

**REPORT OF THE SEMINAR ON  
MOTIVIC COHOMOLOGY  
CIMAT, GUANAJUATO 2010**

PEDRO LUIS DEL ÁNGEL AND PERE PASCUAL GAINZA

A seminar on *Motivic Cohomology* took place at the Centro de Investigación en Matemáticas (CIMAT), at Guanajuato (México) during the month of August 2010. The aim of the seminar was to cover the definition and main properties of motivic cohomology of smooth schemes over a perfect field as defined by V. Voevodsky, and its relation to étale cohomology, Bloch's higher Chow groups and Milnor's algebraic  $K$ -theory, based on the book [MVW]. The seminar was addressed to an audience with a basic knowledge of algebraic geometry and classical motives. It was not possible to cover all proofs of the different results presented along the sessions of the seminar, but we think that it emerged a clear picture of the subject.

We thank all participants by their enthusiasm and specially Abdó Roig, who has read a first draft of this report and contributed with his remarks.

The seminar began on August 2 and ended on September 1. It consisted of ten sessions of one hour and a half according to the following programme and speakers.

Schedule:

August 2	Pedro Luis del Ángel Pere Pascual	A global overview of Motivic Cohomology: aims and scope
August 4	Abdó Roig	Grothendieck topologies
August 9	Pere Pascual	Derived categories
August 11	Herbert Kanarek	From Grothendieck's classical motives to mixed motives
August 16	Pedro Luis del Ángel	The category of mixed realizations
August 18	Pere Pascual	Triangulated category of mixed motives
August 23	Abdó Roig	The fundamental technical result
August 25	Pedro Luis del Ángel Pere Pascual	Weight one motivic cohomology Bloch's higher Chow groups Classical motives versus Voevodsky motives
August 30	Abdó Roig	Triangles in the triangulated motivic category
September 1	Pere Pascual	The étale version of all this stuff

---

Date: September 30, 2010.

The second author was partially supported by DGICYT MTM 2009-09557.

List of Participants

- Lilia Alanís, Master Student, Centro de Investigación en Matemáticas (CIMAT, Mexico)
- José Luis Alonso Velázquez, Master Student, Centro de Investigación en Matemáticas (CIMAT, Mexico)
- Pedro Luis del Ángel, Centro de Investigación en Matemáticas (CIMAT, Mexico)
- Xavier Gómez-Mont, Centro de Investigación en Matemáticas (CIMAT, Mexico)
- Herbert Kanarek, Facultad de Matemáticas, Universidad de Guanajuato (Mexico)
- Miguel López Cruz, Master Student, Universidad Autónoma de México (UNAM, Mexico)
- Isidro Nieto Baños, Universidad de Guanajuato (Mexico)
- Ismael Olmos, Master Student, Centro de Investigación en Matemáticas (CIMAT, Mexico)
- Pere Pascual, Universitat Politècnica de Catalunya (UPC, Spain)
- Abdó Roig, Universitat Politècnica de Catalunya (UPC, Spain)
- Ricardo Vila Freyer, Centro de Investigación en Matemáticas (CIMAT, Mexico)

## RESUME OF THE SESSIONS

We present a resume of the talks. We do not pretend to give a coherent and complete resume, our objective being to describe the general contents of the sessions.

**First session: A global overview.**

This session was a general introduction to motives and motivic cohomology from an historical point of view, stressing their usefulness and the different approximations to the subject appeared in recent years.

We began defining what is a Weil cohomology for smooth varieties over a field  $\mathbf{k}$  and explaining Grothendieck's idea of creating a universal Weil cohomology, introducing the category of rational motives and its expected properties. After that we commented the existence of weight filtrations on Hodge theory and on étale cohomology and gave a nontechnical description of what would be expected for a category of mixed motives.

The final part of the session was dedicated to organize the schedule of the seminar and to propose the different speakers. Due to the different background of the audience we decided to have two preliminary sessions on Grothendieck Topologies and Derived Categories.

