B. Equations to obtain the cell constants

This annex details how the expressions presented in chapter 2 have been obtained. The applied methodologies are not a novelty and they are sufficiently described in basic electromagnetic text books [1]. Here it is not intended to provide a formal demonstration of the expressions but to show the methodology to obtain them.

The relationships between the injected currents and the measured voltage drops when a non-constant IESD four-electrode configuration is used are analyzed for various simple medium geometries. In all the cases, the media are considered homogeneous and isotropic.

B.1. Sheet

For the four-electrode measurement depicted in Figure B.1 the relationship between the applied current (I) and the measured voltage difference at 1-2 can be obtained as follows.

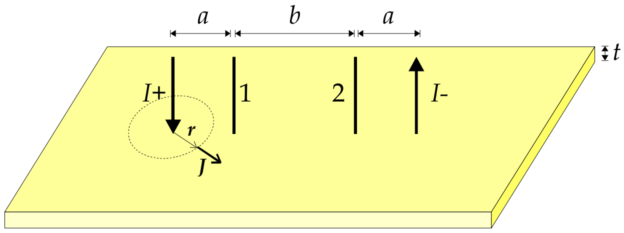

Figure B.1. Schematic representation of the resistance measurement of a sheet by using the four-electrode method. The sheet area is considered to be infinite in relation to the electrode array dimensions.

The voltage difference (ΔV=V_1-V_2) is obtained by applying the superposition principle. That is, for each potential the contribution of I+ and I- is obtained separately:

$$V_1 = V_1\big|_{I+} + V_1\big|_{I-} \quad ; \quad V_2 = V_2\big|_{I+} + V_2\big|_{I-} \quad ; \quad \Delta V = V_1 - V_2 \tag{B.1}$$

If the voltage at ∞ is defined as 0 V, then the voltage at points 1 and 2 can be obtained as follows:

$$V(r) = -\int_{\infty}^{r} \vec{E}(r)\, d\vec{r} \tag{B.2}$$

for each current source (I+ or I-) the electric field (E) is given by the Ohm's law:

$$\vec{E}(r) = \rho \vec{J}(r) = \rho \left(\frac{I}{2\pi r t} \vec{r} \right) \tag{B.3}$$

where J denotes the current density and it is uniformly distributed around each current source.

Thus:

$$V_1\big|_{I+} = -\int_{\infty}^{a} \frac{\rho I}{2\pi r t} dr = -\frac{\rho I}{2\pi t}\left(\ln a - \ln \infty\right) \tag{B.4}$$

$$V_1\big|_{I-} = -\int_{\infty}^{a+b} \frac{\rho(-I)}{2\pi r t} dr = \frac{\rho I}{2\pi t}\left(\ln(a+b) - \ln \infty\right) \tag{B.5}$$

$$V_1 = V_1\big|_{I+} + V_1\big|_{I-} = \frac{\rho I}{2\pi t}\left(\ln(a+b) - \ln a\right) \tag{B.6}$$

and in an equivalent way,

$$V_2 = V_2\big|_{I+} + V_2\big|_{I-} = -\frac{\rho I}{2\pi t}\left(\ln(a+b) - \ln a\right) \tag{B.7}$$

Hence,

$$\Delta V = V_1 - V_2 = \frac{\rho I}{\pi t}\left(\ln\left(\frac{a+b}{a}\right)\right) \tag{B.8}$$

or equivalently

$$\rho = \frac{\pi t}{\ln\left(\frac{a+b}{a}\right)}\frac{V}{I} = kR \tag{B.9}$$

where R would be the measured resistance and k can be defined as the cell constant. Observe that k is independent on the cell size if a=b, this is something that will not be found in the next cases.

