}}(\mathbf{x}) + R_{m+1}(\mathbf{x}, z),
\end{split}
\end{align*}
\]

\[
(3.2.3)
\]

where

\[
P(\xi) = \left\{ \frac{\xi^\alpha}{\alpha!} \right\}; \quad U(\mathbf{x}) = \left\{ \rho^\alpha \frac{\partial |\alpha| u}{\partial x^\mathbf{\alpha}} \right\} \quad |\alpha| = 0, \ldots, m. \tag{3.2.4}
\]

Observe that \( U(\mathbf{x}) \) depends on the exact derivatives of \( u \).
The MLS approach is based on the local approximation of the unknown scalar function \( u \) by \( u^\rho \), see equation (2.2.5). Since in equation (3.2.3) polynomials \( P(\xi) \) are centered and scaled, the MLS interpolant is also centered and scaled,

\[
u(z) \simeq u^\rho(x, z) = P^T \left( \frac{z - x}{\rho} \right) a(x) \quad \text{for } z \text{ near } x.
\]

Then the MLS approach requires the resolution of the normal equations given by (2.2.6), here \( u(x_\xi) \) is substituted by (3.2.3)

\[
M(x)a(x) = \left< P \left( \frac{z - x}{\rho} \right), P^T \left( \frac{z - x}{\rho} \right) U(x) + R_{m+1}(x, z) \right>,
\]

which can be rearranged as

\[
M(x)[a(x) - U(x)] = \sum_{j \in I_n} \phi \left( \frac{x_j - x}{\rho} \right) P \left( \frac{x_j - x}{\rho} \right) R_{m+1}(x, x_j) =: b.
\]

Now, let’s rewrite the r.h.s. of (3.2.3) in a more convenient way. The error term of the Taylor’s formula has the form

\[
R_{m+1}(x, x_j) = \sum_{|\alpha|=m+1} \frac{(x_j - x)^\alpha}{(m + 1)!} \frac{\partial^{|\alpha|}_u}{\partial x^\alpha}(x, x_j), \quad (3.2.6)
\]

substituting (3.2.6) in the definition of vector \( b \), see (3.2.5), produces

\[
b = \sum_{j \in I_n} \phi \left( \frac{x_j - x}{\rho} \right) P \left( \frac{x_j - x}{\rho} \right) \sum_{|\alpha|=m+1} \frac{(x_j - x)^\alpha}{(m + 1)!} \frac{\partial^{|\alpha|}_u}{\partial x^\alpha}(x, x_j).
\]

Each component of the previously defined vector \( b \) is associated to the corresponding component of \( P \), namely the polynomial of degree \( |k| = 0, \ldots, m \) defined as

\[
\xi^k / k! = (\xi_1^{k_1} \xi_2^{k_2} \cdots \xi_{\text{mod}}^{k_{\text{mod}}}) / (k_1! k_2! \cdots k_{\text{mod}}!).
\]
Under these circumstances, each component of \( b \) can be written as

\[
b_k = \sum_{j \in I_n} \phi \left( \frac{x_j - x}{\rho} \right) \frac{(x_j - x)^k}{\rho^{|k|} |k|!} \sum_{|\alpha|=m+1} \frac{(x_j - x)^{\alpha}}{\rho^{k+|\alpha|+1}} \frac{\partial^{k+|\alpha|+1} u}{\partial x^{\alpha}}(x, x_j)
\]

\[
= \frac{\rho^{m+1}}{(m+1)!} \sum_{j \in I_n} \phi \left( \frac{x_j - x}{\rho} \right) \frac{(x_j - x)^k}{\rho^{|k|} |k|!} \sum_{|\alpha|=m+1} \left( \frac{x_j - x}{\rho} \right)^{k+|\alpha|} \frac{\partial^{k+|\alpha|+1} u}{\partial x^{\alpha}}(x, x_j)
\]

\[
= \frac{\rho^{m+1}}{(m+1)!} r_k(x).
\]

(3.2.7)

Thus, the r.h.s. of (3.2.5) becomes

\[
b = \frac{\rho^{m+1}}{(m+1)!} r(x).
\]

(3.2.8)

Substituting (3.2.8) into equation (3.2.5) and assuming that \( M \) is regular,

\[
a(x) - U(x) = \frac{\rho^{m+1}}{(m+1)!} M^{-1}(x) r(x).
\]

On one hand, \( r_k \) is bounded for all \(|k| = 0, \ldots, m\). This can be seen from the definition of \( r_k \), see (3.2.7). Note that for a fixed \( x \), if \( \rho/\lambda \) is constant, \( r_k \) is the sum of products of continuous functions in \( \Omega \). Thus, it is a continuous function in \( \Omega \). Moreover, in every product, there is the weighting function \( \phi \), which has compact support. Since \( r_k \) is a continuous function in a compact support it is bounded by a constant that only depends on \( x \).

On the other hand, matrix \( M \) is also bounded (see Huerta and Fernández-Méndez 2000). Then, if both, \( M \) and \( r_k \), are bounded, a constant \( C(x) \) can be defined as the bound of \( M^{-1}(x) r(x) \) and consequently

\[
|a(x) - U(x)| \leq \frac{\rho^{m+1}}{(m+1)!} C(x)
\]

The previous expression can be divided by \( j^{|k|} \). Then, for each component,

\[
\left| \frac{a_k(x)}{j^{|k|}} - \frac{U_k(x)}{j^{|k|}} \right| \leq \frac{\rho^{m+1-|k|}}{(m+1)!} C(x) \quad \forall |k| = 0, \ldots, m,
\]

(3.2.9)
where $a_k$ and $U_k$ are the components of $a$ and $U$, respectively. Recall that each component of $\mathbf{U}(\mathbf{x})$ depends on the corresponding exact derivatives of $u$, see (3.2.4). Now, observe that each component of $a(\mathbf{x})$ shall depend on the corresponding pseudo-derivatives; that is, for $|k| = 0, \ldots, m$,

$$
\frac{\partial^{|k|} u}{\partial x^k} := \frac{\partial^{|k|} u}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} := \frac{\partial^{|k|} u}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_{z=x} = \frac{a_k(\mathbf{x})}{\rho^{k_1} \cdots \rho^{k_n}}. \quad (3.2.10)
$$

Finally, replacing the definition of $\mathbf{U}(\mathbf{x})$ and $a(\mathbf{x})$ given by (3.2.4) and (3.2.10), one gets the final expression, which completes the proof,

$$
\left\| \frac{\partial^{|k|} u}{\partial x^k} - \frac{\partial^{|k|} u}{\partial x^k} \right\|_{\infty} \leq C(\mathbf{x}) \frac{\rho^{m+1-|k|}}{(m+1)!} \quad \forall |k| = 0, \ldots, m. \quad \square
$$

### 3.3 Pseudo-divergence free condition

#### 3.3.1 Diffuse divergence

Incompressible computations require that the approximating field must be divergence free. That is, the solution $\mathbf{u}(\mathbf{x})$, now a vector $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$, verifies $\nabla \cdot \mathbf{u} = 0$, and the approximation $\mathbf{u}_0(\mathbf{x})$ should also be divergence-free. This condition however depends on the interpolation space. Here, instead of requiring
a divergence-free interpolation, the diffuse divergence of the approximation

\[
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} = 
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} = 
\begin{bmatrix}
ce_0(x) \\
ce_1(x) \\
\vdots \\
ce_l(x) \\
\end{bmatrix}
\]

is imposed equal to zero, that is

\[
\nabla^\delta \cdot \bm{u}^\rho = \sum_{i=1}^{n_{sd}} \frac{\delta \bm{u}^\rho}{\delta x_i} = \sum_{i=1}^{n_{sd}} \frac{\partial \bm{P}^T}{\partial x_i} \bm{a}_i(x) = \left( \nabla \cdot \bm{Q}^T(x) \right) \bm{c}(x) = 0. \quad (3.3.1)
\]

Note that \( \bm{I}_{n_{sd}} \) is the identity matrix of order \( n_{sd} \) and the coefficients have been rearranged as

\[
\begin{bmatrix}
ce_0(x) \\
ce_1(x) \\
\vdots \\
ce_l(x) \\
\end{bmatrix} = 
\begin{bmatrix}
\frac{a_{0,1}}{c_0^T(x)} & \cdots & \frac{a_{0,n_{sd}}}{c_0^T(x)} \\
\frac{a_{1,1}}{c_1^T(x)} & \cdots & \frac{a_{1,n_{sd}}}{c_1^T(x)} \\
\vdots & \cdots & \vdots \\
\frac{a_{l,1}}{c_l^T(x)} & \cdots & \frac{a_{l,n_{sd}}}{c_l^T(x)} \\
\end{bmatrix}.
\]

Equation (3.3.1) must hold at each point \( \bm{x} \) and for any approximation. Thus appropriate approximation functions, \( \bm{Q} \), must be defined in order to verify (3.3.1) and thus ensure asymptotically a divergence-free approximation (i.e., the divergence-free condition is fulfilled as \( \rho \to 0 \)).
3.3.2 A 2D pseudo-divergence free approximation

The previous concepts are particularized for a two-dimensional case and in order to define pseudo-divergence-free approximation functions. Suppose for instance that consistency of order two is desired, then $\mathbf{P}^T = \{1, x_1, x_2, \frac{x_1^3}{2}, x_1 x_2, \frac{x_2^3}{2}\}$, thus

$$Q^T = \begin{pmatrix}
1 & 0 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1 x_2 & 0 & x_2^2/2 & 0 \\
0 & 1 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1 x_2 & 0 & x_2^2/2 & 0
\end{pmatrix} \quad (3.3.2)$$

and

$$e^T = \begin{pmatrix}
a_{0,1} & a_{0,2} & a_{1,1} & a_{1,2} & a_{2,1} & a_{2,2} & a_{3,1} & a_{3,2} & a_{4,1} & a_{4,2} & a_{5,1} & a_{5,2}
\end{pmatrix}. \quad (3.3.3)$$