**Second session: Grothendieck Topologies.**

We reviewed basic material on Grothendieck topologies, which are a fundamental prerequisite for Voevodsky's construction of the derived category of motives. De Rham's theorem identifies singular cohomology of a nice enough topological space with sheaf cohomology for the constant sheaf. This does not work for the Zariski topology, and suggests that if one wants to construct meaningful cohomologies as sheaf cohomology of some constant sheaf, one should generalize the concept of topology. This leads to the formal definition of Grothendieck topology, the definition using covering families, and the properties of the category of sheaves for such a topology were discussed.

We emphasized the relation between the big and small Grothendieck sites, and how one can embed the category of schemes into the category of sheaves on the big site  $\mathbf{Sch}/\mathbf{k}$  for any sub-canonical Grothendieck topology, as this idea plays a major role in Voevodsky's constructions. Finally we discussed the definitions of the three main topologies we needed: Zariski, Étale and Nisnevich, together with the notion of points for all of them.

**Third session: Derived categories.**

The session consisted of a quick review of derived and triangulated categories following the classical book by Gelfand and Manin, [GM]. Derived categories were introduced as the localization category of a category of complexes in an abelian category by the class of quasi-isomorphisms. The two basic results presented were the equivalence with the localization of the homotopy category of complexes, and the subsequent calculus of fractions, and the equivalence with the homotopy category of complexes of injective objects, with suitable bounding hypothesis.

As for triangulated categories, we discussed the Verdier quotient of a triangulated category  $\mathcal{T}$  by a thick subcategory  $\mathcal{N}$  and the representation of a quotient as a suitable triangulated subcategory of  $\mathcal{T}$  as given by the following result, (see Bousfield's localization, [N]).

**Theorem 1.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{N}$  a thick subcategory. The quotient functor  $\pi : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{N}$  admits a right adjoint if and only if for every object  $T$  in  $\mathcal{T}$  there is a distinguished triangle*

$$T_{\mathcal{N}} \longrightarrow T \longrightarrow {}^{\perp}_{\mathcal{N}} T \longrightarrow \Sigma T.$$

*In this case, the induced quotient functor on local objects*

$${}^{\perp}_{\mathcal{N}} \subset \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N},$$

*is an equivalence of categories.*

We also talked about  $t$ -structures on a triangulated category.

#### Fourth session: From Grothendieck classical motive to mixed motives.

We began by reviewing the classical theory of rational motives as introduced by Grothendieck. We defined the category of correspondences  $\mathbf{CSmProj}/\mathbf{k}$  whose objects are the smooth projective varieties and the morphisms from an irreducible  $X$  to  $Y$  are given by the rational Chow groups  $CH^{d_X}(X \times Y) \otimes \mathbb{Q}$ , where  $d_X = \dim X$ ; then we described the pseudo-abelianization to obtain the category of effective rational motives  $CHM^{eff}(\mathbf{k})$ . Finally we introduced the Lefschetz motive  $\mathbb{L}$  and its inversion to obtain the category of rational motives  $CHM(\mathbf{k})$  and the contravariant functor

$$\mathbf{SmProj}/\mathbf{k} \longrightarrow CHM(\mathbf{k}),$$

and established its universal property factorizing Weil cohomologies.

We also reviewed the standard conjectures and using them to modify the commutativity isomorphism in  $CHM(\mathbf{k})$ , we enunciate the following result:

**Theorem 2.** *Assume there is a Weil cohomology  $H^*$  satisfying the standard conjectures. Let  $CHM_{num}(\mathbf{k})$  be the category of motives relative to numerical equivalence, with the modified product. Then  $CHM_{num}(\mathbf{k})$  is a semisimple tannakian category over  $\mathbb{Q}$ , the functor*

$$h : \mathbf{SmProj}/\mathbf{k} \longrightarrow CHM_{num}(\mathbf{k}),$$

*is a universal cohomology theory, and*

$$H^* : CHM_{num}(\mathbf{k}) \longrightarrow \mathbf{Vect}_{\mathbb{Q}}^*,$$

*is a fibre functor.*

The second part of the session was dedicated to introduce Beilinson's conjectures on the existence of "derived" categories of mixed motives  $D_{\mathcal{MM}}(X)$  for all smooth varieties  $X$  over a field  $\mathbf{k}$ ; this categories should satisfy the formalism of Grothendieck's six operations. The category  $D_{\mathcal{MM}}(\mathbf{k})$  would be the category of derived mixed motives and would have a natural  $t$ -structure whose heart  $\mathcal{MM}$  would be the category of mixed motives.

