# Consecutive patterns and statistics on restricted permutations

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# Contents

# Introduction

The subject of *pattern-avoiding permutations*, also called *restricted permutations*, has blossomed in the past decade. A number of enumerative results have been proved, new bijections found, and connections to other fields established. A recent breakthrough [70] (see also [57, 2, 14, 3]) has been the proof of the so-called Stanley-Wilf conjecture, which gives an exponential upper bound on the number of permutations avoiding any given pattern.

However, the study of statistics on restricted permutations started developing very recently, and the interest in this topic is currently growing. On the one hand, the concept of pattern avoidance concerns permutations regarded as words $\pi = \pi_1\pi_2\cdots\pi_n$. On the other hand, concepts such as fixed points or excedances arise when we look at permutations as bijections $\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$. It was not until recently that these two kinds of concepts were studied together.

An unexpected recent result of Robertson, Saracino and Zeilberger [78] gives a new and exciting extension to the classical result that the number of 321-avoiding permutations equals the number of 132-avoiding permutations. They show that one can refine this result by taking into account the number of fixed points in a permutation. Their proof is nontrivial and technically involved. The first part of the work in the present thesis is motivated by this result. A natural question that arises is whether the fact that the number of fixed points has the same distribution in both 321-avoiding and 132-avoiding permutations can be generalized to other statistics and to other patterns. In particular, this gives more interest to the problem of studying the distribution of statistics on pattern-avoiding permutations. Another natural question to consider is whether a bijection between 321-avoiding permutations and 132-avoiding permutations that preserves the number of fixed points can be described. It is somewhat surprising that, before the work on this thesis [31, 33, 36], none of the several known bijections between these two sets of permutations preserved the number of fixed points.

Aside from the study of statistics, there is a variation to the notion of pattern avoidance that started to develop recently. The concept of *generalized pattern* allows the requirement that, for a pattern to occur in a permutation, certain elements have to be in adjacent positions. A particular case of this are the *consecutive patterns*, where all the elements have to be consecutive. The last part of this work is devoted to the study of such patterns.

This thesis is structured as follows. Chapter 1 introduces the basic definitions regarding pattern avoidance, permutation statistics and Dyck paths, as well as some tools to manipulate generating functions. In Chapter 2 we study the distribution of the statistics 'number of fixed points' and 'number of excedances' in permutations avoiding a pattern of length 3. The main result is that the joint distribution of this pair of parameters is the same in 321-avoiding as in 132-avoiding permutations. This generalizes a recent theorem of Robertson, Saracino and Zeilberger. We prove this result by giving a bijection preserving these two statistics. A part of it is based on the Robinson-Schensted-Knuth correspondence. We also show that our bijection preserves additional parameters. The key idea is to introduce a new class of statistics on Dyck paths, based on what we call a *tunnel*.

In Chapter 3 we consider the same pair of statistics in permutations avoiding simultaneously two or more patterns of length 3. We solve all the cases by giving generating functions which enumerate them. Some cases are generalized to patterns of arbitrary length. We also describe the distribution of these parameters in involutions avoiding any subset of patterns of length 3. The main technique consists in using bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that the statistics in permutations that we consider correspond to statistics on Dyck paths which are easier to enumerate.

In Chapter 4 we present a new statistic-preserving family of bijections from the set of Dyck paths to itself. They map statistics that appear in the study of pattern-avoiding permutations into classical statistics on Dyck paths, whose distribution is easy to obtain. In particular, this gives a simple bijective proof of the equidistribution of fixed points in 321- and 132-avoiding permutations. Chapter 5 gives some new interpretations of the Catalan and Fine numbers and a few additional bijections. We consider a class of permutations enumerated by the Catalan numbers, defined in terms of noncrossing matchings of $2n$ points around a circle. We study some of their properties, and we give the distribution of several statistics on them.

In Chapter 6 we consider a different notion of pattern avoidance, with the requirement that the elements forming the pattern have to occur in consecutive positions in the permutation. More generally, we study the distribution of the number of occurrences of consecutive patterns in permutations. We solve the problem in several cases depending on the shape of the subword by obtaining the corresponding bivariate exponential generating functions as solutions of certain linear differential equations with polynomial coefficients. Our method is based on the representation of permutations as increasing binary trees and on symbolic methods.

Finally, Chapter 7 deals with generalized patterns, which extend the notions of both classical and consecutive patterns. For a few patterns we obtain new exact enumerative results. Then we study the asymptotic behavior of the number of permutations in $\mathcal{S}_n$ avoiding a fixed generalized pattern as $n$ goes to infinity. We also give some lower and upper bounds on the number of permutations avoiding certain patterns.

# 1

# Definitions and preliminaries

## 1.1 Permutations

### 1.1.1 Pattern avoidance

We will denote by $[n]$ the set $\{1, 2, \ldots, n\}$, and by $\mathcal{S}_n$ the symmetric group on $[n]$. A permutation $\pi \in \mathcal{S}_n$ will be written in one-line notation as $\pi = \pi_1\pi_2\cdots\pi_n$. We will write the cardinality of a set $A$ as $|A|$. In this section we define the classical notion of pattern avoidance, which will be used throughout most of this thesis. For a definition of *generalized patterns* see Section 7.1.

Let $n$, $m$ be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{S}_m$ be two permutations. We say that $\pi$ *contains* $\sigma$ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\rho(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}) = \sigma$, where $\rho$ is the reduction consisting in relabelling the elements with $\{1, \ldots, m\}$ so that they keep the same order relationships they had in $\pi$. (Equivalently, this means that for all indices $a$ and $b$, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$.) In that case, $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is called an *occurrence* of $\sigma$ in $\pi$. In this context, $\sigma$ is also called a *pattern*.

If $\pi$ does not contain $\sigma$, we say that $\pi$ *avoids* $\sigma$, or that it is $\sigma$*-avoiding*. For example, if $\sigma = 132$, then $\pi = 24531$ contains 132, because the subsequence $\pi_1\pi_3\pi_4 = 253$ has the same relative order as 132. However, $\pi = 42351$ is 132-avoiding. Denote by $\mathcal{S}_n(\sigma)$ the set of $\sigma$-avoiding permutations in $\mathcal{S}_n$. More generally, for any subset $A \subseteq \mathcal{S}_n$ and any pattern $\sigma$, define $A(\sigma) := A \cap \mathcal{S}_n(\sigma)$ to be the set of $\sigma$-avoiding permutations in $A$.

It is a natural generalization to consider permutations that avoid several patterns at the same time. If $\Sigma \subseteq \bigcup_{k\geq 1} \mathcal{S}_k$ is any finite set of patterns, denote by $\mathcal{S}_n(\Sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid simultaneously all the patterns in $\Sigma$. These are also called $\Sigma$-avoiding permutations. For example, if $\Sigma = \{123, 231\}$, then $\mathcal{S}_4(\Sigma) = \{1432, 2143, 3214, 4132, 4213, 4312, 4321\}$.

### 1.1.2 Permutation statistics

Informally speaking, the notion of permutation can be viewed in two different ways. On one hand, a permutation can be regarded as a word $\pi = \pi_1\pi_2\cdots\pi_n$, namely, as a sequence of numbers in some given order. From this description arises the concept of pattern avoidance discussed in the previous subsection. On the other hand, one can regard a permutation $\pi \in \mathcal{S}_n$ as a bijection $\pi : [n] \longrightarrow [n]$. Some concepts such as fixed points or excedances arise when we consider a permutation as a bijection. This double nature of permutations makes it interesting to study some of the following statistics together with the notion of pattern avoidance. There is a lot of mathematical literature devoted to permutation statistics (see for example [30, 42, 44, 46]).

We say that $i$ is a *fixed point* of a permutation $\pi$ if $\pi_i = i$. We say that $i$ is an *excedance* of $\pi$ if $\pi_i > i$. Denote by $\text{fp}(\pi)$ and $\text{exc}(\pi)$ the number of fixed points and the number of excedances of $\pi$ respectively. The distribution of the statistics fp and exc in pattern-avoiding permutations will be one of the main topics of this thesis. An element of a permutation that is neither a fixed point nor an excedance, namely an $i$ for which $\pi_i < i$, will be called a *deficiency*. Permutations without fixed points are also called *derangements*.

We say that $i \leq n-1$ is a *descent* of $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. Similarly, $i \leq n-1$ is an *ascent* of $\pi \in \mathcal{S}_n$ if $\pi_i < \pi_{i+1}$. Denote by $\text{des}(\pi)$ and $\text{asc}(\pi)$ the number of descents and the number of ascents of $\pi$ respectively. A *right-to-left minimum* of $\pi$ is an element $\pi_i$ such that $\pi_i < \pi_j$ for all $j > i$. Similarly, $\pi_i$ is a *left-to-right minimum* of $\pi$ if $\pi_i < \pi_j$ for all $j < i$. A *right-to-left maximum* is an element $\pi_i$ such that $\pi_i > \pi_j$ for all $j > i$.

