Appendix A

Dissipativity and passivity characterization

For the reader's convenience, here statements of several fundamental dissipativity-related theorems which were referenced in this dissertation are collected.

A.1 The nonlinear continuous-time case

Let a system of the form

$$\dot{x} = f(x) + g(x)u \tag{A.1}$$
$$y = h(x) + J(x)u \tag{A.2}$$

where f and h are real vector functions of the state vector x, and g and J are real matrix functions of x. It is supposed that f, g, h, and J have continuous derivatives of all orders. The input u and the output y have the same dimensions, therefore, J is a square matrix.

Theorem A.1 (Hill and Moylan, 1976) *[53] System (A.1)-(A.2) is (Q,S,R)–dissipative if and only if there exist real functions of the state vector x V, l and W, with V continuous and satisfying*

$$V(x) \geq 0, \forall x$$
$$V(0) = 0,$$

such that

$$\frac{\partial V(x)}{\partial x} f(x) = h^T(x)Qh(x) - l^T(x)l(x) \tag{A.3}$$
$$\frac{1}{2} g^T(x) \left[\frac{\partial V(x)}{\partial x}\right]^T = \hat{S}^T(x)h(x) - W^T(x)l(x) \tag{A.4}$$
$$\hat{R}(x) = W^T(x)W(x) \tag{A.5}$$

with

$$\hat{R}(x) = R + J(x)S + S^T J(x) + J^T(x)QJ(x),$$
$$\hat{S}(x) = QJ(x) + S$$

Theorem A.2 (Moylan, 1974) *[116] System (A.1)-(A.2) is passive if and only if there exist real functions of the state vector x V, l and W, with V continuous and satisfying*

$$V(x) \geq 0, \forall x$$

and

$$V(0) = 0$$

such that

$$\frac{\partial V(x)}{\partial x} f(x) = -l^T(x)l(x) \tag{A.6}$$

$$\frac{1}{2} g^T(x) \left[\frac{\partial V(x)}{\partial x}\right]^T = h(x) - W^T(x)l(x) \tag{A.7}$$

$$J(x) + J^T(x) = W^T(x)W(x) \tag{A.8}$$

If J is a constant matrix, then W may be taken to be constant.

Consider a system of the following form,

$$\dot{x} = f(x,u) \tag{A.9}$$

$$y = h(x,u) \tag{A.10}$$

with $x \in \mathscr{X} \subset \Re^n$, $u \in \mathscr{U} \subset \Re^m$, $y \in \mathscr{Y} \subset \Re^m$.

Proposition A.1 (Lin, 1995) *[86] Let* $\Omega = \{x \in \Re^n : L_{f(x,0)}V(x) = 0\}$. *Necessary conditions for (A.9)-(A.10) to be passive with a* $\mathscr{C}^2$ *storage function V are that,*

(i) $L_{f(x,0)}V(x) \leq 0$

(ii) $L_{g_0}V(x) = h^T(x,0), \forall x \in \Omega$

(iii) $\sum_{i=1}^n \frac{\partial^2 f_i}{\partial u^2}(x,0)\frac{\partial V}{\partial x_i} \leq \frac{\partial h^T}{\partial u}(x,0) + \frac{\partial h}{\partial u}(x,0), \forall x \in \Omega$

with $g_0(x) = \frac{\partial f}{\partial u}(x,0) = \left[g_1^0(x), \ldots, g_m^0(x)\right] \in \Re^{n\times m}$, $g_i^0 = \frac{\partial f}{\partial u_i}(x,0) \in \Re^n, 1 \leq i \leq m$.

A.2 The nonlinear discrete-time case

Consider,

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{A.11}$$

$$y(k) = h(x(k)) + J(x(k))u(k) \tag{A.12}$$

where $x \in \Re^n$, $u \in \Re^m$, and $y \in \Re^m$, f, g, h, and J are smooth maps, all of appropriate dimensions, and $f(0) = 0$, $h(0) = 0$.

Proposition A.2 (Lin and Byrnes, 1995) *[87] Let*

$$\Omega = \{x \in \Re^n : V(f^{i+1}(x)) = V(f^i(x)), \forall i \in \mathscr{Z}_+\}$$

for a $\mathscr{C}^2$ storage function V, which is positive definite and $V(0) = 0$. A system of the form (A.11)-(A.12) is passive only if

$$V(f(x)) \leq V(x) \forall x \in \Re^n \tag{A.13}$$

$$\left.\frac{\partial V}{\partial \alpha}\right|_{\alpha=f(x)} g(x) = h^T \forall x \in \Omega \tag{A.14}$$

$$g^T(x) \left.\frac{\partial^2 V}{\partial \alpha^2}\right|_{\alpha=f(x)} g(x) \leq J^T(x) + J(x) \forall x \in \Omega \tag{A.15}$$

Let a discrete-time system of the form,

$$x(k+1) = f(x(k), u(k)), x \in \mathscr{X} \subset \Re^n, u \in \mathscr{U} \subset \Re^m \tag{A.16}$$

$$y(k) = h(x(k), u(k)), y \in \mathscr{Y} \subset \Re^m \tag{A.17}$$

where f and h are smooth maps, and $f(0,0) = 0$, $h(0,0) = 0$.