Associated to the categories  $D_{\mathcal{MM}}(X)$ , we considered two cohomology theories: denote by  $\mathbf{1}$  the constant sheaf for the unit in the category of pure motives and by  $\pi : X \rightarrow \text{Spec } \mathbf{k}$  the structural morphism, then we have

- the geometric cohomology:  $h_{\mathcal{MM}}^*(X) = R^*\pi_*(\mathbf{1}_X)$ , which is a graded object in  $\mathcal{MM}$ .
- the absolute cohomology:  $H_{\mathcal{MM}}^*(X) = R^*\Gamma(X, \mathbf{1}_X)$  which is a graded abelian group.

These cohomologies should be related by a motivic spectral sequence coming from the chain of quasi-isomorphisms

$$R^i\Gamma(X, \mathbf{1}_X) \cong \text{Hom}_{D_{\mathcal{MM}}(X)}(\mathbf{1}_X, \mathbf{1}_X[i]) \cong \text{Hom}_{D_{\mathcal{MM}}(\mathbf{k})}(\mathbf{1}, R\pi_*\mathbf{1}_X[i]) = R\Gamma(\mathbf{1}, R\pi_*\mathbf{1}_X).$$

### Fifth session: The category of mixed realizations.

As a first approximation to the construction of the triangulated categories  $D_{\mathcal{MM}}(X)$ , we dedicated the session to the triangulated category of mixed realizations introduced by Huber, [H], following basic ideas of Deligne and Jannsen who have introduced the tannakian category of mixed realizations  $\mathcal{RM}$ . We began by recalling the three basic realization functors taking part in the definition: singular cohomology, De Rham and Hodge theory and  $\ell$ -adic cohomology; using the Leray spectral sequence to define the filtered complexes associated to open smooth varieties and Deligne's simplicial hyperresolutions for the general case. From this examples we constructed the derived category of mixed realizations  $D_{\mathcal{MR}}$  by a gluing process of categories as the one introduced by Beilinson in his study of absolute Hodge cohomology.

**Theorem 3.** *The category  $D_{\mathcal{MR}}(\mathbf{k})$  is a tensor triangulated category. It has a  $t$ -structure with heart  $\mathcal{MR}$ .*

*There is a contravariant functor*

$$R_{\mathcal{MR}} : \mathbf{SmProj}(\mathbf{k}) \longrightarrow D_{\mathcal{MR}}(\mathbf{k}).$$

Obviously, the functor  $R_{\mathcal{MR}}$  is universal for the cohomology theories involved in the construction of  $D_{\mathcal{MR}}(\mathbf{k})$ .

The absolute cohomology  $H_{\mathcal{MR}}^*(X, j)$  is defined as hom groups in  $D_{\mathcal{MR}}(\mathbf{k})$  and it comes as no surprise that the Hodge and  $\ell$ -adic realizations of this cohomology are Beilinson's absolute Hodge cohomology and Jannsen's continuous  $\ell$ -adic cohomology, respectively. Moreover, we have:

**Theorem 4.** *The functor  $R_{\mathcal{MR}}$  induces a contravariant functor*

$$CHM(\mathbf{k}) \longrightarrow D_{\mathcal{MR}}(\mathbf{k}).$$

### Sixth session: Triangulated category of mixed motives.

In accordance with Deligne's remark, it is easier to look for a *derived* category of mixed motives rather than to the category of mixed motives. Following Voevodsky, (see [VSF]), to define this category we imposed several conditions:

- (1) In order to have good set theoretical constructions, we use Grothendieck's classical idea of identifying schemes with the functor they define on  $\mathbf{Sm}/\mathbf{k}$  via the Yoneda embedding.
- (2) As we want to work with singular motives, we look for a *homology* presentation, rather than a cohomological one. In this respect, the classical Dold-Thom theorem assures that for a  $CW$ -complex there is an isomorphism  $H_*(X; \mathbb{Z}) \cong \pi_*(S^\infty X)$ , where  $S^\infty X$  is the infinite symmetric power of  $X$ . The substitute of this product in the algebraic setting will be the presheaf  $\mathbb{Z}_{tr}(X)$ , which is the free presheaf generated by  $X$  in the category of finite correspondences  $Cor_{\mathbf{k}}$ .
- (3) We impose the Mayer-Vietoris property, taking sheaves in a convenient topology. As the Zariski topology is not compatible with transfers, we take the Nisnevich topology on  $\mathbf{Sm}/\mathbf{k}$ .
- (4) We impose homotopy invariance.