Let $\text{lis}(\pi)$ denote the length of the *longest increasing subsequence* of $\pi$, i.e., the largest $m$ for which there exist indices $i_1 < i_2 < \cdots < i_m$ such that $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_m}$. Equivalently, in terms of forbidden patterns, $\text{lis}(\pi)$ is the smallest $m$ such that $\pi$ avoids $12\cdots(m+1)$. The length of the *longest decreasing subsequence* is defined analogously, and it is denoted $\text{lds}(\pi)$. Define the *rank* of $\pi$, denoted $\text{rank}(\pi)$, to be the largest $k$ such that $\pi(i) > k$ for all $i \leq k$. For example, if $\pi = 63528174$, then $\text{fp}(\pi) = 1$, $\text{exc}(\sigma) = 4$, $\text{des}(\pi) = 4$, $\text{lis}(\pi) = 3$, $\text{lds}(\pi) = 4$ and $\text{rank}(\pi) = 2$.

We say that a permutation $\pi \in \mathcal{S}_n$ is an *involution* if $\pi = \pi^{-1}$. Denote by

$\mathcal{I}_n$ the set of involutions of length $n$.

### 1.1.3 Trivial operations

Often it will be convenient to represent a permutation $\pi \in \mathcal{S}_n$ by an $n \times n$ array with a cross in each one of the squares $(i, \pi_i)$. We will denote by $\text{arr}(\pi)$ the array corresponding to $\pi$. Figure 1.1 shows arr(63528174).

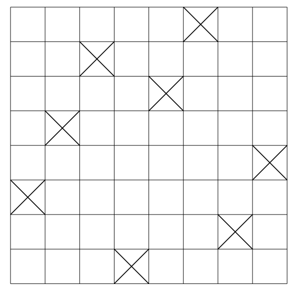

**Figure 1.1** The array of $\pi = 63528174$.

The diagonal from the top-left corner to the bottom-right corner of the array will be referred to as *main diagonal*, and the diagonal perpendicular to it will be called *secondary diagonal*. Note that fixed points of $\pi$ correspond to crosses on the main diagonal of the array, and excedances of $\pi$ are represented by crosses to the right of this diagonal.

Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, define its reversal $\pi^R = \pi_n \ldots \pi_2\pi_1$ and its complementation $\bar{\pi} = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$. The array of $\bar{\pi}$ is obtained from the array of $\pi$ by a flip along a vertical axis, so fixed points (resp. excedances) of $\pi$ correspond to crosses on (resp. to the left of) the secondary diagonal of the array of $\bar{\pi}$. Similarly, define $\widehat{\pi}$ to be the permutation whose array is the one obtained from that of $\pi$ by reflection along the secondary diagonal. Note that reflecting the array of $\pi$ along the main diagonal we get the array of its inverse $\pi^{-1}$. For any set of permutations $\Sigma$, let $\bar{\Sigma}$ be the set obtained by reversing each of the elements of $\Sigma$. Define $\widehat{\Sigma}$ and $\Sigma^{-1}$ analogously. The following trivial lemma will be used in Chapters 2 and 3.

**Lemma 1.1** *Let $\Sigma \subset \bigcup_{k\geq 1} \mathcal{S}_k$ be a finite set of patterns, and let $\pi \in \mathcal{S}_n$. We have that*

(1) $\pi \in \mathcal{S}_n(\Sigma) \iff \bar{\pi} \in \mathcal{S}_n(\bar{\Sigma}) \iff \widehat{\pi} \in \mathcal{S}_n(\widehat{\Sigma}) \iff \pi^{-1} \in \mathcal{S}_n(\Sigma^{-1})$,

(2) $\text{fp}(\widehat{\pi}) = \text{fp}(\pi)$, $\text{exc}(\widehat{\pi}) = \text{exc}(\pi)$,

(3) $\mathrm{fp}(\pi^{-1}) = \mathrm{fp}(\pi)$, $\mathrm{exc}(\pi^{-1}) = n - \mathrm{fp}(\pi) - \mathrm{exc}(\pi)$.

In this thesis we will often be interested in the distribution of the statistics fp and exc among the permutations avoiding a certain pattern or set of patterns. Given any such set $\Sigma$, we define the generating function $F_\Sigma$ as

$$F_\Sigma(x,q,z) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n. \qquad (1.1)$$

If $\Sigma = \{\sigma\}$, we will write $F_\sigma$ instead of $F_{\{\sigma\}}$. The following lemma restates the previous one in terms of generating functions.

**Lemma 1.2** *Let $\Sigma$ be any set of permutations. We have*

(1) $F_{\widehat{\Sigma}}(x,q,z) = F_\Sigma(x,q,z)$,

(2) $F_{\Sigma^{-1}}(x,q,z) = F_\Sigma(x/q, 1/q, qz)$.

*Proof.* To prove (1), consider the bijection between $\mathcal{S}_n(\Sigma)$ and $\mathcal{S}_n(\widehat{\Sigma})$ that maps $\pi$ to $\widehat{\pi}$. The equation follows from parts (1) and (2) of Lemma 1.1.

Equation (2) follows similarly from parts (1) and (3) of the previous lemma, noticing that

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\Sigma^{-1})} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi^{-1})} q^{\mathrm{exc}(\pi^{-1})} z^n =$$

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{n-\mathrm{fp}(\pi)-\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\Sigma)} \left(\frac{x}{q}\right)^{\mathrm{fp}(\pi)} \left(\frac{1}{q}\right)^{\mathrm{exc}(\pi)} (qz)^n.$$

□

If for two sets of patterns $\Sigma_1$ and $\Sigma_2$ we have that $F_{\Sigma_1}(x,q,z) = F_{\Sigma_2}(x,q,z)$ (i.e., the joint distribution of fixed points and excedances is the same in $\Sigma_1$-avoiding as in $\Sigma_2$-avoiding permutations), we will write $\Sigma_1 \approx \Sigma_2$. If we have that $F_{\Sigma_1}(x,q,z) = F_{\Sigma_2}(x/q,1/q,qz)$, we will write $\Sigma_1 \sim_f \Sigma_2$. In this notation, Lemma 1.2 says that $\widehat{\Sigma} \approx \Sigma$ and $\Sigma^{-1} \sim_f \Sigma$.

## 1.2 Dyck paths

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the $x$-axis. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$ the class of all Dyck paths. It is well-known that $|\mathcal{D}_n| = \mathbf{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the $n$-th Catalan number. If $D \in \mathcal{D}_n$, we will write $|D| = n$ to indicate the semilength of $D$. The generating function that enumerates Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} z^{|D|} = \sum_{n\geq 0} \mathbf{C}_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, which we denote by $\mathbf{C}(z)$.

Sometimes it will be convenient to encode each up-step by a letter **u** and each down-step by **d**, obtaining an encoding of the Dyck path as a *Dyck word*. We will use $D$ to refer indistinctively to the Dyck path $D$ or to the Dyck word associated to it. In particular, given $D_1 \in \mathcal{D}_{n_1}$, $\mathcal{D}_2 \in \mathcal{D}_{n_2}$, we will write $D_1D_2$ to denote the concatenation of $D_1$ and $D_2$ (note that, as seen in terms of lattice paths, $D_2$ has to be shifted $2n_1$ units to the right). If $A$ is any sequence of up and down steps, length$(A)$ will denote the number of steps in the sequence. For example, if $A \in \mathcal{D}_n$, then length$(A) = 2n$.

### 1.2.1 Standard statistics

A *peak* of a Dyck path $D \in \mathcal{D}$ is an up-step followed by a down-step (i.e., an occurrence[1] of **ud** in the associated Dyck word). The coordinates of a peak are given by the point at the top of it. A *hill* is a peak at height 1, where the height is the $y$-coordinate of the peak. Denote by $h(D)$ the number of hills of $D$, and by $p_2(D)$ the number of peaks of $D$ of height at least 2. A *valley* of $D$ is a down-step followed by an up-step (i.e., an occurrence of **du** in the associated Dyck word). Denote by va$(D)$ the number of valleys of $D$. Clearly, both $p_2(D) + h(D)$ and va$(D) + 1$ equal the total number of peaks of $D$. A *double rise* of $D$ is an up-step followed by another up-step (i.e., an occurrence **uu** in the Dyck word). Denote by dr$(D)$ the number of double rises of $D$.

An *odd rise* is an up-step in an odd position when the steps are numbered from left to right starting with 1 (or, equivalently, it is an up-step at odd level when the steps leaving the $x$-axis are considered to be at level 1). Denote by or$(D)$ the number of odd rises of $D$. *Even rises* and er$(D)$ are defined analogously. The $x$-coordinate of an odd or even rise is given by the rightmost end of the corresponding up-step.