B.2. Infinite extent medium

In this case the electrodes are completely surrounded by the medium and the injected currents are spherically distributed and not cylindrically distributed as it was the previous case. Thus, the expression B.3 is transformed into:

$$\vec{E}(r) = \rho \vec{J}(r) = \rho\left(\frac{I}{4\pi r^2 t}\vec{r}\right) \tag{B.10}$$

and, following the same procedure than for the previous case (B.4 - B.7), it is obtained,

$$V_1\big|_{I+} = -\int_{\infty}^{a} \frac{\rho I}{4\pi r^2}dr = \frac{\rho I}{4\pi a} \tag{B.11}$$

$$V_1\big|_{I-} = -\int_{\infty}^{a+b} \frac{\rho(-I)}{4\pi r^2}dr = -\frac{\rho I}{4\pi(a+b)} \tag{B.12}$$

$$\Delta V = V_1 - V_2 = \frac{\rho I}{4\pi}\left(\frac{1}{a} - \frac{1}{a+b} - \frac{1}{a+b} - \frac{1}{a}\right) = \frac{\rho I}{4\pi}\frac{2b}{a(a+b)} \tag{B.13}$$

in the case that $a = b = r$ (constant IESD),

$$\Delta V = = \frac{\rho I}{4\pi}\frac{1}{r} \tag{B.14}$$

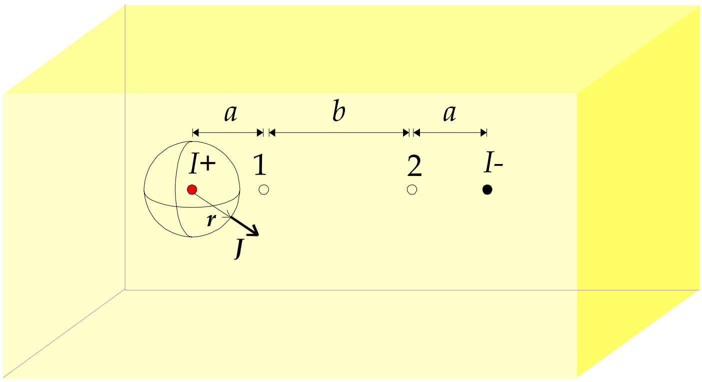

Figure B.2. Schematic representation of the resistance measurement of an infinite extent medium.

B.3. Semi-infinite extent medium

In this case, the electrode array is on the flat interface between the medium of interest and a medium of infinite resistivity such as air.

The expressions are identical to the infinite extent case with the proviso that the current distribution is semispherical and, therefore, the 4π constant in B.10 and the following equations must be replaced by 2π. Thus,

$$\Delta V = V_1 - V_2 = \frac{\rho I}{2\pi}\left(\frac{1}{a} - \frac{1}{a+b} - \frac{1}{a+b} - \frac{1}{a}\right) = \frac{\rho I}{2\pi}\frac{2b}{a(a+b)} \quad \text{(B.15)}$$

B.4. Infinite extent medium with a medium transition.

In the case that there is a medium transition (from medium 1 to medium 2, $\rho_1 \Rightarrow \rho_2$) at a distance x from the electrode array (Figure B.3) the relationship between the voltage difference and the injected current can be determined by using the image method.

The image method [1;2] assumes that the behavior of the electric potentials and electric currents in the medium 1 is equivalent to the case in which the medium 2 does not exist and image current sources ($I'+$, $I'-$) are located with respect to the medium transition. The magnitude of this image sources must assure that the boundary conditions in the medium transition are fulfilled: 1) continuity of potential and 2) continuity of the normal component of the current density. And that implies:

$$I'+ = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} I+ = K\,I+ \quad \text{(B.16)}$$

$$I'- = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} I- = K\,I- \quad \text{(B.17)}$$

where K is usually referred as the reflection coefficient.