From the property (2) it follows immediately that for any smooth  $X$  and any  $F \in Sh_{Nis}(Cor_{\mathbf{k}})$  we have natural isomorphisms

$$Ext_{Sh_{Nis}(Cor_{\mathbf{k}})}^i(\mathbb{Z}_{tr}(X), F) \cong H_{Nis}^i(X, F), \quad i \geq 0.$$

With all these preliminaries, we define the triangulated category of mixed motives as

$$DM_{Nis}^{eff,-}(\mathbf{k}) := D^-(Sh_{Nis}(Cor_{\mathbf{k}}))[W_{\mathbb{A}}^{-1}],$$

where  $W_{\mathbb{A}}$  is the class of morphisms whose cone is in the thick localizing subcategory generated by the complexes  $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{tr}(X)$ .

Moreover, starting with the product

$$\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y),$$

a tensor structure  $\otimes_{tr}$  was introduced in  $DM_{Nis}^{eff,-}(\mathbf{k})$ , so that we obtained a tensor triangulated category.

Sending a smooth variety  $X$  to  $\mathbb{Z}_{tr}(X)$  defines a covariant functor  $M : \mathbf{Sm}(\mathbf{k}) \rightarrow DM_{Nis}^{eff,-}(\mathbf{k})$ .

### Seventh session: The fundamental technical result.

The Verdier quotient defining  $DM_{Nis}^{eff,-}(\mathbf{k})$  admits Bousfield localization. The right adjoint can be concretely defined as  $TotC_*(K)$  where  $K$  is a complex of presheaves with transfers and the functor  $C_*$  sends a single presheaf with transfers to the (chain) complex of presheaves

$$C_n F(U) = F(U \times \Delta^n).$$

The functor  $TotC_*$  induces an equivalence between the derived category of effective motives  $DM_{Nis}^{eff,-}(\mathbf{k})$  and the subcategory of  $\mathbb{A}^1$ -local complexes of sheaves  $\mathcal{L}_{\mathbb{A}^1} \subset C^-(Sh_{Nis}(Cor_{\mathbf{k}}))$ .

Now, there are two fundamental results, one characterizing the  $\mathbb{A}^1$ -local complexes of sheaves, and the other relating Nisnevich and Zariski hypercohomologies for such  $\mathbb{A}^1$ -local complexes.

**Theorem 5.** *Let  $\mathbf{k}$  be a perfect field. A presheaf with transfers  $F$  is homotopy invariant if, and only if all its cohomology presheaves  $X \mapsto H^n(X, F_{Nis})$  are homotopy invariant.*

A consequence of this is that  $\mathbb{A}^1$ -local complexes are exactly the complexes with homotopy invariant cohomology sheaves.

**Theorem 6.** *Let  $F$  be a homotopy invariant presheaf with transfers. If  $F(E) = 0$  for every field  $E$ , then  $F_{Zar} = 0$ .*

Note that in general, sheaf theory would say that  $F_{Zar} = 0$  if, and only if  $F(S) = 0$  for every local scheme  $S$ . The homotopy invariance hypothesis allows us to check only for fields (generic points of the local  $S$ ). This allows us to transfer results from the Nisnevich to the Zariski topology. For instance, if a homotopy invariant presheaf with transfers  $F$  is such that  $F_{Nis} = 0$ , then  $F(E) = 0$  for every field, as the fields are Henselian, and then  $F_{Zar} = 0$  by the theorem.