A *return* of a Dyck path is a down-step landing on the $x$-axis. An *arch* is

[1] In the context of Dyck words, the letters have to appear in consecutive positions to form an *occurrence* of a subword.

a part of the path joining two consecutive points on the $x$-axis. Clearly for any $D \in \mathcal{D}_n$ the number of returns equals the number of arches. Denote it by $\mathrm{ret}(D)$. Define the $x$-coordinate of an arch as the $x$-coordinate of its leftmost point.

The *height* of $D$ is the $y$-coordinate of the highest point of the path. Denote by $\mathcal{D}^{\leq k}$ the set of Dyck paths of height at most $k$. For any $D \in \mathcal{D}_n$, define $\nu(D)$ to be the height of the middle point of $D$, that is, the $y$-coordinate of the intersection of the vertical line $x = n$ with the path. For example, if $D \in \mathcal{D}_8$ is the path in Figure 1.2, then $h(D) = 1$, $p_2(D) = 4$, $\mathrm{va}(D) = 4$, $\mathrm{dr}(D) = 3$, $\mathrm{or}(D) = 5$, $\mathrm{er}(D) = 3$, $\mathrm{ret}(D) = 2$, $\nu(D) = 2$, and its height is 3.

Define a *pyramid* to be a Dyck path that has only one peak, that is, a path of the form $\mathbf{u}^k\mathbf{d}^k$ with $k \geq 1$ (here the exponent indicates the number of times the letter is repeated). For a Dyck path $D \in \mathcal{D}_n$, denote by $D^*$ the path obtained by reflection of $D$ from the vertical line $x = n$. We say that $D$ is *symmetric* if $D = D^*$. Denote by $\mathcal{D}s \subset \mathcal{D}$ the subclass of symmetric Dyck paths.

### 1.2.2 Tunnels

Here we introduce a new class of statistics on Dyck paths that will become very useful for the study of statistics on permutations avoiding patterns of length 3. They are based on the notion of tunnel of a Dyck path.

For any $D \in \mathcal{D}$, define a *tunnel* of $D$ to be a horizontal segment between two lattice points of $D$ that intersects $D$ only in these two points, and stays always below $D$. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, where $B \in \mathcal{D}$ (no restrictions on $A$ and $C$). In the decomposition, the tunnel is the segment that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$. If $D \in \mathcal{D}_n$, then $D$ has exactly $n$ tunnels, since such a decomposition can be given for each up-step $\mathbf{u}$ of $D$. The *length* of a tunnel is just its length as a segment, and the *height* is its $y$-coordinate. It will be useful to define the *depth* of a tunnel $T$ as $\mathrm{depth}(T) := \frac{1}{2}\mathrm{length}(T) - \mathrm{height}(T) - 1$.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the $x$-coordinate of its midpoint (as a segment) is $n$, that is, the tunnel is centered with respect to the vertical line through the middle of $D$. In terms of the decomposition of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to $A$ and $C$ having the same length, namely, $\mathrm{length}(A) = \mathrm{length}(C)$. Alternatively, this can be taken as a definition of centered tunnel. Denote by $\mathrm{ct}(D)$ the number of centered tunnels of $D$.

A tunnel of $D \in \mathcal{D}_n$ is called a *right tunnel* if the $x$-coordinate of its midpoint is strictly greater than $n$, that is, the midpoint of the tunnel is to the right of the vertical line through the middle of $D$. In terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that $\text{length}(A) > \text{length}(C)$. Denote by $\text{rt}(D)$ the number of right tunnels of $D$. In Figure 1.2, there is one centered tunnel drawn with a solid line, and four right tunnels drawn with dotted lines. Similarly, a tunnel is called a *left tunnel* if the $x$-coordinate of its midpoint is strictly less than $n$. Denote by $\text{lt}(D)$ the number of left tunnels of $D$. Clearly, $\text{lt}(D) + \text{rt}(D) + \text{ct}(D) = n$ for any $D \in \mathcal{D}_n$.

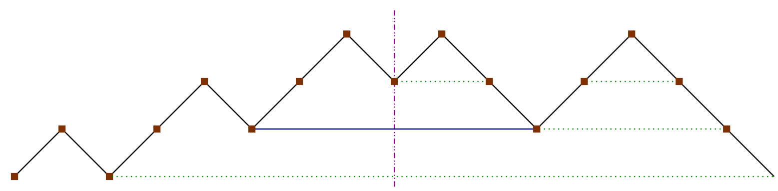

**Figure 1.2** One centered and four right tunnels.

We will distinguish between right tunnels of $D \in \mathcal{D}_n$ that are entirely contained in the half plane $x \geq n$ and those that cross the vertical line $x = n$. These will be called *right-side tunnels* and *right-across tunnels*, respectively. In terms of Dyck words, a decomposition $D = A\mathbf{u}B\mathbf{d}C$ corresponds to a right-side tunnel if $\text{length}(A) \geq n$, and to a right-across tunnel if $\text{length}(C) < \text{length}(A) < n$. In Figure 1.2 there are three right-side tunnels and one right-across tunnel. *Left-side tunnels* and *left-across tunnels* are defined analogously.

For any $D \in \mathcal{D}$, we define a *multitunnel* of $D$ to be a horizontal segment between two lattice points of $D$ such that $D$ never goes below it. In other words, a multitunnel is just a concatenation of tunnels, so that each tunnel starts at the point where the previous one ends. Similarly to the case of tunnels, multitunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = ABC$, where $B \in \mathcal{D}$ is not empty. In the decomposition, the multitunnel is the segment that connects the initial and final points of $B$.

A multitunnel of $D \in \mathcal{D}_n$ is called a *centered multitunnel* if the $x$-coordinate of its midpoint (as a segment) is $n$, that is, the tunnel is centered with respect to the vertical line through the middle of $D$. In terms of the decomposition $D = ABC$, this is equivalent to saying that $A$ and $C$ have the same length. Denote by $\text{cmt}(D)$ the number of centered multitunnels of $D$.

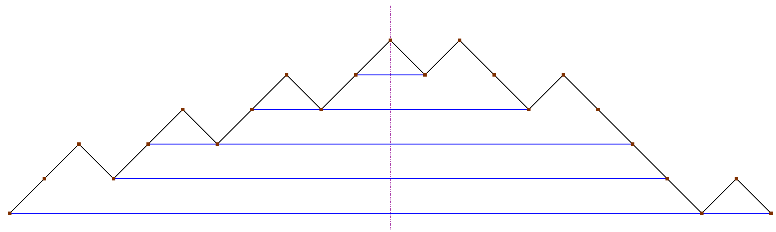

**Figure 1.3** Five centered multitunnels, two of which are centered tunnels.

## Additional interpretations of centered tunnels

Through the numerous known bijections between Dyck paths and other combinatorial objects counted by the Catalan numbers, the new statistics that we defined on Dyck paths give rise to corresponding statistics in other objects. Here we give a couple of examples that were suggested by Emeric Deutsch.

It is known [90, Exercise 6.19(n)] that the diagrams of $n$ nonintersecting chords joining $2n$ points on the circumference of a circle are in bijection with $\mathcal{D}_n$. We can draw these points as the vertices of a regular $2n$-gon, and the chords as straight segments, so that one of the diagrams has $n$ horizontal chords. The bijection to Dyck paths can be described as follows. Starting counterclockwise from the topmost vertex on the left, for each vertex draw an up-step in the path if the chord from that vertex is encountered for the first time, and a down-step otherwise. By means of this bijection, horizontal chords of the diagram correspond precisely to centered tunnels of the Dyck path (see Figure 1.4).

More generally, if we number the vertices of the $2n$-gon from 1 to $2n$ in the order in which they are read by the bijection, then, for $1 \leq i \leq n$, the chords parallel to the line between vertices $i$ and $i+1$ correspond to tunnels of the Dyck path with midpoint at $x = i$ or at $x = n + i$.

Another class of objects in bijection with $\mathcal{D}_n$ is the set of plane trees with $n+1$ vertices. Consider the bijection described in [90, Exercise 6.19(e)]. Now, given a plane tree on $n+1$ vertices, label the vertices with integers from 0 to $n$ in preorder (depth-first search) from left to right. Next, label the vertices again from 0 to $n$, but now in preorder from right to left. Then, the vertices other than the root for which the two labels coincide correspond

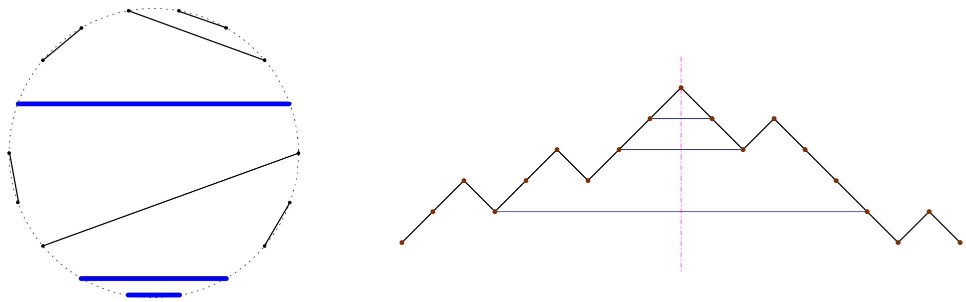

**Figure 1.4** A bijection between nonintersecting chord diagrams and Dyck paths.

to centered tunnels in the Dyck path. Besides, right tunnels correspond precisely to vertices for which the second label is less than the first one.