The pseudo-divergence-free condition (3.3.1) is, in this case, written as

$$\nabla^\delta \cdot \mathbf{u}^\theta = \frac{\partial \mathbf{P}^T}{\partial x_1} \mathbf{a}_1 + \frac{\partial \mathbf{P}^T}{\partial x_2} \mathbf{a}_2 = 0,$$

which implies

$$(a_{1,1} + a_{2,2}) + x_1(a_{3,1} + a_{4,2}) + x_2(a_{5,1} + a_{5,2}) = 0,$$

and consequently,

$$a_{1,1} + a_{2,2} = 0, \quad a_{3,1} + a_{4,2} = 0, \quad a_{5,1} + a_{5,2} = 0.$$

The influence of these three restrictions in the approximation functions (3.3.2) can be viewed as follows

$$\begin{pmatrix}
1 & 0 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1 x_2 & 0 & x_2^2/2 & 0 \\
0 & 1 & -x_2 & x_1 & 0 & 0 & -x_1 x_2 & x_1^2/2 & -x_2^2/2 & 0 & 0 & 0
\end{pmatrix}, \quad (3.3.4)$$

where one should note that the coefficients in the $x_1$ and $x_2$ directions are now coupled and that the total number of degrees of freedom has decreased.
3.3.3 The pseudo-divergence-free EFG method

Using (3.3.4), let $Q_\delta$ be the new approximation matrix (where obviously the unnecessary columns have been removed). The approximation is then defined as

$$u(z) \simeq u_\delta(x, z) = \begin{pmatrix} u_1^\delta(x, z) \\ u_2^\delta(x, z) \end{pmatrix} = Q_\delta^T(z) c(x). \quad (3.3.5)$$

The vector version of the discrete scalar product defined in (2.2.2),

$$<u, v> = \sum_{i \in I_m} \phi(x, x_i) \, u_i^T(x_i) \, v(x_i)$$

allows now to reproduce the MLS approximation. Thus at each point $x$ the normal equations should be solved, see (2.2.6),

$$M(x) \, c(x) = <u, Q_\delta> \quad \text{with} \quad M(x) := <Q_\delta, Q_\delta>.$$ 

Thus, as previously, the coefficients $c$ are substituted in (3.3.5) and the approximation is particularized at $z = x$. Then, equation (2.2.8) becomes

$$u(x) \simeq u_\delta(x) = Q_\delta^T(x) \, c(x) = Q_\delta^T(x) \, M^{-1}(x) \, <u, Q_\delta>,$$

and a final expression can be found as

$$u_\delta(x) = \sum_{i \in I_m} N_i^\delta(x) \, u(x_i) = \sum_{i \in I_m} \left[ \phi(x, x_i) \, Q_\delta^T(x) \, M^{-1}(x) \, Q_\delta(x_i) \right] u(x_i).$$

It is important to note that the matrix of approximation functions $N_i^\delta$ is now a full matrix not a diagonal one as standard EFG would induce in this non scalar problem. This is due to the fact that the two components of the solution are linked by the incompressibility restriction.
3.4 Modal analysis

3.4.1 Preliminaries

The modal analysis presented here follows the same rationale originally presented by Huerta and Fernández-Méndez (2001). It is restricted to small deformations, namely $\nabla u$, where $u$ is the displacement and $\nabla s$ the symmetric gradient, i.e. $\nabla s = \frac{1}{2}(\nabla^T + \nabla)$. Moreover, linear elastic isotropic materials under plane strain conditions are considered. Dirichlet boundary conditions are imposed on $\Gamma_D$, a traction $\mathbf{h}$ is prescribed along the Neumann boundary $\Gamma_N$ and there is a body force $\mathbf{f}$. Thus, the problem that needs to be solved may be stated as:

\[
\text{solve for } u \in [H^1_{\Gamma_D}]^2 \text{ such that } \quad \begin{align*}
\frac{E}{1 + \nu} \int_\Omega \nabla^s \mathbf{v} : \nabla^s u d\Omega + \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \int_\Omega \left( \nabla \cdot \mathbf{v} \right) \left( \nabla \cdot u \right) d\Omega \\
= \int_\Omega \mathbf{f} \cdot \mathbf{v} d\Omega + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} d\Gamma \quad \forall \mathbf{v} \in [H^1_{\Gamma_D}]^2 \end{align*} \quad (3.4.1)
\]

In this equation, the standard vector subspaces of $H^1$ are employed for the solution $\mathbf{u}$

\[
[H^1_{\Gamma_D}]^2 := \{ \mathbf{u} \in [H^1]^2 \mid \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D \}
\]

(Dirichlet conditions, $\mathbf{u}_D$, are automatically satisfied) and for the test functions $\mathbf{v}$

\[
[H^1_{0,\Gamma_D}]^2 := \{ \mathbf{v} \in [H^1]^2 \mid \mathbf{v} = 0 \text{ on } \Gamma_D \}
\]

(zero values are imposed along $\Gamma_D$).

This equation shows the inherent difficulties of the incompressible limit. The standard a priori error estimate emanating from (3.4.1) and based on the energy norm, which is induced by the LHS of (3.4.1), is

\[
\| \mathbf{u} - \mathbf{u}_h \| \leq \inf_{\mathbf{w} \in S_h} \| \mathbf{u} - \mathbf{w} \| \leq C_{u,\nu,p} h^{l(p)} \quad (3.4.2)
\]
where $\mathcal{S}_h$ is the finite dimensional subspace of $[H^1_\text{div}]^2$ in which the approximation $\mathbf{u}_h$ is sought, $C_{u,p}$ is a constant independent of $h$ (characteristic size of the mesh), and $f(p)$ is a positive monotone function of $p$ (degree of the polynomials used for the approximation). The subindices of the constant $C$ indicate that it depends on the Poisson ratio, the order of the interpolation and the exact solution itself.

From (3.4.1) one can observe the difficulties of the energy norm to produce a small infimum in (3.4.2) for values of $\nu$ close to 0.5. In fact, in order to have finite values of the energy norm the divergence-free condition must be enforced in the continuum case, i.e. $\nabla \cdot \mathbf{u} = 0$ for $\mathbf{u} \in [H^1_\text{div}]^2$, and also in the finite dimensional space, i.e. $\nabla \cdot \mathbf{u}_h = 0$ for $\mathbf{u}_h \in \mathcal{S}_h \subset [H^1_\text{div}]^2$. In fact, locking will occur when the approximation space $\mathcal{S}_h$ is not rich enough for the approximation to verify the divergence-free condition.

Under these conditions, it is evident that locking may be studied from the LHS of (3.4.1). This is the basis for the modal analysis of locking. The discrete eigenfunctions (the eigenvectors) corresponding to the LHS of (3.4.1) are computed because they completely describe, in the corresponding space, the behavior of the bilinear operator induced by this LHS.

In computational mechanics it is standard to write the strain, $\mathbf{\varepsilon}$, and the stress, $\mathbf{\sigma}$, tensors in vector form (Belytschko, Liu and Moran 2000). Moreover, under the assumptions already discussed, they are related as

$$
\mathbf{\varepsilon} = \mathbf{Bd}, \quad \mathbf{\sigma} = \mathbf{C} \mathbf{\varepsilon}, \quad \mathbf{C} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{pmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{pmatrix}.
$$

Where $\mathbf{d}$ is the vector of nodal displacements (the coefficients corresponding to the approximation $\mathbf{u}_h$ in the base of $\mathcal{S}_h$), and $\mathbf{B}$ is the standard matrix relating
displacements and strains. Then, the stiffness matrix can be computed as usual,

\[ K = \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega. \]

The modal analysis presented in the following is based on \( K \), which is naturally related to the energy norm in the finite dimensional interpolation space, \( \mathcal{S}_h \), defined by the finite elements employed (and characterized by \( \mathbf{B} \)).

### 3.4.2 Comparing EFG and pseudo-divergence-free EFG

The incompressible limit is studied by evaluating the eigenvalues associated to each mode as the Poisson ratio, \( \nu \), tends to 0.5. Following the procedure proposed by Huerta and Fernández-Méndez (2001) the logarithm of the eigenvalue is plotted as a function of the logarithm of \( 0.5 - \nu \). Then each mode is classified in three groups:

1) modes that do not present any locking behavior,
2) modes that do have physical locking, that is the eigenvalue goes to infinity because it is a volumetric mode, and
3) modes associated to non-physical locking, that is the eigenvalue goes to infinity but there is no volume variation.

In the last case, the displacement field conserves the total area but suffers from non-physical locking. The interpolation space is not rich enough to ensure the divergence-free condition.

In fact, these last modes do verify that

\[ \int_{\Omega} \nabla \cdot \mathbf{u}_h \, d\mathbf{x} = 0, \]

but do not comply with the divergence-free condition locally (at each interior point). This is clearly a non-physical locking behavior.

The modal analysis is performed on a distribution of \( 3 \times 3 \) particles and for bilinear consistency, that is \( \mathbf{P} = \{1, x_1, x_2, x_1 \, x_2\}^T \). Figures 3.1 and 3.2 show the
Locking in the incompressible limit: pseudo-divergence-free EFG

modes already classified for two different dilation parameters, $\rho/h = 1.2$ and $2.2$.

Non-locking  |  Physical locking  |  Non-physical locking
\hline
\includegraphics[width=0.3\textwidth]{figure3_1.png} & \includegraphics[width=0.3\textwidth]{figure3_1.png} & \includegraphics[width=0.3\textwidth]{figure3_1.png} \\
\includegraphics[width=0.3\textwidth]{figure3_2.png} & \includegraphics[width=0.3\textwidth]{figure3_2.png} & \includegraphics[width=0.3\textwidth]{figure3_2.png} \\
\includegraphics[width=0.3\textwidth]{figure3_3.png} & \includegraphics[width=0.3\textwidth]{figure3_3.png} & \includegraphics[width=0.3\textwidth]{figure3_3.png} \\

Figure 3.1: Modes for a $3 \times 3$ distribution, bilinear consistency and $\rho/h = 1.2$.