One then defines the complexes of sheaves  $\mathbb{Z}(n)$  as

$$\mathbb{Z}(q) = C_*(\mathbb{Z}_{tr}(\mathbb{P}^q)/\mathbb{Z}_{tr}(\mathbb{P}^{q-1}))$$

which can be motivated either homotopically, or looking at the decomposition of the Chow motive of  $\mathbb{P}^n$  via the decomposition of the diagonal. This leads to the definition of motivic cohomology:

$$H^n(X, \mathbb{Z}(q)) = Hom_{DM_{Nis}^{eff, -}(\mathbf{k})}(\mathbb{Z}_{tr}(X), \mathbb{Z}(q)).$$

When  $X$  is smooth, by the previous technical results this coincides with Zariski hypercohomology, that is

$$H^n(X, \mathbb{Z}(q)) = \mathbb{H}_{Zar}^n(X, \mathbb{Z}(q)).$$

### **Eighth session: Weight one motivic cohomology; Chow groups.**

The motivic complexes  $\mathbb{Z}(q)$  can be expressed in a different form involving smash products of multiplicative groups, which sometimes is more convenient. We define  $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$  as the cokernel

$$\bigoplus_i \mathbb{Z}_{tr}(\mathbb{G}_m^{\times q-1}) \rightarrow \mathbb{Z}_{tr}(\mathbb{G}_m^{\times q}) \rightarrow \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \rightarrow 0$$

where the first map is the sum of the inclusions that put the marked point 1 at position  $i$  and the identity everywhere else.

**Proposition 7.** *There is a quasi-isomorphism of complexes of Zariski sheaves*

$$\mathbb{Z}(q) \simeq C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

Using this version of  $\mathbb{Z}(q)$  and a spectral sequence argument, it is easy to prove that the motivic cohomology groups  $H^n(X, \mathbb{Z}(q))$  vanish for  $n > q + \dim X$ .

One can give a rather explicit model for  $\mathbb{Z}(1)$  which allows to do explicit computations of motivic cohomology in weight one.

**Theorem 8.** *There is a quasi-isomorphism of presheaves with transfers*

$$\mathbb{Z}(1) \simeq \mathcal{O}^*[-1].$$

This can be shown from an explicit computation of the kernel  $\mathcal{M}^*(\mathbb{P}^1, 0, \infty)$  in the exact sequence of presheaves with transfers

$$0 \rightarrow \mathcal{M}^*(\mathbb{P}^1, 0, \infty) \rightarrow \mathbb{Z}_{tr}(\mathbb{A}^1 \setminus 0) \rightarrow \mathbb{Z} \oplus \mathcal{O}^* \rightarrow 0$$

and showing it is acyclic.

We also discussed the relation of motivic cohomology with Bloch's higher Chow groups. We proved that the groups  $CH^i(X, j)$  are compatible with finite correspondences and hence they define a presheaf with transfers; this is not evident, since the complexes  $z^i(X, j)$  which are used to define these groups are not presheaves with transfers. The main result is:

**Theorem 9.** *For any smooth projective variety over a field  $\mathbf{k}$  of characteristic zero there are isomorphisms*

$$H^n(X, \mathbb{Z}(i)) \cong CH^i(X, 2i - n),$$

for any  $n, i$ .

As an easy consequence we deduce that

$$H^n(X, \mathbb{Z}(i)) = 0, \quad n > 2i.$$

Moreover, it follows that  $H^{2i}(X, \mathbb{Z}(i)) = CH^i(X)$ . In fact we have:

**Theorem 10.** *Sending a smooth projective variety  $X$  of dimension  $d$  to  $M(X)$  extends to a full embedding*

$$i : CHM^{eff}(\mathbf{k})^{op} \longrightarrow DM_{Nis}^{eff, -}(\mathbf{k}),$$

such that

$$i(h(X)(-r)) = M(X)(r).$$

### Ninth session: Triangles in the triangulated motivic category.

Triangles in the derived category  $DM_{Nis}^{eff, -}(\mathbf{k})$  give rise to long exact sequences of abelian groups, just taking  $Hom(-, F)$ . This produces long exact sequences for every theory representable in  $DM_{Nis}^{eff, -}(\mathbf{k})$ , like motivic cohomology or étale cohomology. Some interesting triangles and isomorphisms in  $DM_{Nis}^{eff, -}(\mathbf{k})$  are the following:

- (1) **Homotopy Invariance:** This property was essentially forced by definition. for every variety  $X$  one has an isomorphism

$$M(X) \simeq M(X \times \mathbb{A}^1).$$

- (2) **Mayer-Vietoris:** This is the second property forced by definition. For every  $X$  and Zariski open cover  $U, V \subset X$ , there is the following exact triangle.

$$M(U \cap V) \rightarrow M(U) \oplus M(V) \rightarrow M(X) \rightarrow M(U \cap V)[1]$$

- (3) **Vector bundles:** The previous two properties together easily imply that for any vector bundle  $E \rightarrow X$ , the induced map on motives is an isomorphism, that is,  $M(E) \simeq M(X)$ .