# 1.3 Combinatorial classes and generating functions

Here we direct the reader to [41] and [83] for a detailed account on combinatorial classes and the symbolic method.

## 1.3.1 Ordinary generating functions

Let $\mathcal{A}$ be a class of *unlabelled* combinatorial objects and let $|\alpha|$ be the size of an object $\alpha \in \mathcal{A}$. If $\mathcal{A}_n$ denotes the objects in $\mathcal{A}$ of size $n$ and $a_n = |\mathcal{A}_n|$, then the *ordinary generating function* of the class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} = \sum_{n \geq 0} a_n z^n.$$

In our context, the size of a Dyck path is simply its semilength. From now on we will use the acronym GF as a shorthand for the term *generating function* or, more specifically, *ordinary generating function*.

There is a direct correspondence between set theoretic operations (or "constructions") on combinatorial classes and algebraic operations on GFs. Table 1.1 summarizes this correspondence for the operations that are used in this work. There "union" means union of disjoint copies, "product" is the usual cartesian product, and "sequence" forms an ordered sequence in the usual sense. Enumerations according to size and auxiliary parameters

| *Construction* | | *Operation on GFs* |
|---|---|---|
| Union | $\mathcal{A} = \mathcal{B} + \mathcal{C}$ | $A(z) = B(z) + C(z)$ |
| Product | $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ | $A(z) = B(z)C(z)$ |
| Sequence | $\mathcal{A} = \mathrm{Seq}(\mathcal{B})$ | $A(z) = \frac{1}{1-B(z)}$ |

**Table 1.1** The basic combinatorial constructions and their translation into ordinary generating functions.

$\chi_1, \chi_2, \ldots, \chi_r$ are described by multivariate (ordinary) GFs,

$$A(u_1, u_2, \ldots, u_r, z) = \sum_{\alpha \in \mathcal{A}} u_1^{\chi_1(\alpha)} u_2^{\chi_2(\alpha)} \cdots u_r^{\chi_r(\alpha)} z^{|\alpha|}.$$

Throughout this thesis the variable $z$ is reserved for marking the length of a permutation and the semilength of a Dyck path, $x$ is used for marking the number of fixed points of a permutation and the number of centered tunnels or tunnels of depth 0 of a Dyck path, and $q$ is the variable that marks the number of excedances of a permutation and the number of right tunnels or tunnels of negative depth of a Dyck path, unless otherwise stated.

### 1.3.2 Exponential generating functions

Let now $\mathcal{A}$ be instead a class of *labelled* combinatorial objects and let $|\alpha|$ be the size of an object $\alpha \in \mathcal{A}$ as before. Let $\mathcal{A}_n$ denote the objects in $\mathcal{A}$ of size $n$ and let $a_n = |\mathcal{A}_n|$ again. The *exponential generating function*, EGF for short, of the class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n \geq 0} a_n \frac{z^n}{n!}.$$

In our context, the size of a permutation is simply its length.

Table 1.2 summarizes the correspondence between set-theoretic operations on labelled combinatorial classes and algebraic operations on EGFs. There "labelled product" is the usual cartesian product enriched with the relabelling operation, and "set" forms sets in the usual sense. Particularly important for us is the construction "boxed product" $\mathcal{A} = \mathcal{B}^{\Box} * \mathcal{C}$, which corresponds to the subset of $\mathcal{B} \star \mathcal{C}$ (the usual labelled product) formed by those pairs in which the smallest label lies in the $\mathcal{B}$ component.

| *Construction* | | *Operation on GF* |
|---|---|---|
| Union | $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ | $A(z) = B(z) + C(z)$ |
| Labelled product | $\mathcal{A} = \mathcal{B} \star \mathcal{C}$ | $A(z) = B(z)C(z)$ |
| Set | $\mathcal{A} = \Pi(\mathcal{B})$ | $A(z) = \exp(B(z))$ |
| Boxed product | $\mathcal{A} = \mathcal{B}^{\Box} \star \mathcal{C}$ | $A(z) = \int_0^z (\frac{d}{dt}B(t)) \cdot C(t)dt$ |

**Table 1.2** The basic combinatorial constructions and their translation into exponential generating functions.

Enumerations according to size and an auxiliary parameter $\chi$ are described by bivariate (exponential) generating functions, or BGFs,

$$A(u, z) = \sum_{\alpha \in \mathcal{A}} u^{\chi(\alpha)} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n,k \geq 0} A_{n,k} u^k \frac{z^n}{n!},$$

with $A_{n,k}$ the number of objects of size $n$ with $\chi$-parameter equal to $k$. Exponential generating functions are used in Chapter 6. There, the variable $z$ is reserved for marking the length of a permutation, and the variable $u$ is used mostly for marking occurrences of a subword. All derivatives in that chapter are taken with respect to $z$.

### 1.3.3 The Lagrange inversion formula

The Lagrange inversion formula (see for example [90, Theorem 5.4.2]) is a useful tool that provides a way to compute the coefficients of a generating function if it satisfies an equation of a certain form.

**Theorem 1.3 ([90])** *Let $G(x) \in \mathbb{C}[[x]]$ be a formal power series such that $G(0) \neq 0$, and let $f(x)$ be defined by $f(x) = xG(f(x))$. Then, for any $k, n \in \mathbb{Z}$,*

$$n[x^n]f(x)^k = k[x^{n-k}]G(x)^n,$$

*where $[z^n]A(z)$ denotes the coefficient of $z^n$ in the expansion of $A(z)$.*

### 1.3.4 Chebyshev polynomials

Chebyshev polynomials of the second kind are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. It can be checked that $U_r(t)$ is a polynomial of degree $r$ in $t$ with integer coefficients, and that the following recurrence holds:

$$\begin{cases} U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for all } r \geq 2, \\ U_0(t) = 1,\, U_1(t) = 2t. \end{cases} \tag{1.2}$$

Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory. The relation between restricted permutations and Chebyshev polynomials was discovered for the first time by Chow and West in [20], and later by Mansour and Vainshtein [66, 67], and Krattenthaler [59].

## 1.4 Patterns of length 3

For the case of patterns of length 3, it is known [58] that regardless of the pattern $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = \mathbf{C}_n$, the $n$-th Catalan number. While the equalities $|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$ and $|\mathcal{S}_n(321)| = |\mathcal{S}_n(123)|$ are straightforward by reversal and complementation operations, the equality $|\mathcal{S}_n(321)| = |\mathcal{S}_n(132)|$ is more difficult to establish. Bijective proofs of this fact are given in [59, 75, 84, 94]. However, none of these bijections preserves either of the statistics fp or exc.

Patterns $\sigma$ and $\sigma'$ are said to be in the same *Wilf-equivalence* class if $|\mathcal{S}_n(\sigma)| = |\mathcal{S}_n(\sigma')|$ for all $n$. Partial results on the classification of forbidden patterns can be found in [5, 12, 13, 86, 87, 88].

### 1.4.1 Equidistribution of fixed points

It was not until recently that the concept of pattern avoidance, which regards a permutation as a word, was studied together with a statistic arising from viewing a permutation as a bijection. In the recent paper [78], Robertson, Saracino and Zeilberger consider restricted permutations with respect to the number of fixed points, obtaining the following refinement of the fact that $|\mathcal{S}_n(321)| = |\mathcal{S}_n(132)|$.

**Theorem 1.4 ([78])** *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$*, for any* $0 \leq i \leq n$*.*

Their proof is nontrivial and technically involved. In the same paper, they study the distribution of fixed points for all six patterns of length 3.

Two questions arise naturally with this result in sight. The first one is whether there exists a simple bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$ that preserves the number of fixed points. This would give a better understanding of why fixed points are equidistributed in both sets of pattern-avoiding permutations. There does not seem to be an intuitive reason why Theorem 1.4 holds, especially since from the definitions fixed points do not seem to be related to the notion of pattern avoidance. The second question is whether this theorem can be generalized to other statistics or to other patterns. These two issues are discussed in the next chapter.

