Appendix A

Homogeneous intermittency model

A.1 The $\beta$-model

The $\beta$-model was introduced by Frisch et al. (1978). The idea behind this model comes from the Richardson cascade, in which at each level of the cascade the energy of the large eddies $L$ (mother) is uniformly distributed over the other eddies of size $\ell$ (daughters) as

$$\ell_n = L \ell^n,$$  \hspace{1cm} (A.1)

where $n = 0, 1, 2, \ldots$ and $0 < \ell < 1$. The fraction $P_n$ of this decrease of the eddies, which has a factor $\beta$ ($0 < \beta < 1$), can be defined as

$$P_n = \beta^n = \left( \frac{\ell_n}{L} \right)^{(3-D)},$$ \hspace{1cm} (A.2)

where

$$3 - D = \frac{\ln \beta}{\ln \ell}.$$ \hspace{1cm} (A.3)

The parameter $D$ is called a fractal dimension, and it expresses how the number of active eddies scales with $L$ and is related to the number of offspring.

Frisch (1995) in his book interpreted this probability $P_n$, which has within three objects, a point, a curve and surface having respectively the dimension $D$ of zero, one and two (figure A.1), and he states that the probability $P_n$ that a sphere of
radius $\ell_n$ (small), which is chosen in the center of the cube with a random uniform distribution, will intersect such an object is given in all cases by

$$P_n \propto \ell_n^{-D}. \tag{A.4}$$

![Unit cube diagram](image_url)

Figure A.1: The probability that a sphere of radius of $\ell$ encounters an object of dimension $D$ behaves as $\ell_n^{-D}$.

It is appropriate now to explain the scaling laws of the $\beta$-model, which are derived by adapting the standard Kolmogorov theory (K41). Denote again by $v_n$ the typical
velocity difference over a distance $\ell_n$ within an active eddy of size $\ell$. We define the energy per unit mass on scales $\ell_n$ as

$$E_n \propto P_n v_n^2 = v_n^2 \left( \frac{\ell_n}{L} \right)^{3-D}. \quad (A.5)$$

The rate of energy transfer from $n$-eddy to $(n+1)$-eddy in the inertial range is defined according to the Kolmogorov theory K41

$$\epsilon_n \propto \frac{E_n}{\ell_n} \propto P_n \frac{v_n^3}{\ell_n} \propto \langle \epsilon \rangle. \quad (A.6)$$

From equations (A.5) and (A.6), we obtain new form of $v_n$ and $E_n$ that take into account the spatial structure of the fractal dissipation field

$$v_n \propto \langle \epsilon \rangle^{\frac{2}{3}} \ell_n^{\frac{1}{3}} \left( \frac{\ell_n}{L} \right)^{-\frac{1}{2(3-D)}}. \quad (A.7)$$

and

$$E_n \propto \langle \epsilon \rangle^{\frac{2}{3}} \ell_n^{\frac{1}{3}} \left( \frac{\ell_n}{L} \right)^{\frac{1}{2}(3-D)} . \quad (A.8)$$

From equation (A.7) it is clear that the velocity field has the scaling exponent

$$h = \frac{1}{3} - \frac{3-D}{3}, \quad (A.9)$$
and the structure function of order $p$ is written as

$$S_p(\ell_n) = P_n v_n^p \propto v_n^p \left( \frac{\ell_n}{L} \right)^{\xi_p}, \quad (A.10)$$

with

$$\xi_p = \frac{p}{3} + (3-D)(1 - \frac{p}{3}). \quad (A.11)$$

Using the equation (A.11), the scaling exponent $\xi_2$ of the second order structure function is $\frac{2}{3} + \frac{2-D}{3}$, therefore the energy spectrum is given as

$$E(k) \propto k^{-\left(\frac{\xi_2}{3} + \frac{2-D}{3}\right)}, \quad (A.12)$$

which is derived as a correction to the $k^{-\frac{5}{3}}$ law of the Kolmogorov theory (K41).
A.2 The random $\beta$-model

The random $\beta$-model was introduced by Benzi et al. (1984), assuming that the contraction factors $\beta$ are independent random variables, and can take different values for each eddy of size $\ell_n$. The $\beta_n(i)$'s are distributed according to a given probability distribution $P(\beta)$.