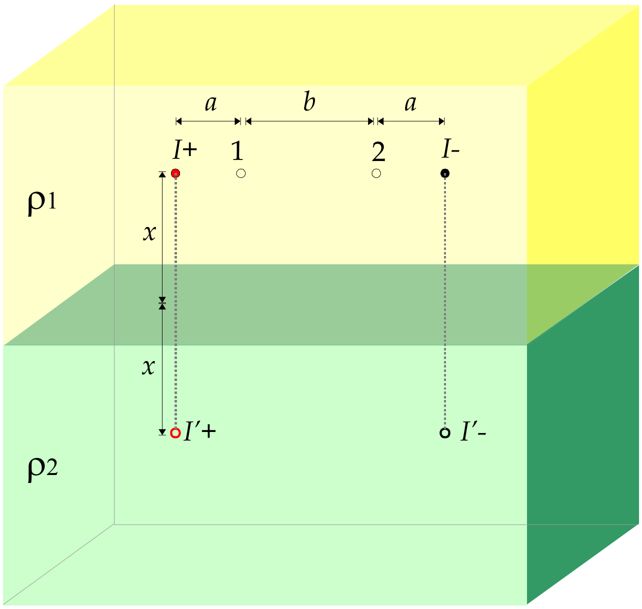

Figure B.3. Schematic representation of the image method for a single medium transition.

Thus, now each point sees the contribution of four current sources instead of two.

$$V_1 = V_1\big|_{I+} + V_1\big|_{I-} + V_1\big|_{I'+} + V_1\big|_{I'-} \quad ; \quad V_2 = V_2\big|_{I+} + V_2\big|_{I-} + V_2\big|_{I'+} + V_2\big|_{I'-} \tag{B.18}$$

$$V_1 = \frac{\rho_1 I}{4\pi}\left(\frac{1}{a} - \frac{1}{a+b} + \frac{K}{\sqrt{a^2+4x^2}} - \frac{K}{\sqrt{(a+b)^2+4x^2}}\right) \tag{B.19}$$

$$V_2 = \frac{\rho_1 I}{4\pi}\left(-\frac{1}{a} + \frac{1}{a+b} - \frac{K}{\sqrt{a^2+4x^2}} + \frac{K}{\sqrt{(a+b)^2+4x^2}}\right) \tag{B.20}$$

$$\Delta V = V_1 - V_2 = \frac{\rho_1 I}{4\pi}\left(\frac{2}{a} - \frac{2}{a+b} + \frac{2K}{\sqrt{a^2+4x^2}} - \frac{2K}{\sqrt{(a+b)^2+4x^2}}\right) \tag{B.21}$$

or, using he same formulation than Robillard and Poussart [3],

$$\rho' = \rho\left(1 + K\frac{G'}{G}\right) \tag{B.22}$$

$$G = \left(\frac{1}{a} - \frac{1}{a+b}\right) \tag{B.23}$$

$$G' = \left(\frac{1}{\sqrt{4x^2+a^2}} - \frac{1}{\sqrt{4x^2+(a+b)^2}}\right) \tag{B.24}$$

where ρ' is the measured resistivity and ρ would be the measured resistivity if the medium 1 was infinite extent.

It must be said that the it is possible to obtain the same results without applying the image method as it is show in [4].

B.5. Semi-infinite extent medium with a medium transition

In the case that the electrode array is on the surface of a medium that contains a medium transition at distance x, the solution is more complex since infinite reflections must be considered: each current source on the medium 1 surface, for instance $I+$, is 'reflected' on the medium transition ($I'+_1$) with magnitude $K.I+$ but, this reflection ($I'+_1$), in turn, is reflected on the medium 1 surface because it also means a medium transition ($\rho_1 \Rightarrow \rho_{DIELECTRIC}$, reflection coefficient = K' =1), then, this reflection ($I''+_1$) is reflected on the transition 1-2 with an apparent image at $2x$ and this image ($I'+_2$) is reflected on the medium 1 surface ($I''+_2$) ...