# 2

# Fixed points and excedances in permutations avoiding one pattern of length 3

Here we consider $\sigma$-avoiding permutations for every pattern $\sigma \in \mathcal{S}_3$, and we study the distribution of the statistics 'number of fixed points' and 'number of excedances' on them. The work in this chapter is motivated in large part by Theorem 1.4, and more precisely by a generalization of it, namely Theorem 2.3, which is the main result of this chapter. We will show that the joint distribution of the number of fixed points and the number of excedances is the same in $\mathcal{S}_n(321)$ as in $\mathcal{S}_n(132)$. In other words, we have that for any $0 \leq i, j \leq n$,

$$|\{\pi \in \mathcal{S}_n(321) : \ \mathrm{fp}(\pi) = i, \ \mathrm{exc}(\pi) = j\}| \\ = |\{\pi \in \mathcal{S}_n(132) : \ \mathrm{fp}(\pi) = i, \ \mathrm{exc}(\pi) = j\}|.$$

In terms of the generating functions $F_\sigma$ defined in equation (1.1), this result can be expressed equivalently as $F_{321}(x, q, z) = F_{132}(x, q, z)$. A bijective proof of this theorem is given in Section 2.2, where we also obtain an expression for this GF. In Sections 2.3 and 2.4 we consider permutations avoiding each of the remaining patterns of length 3, giving the distribution of the statistics fp and exc in all cases except for the pattern 123, for which we can only give partial results regarding fp.

One of the main tools in the this chapter and the next one will be a bijection between 132-avoiding permutations and Dyck paths that we denote $\varphi$. This bijection is presented in Section 2.1, where several of its properties are studied.

It is well known that for any $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = \mathbf{C}_n$. By Lemma 1.2, we have that $132 \approx 213$, and that $231 \sim_f 312$. These are the only equivalences that follow from the trivial bijections. Together with the just mentioned fact that $321 \approx 132$ (see Section 2.2), we have the following equivalence classes of patterns of length 3 with respect to fixed points and excedances:

**a)** $132 \approx 213 \approx 321$
**b)** 123
**c)** 231 $\sim_f$ **c')** 312

## 2.1 The bijection $\varphi$

In this section we define a bijection $\varphi$ between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$. This bijection will be used extensively throughout this work, because of its convenient properties.

Given any permutation $\pi \in \mathcal{S}_n$, consider its array $\mathrm{arr}(\pi)$ as defined in Section 1.1.3. The *diagram* of $\pi$ can be obtained from it as follows. For each cross, shade the cell containing it and the squares that are due south and due east of it. The diagram is defined as the region that is left unshaded. It is shown in [74] that this gives a bijection between $\mathcal{S}_n(132)$ and Young diagrams that fit in the shape $(n-1, n-2, \ldots, 1)$. Consider now the path determined by the border of the diagram of $\pi$, that is, the path with *north* and *east* steps that goes from the lower-left corner to the upper-right corner of the array, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. Define $\varphi(\pi)$ to be the Dyck path obtained from this path by reading an up-step for each north step and a down-step for each east step (that is, we rotate it 45°). Since the path in the array does not go below the diagonal, $\varphi(\pi)$ does not go below the $x$-axis. Figure 2.1 shows an example when $\pi = 67435281$.

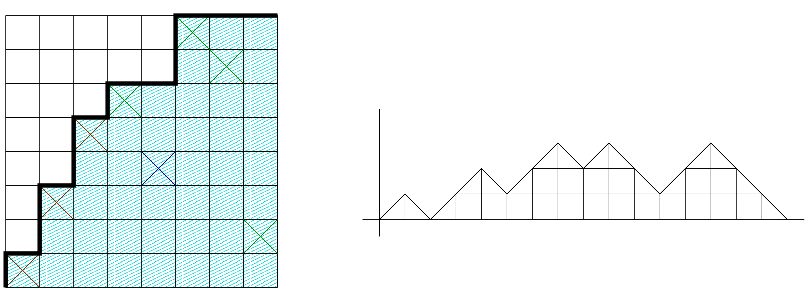

**Figure 2.1** The bijection $\varphi$.

The bijection $\varphi$ is essentially the same bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ given by Krattenthaler [59] (see also [43]), up to reflection of the path from a vertical line.

Next we define the inverse map $\varphi^{-1} : \mathcal{D}_n \longrightarrow \mathcal{S}_n(132)$. Given a Dyck path $D \in \mathcal{D}_n$, the first step needed to reverse the above procedure is to transform $D$ into a path $U$ from the lower-left corner to the upper-right corner of an $n \times n$ array, not going below the diagonal connecting these two corners. Then, the squares to the left of this path form a Young diagram contained in the shape $(n-1, n-2, \ldots, 1)$, and we can shade all the remaining squares. From this diagram, the permutation $\pi \in \mathcal{S}_n(132)$ can be recovered as follows: row by row, put a cross in the leftmost shaded square such that there is exactly one cross in each column. Start from the top and continue downward until all crosses are placed.

The bijection $\varphi$ is useful here because it transforms fixed points and excedances of the permutation into centered tunnels and right tunnels of the Dyck path respectively. These two properties, along with a few more that will be used in upcoming chapters, are shown in the next proposition. Recall the definitions from Sections 1.1.2 and 1.2.1. Denote by $\mathrm{nlis}(\pi)$ the number of increasing subsequences of $\pi$ of length $\mathrm{lis}(\pi)$.

**Proposition 2.1** *The bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ satisfies*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$,

(3) $\mathrm{des}(\pi) = \mathrm{va}(\varphi(\pi))$,

(4) $\mathrm{lis}(\pi) =$ *height of* $\varphi(\pi)$,

(5) $\mathrm{nlis}(\pi) = \#\{$*peaks of* $\varphi(\pi)$ *at maximum height*$\}$,

(6) $\mathrm{lds}(\pi) = \#\{$*peaks of* $\varphi(\pi)\}$,

(7) $\mathrm{rank}(\pi) = \frac{1}{2}\big(n - \nu(\varphi(\pi))\big)$,

*for all* $\pi \in \mathcal{S}_n(132)$.

*Proof.* For the proof of the first six equalities, instead of using $D = \varphi(\pi)$, it will be convenient to consider the associated path $U$ from the lower-left corner to the upper-right corner of $\mathrm{arr}(\pi)$ with north and east steps. We will talk about tunnels of $U$ to refer to the corresponding tunnels of $D$ under this trivial transformation.

We now show how to associate a unique tunnel of $D$ to each cross of the array $\mathrm{arr}(\pi)$. Observe that given a cross in position $(i,j)$, $U$ has a north step in row $i$ and an east step in column $j$. In $D$, these two steps correspond to steps $\mathbf{u}$ and $\mathbf{d}$ respectively, so they determine a decomposition $D = A\mathbf{u}B\mathbf{d}C$ (see Figure 2.2), and therefore a tunnel of $D$ (it is not hard to see that $\mathbf{u}$ and $\mathbf{d}$ are at the same level). According to whether the cross was to the left of, to the right of, or on the main diagonal or $\mathrm{arr}(\pi)$, the associated tunnel will be respectively a left, right, or centered tunnel of $D$. Thus, fixed points give centered tunnels and excedances give right tunnels.

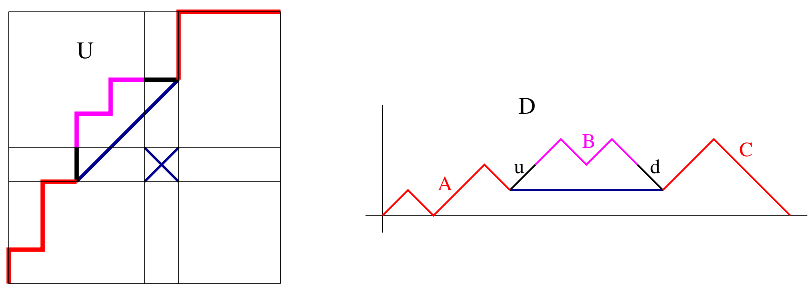


**Figure 2.2** A cross and the corresponding tunnel.

To show (3), observe that from the description of $\varphi^{-1}$, a sequence of consecutive north steps of $U$ gives rise to an increasing run of crosses in the rows of $\mathrm{arr}(\pi)$ where those steps lie. Descents of the permutation occur precisely in the rows of the array where there is a north step of $U$ that is preceded by an east step. And these are just the valleys of $\varphi(\pi)$.

Property (4) is shown in [59], but here we give a more graphical proof. Given an increasing subsequence of $\pi$, consider the crosses of $\mathrm{arr}(\pi)$ that form such subsequence. The tunnels of $\varphi(\pi)$ corresponding to these crosses are all at different heights, and their projections on the $x$-axis are nested intervals (i.e., pairwise contained in each other). Reciprocally, any tower of tunnels of $\varphi(\pi)$ whose projections on the $x$-axis are nested corresponds to an increasing subsequence of $\pi$. The maximum number of tunnels in such a tower is the height of the path, so (4) follows. Furthermore, the number of such towers having as many tunnels as possible equals the number of peaks of $\varphi(\pi)$ at maximum height (the highest tunnel of the tower determines the peak), which proves (5).

Part (6) follows from the description of $\varphi^{-1}$ and the observation that the crosses of the $\mathrm{arr}(\pi)$ located in the positions of the peaks (inner corners) of $U$ form a decreasing subsequence of $\pi$ of maximum length.