Non-locking  |  Physical locking  |  Non-physical locking
\hline
\includegraphics[width=0.3\textwidth]{figure3_4.png} & \includegraphics[width=0.3\textwidth]{figure3_4.png} & \includegraphics[width=0.3\textwidth]{figure3_4.png} \\
\includegraphics[width=0.3\textwidth]{figure3_5.png} & \includegraphics[width=0.3\textwidth]{figure3_5.png} & \includegraphics[width=0.3\textwidth]{figure3_5.png} \\
\includegraphics[width=0.3\textwidth]{figure3_6.png} & \includegraphics[width=0.3\textwidth]{figure3_6.png} & \includegraphics[width=0.3\textwidth]{figure3_6.png} \\

Figure 3.2: Modes for a $3 \times 3$ distribution, bilinear consistency and $\rho/h = 2.2$.

Figure 3.3 compares the eigenvalues obtained by standard EFG and the pseudo-divergence-free interpolation for two particular non-physical locking modes. Moreover, three values of ratio $\rho/h$ are also compared, namely $1.2$, $2.2$ and $3.2$. 
Locking in the incompressible limit: pseudo-divergence-free EFG

Figure 3.3: Comparison between EFG and pseudo-divergence-free interpolations: variation of the eigenvalue as $\nu$ goes to 0.5 for two non-physical locking modes with $\rho/h = 1.2, 2.2$ and 3.2.

Figure 3.4: Comparison between EFG and pseudo-divergence-free interpolations: variation of the maximum eigenvalue as $\nu$ goes to 0.5.
Note that the pseudo-divergence-free approximation has not suppressed the non-physical locking modes. Thus, for a fixed dilation parameter $\rho$ variations on the ratio $\rho/h$ do not suppress locking. Indeed, the influence of locking is reduced because the eigenvalue is decreased. That is, the energy associated to the locking mode is decreased and this attenuates the volumetric locking. Nevertheless, in the incompressible limit, locking will still be present and it may induce useless numerical results.

This results should be expected. The convergence of the diffuse derivative, see (3.2.1), is ensured as $\rho$ approaches zero for a ratio $\rho/h$ kept constant. In other words, convergence is ensured as the interpolation is refined.

This is analyzed in Figure 3.4 for the non-physical locking mode that presents the largest eigenvalue (the first mode to spoil the approximation). These results are obtained for the “worst” dilation parameter, $\rho/h = 1.2$; that is, the one that induces results more similar to bilinear finite elements.

Four different values of $\rho$ are tested, $\rho = 0.60, 0.24, 0.15$ and 0.05. It is important to note that as $\rho$ decreases the eigenvalue also decreases (and drastically, the scale is logarithmic). Thus, as $\rho$ decreases the influence of locking attenuates.

Moreover, and more importantly, the slope of the curve also decreases as $\rho$ goes to zero (note that for standard EFG the slope remains constant). Thus, in the limit, as expected, the interpolation is divergence-free. Note that for $\rho = 0.05$ and $\nu = 0.5 - 10^{-11}$ the eigenvalue has been reduced in more than three orders of magnitude.
3.5 Numerical examples

3.5.1 The cantilever beam

As shown in Figure 3.5, a beam with linear isotropic material under plane strain conditions and with a parabolic traction applied to the free end is considered. This is a well-known example studied, for instance, by Hughes (2000) and Dolbow and Belytschko (1999b). Displacements in both directions are prescribed at \( \Gamma_D \). The prescribed displacements and the applied traction are such that the solution is known:

\[
\begin{align*}
    u_1 &= -2 \frac{1 - \nu^2}{E} y \left( (48 - 3x_1)x_1 + (2 + \frac{\nu}{1 - \nu})(x_2^2 - 0.25) \right), \\
    u_2 &= 2 \frac{1 - \nu^2}{E} \left( \frac{\nu}{1 - \nu} x_2^2 (8 - x_1) + (4 + \frac{\nu}{1 - \nu}) \frac{x_1}{4} + (24 - x_1) x_1^2 \right), \\
    \sigma_{11} &= -12x_2(8 - x_1), \quad \sigma_{22} = 0, \quad \sigma_{12} = 6(0.25 - x_2^2).
\end{align*}
\]

The problem is solved with uniform distributions of particles. Figure 3.6 shows the relative \( L_2 \) error in displacements for \( \nu = 0.3, 0.5 - 10^{-4} \) and \( 0.5 - 10^{-6} \). Results are shown for bilinear consistency and \( \rho/h = 2.2 \). The EFG results are compared with the pseudo-divergence-free approximation. For EFG the typical...
convergence rates are obtained when $\nu = 0.3$, but, as expected, results degrade as $\nu$ approaches the incompressible limit 0.5. However, the pseudo-divergence free approximation is able to reproduce the theoretical rate of convergence even for a nearly incompressible case $\nu = 0.5 - 10^{-6}$ and a moderately fine discretization ($h < 0.25$, i.e. $\rho < 0.55$).

### 3.5.2 The plate with a hole

The stress field in an infinite plate with a hole subject to a far-field unit traction in the $x$ direction is (Dolbow and Belytschko 1999b):

$$
\sigma_{xx} = 1 - \frac{\alpha^2}{r^2} \left( \frac{3}{2} \cos(2\theta) + \cos(4\theta) \right) + \frac{3\alpha^4}{2r^4} \cos(4\theta)
$$
where $a = 1$ is the hole radius, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. The bounded upper quadrant shown in Figure 3.7 is used to solve the problem. Symmetry conditions are imposed in $x = 0$ and $y = 0$ and the tractions of the exact solution are considered in $\Gamma$.

Figures 3.8 and 3.9 show the relative $L_2$ error norm with $\nu = 0.3, 0.5 - 10^{-1}$ and $0.5 - 10^{-6}$. When $\nu = 0.3$ typical convergence is obtained for EFG but it suffers from locking when the incompressible limit is approached. The improved method maintains the convergence rate even with $\nu = 0.5 - 10^{-6}$. 
Figure 3.8: Hole problem with bilinear consistency and $\rho/h = 3.2$

Figure 3.9: Hole problem with biquadratic consistency and $\rho/h = 3.2$
3.6 Stokes problem

It is well known that the study of viscous incompressible flows presents similar difficulties as those found in incompressible solid mechanics. Thus, here, the pseudo-divergence-free method is also used to solve the Stokes problem. Continuous and discrete spaces for Stokes equations are subject to an inf-sup condition (Girault and Raviart 1986). This stability requirement is evidenced in practical computations by the existence of spurious pressure modes. The pseudo-divergence-free velocity field and the pressure field employed here comply the LBB condition asymptotically.

3.6.1 Statement of the problem

Let $\Omega$ denote an open bounded region of $\mathbb{R}^2$ with boundary $\partial \Omega$. The 2D Stokes problem in $\Omega$ seeks a velocity field $\mathbf{u} = (u_1, u_2)$ and a pressure field $p$ such that:

$$\begin{cases}
-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u} = \mathbf{g} & \text{on } \partial \Omega,
\end{cases} \quad (3.6.1)$$

where $\nu$ is the viscosity of the fluid and $\mathbf{f}$ is the body force (see Donea and Huerta 2003).

3.6.2 Weak form

Given the problem defined in (3.6.1) with $\mathbf{u} \in \mathbf{V}$ and $p \in \mathcal{Q}$, where $\mathbf{V} := [\mathcal{H}^1(\Omega)]^2$ and $\mathcal{Q} := L_2(\Omega)$, the weak form of the Stokes problem, taking $\mathbf{g} = 0$, is: find $\mathbf{u} \in \mathbf{V}, p \in \mathcal{Q}$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times \mathcal{Q},$$
where we define forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ as

$$a(u, v) := \int_{\Omega} \nabla v : \nu \nabla u \, d\Omega = \nu (\nabla u, \nabla v),$$

$$b(v, p) := - \int_{\Omega} p \nabla \cdot v \, d\Omega = - (p, \nabla \cdot v).$$

Note that $(\cdot, \cdot)$ denotes the standard $\mathcal{L}_2(\Omega)$-scalar product.

We now turn to the consideration of an approximate discrete solution of the problem. Let $V_\rho$ and $Q_\rho$ denote finite dimensional subspaces of $\mathcal{V}$ and $\mathcal{Q}$ respectively. The index $\rho$ refers to a characteristic measure of the support of the interpolation functions; it is related to the characteristic measure between particles, $h$. The discrete version of the problem, which in this case uses Nitsche’s method (see Arnold, Brezzi, Cockburn and Marini 2001/02, Babuška, Banerjee and Osborn 2002a, Becker 2002, Stenberg 1995, Fernández-Méndez and Huerta 2004) to impose boundary conditions, reads: find $u^\rho \in V_\rho, p^\rho \in Q_\rho$ such that, for $(v^\rho, q^\rho) \in V_\rho \times Q_\rho$,

$$a(u^\rho, v^\rho) + b(v^\rho, p^\rho) + b(u^\rho, q^\rho)$$

$$= (f, v^\rho) - (g, \nu \partial_n v^\rho - q^\rho n)_{\partial \Omega} + \nu \gamma \frac{\rho}{\rho} (u^\rho, v^\rho)_{\partial \Omega}$$

Now, $(\cdot, \cdot)_{\partial \Omega}$ denotes the $\mathcal{L}_2(\partial \Omega)$-scalar product. Finally, the scalar $\gamma$ is an arbitrary positive parameter that has to be chosen big enough in order to guarantee stability. Here an eigenvalue problem is solved as proposed by Griebel and Schweitzer (2002).