Proposition A.3 (Lin, 1995) *[86] Let $\Omega_d = \{x \in \Re^n : V(f_0(x)) = V(x)\}$. A system of the form (A.16)-(A.17) is passive with a $\mathscr{C}^r (r \geq 2)$ storage function V, with $V(0) = 0$ only if*

$$V(f_0(x)) \leq V(x) \forall x \in \Re^n \tag{A.18}$$

$$\left.\frac{\partial V}{\partial \alpha}\right|_{\alpha=f_0(x)} g_0(x) = h^T(x,0) \forall x \in \Omega_d \tag{A.19}$$

$$g_0^T(x) \left.\frac{\partial^2 V}{\partial \alpha^2}\right|_{\alpha=f_0(x)} g_0(x) \leq \frac{\partial h}{\partial u}(x,0) + \frac{\partial h^T}{\partial u}(x,0) \forall x \in \Omega_d, \tag{A.20}$$

with

$$\begin{aligned} f_0(x) &= f(x,0) \in \Re^n, \\ g_i^0 &= \frac{\partial f}{\partial u_i}(x,0) \in \Re^n, 1 \leq i \leq m, \\ g_0(x) &= \frac{\partial f}{\partial u}(x,0) = \left[g_1^0(x), \ldots, g_m^0(x)\right] \in \Re^{n \times m}. \end{aligned} \tag{A.21}$$

Definition A.1 (Sëngor, 1995) *[151] A dynamical discrete-time system is a dynamical energy system if there exists a function $s(y,u)$, called the supply rate or the power input function, such that the associated consumed energy is defined by*

$$e(K, K_0, u, y) = \sum_{K_0}^{K-1} s(u(\tau), y(\tau)) \tag{A.22}$$

Definition A.2 (Sëngor, 1995) *[151] ψ is a gradient-like function if and only if there exists $B : \mathscr{X} \to \Re^n$ and $C : \mathscr{X} \to \Re^{n \times n}$, such that*

$$\psi(\hat{x}) - \psi(x) \equiv B(x)^T (\hat{x} - x) + (\hat{x} - x)^T C(x)(\hat{x} - x), \forall \hat{x}, x \in \mathscr{X}$$

Theorem A.3 (Sĕngor, 1995) *[151] Consider the system (A.16)-(A.17) and the function (A.22). Then a gradient-like function* $\psi : \Re^n \to \Re^+$ *is a conservative potential function for the given system if and only if*

$$B(x)^T[f(x,u)-x]+[f(x,u)-x]^T C(x)[f(x,u)-x] = s(u,y(x,u)),\, \forall (x,u) \in \mathscr{X} \times \mathscr{U}$$

Theorem A.4 (Sĕngor, 1995) *[151] Consider the system (A.16)-(A.17) and the function (A.22). Then a gradient-like function* $\psi : \Re^n \to \Re^+$ *is an internal energy function for the given system if and only if*

$$B(x)^T[f(x,u)-x]+[f(x,u)-x]^T C(x)[f(x,u)-x] \le s(u,y(x,u)),\, \forall (x,u) \in \mathscr{X} \times \mathscr{U}$$

Theorem A.5 (Sĕngor, 1995) *[151] Every dynamical energy system with a conservative potential function is lossless.*

Theorem A.6 (Sĕngor, 1995) *[151] A system is dissipative if and only if there exists an internal energy function.*

Theorem A.7 (Sĕngor, 1995) *[151] A gradient-like function* $\psi : \Re^n \to \Re^+$ *is a conservative potential energy for the system (A.11)-(A.12) with* $s = y^T Qy + 2y^T Su + u^T Ru$ *if and only if*

$$B(x)^T[f(x)-x]+[f(x)-x]^T C(x)[f(x)-x] = h^T(x)Qh(x) \tag{A.23}$$

$$g^T(x)\left\{B(x)+\left[C^T(x)+C(x)\right][f(x)-x]\right\} = 2\left[QJ(x)+S\right]^T h(x) \tag{A.24}$$

$$R+J^T(x)S+S^T J(x)+J^T(x)QJ(x)-g(x)^T C(x)g(x) = 0 \tag{A.25}$$

Theorem A.8 (Sĕngor, 1995) *[151] A gradient-like function* $\psi : \Re^n \to \Re^+$ *is an internal energy function for the system (A.11)-(A.12) with* $s = y^T Qy + 2y^T Su + u^T Ru$ *if and only if there exist real functions l, m, W, all of appropriate dimensions, satisfying* $\forall x \in \mathscr{X}$

$$B(x)^T[f(x)-x]+[f(x)-x]^T C(x)[f(x)-x] = \hat{Q} - l^T(x)l(x) - m^T(x)m(x) \tag{A.26}$$

$$g^T(x)\left\{B(x)+\left[C^T(x)+C(x)\right][f(x)-x]\right\} = \hat{S} - 2W^T(x)l(x) \tag{A.27}$$

$$\hat{R} - g^T(x)C(x)g(x) = W^T(x)W(x) \tag{A.28}$$

with

$$\hat{Q} = h^T(x)Qh(x)$$
$$\hat{S} = 2\left[QJ(x)+S\right]^T h(x)$$
$$\hat{R} = R+J^T(x)S+S^T J(x)+J^T(x)QJ(x)$$