- (4) **Projective bundles:** Another related result is a projective bundle formula. For a vector bundle  $E \rightarrow X$  of rank  $r + 1$ , we have the isomorphism

$$M(\mathbb{P}E) \simeq \bigoplus_{i=0}^r M(X) \otimes \mathbb{Z}(i)[2i].$$

- (5) **Blow-up's:** There is also a blow-up formula. Assume the base field  $k$  satisfies resolution of singularities. Take a pullback square

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

such that  $p$  is proper,  $i$  is a closed embedding and the restriction of  $p$  to the open complement of  $Z$  is an isomorphism. This is called an abstract blow-up. Then there is an exact triangle

$$M(Z') \rightarrow M(X') \oplus M(Z) \rightarrow M(X) \rightarrow M(Z')[1].$$

This follows from the homotopy invariance property, which gives the triangles Nisnevich locally, and a Nisnevich version of the Mayer-Vietoris property used to glue the local triangles together.

- (6) **Gysin:** For a smooth variety  $X$  and a smooth closed subvariety  $Z \subset X$  of codimension  $c$ , there is a Gysin exact triangle

$$M(X \setminus Z) \rightarrow M(X) \rightarrow M(Z)(c)[2c] \rightarrow M(X \setminus Z)[1].$$

### Tenth session: The étale version of all this stuff.

We began with the observation that the étale topology is compatible with finite correspondences, so there is an étale topology on  $Cor_{\mathbf{k}}$  and, for any ring  $R$ , it makes sense to define

$$DM_{\acute{e}t}^{eff,-}(\mathbf{k}; R) := D^-(Sh_{\acute{e}t}(Cor_{\mathbf{k}}); R)[W_{\mathbb{A}^1}^{-1}].$$

In order to characterize the  $\mathbb{A}^1$ -local étale complexes we distinguished two separate cases:

*1st case:*  $R = \mathbb{Z}/n$ , with  $n$  prime with the characteristic of  $\mathbf{k}$ .

In that case Suslin's rigidity theorem for torsion sheaves allows one to characterize  $\mathbb{A}^1$ -local complexes as the complexes of étale sheaves of  $\mathbb{Z}/n$ -modules with homotopy invariant cohomology sheaves. From this, it follows that the category  $DM_{\acute{e}t}^{eff,-}(\mathbf{k}; \mathbb{Z}/n)$  is equivalent to the category of  $\mathbb{A}^1$ -local complexes  $\mathcal{L}(\mathbb{Z}/n)_{\mathbb{A}^1}$ . Even more, applying once more Suslin's rigidity theorem, we proved the following result:

**Theorem 11.** *Let  $G = Gal(\mathbf{k}_{sep}/\mathbf{k})$  and  $D^-(G, \mathbb{Z}/n)$  the derived category of  $\mathbb{Z}/n$ -modules with a  $G$ -action. Then the natural functors*

$$D^-(G, \mathbb{Z}/n) \longrightarrow \mathcal{L}_{\mathbb{A}^1} \longrightarrow DM_{\acute{e}t}^{eff,-}(\mathbf{k}; \mathbb{Z}/n),$$

*are equivalences of tensor triangulated categories.*

2nd case:  $R = \mathbb{Q}$ .