### 3.6.3 Analytical test

We consider a test problem with an analytical polynomial solution on the unit square, see Oden and Jacquotte (1984). Homogeneous Dirichlet boundary condi-
tions are imposed on the whole boundary, and the theoretical rates of convergence, recall equation (3.2.1), shall be recovered numerically.

We consider the Stokes problem presented in (3.6.1) with \( \partial \Omega \), a polynomial force \( f \) is imposed in order to ensure the following solution of the problem:

\[
\begin{align*}
    u_1(x, y) &= x^2(1 - x)^2(2y - 6y^2 + 4y^3) \\
    u_2(x, y) &= (-2x + 6x^2 - 4x^3)y^2(1 - y)^2 \\
    p(x, y) &= x(1 - x)
\end{align*}
\]

We solve this problem with the pseudo-divergence-free MLS method and using \( \rho/h = 1.2 \) with a bilinear base to approximate both velocity and pressure.

The convergence results are shown in Figure 3.10. The velocity convergence rates for standard EFG and for the pseudo-divergence-free method are, as expected, similar. However, convergence in pressure is far from optimal in EFG, whereas it presents the theoretical slope in the proposed method. Recall that Eq (3.2.1) indicates that diffuse derivatives converge to the actual derivatives as \( \rho \to 0 \) (\( \rho/h = \text{cst} \)). Since we use a bilinear base (\( i.e. \ m = 1 \)) the convergence behaves as \( \rho^1 \). This means that if we double the number of particles (\( i.e. \), if we divide \( \rho \) by 2) then the divergence must be at least divided by two. Figure 3.10 shows exactly this behavior.

3.6.4 Driven cavity flow problem (leaky)

Now we consider Stokes problem, equations (3.6.1), with \( \Omega = [0, 1] \times [0, 1] \), \( f = (0, 0)^T \), \( g = (0, 0)^T \) on \( \partial \Omega \setminus \{y = 1\} \) and \( g = (1, 0)^T \) on \( \partial \Omega \cap \{y = 1\} \). We solve this well-known benchmark problem with the pseudo-divergence-free method using \( \rho/h = 2.1 \) and a biquadratic base to approximate velocity and pressure. Streamlines, pressure distribution and divergence of \( \mathbf{u} \) are depicted in Figure

3.10.
Reasonable results are obtained in spite of the equal order interpolation for velocity and pressure. No spurious pressure modes are observed. Figure 3.11 shows the presence of spurious pressure modes with the EFG method.

3.7 Conclusions

A novel improved formulation of the Element Free Galerkin method is proposed in order to alleviate volumetric locking. It is based on a pseudo-divergence-free interpolation. Using the concept of diffuse derivatives an a convergence theorem of these derivatives to the ones of the exact solution, the new approximation proposed is obtained imposing a zero diffuse divergence. This implies that the method verifies asymptotically the incompressibility condition. Moreover, the imposition of the incompressibility restriction can be done a priori (i.e. before the computation). This means that the main difference between standard EFG and the improved method is how the interpolation basis is chosen. Modal analysis and numerical results for two classical benchmark tests in solids corroborate that, as expected, diffuse derivatives converge to the derivatives of the exact solution.
Figure 3.11: EFG solution for a uniform distribution of 11x11 (top), 21x21 (middle) and 41x41 (bottom) particles.
Figure 3.12: Pseudo-divergence-free solution for a uniform distribution of 11x11(top), 21x21(middle) and 41x41(bottom) particles.
when the discretization is refined (for a fixed dilation parameter) and, of course, that diffuse divergence converges to the exact divergence with the expected theoretical rate. The typical convergence rate is degraded as the incompressible limit approaches for standard EFG but with the improved method good results are obtained even for a nearly incompressible case and a moderately fine discretization. The improved method has also been used to solve the Stokes equations. In this case the LBB condition is not explicitly satisfied because the pseudo-divergence-free approximation is employed. Reasonable results are obtained in spite of the equal order interpolation for velocity and pressure.
CSPH for explicit dynamic problems
Chapter 4

CSPH for explicit dynamic problems

4.1 Introduction

In its original form the method SPH had several weak points, described in detail in a study made by Swegle et al. (1995) and also by Belytschko, Guo, Liu and Xiao (2000). These problems consisted on lack of consistency, tension instability and the presence of zero energy modes in the numeric solution.

Johnson, Stryk and Beissel (1996) developed the normalized smoothing method obtaining lineal consistency. Bonet and Kulasegaram (2000) developed the Corrected Smooth Particle Hydrodynamics (CSPH) method that allows to obtain lineal consistency in the interpolation of the function and in the interpolation of the gradient. Consistency is achieved introducing corrections in the kernel functions and in their derivatives.

Concerning the tension instability, Bonet and Kulasegaram (2001) proved that a Lagrangian formulation removes completely the instability. However, zero energy modes still remain in the Lagrangian formulation. Moreover, in problems with large distortions a Lagrangian formulation will require updates of the refer-
ence configuration. When such updates are incorporated zero energy modes are more likely to be activated. The objective of next chapter is to develop an updated Lagrangian formulation which allows to carry out updates of the reference configuration without suffering the appearance of spurious modes.

The problem of zero energy modes is still open. In the literature two types of solutions are used: dissipate spurious modes in a way conceptually similar to the one used in finite elements for hourglass modes or otherwise use an alternative discretization which does not evaluate the variables and their derivatives in the same points. For example, Gray, Monaghan and Swift (2001) preclude instability introducing an artificial stress (but that introduces also small errors in the solution) and Randles and Libersky (2000) use different sets of particles to interpolate different fields generating the denominated stress points.

In next section Eulerian CSPH formulation is revised. Some numerical examples will show the appearance of tension instability. In Section 4.2.4 an stability analysis is performed which gives a mathematical understanding of tension instability. Finally, in Section 4.3 Lagrangian CSPH formulation is revised for large strains dynamic problems and some numerical examples are performed. These numerical examples will show the abilities and also the limits of the Lagrangian formulation.

4.2 CSPH Eulerian Formulation

4.2.1 Continuum equations

Figure 4.1 shows the movement of a deformable body. The movement can be described mathematically by a time dependent mapping \( \varphi \) between the initial position \( X \) and the current position \( \bar{x} \) of the particles as,

\[
\bar{x} = \varphi(X, t).
\]  

(4.2.1)
For a fixed value of time $t$, the previous equation represents a mapping between the undeformed body and the deformed one. Moreover, for a given point $X$, equation (4.2.1) describes the trajectory of this point as a function of time. In large deformations analysis no hypothesis is made on the magnitude of the displacements $x - X$. In fact, the displacements can be of the order or even larger than the initial dimensions of the body.

Velocity is defined as the material time derivative of the motion, as

$$v(X,t) = \frac{\partial \varphi(X,t)}{\partial t}.$$  

Observe that velocity is an Eulerian (spatial) vector despite the fact that the equation has been expressed in terms of the material coordinates of the particle $X$. In fact, the velocity can be more consistently expressed as a function of the spatial position $x$ and time as,

$$v(x,t) = v(\varphi^{-1}(x,t),t).$$  \hfill (4.2.2)
The derivative of (4.2.2) with respect to the spatial coordinates defines the 
velocity gradient tensor \( \mathbf{l} \) as,
\[
\mathbf{l} = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial \mathbf{x}} = \nabla \mathbf{v}.
\] (4.2.3)

This is a spatial tensor which gives the relative velocity of a particle currently at 
point \( q \) with respect to a particle currently at \( p \) with elemental vector \( d\mathbf{x} = \mathbf{x}_q - \mathbf{x}_p \) 
as,
\[
d\mathbf{v} = \mathbf{l} \, d\mathbf{x}.
\]

The symmetric part of \( \mathbf{l} \) is known as the rate of deformation tensor \( \mathbf{d} \) given as,
\[
\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T).
\]

In order to establish constitutive models in terms of stress rates it is essential 
to derive stress rate measures that are objective. One of the most common of these 
rates is the Truesdell stress rate \( \mathbf{\sigma}^\tau \) tensor which is defined as,
\[
\mathbf{\sigma}^\tau = \dot{\mathbf{\sigma}} - \mathbf{l} \mathbf{\sigma} - \mathbf{\sigma} \mathbf{l}^T + (\text{tr} \, \mathbf{l}) \mathbf{\sigma}.
\]

### 4.2.2 Eulerian CSPH

Let us consider a discretized body using SPH particles. The velocity gradient 
tensor, defined in (4.2.3), can be evaluated now in a certain particle \( j \) in terms of 
the current positions as
\[
\mathbf{l}_j = \nabla \mathbf{v}_j = \sum_k \mathbf{v}_k \otimes \mathbf{g}_k(\mathbf{x}_j),
\] (4.2.4)

where \( \nabla \) indicates the gradient respect to the current configuration and where the 
functions \( \mathbf{g}_k(\mathbf{x}_j) \) are the corrected kernel gradients. They are corrected using 
equation (2.2.4) in the current configuration, that is,
\[
\mathbf{g}_k(\mathbf{x}_j) = V_k \, \nabla \phi \left( \frac{\mathbf{x}_j - \mathbf{x}_k}{\rho} \right),
\]
where $V_k$ is the current volume associated to particle $k$.

A similar derivation to that followed in (4.3.4-4.3.5) applied to the current configuration yields the corresponding Eulerian counterpart for the internal forces as,

$$T_i = \sum_j V_j \sigma_{ij}(x_j). \quad (4.2.5)$$

It is important to observe that in equation (4.2.5) the kernel derivatives, $g_i(x_j)$, depend on the current position of the particles. This implies that corrections are calculated each time step.