However, the geometrical structure of intermittency does not possess a global dilatation invariance (Benzi et al. (1984)) (figure A.2).

![Diagram of the $\beta$-model and random $\beta$-model](image)

Figure A.2: (a) Schematic view of the $\beta$-model and compared with the random $\beta$-model (b).

Considering then that an eddy of size $\ell_{j+1} = \ell_j/2$ has a fraction $B_j = \sum_i \beta_j(i)/N_j$ of the volume occupied by its hypercube “mother”, it follows that the number of
active eddies after \( n \) steps is given by

\[
N_n = 2^{3n} \prod_{j=1}^{n} \beta_j
\]  

(A.13)

if we denote by \( \ell_n(k) \), \( k = 1, \ldots, N_n \), the \( N_n \) active eddies at the \( n \)th step. Each \( \ell_n(k) \) generated eddies of size \( \ell_{n+1}(k) \), where \( k \) indicates their origin. Taking into account that the energy dissipation rate for \( \ell_n(k) \) and \( \ell_{n+1}(k) \) eddies is constant then,

\[
v_n^3(k)/\ell_n(k) = \beta_{n+1}(k)v_{n+1}(k)/\ell_{n+1}(k).
\]  

(A.14)

Iterating this equation with a particular history of random \( \beta \)'s \((\beta_1, \ldots, \beta_n)\) leads to

\[
v_n \sim \ell_n^{-1/3} \left( \prod_{i=1}^{n} \beta_i \right)^{-1/3},
\]  

(A.15)

from equation (A.15) the structure function can be written as:

\[
S_p(\ell_n) \sim \ell_n^{p/3} \int \prod_{i=1}^{n} \beta_i^{1-p/3} P(\beta_1, \ldots, \beta_n) d\beta_i,
\]  

(A.16)

and because there are no correlations between different steps of the fragmentation process, it follows that

\[
P(\beta_1, \ldots, \beta_n) = \prod_{i=1}^{n} P(\beta_i).
\]  

(A.17)

Therefore, the structure function is rewritten as

\[
S_p(\ell_n) \sim \ell_n^{p/3} \prod_{i=1}^{n} \int \beta_i^{1-p/3} P(\beta_i) d\beta_i = \ell_n^{p/3} \langle \beta^{1-p/3} \rangle^n,
\]  

(A.18)

where \( \langle \ldots \rangle \) is the average over the distribution \( P(\beta) \). Taking the last step \( \ell_n = 2^{-2} \), then

\[
S_p(\ell_n) \sim \ell_n^{\frac{p}{3} - \log_2(\langle \beta^{1-\frac{p}{3}} \rangle)}.
\]  

(A.19)

From equation (A.19), it follows that the scaling exponents \( \xi_p \) for the random \( \beta \)-model can be written as,

\[
\xi_p = \frac{p}{3} - \log_2(\langle \beta^{1-\frac{p}{3}} \rangle).
\]  

(A.20)
This scaling exponents $\xi_p$ in general will be a nonlinear function of $p$. The probability distribution $P(\beta)$ is based in principle on the knowledge of all the $\beta$ moments and all the scaling exponents $\xi_p$.

It was assumed by Benzi et al. (1984) that an active eddy can generate either velocity sheets ($\beta = 0.5$) or space-filling eddies ($\beta = 1$). Therefore, the probability distribution $P(\beta)$ was supposed to have a form such as,

$$P(\beta) = \chi \delta(\beta - 0.5) + (1 - \chi) \delta(\beta - 1),$$

(A.21)

where $\chi$ is a free parameter. Benzi et al. (1984) found that this probability distribution function $P(\beta)$ leads, with $\chi = \frac{1}{2}$, to a good fit to the available experimental data.