To prove the last equality of the proposition, notice that $\mathrm{rank}(\pi)$ is the largest $m$ such that an $m \times m$ square fits in the upper-left corner of the diagram of $\pi$. Therefore, the height of $\varphi(\pi)$ at the middle is exactly $\nu(\varphi(\pi)) = n - 2\,\mathrm{rank}(\pi)$. □

### 2.1.1 More properties

We have seen what fixed points and excedances in 132-avoiding permutations are mapped to by $\varphi$. To study these statistics in 312-avoiding permutations, it will be convenient to first apply the complementation operation that maps $\pi$ to $\bar{\pi}$. Table 2.1 summarizes the correspondences of $\varphi$ that we will use to study fixed points and excedances. Recall from Section 1.2.2 that the depth of a tunnel $T$ is defined as $\mathrm{depth}(T) := \frac{1}{2}\mathrm{length}(T) - \mathrm{height}(T) - 1$.

| In the permutation $\pi$ | In the array of $\pi$ | In the Dyck path $\varphi(\pi)$ |
|---|---|---|
| fixed points of $\pi$ | crosses on the main diagonal | centered tunnels |
| excedances of $\pi$ | crosses to the right of the main diagonal | right tunnels |
| fixed points of $\bar{\pi}$ | crosses on the secondary diagonal | tunnels of depth 0 |
| excedances of $\bar{\pi}$ | crosses to the left of the secondary diagonal | tunnels of negative depth |

**Table 2.1** Behavior of $\varphi$ on fixed points and excedances.

The correspondences between the first two columns are clear as we saw in Section 1.1.3. The first two rows of the table have been discussed in Proposition 2.1. Here we repeat the same reasoning from the proof of that proposition to show how $\varphi$ maps crosses on the secondary diagonal to tunnels of depth 0, and crosses to the left of the secondary diagonal to tunnels of negative depth.

Again, instead of using $D = \varphi(\pi)$, it will be convenient to consider the path $U$ from the lower-left corner to the upper-right corner of the array of $\pi$, and to talk about tunnels of $U$ to refer to the corresponding tunnels of $D$ under this trivial transformation.

Recall how in the proof of Proposition 2.1 we associated a unique tunnel $T$ of

$D$ to each cross $X$ of $\mathrm{arr}(\pi)$. Given a cross $X = (i, j)$, $U$ has a north step in row $i$ and an east step in column $j$. These two steps in $U$ correspond to steps **u** and **d** in $D$, respectively, so they determine a decomposition $D = A\mathbf{u}B\mathbf{d}C$ (see Figure 2.2), and therefore a tunnel $T$ of $D$.

The distance between these two steps determines the length of $T$, and the distance from these steps to the secondary diagonal of the array determines the height of $T$. In order for the corresponding cross to lie on the secondary diagonal, the relation between these two quantities must be $\frac{1}{2}\mathrm{length}(T) = \mathrm{height}(T)+1$, which is equivalent to $\mathrm{depth}(T) = 0$, by the definition of depth. The depth of $T$ indicates how far from the secondary diagonal $X$ is. The cross lies to the left of the secondary diagonal exactly when $\mathrm{depth}(T) < 0$. This justifies the last two rows of the table.

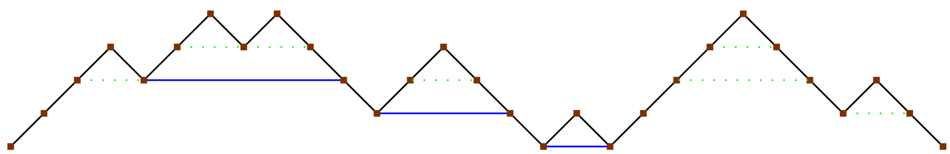

**Figure 2.3** Three tunnels of depth 0 and seven tunnels of negative depth.

We define two new statistics on Dyck paths. For $D \in \mathcal{D}$, let $\mathrm{td}_0(D)$ be the number of tunnels of depth 0 of $D$, and let $\mathrm{td}_{<0}(D)$ be the number of tunnels of negative depth of $D$. In Figure 2.3, there are three tunnels of depth 0 drawn with a solid line, and seven tunnels of negative depth drawn with dotted lines. Let us state these results as a lemma, which partially overlaps with Proposition 2.1.

**Lemma 2.2** *Let $\pi \in \mathcal{S}_n(132)$, $\rho \in \mathcal{S}_n(312)$. We have*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$,

(3) $\mathrm{fp}(\rho) = \mathrm{td}_0(\varphi(\bar{\rho}))$,

(4) $\mathrm{exc}(\rho) = \mathrm{td}_{<0}(\varphi(\bar{\rho}))$.

## 2.2 a) $132 \approx 213 \approx 321$

This is the most interesting case of permutations avoiding one pattern of length 3. By Lemma 1.2 we have that $132 \approx 213$. On the other hand, Theorem 1.4 says that $132 \sim_f 321$, which is a surprising and nontrivial result. Here we give a generalization of this fact, namely that $132 \approx 321$. This result is stated in the following theorem, for which we give a combinatorial proof.

**Theorem 2.3** *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *and* $\mathrm{exc}(\pi) = j$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *and* $\mathrm{exc}(\pi) = j$*, for any* $0 \leq i, j \leq n$*.*

We present a bijection between 321- and 132-avoiding permutations that preserves the number of fixed points and the number of excedances. Our bijection is a composition of two slightly modified known bijections into Dyck paths, and the result follows from a new analysis of these bijections. One of them is the bijection $\varphi$ from Section 2.1. The other one is based on the Robinson-Schensted-Knuth correspondence, and from it stems the difficulty of the analysis.

We also show that the length of the longest increasing subsequence in 321-avoiding permutations corresponds to a statistic in 132-avoiding permutations that we call *rank*, which further refines Theorem 2.3. This proof is joint work with Igor Pak [36].

This section is structured as follows. The description of the main bijection is done in Subsection 2.2.1, where the new part is a bijection from 321-avoiding permutations to Dyck paths. In Subsection 2.2.2 we establish properties of this bijection. Subsection 2.2.3 contains proofs of two technical lemmas. In Subsection 2.2.4 we give an expression for the generating function $F_{321}(x, q, z)$.

### 2.2.1 A bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$ preserving fixed points and excedances

Here we present a bijection that is a composition of two bijections into Dyck paths. In fact, this will prove the following generalization of Theorem 2.3:

**Theorem 2.4** *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$*,* $\mathrm{exc}(\pi) = j$ *and* $\mathrm{lis}(\pi) = k$ *equals the number of* 132*-avoiding* *permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ *and* $\mathrm{rank}(\pi) = n - k$, *for any* $0 \leq i, j, k \leq n$.

We establish a bijection $\Theta : \mathcal{S}_n(321) \longrightarrow \mathcal{S}_n(132)$ which respects the statistics as above. While $\Theta$ is not hard to define, its analysis is less straightforward and will occupy much of the section. The bijection $\Theta$ is the composition of two bijections, one from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$, and another one from $\mathcal{D}_n$ to $\mathcal{S}_n(132)$. The second one is just the inverse of the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ presented in Section 2.1. The first one is described next.

We define the bijection $\Psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ in two steps. Given $\pi \in \mathcal{S}_n(321)$, we start by applying the Robinson-Schensted-Knuth correspondence to $\pi$ [90, Section 7.11] (see also [58, 82]). This correspondence gives a bijection between the symmetric group $\mathcal{S}_n$ and pairs $(P, Q)$ of *standard Young tableaux* of the same shape $\lambda \vdash n$. For $\pi \in \mathcal{S}_n(321)$ the algorithm is particularly easy because in this case the tableaux $P$ and $Q$ have at most two rows. The *insertion tableau* $P$ is obtained by reading $\pi$ from left to right and, at each step, inserting $\pi_i$ to the partial tableau obtained so far. Assume that $\pi_1, \ldots, \pi_{i-1}$ have already been inserted. If $\pi_i$ is larger than all the elements on the first row of the current tableau, place $\pi_i$ at the end of the first row. Otherwise, let $m$ be the leftmost element on the first row that is larger than $\pi_i$. Place $\pi_i$ in the square that $m$ occupied, and place $m$ at the end of the second row (in this case we say that $\pi_i$ *bumps* $m$). The *recording tableau* $Q$ has the same shape as $P$ and is obtained by placing $i$ in the position of the square that was created at step $i$ (when $\pi_i$ was inserted) in the construction of $P$, for all $i$ from 1 to $n$. We write $\mathrm{RSK}(\pi) = (P, Q)$.