### 4.2.3 Numerical Examples

In order to illustrate the ability and limitations of Eulerian CSPH formulation various numerical examples are presented. In all the following examples an hyperelastic nearly incompressible Neo-Hookean material is used. Here, a schematic review of the properties of such material is provided, see Bonet and Wood (1997).

\[ \exists \Psi(X,F) \text{ s.t. } P = \frac{\partial \Psi}{\partial F} \]

\[ \int_{t_0}^{t} P : \dot{F} dt = \Psi(X, F(X, t)) - \Psi(X, F(X, t_0)) \]

\[ S = 2 \frac{\partial \Psi}{\partial C} \]

▷ Hyperelastic

- $\hat{C} = (\det C)^{-1/3} C$
- $\hat{\Psi}(C) = \Psi(\hat{C}) + \frac{1}{2} k(J - 1)^2$
- $S = 2 \frac{\partial \hat{\Psi}(C)}{\partial C} + pJC^{-1}$

▷ Nearly Incompressible
Neo-Hookean

- $\Psi(C) = \frac{1}{2}\mu(\text{tr} C - 3)$
- $S = \mu (\det C)^{-1/3}(I - \frac{1}{3}(\text{tr} C) C^{-1}) + pJ C^{-1}$

**Bending test of hyperelastic material**

This example consists in the simulation of a three-dimensional problem with large deformations using an hyperelastic material. Let’s consider a nearly incompressible neo-Hookean cylinder travelling with an initial speed of $1.88\text{m/s}$ which is suddenly fixed at its base (see Figure 4.2). The initial radius is $0.32\text{m}$ and the length $3.24\text{m}$. The shear modulus is taken as $0.3571\text{MN/m}^2$ and the bulk modulus is $1.67\text{MN/m}^2$.

Figure 4.3 shows that numerical fracture is observed in areas of high tension. This is the typical behavior of tension instability: wherever high tensile stresses
are developed the material fractures artificially and particles cluster together.

Figure 4.3: Eulerian results for the cylinder problem.

**Rubber rings collision**

This is a classical benchmark test for tension instability. This test consists in simulating the collision of two rubber rings. These rings should bounce off each other without disintegration.

We use a neo-Hookean hyperelastic material as in previous example. The problem consists in the impact of two circular rings coming together at a relative speed of 1.18m/s. The exterior radius of each ring is 4cm and interior radius 3cm. Figure [4.4](#) shows the numerical results. Again, artificial fracture is observed in areas of high tension. In fact, calculations were performed only with one rubber ring and symmetry conditions.
4.2.4 Stability analysis: Tension Instability

Typically, stability is analyzed in the context of a time integration scheme. However, as highlighted Bonet and Kulasegaram (2001), introducing a time stepping scheme is an unnecessary complication, as the real problem lies with the existence of negative (tension instability) or zero (spurious modes) eigenvalues in the tangent stiffness of the discretized body. In this section the stability analysis of the Eulerian formulation is performed.

Let’s consider a 1D bar discretized by a given number of particles as shown in Figure 4.5. For simplicity particle spacing will be assumed to be uniform and
only immediate neighbors of a given particle will contribute to the internal force
evaluation at this particle.

\[ \phi((x - x_i)/\rho) \]

Figure 4.5: 1D Eulerian CSPH

An Eulerian formulation yields an equation for the internal forces in 1D as, see (4.2.5),

\[ T_i = \sum_j l_j \sigma_j g_i(x_j). \quad (4.2.6) \]

The tangent stiffness matrix can now be obtained by differentiation of (4.2.6) with respect to nodal positions using for instance the linear constitutive relationship \( \sigma_i = \kappa(J_i - 1) \). It is crucial that the variations of all terms in (4.2.6) with nodal positions are accounted for. After simple but lengthy algebra, the following
expression is obtained for the case with uniform spacing \( h \),

\[
K_{i,i+2} = -\frac{\alpha}{4h} - \frac{h}{8}\sigma''_i
\]

\[
K_{i,i+1} = \frac{\beta h^2}{4\alpha}\sigma''_i
\]

\[
K_{i,i} = \frac{\alpha}{2h} + \frac{h}{4}\sigma''_i - \frac{\beta h^2}{2\alpha}\sigma''_i
\]

\[
K_{i,i-1} = \frac{\beta h^2}{4\alpha}\sigma''_i
\]

\[
K_{i,i-2} = -\frac{\alpha}{4h} - \frac{h}{8}\sigma''_i.
\]

Where \( h \) is the uniform particle spacing and the parameters \( \alpha \) and \( \beta \) are:

\[
\alpha = \phi'(x_{i-1}) = -\phi'(x_{i+1})
\]

\[
\beta = \phi''(x_{i-1}) = \phi''(x_{i+1}).
\]

Note that the mode \((-1)^j\) is an eigenvector of the stiffness matrix. In fact, multiplying the matrix by the mode one obtains,

\[
\sum_j K_{ij}(-1)^j = -\frac{\beta h^2}{\alpha}\sigma''(-1)^j.
\]

From this equation is clear that stress distributions with a positive second derivative lead to a stiffness matrix with negative eigenvalues and, obviously, the formulation becomes unstable.

If a similar analytical analysis is performed with the standard SPH formulation it is found that negative eigenvalues appear when the rod is in tension (positive \( \sigma \)). That’s why the instability is called tension instability.

4.2.5 Conclusions

It has been shown in this section that the Eulerian formulation is not suitable because when involving tensile stress the tension instability appears. However,
fluids usually remain in compression, at least in absolute terms, and this problem will not arise. That’s why Eulerian formulation has been widely used in fluid problems with good results.

4.3 CSPH Lagrangian Formulation

4.3.1 Continuum equations

The deformation gradient $F$, defined as

$$ F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, $$

is a quantity of interest in the study of large deformations because it is present in all those equations that relate magnitudes in the initial configuration with their corresponding ones in the final configuration. The volume change of the continuum can be obtained in terms of the Jacobian

$$ J = \det F = \frac{dV}{dV^0}, $$

where $dV^0$ and $dV$ represent the initial and current volume elements. The Cauchy equation (balance of moments equation) for the deformable body reads

$$ \nabla \cdot \mathbf{\sigma} + \zeta \mathbf{f} = \zeta \mathbf{\alpha}, $$

where $\zeta$ is the material density, $\mathbf{\alpha}$ is the acceleration, $\mathbf{f}$ is external force for volume unit and $\mathbf{\sigma}$ is the Cauchy stress tensor. Recall that,

$$ \mathbf{P} = J \mathbf{\sigma} \mathbf{F}^{-T}, $$

where $\mathbf{P}$ is the first Piola-Kirchhoff tensor.
4.3.2 Lagrangian CSPH

Let us consider a discretized body using SPH particles. The deformation gradient, defined in (4.3.1), can be evaluated now at a certain particle $j$ in terms of the current positions as

$$\mathbf{F}_j = \nabla \alpha \varphi = \sum_k \mathbf{x}_k \otimes \mathbf{G}_k(X_j), \quad (4.3.3)$$

where $\nabla \alpha$ indicates the gradient respect to the initial configuration and where the functions $\mathbf{G}_k(X_j)$ are the corrected kernel gradients. They are corrected using (2.2.4) in the initial reference configuration, that is,

$$\mathbf{G}_k(X_j) = V_k \nabla \frac{\alpha \varphi}{\rho} \left( \frac{X_j - X_k}{\rho} \right).$$

In order to find general equations for the internal forces using a Lagrangian CSPH formulation, consider the equation of the internal virtual work in the reference configuration in terms of the first Piola-Kirchhoff tensor, $\mathbf{P}$,

$$\delta w_{\text{int}} = \int_{V_0} \mathbf{P} : \delta \mathbf{F} dV^0 \simeq \sum_j V_j^0 \mathbf{p}_j : \delta \mathbf{f}_j. \quad (4.3.4)$$

The variation of the virtual deformation gradient emerges from equation (4.3.3) as

$$\delta \mathbf{F}_j = \sum_k \delta \mathbf{x}_k \otimes \mathbf{G}_k(X_j),$$

where after substituting in $t_0$ the equation (4.3.4), leads to the expression of the internal virtual work

$$\delta w_{\text{int}} = \sum_j V_j^0 \mathbf{p}_j : \delta \mathbf{f}_j = \sum_j V_j^0 \mathbf{p}_j : \left( \sum_k \delta \mathbf{x}_k \otimes \mathbf{G}_k(X_j) \right)$$

$$= \sum_k \delta \mathbf{x}_k \cdot \left( \sum_j V_j^0 \mathbf{p}_j \mathbf{G}_k(X_j) \right).$$
This expression allows the vector of internal forces corresponding to a certain particle $i$ to be identified as:

$$
T_i = \sum_j V_j^0 P_j G_i(X_j).
$$

(4.3.5)

It is important to observe that in equation (4.3.5) the kernel derivatives, $G_i(X_j)$, are fixed in the reference configuration and therefore they do not depend on the current positions of the particles. This bears that corrections are only calculated at the beginning reducing the computational cost.

### 4.3.3 Numerical Examples

In order to illustrate the ability and limitations of Lagrangian CSPH formulation various numerical examples are solved, see also Bonet and Kulasegaram (2001).

**Bending test of hyperelastic material**

Problem presented in Section 4.2.3 is now solved using a Lagrangian CSPH formulation. The improved quality of the results obtained can be seen in Figure 4.6. The bar oscillates from initial position to maximum deformation and then back to initial position as expected. The stress component $\sigma_{zz}$ is shown where $z$ is the height component. The cylinder deformation is simulated with good results even in the presence of high tension.

**Rubber rings collision**

The problem presented in Section 4.2.3 is solved again using a Lagrangian CSPH formulation. In Figure 4.7 the results are shown. The rings collide, bounce and then oscillate with no fracture. In fact, calculations were performed only with one rubber ring and symmetry conditions.
Figure 4.6: Oscillation from initial position to maximum deformation. Stress component $\sigma_{zz}$ is shown.