Thus,

$$V_1 = \frac{\rho_1 I}{2\pi} \cdot \left(\frac{1}{a} - \frac{1}{a+b} + \frac{2K}{\sqrt{a^2 + 4x^2}} - \frac{2K}{\sqrt{(a+b)^2 + 4x^2}} + \frac{2K^2}{\sqrt{a^2 + 4(2x)^2}} - \frac{2K^2}{\sqrt{(a+b)^2 + 4(2x)^2}} + \ldots \right) \tag{B.25}$$

$$V_1 = \frac{\rho_1 I}{2\pi} \left(\left(\frac{1}{a} - \frac{1}{a+b} \right) + 2 \left(\sum_{n=1}^{\infty} K^n \left[\frac{1}{\sqrt{a^2 + 4n^2x^2}} - \frac{1}{\sqrt{(a+b)^2 + 4n^2x^2}} \right] \right) \right) \tag{B.26}$$

$$\Delta V = \frac{\rho_1 I}{2\pi} \left(2\left(\frac{1}{a} - \frac{1}{a+b} \right) + 4 \left(\sum_{n=1}^{\infty} K^n \left[\frac{1}{\sqrt{a^2 + 4n^2x^2}} - \frac{1}{\sqrt{(a+b)^2 + 4n^2x^2}} \right] \right) \right) \tag{B.27}$$

Observe that in the case that $x \to \infty$,

$$\begin{aligned} \Delta V &= \frac{\rho_1 I}{2\pi} \left(2\left(\frac{1}{a} - \frac{1}{a+b} \right) + 4\left(\frac{1}{a} - \frac{1}{a+b} \right) \sum_{n=1}^{\infty} K^n \right) = \\ &= \frac{\rho_1 I}{2\pi} \left(2\left(\frac{1}{a} - \frac{1}{a+b} \right) + 4\left(\frac{1}{a} - \frac{1}{a+b} \right)\left(\frac{K}{1-K} \right) \right) = \frac{\rho_1 I}{2\pi} \left(2\left(\frac{\rho_2}{\rho_1} \right)\left(\frac{1}{a} - \frac{1}{a+b} \right) \right) = \\ &= \frac{\rho_2 I}{2\pi} \left(2\left(\frac{1}{a} - \frac{1}{a+b} \right) \right) \end{aligned} \tag{B.28}$$

which is equivalent to the equation B.15

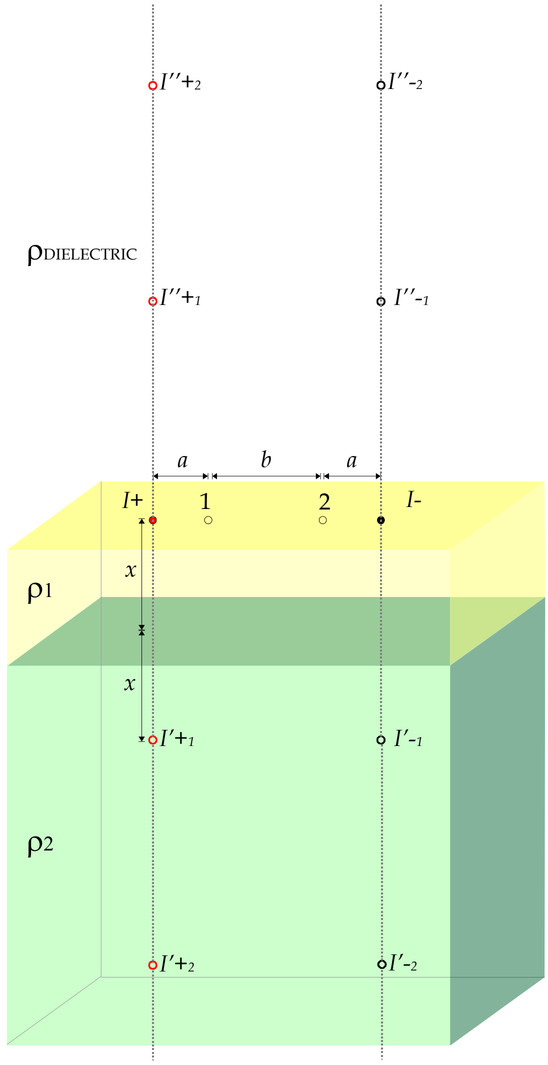

Figure B.4. Schematic representation of the image method when the electrode array is on the surface of a medium (1) that contains a medium transition at distance x.

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