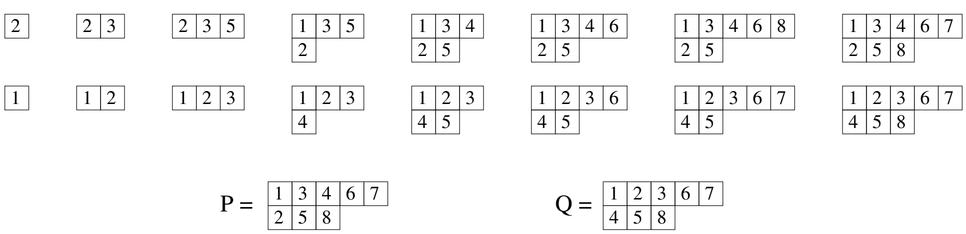

**Figure 2.4** Construction of the RSK correspondence $\mathrm{RSK}(\pi) = (P, Q)$ for $\pi = 23514687$.

Now, the first half of the Dyck path $\Psi(\pi)$ is obtained by adjoining, for $i$ from 1 to $n$, an up-step if $i$ is on the first row of $P$, and a down-step if it is on the second row. Let $A$ be the corresponding word of **u**'s and **d**'s. Similarly, let $B$ be the word obtained from $Q$ in the same way. We define $\Psi(\pi)$ to be the Dyck path obtained by the concatenation of the word $A$ and the word $B$ written backwards. For example, from the tableaux $P$ and $Q$ as in Figure 2.4 we get the Dyck path shown in Figure 1.2. The following proposition, which is proved in Section 2.2.2, summarizes some properties of this bijection $\Psi$:

**Proposition 2.5** *The bijection* $\Psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ *satisfies*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\Psi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\Psi(\pi))$,

(3) $\mathrm{lis}(\pi) = \frac{1}{2}\big(n + \nu(\Psi(\pi))\big)$,

*for all* $\pi \in \mathcal{S}_n(321)$.

Suppose $\mathrm{RSK}(\pi) = (P, Q)$ for any $\pi \in \mathcal{S}_n$. A fundamental and highly nontrivial property of the RSK correspondence is the *duality*: $\mathrm{RSK}(\pi^{-1}) = (Q, P)$ [90, Section 7.13]. The classical *Schensted's Theorem* states that $\mathrm{lis}(\pi)$ is equal to the length of the first row of the tableau $P$ (and $Q$). Both results are used in the proof of Proposition 2.5.

Now Theorem 2.4 follows easily from this proposition together with Proposition 2.1.

*Proof of Theorem 2.4.* Propositions 2.5 and 2.1 imply that $\Theta = \varphi^{-1} \circ \Psi$ is a bijection from $\mathcal{S}_n(321)$ to $\mathcal{S}_n(132)$ which satisfies

$$\mathrm{fp}(\Theta(\pi)) = \mathrm{ct}(\Psi(\pi)) = \mathrm{fp}(\pi),$$

$$\mathrm{exc}(\Theta(\pi)) = \mathrm{rt}(\Psi(\pi)) = \mathrm{exc}(\pi),$$

$$\mathrm{rank}(\Theta(\pi)) = \frac{1}{2}\big(n - \nu(\Psi(\pi))\big) = n - \frac{1}{2}\big(n + \nu(\Psi(\pi))\big) = n - \mathrm{lis}(\pi)\,.$$

This implies the result. □

### 2.2.2 Properties of $\Psi$

In this subsection we prove Proposition 2.5, which describes the properties of $\Psi$ that we need.

Let us first consider only fixed points in a permutation $\pi \in \mathcal{S}_n$. Let $\pi \in \mathcal{S}_n(321)$ and assume that $\pi_i = i$. Then $\pi_1\pi_2\cdots\pi_{i-1}$ is a permutation of

$\{1, 2, \ldots, i-1\}$, and $\pi_{i+1}\pi_{i+2}\cdots\pi_n$ is a permutation of $\{i+1, i+2, \ldots, n\}$. Indeed, if $\pi_j > i$ for some $j < i$, then necessarily $\pi_k < i$ for some $k > i$, and $\pi_j\pi_i\pi_k$ would be an occurrence of 321.

Therefore, when we apply RSK to $\pi$, the elements $\pi_i, \pi_{i+1}, \ldots, \pi_n$ will never bump any of the elements $\pi_1, \pi_2, \ldots, \pi_{i-1}$. In particular, the subtableaux of $P$ and $Q$ determined by the entries that are smaller than $i$ will have the same shape. Furthermore, when the elements greater than $i$ are placed in $P$ and $Q$, the rows in which they are placed do not depend on the subpermutation $\pi_1\pi_2\cdots\pi_{i-1}$. Note also that $\pi_i = i$ will never be bumped, and it will occupy the same position in the first row of $P$ and $Q$.

When the Dyck path $\Psi(\pi)$ is built from $P$ and $Q$, this translates into the fact that the steps corresponding to $\pi_i$ in $P$ and to $i$ in $Q$ will be respectively an up-step in the first half and a down-step in the second half, both at the same height and at the same distance from the center of the path. Besides, the part of the path between them will be itself the Dyck path corresponding to $(\pi_{i+1}-i)(\pi_{i+2}-i)\cdots(\pi_n-i)$. So, the fixed point $\pi_i = i$ determines a centered tunnel in $\Psi(\pi)$. It is clear that the converse is also true, that is, every centered tunnel comes from a fixed point. This shows that $\mathrm{fp}(\pi) = \mathrm{ct}(\Psi(\pi))$, proving the first part of Proposition 2.5.

Let us now consider excedances in a permutation $\pi \in \mathcal{S}_n(321)$. Our goal is to show that the excedances of $\pi$ correspond to right tunnels of $\Psi(\pi)$. The first observation is that we can assume without loss of generality that $\pi$ has no fixed points. Indeed, if $\pi_i = i$ is a fixed point of $\pi$, then the above reasoning shows that we can decompose $\Psi(\pi) = A\mathbf{u}B\mathbf{d}C$, where $AC$ is the Dyck path $\Psi(\pi_1\pi_2\cdots\pi_{i-1})$ and $B$ is a translation of the Dyck path $\Psi((\pi_{i+1}-i)\cdots(\pi_n-i))$. But we have that $\mathrm{exc}(\pi) = \mathrm{exc}(\pi_1\pi_2\cdots\pi_{i-1}) + \mathrm{exc}((\pi_{i+1}-i)\cdots(\pi_n-i))$ and $\mathrm{rt}(A\mathbf{u}B\mathbf{d}C) = \mathrm{rt}(AC) + \mathrm{rt}(B)$, so in this case the result holds by induction on the number of fixed points. Note also that the above argument showed that $\mathrm{fp}(\pi) = \mathrm{fp}(\pi_1\pi_2\cdots\pi_{i-1}) + \mathrm{fp}((\pi_{i+1}-i)\cdots(\pi_n-i)) + 1$ and $\mathrm{ct}(A\mathbf{u}B\mathbf{d}C) = \mathrm{ct}(AC) + \mathrm{ct}(B) + 1$.

Suppose that $\pi \in \mathcal{S}_n(321)$ has no fixed points. We will use the fact that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements (in this case, the deficiencies) are increasing (see e.g. [74]). Denote by $X_i := (i, \pi_i)$ the crosses of the array representation of $\pi$. To simplify the presentation, we will refer indistinctively to $i$ or $X_i$, hoping this does not lead to confusion. For example, we will say "$X_i$ is an excedance", etc.

Define a matching between the excedances and the deficiencies of $\pi$ by the following algorithm. Let $i_1 < i_2 < \cdots < i_k$ be the positions of the excedances of $\pi$ and let $j_1 < j_2 < \cdots < j_{n-k}$ be the deficiencies. Note that from the previous paragraph we know that $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k}$ and $\pi_{j_1} < \pi_{j_2} < \cdots < \pi_{j_{n-k}}$.

**Matching Algorithm**

(1) Initialize $a := 1$, $b := 1$.

(2) Repeat until $a > k$ or $b > n - k$:
   (a) If $i_a > j_b$, then $b := b + 1$. ($X_{j_b}$ is not matched.)
   (b) Else if $\pi_{i_a} < \pi_{j_b}$, then $a := a + 1$. ($X_{i_a}$ is not matched.)
   (c) Else, match $X_{i_a}$ with $X_{j_b}$; $a := a + 1$, $b := b + 1$.

(3) Output the matching sequence.

**Example.** Let $\pi = (4, 1, 2, 5, 7, 8, 3, 6, 11, 9, 10)$ as in Figure 2.5 below. We have $i_1 = 1$, $i_2 = 4$, $i_3 = 5$, $i_4 = 6$, $i_5 = 9$, and $j_1 = 2$, $j_2 = 3$, $j_3 = 7$, $j_4 = 8$, $j_5 = 10$, $j_6 = 11$. In the first execution of the loop in step (2) of the algorithm, neither $i_1 > j_1$ nor $\pi_{i_1} < \pi_{j_1}$ hold, so $X_{i_1} = (1, 4)$ and $X_{j_1} = (2, 1)$ are matched. Now we repeat the loop with $a = b = 2$, and since $i_2 > j_2$, we are in the case given by (2a) ($X_{j_2} = (3, 2)$ is not matched). In the next iteration, $a = 2$ and $b = 3$, so we match $X_{i_2} = (4, 5)$ and $X_{j_3} = (7, 3)$. Now we have $a = 3$ and $b = 4$, so we match $X_{i_3} = (5, 7)$ and $X_{j_4} = (8, 6)$. The values of $a$ and $b$ in the next iteration are 4 and 5 respectively, so we are in the case of (2b), $\pi_{i_4} = 8 < 9 = \pi_{j_5}$, and $X_{i_4} = (6, 8)$ is unmatched. Now $a = b = 5$, and we match $X_{i_5} = (9, 11)$ and $X_{j_5} = (10, 9)$. The matching algorithm ends here because now $a = 6 > 5 = k$.