Figure 4.7: The collision of two rubber rings using a totally Lagrangian CSPH algorithm. Stress component $\sigma_{xx}$ is shown.
Punch test

Previous examples have been solved using a totally lagrangian approach with very good results. Here an example with large distorsions is presented, see Huerta and Casadei (1994); Fernández-Méndez, Bonet and Huerta (n.d.).

The problem consists in an hyper-elastic material equal to the one used in previous examples with shear modulus taken as \(0.3571\text{MN/m}^2\) and the bulk modulus \(1.67\text{MN/m}^2\). The body is deformed by a rigid frictionless tool with a prescribed speed. Only a quarter of the domain (a rectangular region of 3cm by 1cm) is studied because two axes of symmetry are present. A schematic statement of this problem is presented in Figure 4.8.

Figure 4.8: Schematic statement of the Punch test

The punch test is solved using a Lagrangian CSPH formulation but this time results are less good, see Figure 4.9. In the first time steps the method simu-
lates properly the expected behavior, but as time goes on this problem involves large distortions and the Lagrangian formulations is not able to simulate them. Neighboring particles in the initial configuration can move very far away so the Lagrangian approximation becomes worse and worse. In order to circumvent this problem some updates of the reference configuration should be done. In next chapter a deep study of the updated Lagrangian formulation is presented.

Figure 4.9: Punch test using a totally Lagrangian CSPH algorithm. Stress component $\sigma_{zz}$ is represented.
4.3.4 Stability analysis: spurious modes

In order to compare the properties of the above Lagrangian formulation against the Eulerian counterparts discussed in [4.2], it is useful to revisit the one-dimensional wave propagation case discussed in [4.2.4]. For this purpose consider a 1-D bar discretized by a given number of points which deforms from initial to final configurations as shown in Figure 4.10. Note that in the case of a Lagrangian SPH formulation, the kernel functions are fixed at the initial or reference configuration. For simplicity, and in line with previous assumptions, the initial particle spacing will be assumed to be uniform and only immediate neighbors of a given particle will contribute to the internal force evaluation at this particle.

![Figure 4.10: 1D Lagrangian CSPH](image)

Given the one-dimensional nature of the problem and taking the area as constant and equal to one (in the reference configuration), the deformation gradient is
simply given as

\[ F_i = \sum_j x_j G_j(X_i). \] (4.3.6)

For the simple uniformly spaced reference configuration considered, the linearly corrected gradient terms are simply

\[ G_{i-1}(X_i) = -\frac{1}{2h}; \quad G_i(X_i) = 0; \quad G_{i+1}(X_i) = \frac{1}{2h}; \] (4.3.7)

which upon substitution into (4.3.6) leads to

\[ F_i = \frac{x_{i+1} - x_{i-1}}{2h}. \]

Consider now the internal force equation in the current 1-D context where it is easy to show that the First Piola-Kirchhoff stress is identical to the Cauchy stress given that:

\[ P = J \sigma F^{-1} = \sigma. \]

Using this identity together with equation (4.3.7) for the gradient functions enables the internal force at point \( i \) to be obtained as

\[ T_i = \frac{\sigma_{i-1} - \sigma_{i+1}}{2}. \]

Using the linear constitutive relationship \( \sigma_i = \kappa(J_i - 1) \) the tangent stiffness matrix terms are now easily evaluated to give,

\[ K_{i,i+2} = -\frac{\kappa}{4h}, \]
\[ K_{i,i+1} = 0, \]
\[ K_{i,i} = \frac{\kappa}{2h}, \]
\[ K_{i,i-1} = 0, \]
\[ K_{i,i-2} = -\frac{\kappa}{4h}. \]
Finally a simple calculation shows that the eigenvalue associated to the alternating eigenvector \((-1)^j\) now vanishes as

\[ \sum_j K_{ij} (-1)^j = 0. \]

The above equation implies that this alternating mode is now a mechanism instead of a mode with a possible negative eigenvalue. Consequently, the algorithm should be stable but, in the absence of artificial viscosity, uncontrolled oscillations may emerge during the computations.

### 4.3.5 Conclusions

Lagrangian formulation completely eliminates tension instability and as kernel derivatives are fixed at the reference configuration the computational cost is lower than in the Eulerian one. However, for problems with large distortions a Lagrangian formulation will need updates of the reference configuration.

It is important to realize that although tension instability is removed, the Lagrangian formulation still suffers from the presence of zero energy modes. This modes are more likely to be activated when updates are performed. In next chapter the updated Lagrangian formulation is presented, analyzed, some stabilizations are proposed and finally numerical results are shown.
Chapter 5

Stabilized Updated Lagrangian CSPH for explicit dynamic problems

In this chapter an updated Lagrangian formulation is proposed. To update the Lagrangian formulation the incremental approach that is shown in Figure 5.1 is proposed. Configuration $\mathbf{x}^*$ will be the new reference configuration for the next time steps.

It has been observed that this updated formulation suffers from similar numerical fracture to the Eulerian case. A modal analysis has proven that there exist zero energy modes. In the next sections the updated Lagrangian method is exposed in detail, a stability analysis is performed and finally a stabilization technique is incorporated to preclude spurious modes. Finally, some numerical examples and conclusions are presented.
5.1 Standard Formulation

The updated Lagrangian formulation consists of a multiplicative incremental approach as illustrated in Figure 5.1. Configuration \( \mathbf{x}' \) will be the new reference configuration for the next time steps. This means that a new neighbor search must be done in configuration \( \mathbf{x}' \) and that corrections of the kernel and its derivatives must be recalculated.

It is important to observe that the deformation gradient \( \mathbf{F} \) is stored as an internal state variable and only \( \mathbf{f} \) (the deformation gradient between the new reference configuration and the final one) is calculated each time step.

Let’s see how the update affects the internal forces calculation. Recall that the deformation gradient is given by

\[
\mathbf{F}^n = f \mathbf{F}' = \frac{\partial \mathbf{x}^n}{\partial \mathbf{x}'} \bigg|_j \mathbf{F}'_j = \left( \sum_k \mathbf{x}_k^n \otimes V_k \nabla_r \phi \left( \frac{\mathbf{x}_j^n - \mathbf{x}_k^n}{\rho} \right) \right) \mathbf{F}'_j, \quad (5.1.1)
\]

where it is important to notice that functions \( g_k(\mathbf{x}'_j) \) are the corrected kernel gra-
dients in the new reference configuration. From equation (5.1.1) it follows that
the variation of the virtual deformation gradient will be

\[ \delta \mathbf{F}^n_j = \left( \sum_k \delta \mathbf{v}_k \otimes \mathbf{g}_k(\mathbf{x}_j^r) \right) \mathbf{F}^r_j. \]

So the internal virtual work expressed in the initial configuration is

\[ \delta w_{int} = \int_{V_0} \mathbf{P}^n : \delta \mathbf{F}^n dV^0 = \sum_j V_j^0 \mathbf{P}^n_j : \delta \mathbf{F}^n_j = \sum_j V_j^0 \mathbf{P}^n_j : (\delta \mathbf{f}_j \mathbf{F}^r_j). \]

Recalling the matricial property \( \mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B}) \) we have

\[ \mathbf{P}^n_j : (\delta \mathbf{f}_j \cdot \mathbf{F}^r_j) = \text{tr}(\mathbf{P}^n_j \mathbf{F}^r_j^T \delta \mathbf{f}_j^T) = (\mathbf{P}^n_j \mathbf{F}^r_j^T) : \delta \mathbf{f}_j, \]

so internal virtual work can be written as

\[ \delta w_{int} = \sum_j V_j^0 \mathbf{P}^n_j \mathbf{F}^r_j^T : \delta \mathbf{f}_j = \sum_j \left( V_j^0 \mathbf{P}^n_j \mathbf{F}^r_j^T \cdot \left( \sum_k \delta \mathbf{v}_k \otimes \mathbf{g}_k(\mathbf{x}_j^r) \right) \right). \]

Recalling now the matricial and vectorial property \( \mathbf{A} : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{A} \mathbf{v} \) we have

\[ \delta w_{int} = \sum_k \delta \mathbf{v}_k \cdot \left( \sum_j V_j^0 \mathbf{P}^n_j \mathbf{F}^r_j^T \mathbf{g}_k(\mathbf{x}_j^r) \right) \]

and therefore in a certain time, \( t = n \), using as reference configuration \( \mathbf{x}^r \), we can easily identify the internal forces vector in a certain particle \( i \) as:

\[ \mathbf{T}^{n,r}_i = \sum_j V_j^0 \mathbf{P}^n_j (\mathbf{F}^r_j)^T \mathbf{g}_i(\mathbf{x}_j^r). \]
5.2 1D Analytical Stability Analysis

An analytical stability analysis for the updated Lagrangian formulation is presented in this section. Let’s consider a 1D bar discretized by a given number of particles which deforms from reference, \( X^r \), to final, \( x \), configurations as shown in Figure 5.2. Note that in the case of an updated lagrangian CSPH formulation, the kernel functions are fixed at the reference configuration which can be any configuration between initial and current ones. For simplicity particle spacing will be assumed to be uniform in the reference configuration and only immediate neighbors of a given particle will contribute to the internal force evaluation at this particle.

\[
\phi((X-X^r)/\rho)
\]

Figure 5.2: 1D Updated Lagrangian CSPH

Given the one-dimensional nature of the problem and taking the area as constant and equal to one (in the reference configuration), the deformation gradient is
simply given as

\[ F_i = f_i F_i^r = \left( \sum_j x_j g_j(X_i^r) \right) F_i^r. \]  

(5.2.1)