An easy argument with spectral sequences permits to prove that for any smooth variety  $X$  and any homotopy invariant complex of  $\mathbb{Q}$ -modules of étale sheaves with transfers  $F$  there is an isomorphism

$$H_{\acute{e}t}^*(X, F) \cong H_{Nis}^*(X, F).$$

According to this isomorphism, one can use the fundamental theorem 5 to characterize the  $\mathbb{A}^1$ -local complexes in  $C^-(Sh_{\acute{e}t}(Cor_{\mathbf{k}}, \mathbb{Q}))$  and deduce an equivalence of categories

$$\mathcal{L}(\mathbb{Q})_{\mathbb{A}^1} \cong M_{\acute{e}t}^{eff, -}(\mathbf{k}; \mathbb{Q}).$$

Moreover, as the sheaves  $\mathbb{Z}_{tr}(X)$  generate  $M_{\acute{e}t}^{eff, -}(\mathbf{k}; \mathbb{Q})$  and the morphism from them to a complex  $F$  are given by cohomology groups  $H_{\acute{e}t}^*(X, F)$ , one easily deduces:

**Theorem 12.** *The functor  $\pi : (\mathbf{Sm}/\mathbf{k})_{\acute{e}t} \rightarrow (\mathbf{Sm}/\mathbf{k})_{Nis}$ , induces an equivalence of categories*

$$\pi^* : DM_{Nis}^{eff, -}(\mathbf{k}; \mathbb{Q}) \cong DM_{\acute{e}t}^{eff, -}(\mathbf{k}; \mathbb{Q}).$$

We also treated the étale version of the motivic sheaves  $\mathbb{Z}(n)$  and proved that by general homological methods one can extend the quasi-isomorphism  $\mu_n \rightarrow \mathbb{Z}/n(1)$ , which is the analogous of Theorem 8 with finite coefficients, to a quasi-isomorphism of complexes of sheaves

$$\mu_n^{\otimes r} \rightarrow \mathbb{Z}/n(r).$$

As a consequence we deduced the isomorphism between the étale motivic cohomology and the ordinary étale cohomology when taken with finite coefficients:

**Theorem 13.** *For any smooth  $\mathbf{k}$ -variety, there is a natural isomorphism*

$$H_{\acute{e}t}^*(X, \mathbb{Z}/n(r)) \cong H_{\acute{e}t}^*(X, \mu_n^{\otimes r}).$$

Finally we presented informally the Beilinson-Lichtenbaum conjectures relating motivic cohomology with finite coefficients to étale cohomology, and also its relation to the Bloch-Kato conjecture.

## REFERENCES

- [A] André, Y., *Une introduction aux motifs*. Panoramas et Synthèses, **17**. Soc. Math. France, 2004.
- [GM] I. Gelfand, Y. Manin, *Methods of Homological Algebra*. Springer Verlag, 1988.
- [GN] F. Guillén, V. Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, Publ. Math. de l'IHES, **95**, (2002), 1–91.
- [H] Huber, A., *Mixed motives and their realizations in Derived Categories*. Springer LNM, **1604**, 1995.
- [L1] Levine, M., *Mixed motives*, AMS Math. Surveys and Monographs **57**, 1998.
- [L2] Levine, M., *Six lectures on motives*. In **Some recent results in Algebraic K-theory**. ICTP lecture notes, **23** (2008), 131-225.
- [MVW] Mazza, C., Voevodsky, V., Weibel, C., *Motivic cohomology*. Clay Monographs in Math. AMS, 2006.
- [N] Neeman, A., *Triangulated categories*. Annals of Math. Studies, **148**. Princeton UP, 2001.
- [S] Scholl, A., *Classical motives*. In **Motives**, Proc. Symp. in Pure Math. **55** (1994), 163-187.
- [V] Voevodsky, V., Homotopy theory of simplicial sheaves in completely decomposed topologies. Journal of Pure and Appl. Algebra, **214**. (2010), 1384-1398.
- [VSF] Voevodsky, V., Suslin, A., Friedlander, E., *Cycles, transfers and motivic homotopy theories*. Annals of Math. Studies, **143**. Princeton UP, 2000.

(Pedro Luis del Ángel) CIMAT, JALISCO S/N. COL. VALENCIANA. CP 36240, GUANAJUATO (MÉXICO).

(P. Pascual) DEPARTAMENT DE MATEMÀTICA APLICADA 1, UNIVERSITAT POLITÈCNICA DE CATALUNYA, DIAGONAL 647, 08028 BARCELONA (SPAIN).

*E-mail address:* `pere.pascual@upc.es`