For the simple uniformly spaced reference configuration considered, the linearly corrected gradient terms \( g \) are simply

\[ g_{i-1}(X_i^r) = \frac{1}{2h}, \quad g_i(X_i^r) = 0; \quad g_{i+1}(X_i^r) = \frac{1}{2h}; \]  

(5.2.2)

which upon substitution into (5.2.1) leads to

\[ F_i = \frac{x_{i+1} - x_{i-1}}{2h} F_i^r. \]

Consider now the internal force equation in the current 1D context

\[ T_i^{n,r} = \sum_j V_j^0 F_j^r \vec{g}_i(X_j^r) = \sum_j V_j^r \sigma_j \vec{f}_j^r \vec{g}_i(X_j^r). \]

Using this identity together with (5.2.2) for the gradient functions enables the internal force at point \( i \) to be obtained as

\[ T_i = \frac{V_{i-1}^n \sigma_{i-1} f_{i-1}^1 - V_{i+1}^n \sigma_{i+1} f_{i+1}^1}{2h}. \]

The internal force vector is only a function of the current nodal positions via the stress values. Using the linear constitutive relationship, \( \sigma = \kappa (J_i - 1) \), the tangent stiffness matrix terms are now easily evaluated to give

\[ K_{i,i+2} = \frac{-V^n \kappa}{(x_{i+2} - x_{i})^2} \]

\[ K_{i,i+1} = 0 \]

\[ K_{i,i} = \frac{V^n \kappa}{(x_i - x_{i-2})^2} + \frac{V^n \kappa}{(x_{i+2} - x_{i})^2} \]

\[ K_{i,i-1} = 0 \]

\[ K_{i,i-2} = \frac{-V^n \kappa}{(x_i - x_{i-2})^2} \]
Finally a simple calculation shows that the eigenvalue associated to the alternating eigenvector \((-1)^j\) now vanishes as
\[
\sum_j K_{ij} (-1)^j = 0.
\]
The above equation implies that this alternating mode is now a mechanism instead of a mode with a possible negative eigenvalue. Consequently, the algorithm should be stable but, in the absence of artificial viscosity, undamped oscillations may emerge during the computations.

### 5.3 1D Numerical Tests

The previous section has proven the existence of mechanisms in the updated Lagrangian formulation (as well as in the totally Lagrangian) associated to zero energy modes. Next, 1D numerical tests will be performed in order to verify in these formulations if the mechanisms are activated or not.

The totally Lagrangian and the updated Lagrangian formulations will be used to solve the elastic 1D bar problem described in Figure 5.3, see Dyka and Ingel (1995). The bar is fixed at the left end A and the right quarter of the bar is given an initial velocity of \(v_0 = 5m/s\) thus putting the bar in tension initially. Standard SPH methods cannot solve this problem due to tension instability that immediately develops.

![Figure 5.3: 1D bar problem statement.](image.png)
The problem is solved using an uniform distribution of particles. As shown in Figure 5.3 the CSPH particle distribution is very coarse with only 40 uniform particles used in the model. Figure 5.4 presents the displacement time history of the right end B for the totally Lagrangian formulation and the updated Lagrangian one with updates every 3 time steps. Figure 5.5 indicates the predicted time history for the velocity of the right end B. It becomes clear from Figures 5.4 and 5.5 that in the updated Lagrangian formulation the mechanisms are activated and they spoil the solution.

![Graph showing displacement history for right end of the bar (point B). Totally Lagrangian (left) and updated Lagrangian (right) formulations.]

Figure 5.4: Displacement history for the right end of the bar (point B). Totally Lagrangian (left) and updated Lagrangian (right) formulations.

### 5.4 Stabilization techniques

It is necessary to eliminate the observed mechanisms if an updated formulation has to be used. Next, a new stabilization technique is proposed.
5.4.1 Hessian’s Difference Stabilization

This method is based on the addition of a higher order derivative term to the corrected gradient of the trial function ($\varphi$). The added term is the difference of two Hessians which will converge to zero as the particle distribution is refined:

$$
\nabla^2 \varphi^i(x^i) := \sum_j \varphi_j g_j(x^i) + h [\mathcal{H}_\varphi(x^i) - \nabla(\nabla \varphi(x^i))] \eta 
$$

(5.4.1)

where $\eta = (\eta_x, \eta_y, \eta_z)^T$ is a non-dimensional stabilization vector of parameters, $h$ is related with a characteristic length of the discretization and $\mathcal{H}_\varphi(x^i)$ represents the Hessian of $\varphi$ which in the context of CSPH can be obtained as,

$$
\mathcal{H}_\varphi(x^i) := \sum_k V^i_k \varphi_k \tilde{\mathcal{H}}_{\varphi_k}(x^i)
$$

where $\tilde{\mathcal{H}}_{\varphi_k}$ is the kernel Hessian linearly corrected. To obtain linear reproducibility, $\mathcal{H}_{\varphi_k}$ is corrected by means of two terms, namely a matrix $\mathcal{B}$ and a third order
tensor $\mathbf{A}$ as,
\[
\tilde{\mathbf{H}}_{\phi_i}(\mathbf{x}_i^r) = \mathbf{H}_{\phi_i}(\mathbf{x}_i^r) + \delta_{ik} \mathbf{B}_i + \mathbf{A}_i(\mathbf{x}_i^r - \mathbf{x}_k^r).
\]
Correction terms $\mathbf{B}$ and $\mathbf{A}$ are determined enforcing that constant and linear functions must have null Hessian, that is:
\[
\triangleright \sum_k V_k^r \tilde{\mathbf{H}}_{\phi_i}(\mathbf{x}_i^r) = \mathbf{0},
\]
\[
\triangleright \sum_k V_k^r \tilde{\mathbf{H}}_{\phi_i}(\mathbf{x}_i^r) \mathbf{x}_k^r = \mathbf{0}.
\]
This enables $\mathbf{B}_i$ and $\mathbf{A}_i$ to be obtained as:
\[
\mathbf{A}_i = \left[ \sum_k V_k^r \mathbf{H}_{\phi_i}(\mathbf{x}_i^r) \otimes (\mathbf{x}_k^r - \mathbf{x}_i^r) \right]^{-1} \left[ \sum_k V_k^r (\mathbf{x}_k^r - \mathbf{x}_i^r) \otimes (\mathbf{x}_k^r - \mathbf{x}_i^r) \right],
\]
\[
\mathbf{B}_i = \frac{1}{V_i^r} \left[ - \sum_k V_k^r \mathbf{H}_{\phi_i}(\mathbf{x}_i^r) + V_k^r \mathbf{A}_i(\mathbf{x}_k^r - \mathbf{x}_i^r) \right].
\]
Rewriting equation (5.4.1) in expanded form gives,
\[
\sum_k \varphi_k \tilde{g}_k(\mathbf{x}_i^r) = \sum_k \varphi_k \mathbf{g}_k(\mathbf{x}_i^r) + h \left[ \sum_k V_k^r \varphi_k \tilde{\mathbf{H}}_{\phi_i}(\mathbf{x}_i^r) - \sum_k \varphi_k \left( \sum_l \mathbf{g}_k(\mathbf{x}_i^r) \otimes \mathbf{g}_l(\mathbf{x}_i^r) \right) \right] \mathbf{\eta}.
\]
Hence $\tilde{g}_k(\mathbf{x}_i^r)$ can be written as:
\[
\tilde{g}_k(\mathbf{x}_i^r) = \mathbf{g}_k(\mathbf{x}_i^r) + h \left[ V_k^r \tilde{\mathbf{H}}_{\phi_i}(\mathbf{x}_i^r) - \left( \sum_l \mathbf{g}_k(\mathbf{x}_i^r) \otimes \mathbf{g}_l(\mathbf{x}_i^r) \right) \right] \mathbf{\eta}. \quad (5.4.2)
\]
Equation (5.4.2) represents the complete form of corrected gradient of kernel which consists of kernel correction and stabilization.

This stabilization is introduced in the updated Lagrangian formulation. Results for the 1D bar problem introduced in Section 5.3 taking $\mathbf{\eta} = (0.3, 0.3, 0.3)^T$ can be seen in Figure 5.6.
5.4.2 Standard Stabilization

The main idea of this stabilization technique consists in adding a stabilization term directly in the weak form of the problem. Recall the Cauchy equation (4.3.2) for the deformable body:

\[ \nabla \cdot \sigma + \zeta f = \zeta a, \]

which is the strong form of the problem.

The following discretization in time is used:

\[ a^n = \frac{a^{n+1} - 2a^n + a^{n-1}}{(\Delta t)^2} \]

which after substitution in the strong form of the problem:

\[ \nabla \cdot \sigma(a^n) - \zeta \frac{a^{n+1} - 2a^n + a^{n-1}}{(\Delta t)^2} = 0. \]

The differential operator associated to \( a^n \) reads:

\[ \mathcal{L}(a^n) = \nabla \cdot \sigma(a^n) + 2 \zeta \frac{a^n}{(\Delta t)^2}. \]
The standard way to stabilize is adding to the weak Galerkin form the term,
\[ \int_{\Omega} P(w) \tau R(x^n) d\Omega \]
where \( P(w) := L(w), \quad R(x^n) := \nabla \cdot \sigma(x^n) - \zeta a^n \) as the residual of the problem and \( \tau \) is the stabilization parameter.

For simplicity we take \( f = 0 \). By means of weighted residuals the weak form reads as:
\[ \int_T w^T \sigma d\Gamma - \int_{\Omega} \text{tr} (\sigma \nabla w) d\Omega - \int_{\Omega} w^T \zeta a d\Omega = 0 \quad \forall w. \quad (5.4.3) \]

After discretization the problem reads
\[ -\mathbf{T}_i^n - m_i \mathbf{a}_i^n + \tau \sum_j V_j \mathcal{P}(\phi_i(x_j^n)) \mathcal{R}(x_j^n) = 0 \]

which can be written as
\[ m_i \mathbf{a}_i^n + \tau \sum_j V_j \mathcal{P}(\phi_i(x_j^n)) \zeta \mathbf{a}_j^n = -\mathbf{T}_i^n + \tau \sum_j V_j \mathcal{P}(\phi_i(x_j^n)) \nabla \cdot \sigma(x_j^n). \]

In order to find the acceleration three uncoupled linear systems (one for each direction) must be solved.

This stabilization is introduced to the updated Lagrangian formulation. Next, results for the 1D bar problem introduced in Section 5.3 taking \( \eta = 0.1 \) can be seen in Figure 5.7.

5.4.3 Dissipative Potential Stabilization

The key idea of this stabilization is to introduce viscosity via a dissipative potential as,
\[ \Pi(v_i) = \sum_j V_j \eta \mu \Delta t \rho^2 (L v_j - \nabla \cdot (\nabla (v_j)))^2, \quad (5.4.4) \]
where $\eta$ is the stabilization parameter, $\mu$ the material shear modulus $\Delta t$ the time step of the explicit time integration scheme and $\mathcal{L}v_j$ the CSPH interpolation of the velocities Laplacian. Working in the discrete space the Laplacian of the velocities is not the same as the divergence of the gradient of the velocities. However, as the particle distribution is refined they converge.

Let’s find the force term that must be added to stabilize. Using CSPH interpolation:

$$\mathcal{L}v_j = \sum_k v_k \mathcal{L}_k(x_j)$$

$$\nabla \cdot (\nabla (v_j)) = \sum_k v_k \text{tr} \left( \sum_l g_k(x_l) \otimes g_l(x_j) \right)$$

where $\mathcal{L}_k$ is the Laplacian of the kernel function centered at particle $k$ and in order to use a more friendly notation the expression $\sum_l g_k(x_l) \otimes g_l(x_j)$ has been defined as $H_k(x_j)$.

Figure 5.7: Displacement (left) and velocity (right) history for the right end of the bar using standard stabilization.
Substituting the previous interpolations in equation (5.4.4) we obtain:

$$\Pi(\mathbf{v}_i) = \sum_j V_j \eta \mu \Delta t \rho^2 \left( \sum_k \mathbf{v}_k \mathbf{L}_k(x_j) - \sum_k \mathbf{v}_k \mathbf{tr}(\mathbf{H}_k(x_j)) \right)^2.$$ 

Finally, the forces term associated to particle $i$ to stabilize is:

$$\mathbf{T}^{\text{stab}}_i = \frac{\partial \Pi(\mathbf{v}_i)}{\partial \mathbf{v}_i} = 2 \sum_j V_j \eta \mu \Delta t \rho^2 \left( \sum_k \mathbf{v}_k \mathbf{L}_k(x_j) - \sum_k \mathbf{v}_k \mathbf{tr}(\mathbf{H}_k(x_j)) \right) (\mathbf{L}_i(x_j) - \mathbf{tr}(\mathbf{H}_i(x_j))).$$

This stabilization is introduced to the updated Lagrangian formulation. Next, results for the 1D bar problem introduced in Section 5.3 taking $\eta = -5e - 3$ can be seen in Figure 5.8.

Figure 5.8: Displacement (left) and velocity (right) history for the right end of the bar using Dissipative Potential stabilization.
5.5 Numerical Examples

5.5.1 Bending test

The problem presented in Section 4.2.3 is solved using an updated Lagrangian Hessian Difference stabilized formulation. Updates are performed every 3 time steps (which is much more than what’s needed) in order to ensure updates are working properly. Results for $\eta = 1.5$ can be seen in Figure 5.9. Time $t = 0.9$ s

Figure 5.9: Cylinder deformation using an updated CSPH Lagrangian stabilized formulation.

is achieved with 8633 time steps. So 2877 updates have been performed. Good
results similar to the ones obtained with the Lagrangian formulation are obtained.

5.5.2 Rubber rings collision.

The problem presented in Section 4.2.3 is solved now using an updated Lagrangian Hessian Difference stabilized formulation. Updates are performed every 3 time steps. Results for $\eta = 0.3$ can be seen in Figure 5.10.

Time $= 0.00$ s

Time $= 0.01$ s

Time $= 0.02$ s

Time $= 0.03$ s

Time $= 0.04$ s

Time $= 0.05$ s

Figure 5.10: Rubber rings using an updated CSPH Lagrangian stabilized formulation.

Time $t = 0.05s$ is achieved with 1333 time steps. So 444 updates have been performed. Again, good results similar to the ones obtained with the Lagrangian
formulation are obtained.

### 5.5.3 Punch test

Problem presented in Section 4.3.3 is solved again using the Hessian Difference stabilized updated Lagrangian formulation. Updates are performed every 3 time steps. Results for $\eta = 1.4$ can be seen in Figure 5.11.

![Figure 5.11: Punch using an updated CSPH Lagrangian stabilized formulation.](image)

#### 5.6 Conclusions

In this chapter an updated Lagrangian stabilized formulation has been proposed. Its behavior has been tested in benchmark tests for tensile instability and also
for large distortions problems. The updated Lagrangian formulation behaves as the totally Lagrangian one and also permits to solve problems involving large distortions.

On one hand, as updates are only performed when needed the updated formulation has a low computational cost as his totally Lagrangian counterpart. On the other hand, updates can be performed as often as wanted, in fact an update can be performed every time step leading to a formulation “similar” to the Eulerian one in which the reference configuration is the current state.
Chapter 6

Summary

First of all, this thesis dedicates one chapter to the state of the art of mesh-free methods. The main reason is that there are many mesh-free methods that can be found in the literature which can be based on different ideas and with different properties. There is a real need of classifying, ordering and comparing these methods: in fact, the same or almost the same method can be found with different names in the literature.

Secondly, a novel improved formulation of the Element Free Galerkin method is proposed in order to alleviate volumetric locking. It is based on a pseudo-divergence-free interpolation. Using the concept of diffuse derivatives an a convergence theorem of these derivatives to the ones of the exact solution, the new approximation proposed is obtained imposing a zero diffuse divergence. In this way, it is guaranteed that the method verifies asymptotically the incompressibility condition and in addition the imposition can be done a priori. This means that the main difference between standard EFG and the improved method is how the interpolation basis is chosen. Modal analysis and numerical results for two classical benchmark tests in solids corroborate that, as expected, diffuse derivatives converge to the derivatives of the exact solution when the discretization is refined (for
a fixed ratio of dilation parameter by interparticle distance) and, of course, that
diffuse divergence converges to the exact divergence with the expected theoretical
rate. For standard EFG the typical convergence rate is degraded as the incompress-
ible limit is approached but with the improved method good results are obtained
even for a nearly incompressible case and a moderately fine discretization. The
improved method has also been used to solve the Stokes equations. In this case
the LBB condition is not explicitly satisfied because the pseudo-divergence-free
approximation is employed. Reasonable results are obtained in spite of the equal
order interpolation for velocity and pressure.

Finally an updated Lagrangian stabilized formulation has been proposed. Its
behavior has been tested in benchmark tests for tensile instability and also for
large distortions problems. The updated Lagrangian formulation behaves as the
totally Lagrangian one and also permits to solve problems involving large dis-
trctions. On one hand, as updates are only performed when needed the updated
formulation has a low computational cost as his totally Lagrangian counterpart.
On the other hand, updates can be performed as often as wanted, in fact an up-
date can be performed every time step leading to a formulation “similar” to the
Eulerian one in which the reference configuration is the current state.
Chapter 7

Future Developments

7.1 A posteriori error estimation and adaptivity

For any problem, adaptivity is an essential tool to obtain numerical solutions with an accuracy prescribed a priori. Adaptivity in finite elements has been widely studied, however, in mesh-free methods it is still an open topic. Huerta, Rodríguez-Ferran, Díez and Sarrate (1999) present a general overview of adaptive strategies.

The two main ingredients in an adaptive process are:

- A tool for assessing the error of the solution computed with a given mesh: for instance a posteriori error estimation.

- An algorithm to define a new spatial discretization

An a posteriori error estimation technique is essential for any reliable error driven adaptive scheme.

The second one has the main purpose of increase (or decrease) the richness of the interpolation according to the output of the error assessment. Three main types of strategies may be used:
• $h$-adaptivity (in a FE context it consists in changing the size of the elements, in mesh-free methods it consists in changing the dilation parameter)

• $p$-adaptivity (in a FE context the degree of the interpolating polynomials is changed, in mesh-free methods it consists in changing the order of consistency)

• $r$-adaptivity (in a FE context it consists on relocating nodes without changing connectivities, mesh-free methods do not require connectivities)

Parés, Díez and Huerta (n.d.) present a new residual-type flux-free error estimator for the FEM. It estimates upper and lower bounds of the error in energy norm. The proposed approach precludes the main drawbacks of standard residual type estimators, circumvents the need of flux-equilibration and results in a simple implementation that uses standard resources available in finite element codes. This is specially interesting for 3D applications where the implementation of this technique is as simple as in 2D. The bounds for the energy norm of the error are used to produce upper and lower bounds of linear functional outputs, representing quantities of engineering interest. The presented estimators demonstrate their efficiency in numerical tests producing sharp estimates both for the energy and the quantities of interest.

Future work focus in study the applicability of the error estimator proposed by Parés et al. (n.d.) to the EFG Method. First difficulty is that in particle methods we don’t have a non overlapping discretization of the domain. Another problem to overcome is that EFG approximation spaces don’t have the same properties as FEM ones. However, after overcoming all that difficulties, it seems easy to apply the error estimator not only in $h$-adaptivity but also in $p$-adaptivity. When EFG method is used $p$-adaptivity consists only in changing the reproducibility order so the same particle distribution can be used.
Bibliography