An informal, more geometrical description of the matching algorithm is the following. For each pair of crosses of the array (seen as embedded in the plane), consider the line that their centers determine. If one of these lines has positive slope and leaves all the remaining crosses to the right, match the two crosses that determine it, and delete them from the array. If there is no line with these properties, delete the cross that is closer to the upper-left corner of the array (it is unmatched). Repeat the process until no crosses are left.

Now we consider the matched excedances on one hand and the unmatched ones on the other. We summarize rather technical results in the following

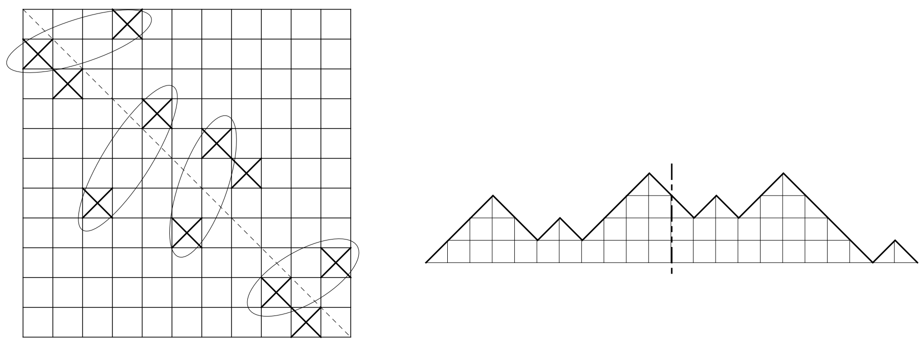

**Figure 2.5** Matching for $\pi = (4, 1, 2, 5, 7, 8, 3, 6, 11, 9, 10)$, and $\Psi(\pi)$.

two lemmas, which are proved in Section 2.2.3. Recall the definitions of right-side and left-side tunnels from Section 1.2.2.

**Lemma 2.6** *The following quantities are equal:*

(1) *the number of matched pairs* $(X_i, X_j)$, *where* $X_i$ *is an excedance and* $X_j$ *a deficiency;*

(2) *the length of the second row of* $P$ *(or* $Q$*);*

(3) *the number of right-side tunnels of* $\Psi(\pi)$*;*

(4) *the number of left-side tunnels of* $\Psi(\pi)$*;*

(5) $\frac{1}{2}\big(n - \nu(\Psi(\pi))\big)$*;*

(6) $n - \mathrm{lis}(\pi)$.

Note that (5)=(6) implies that $\mathrm{lis}(\pi) = \frac{1}{2}\big(n + \nu(\Psi(\pi))\big)$, which is the third part of Proposition 2.5.

**Lemma 2.7** *The number of unmatched excedances (resp. deficiencies) of* $\pi$ *equals the number of right-across (resp. left-across) tunnels of* $\Psi(\pi)$.

Since each excedance of $\pi$ either is part of a matched pair $(X_i, X_j)$ or is unmatched, Lemmas 2.6 and 2.7 imply that the total number $\mathrm{exc}(\pi)$ of excedances equals the number of right-side tunnels of $\Psi(\pi)$ plus the number of right-across tunnels, which is $\mathrm{rt}(\Psi(\pi))$. This implies the second part of Proposition 2.5.

To summarize, we will have shown after proving these lemmas that the bijection $\Psi$ satisfies all three properties described in Proposition 2.5, which completes the proof. □

### 2.2.3 Properties of the matching algorithm

In this subsection we prove the two lemmas above. We also give a more direct description of the bijection $\Psi$ using the matching algorithm, without referring explicitly to RSK.

*Proof of Lemma 2.6.* From the descriptions of the RSK algorithm and the matching, it follows that an excedance $X_i$ and a deficiency $X_j$ are matched with each other precisely when $\pi_j$ bumps $\pi_i$ when RSK is performed on $\pi$, and that these are the only bumpings that take place. Indeed, an excedance never bumps anything because it is larger than the elements inserted before. On the other hand, when a deficiency $X_j$ is inserted, it bumps the smallest element larger than $\pi_j$ which has not been bumped yet (which corresponds to an excedance that has not been matched yet), if such an element exists. This proves the equality (1)=(2).

To see that (2)=(3), observe that right-side tunnels correspond to up-steps in the right half of $\Psi(\pi)$, which by the construction of the bijection $\Psi$ correspond to elements on the second row of $Q$. The equality (3)=(5) follows easily by counting the number of up-steps and down-steps of the right half of the path. The equality (4)=(5) is analogous.

Finally, Schensted's Theorem states that the size of the first row of $P$ equals the length of a longest increasing subsequence of $\pi$ (see [81] or [90, Section 7.23]). This implies that (2)=(6), which completes the proof. □

The reasoning used in the above proof gives a nice equivalent description of the recording tableau $Q$ in terms of $\mathrm{arr}(\pi)$ and the matching. Read the rows of the array from top to bottom. For $i$ from 1 to $n$, place $i$ on the first row of $Q$ if $X_i$ is an excedance or it is unmatched, and place $i$ on the second row if $X_i$ is a matched deficiency. In the construction of the right half of $\Psi(\pi)$, this translates into drawing the path from right to left while reading the array from top to bottom, adjoining an up-step for each matched deficiency and a down-step for each other kind of cross.

To get a similar description of the tableau $P$, we use duality. By construction of the matching algorithm, the matching in the output is invariant under transposition of the array (reflection along the main diagonal). Recall the duality of the RSK correspondence: if $\mathrm{RSK}(\pi) = (P, Q)$, then $\mathrm{RSK}(\pi^{-1}) = (Q, P)$ (see e.g. [90, Section 7.13]). Therefore, the tableau $P$ can be obtained by reading the columns of the array of $\pi$ from left to right and placing integers in $P$ according to the following rule. For each column $j$, place $j$ on the first row of $P$ if the cross in column $j$ is a deficiency or it is unmatched. Similarly, place $j$ on the second row if the cross is a matched excedance. Equivalently, the left half of $\Psi(\pi)$, from left to right, is obtained by reading the array from left to right and adjoining a down-step for each matched excedance, and an up-step for each of the remaining crosses.

In particular, when the left half of the path is constructed in this way, every matched pair $(X_i, X_j)$ produces an up-step and a down-step, giving the latter a left-side tunnel. Similarly, in the construction of the right half of the path, a matched pair gives a right-side tunnel. Observe that these are again the equalities (1)=(3)=(4) in Lemma 2.6.

*Proof of Lemma 2.7.* It is enough to prove it only for the case of excedances. The case of deficiencies follows from it by considering $\pi^{-1}$ and noticing that $\Psi(\pi^{-1}) = \Psi(\pi)^*$. Indeed, by duality $\mathrm{RSK}(\pi^{-1}) = (Q, P)$, so $Q$ gives rise to the first half of $\Psi(\pi^{-1})$ and $P$ to the second, so the path that we obtain is the reflection of $\Psi(\pi)$ in a vertical axis through the middle of the path. Let $X_k$ be an unmatched excedance of $\pi$. We use the above description of $\Psi(\pi)$ in terms of the array and the matching. Each cross $X_i$ produces a step $r_i$ in the right half of the Dyck path and another step $\ell_i$ in the left half. Crosses above $X_k$ produce steps to the right of $r_k$, and crosses to the left of $X_k$ produce steps to the left of $\ell_k$. In particular, there are $k-1$ steps to the right of $r_k$, and $\pi_k - 1$ steps to the left of $\ell_k$. Note that since $X_k$ is an excedance and $\pi$ is 321-avoiding, all the crosses above it are also to the left of it. Consider the crosses that lie to the left of $X_k$. They can be of the following four kinds:

- *Unmatched excedances* $X_i$. They will necessarily lie above $X_k$, because the subsequence of excedances of $\pi$ is decreasing. Each one of these crosses contributes an up-step to the left of $\ell_k$ and down-step to the right of $r_k$.

- *Unmatched deficiencies* $X_j$. They also have to lie above $X_k$, otherwise

$X_k$ would be matched with one of them. So, each such $X_j$ contributes an up-step to the left of $\ell_k$ and down-step to the right of $r_k$.

- *Matched pairs* $(X_i, X_j)$ (i.e. $X_i$ is an excedance and $X_j$ a deficiency), *where both* $X_i$ *and* $X_j$ *lie above* $X_k$. Both crosses together will contribute an up-step and a down-step to the left of $\ell_k$, and an up-step and a down-step to the right of $r_k$.

- *Matched pairs* $(X_i, X_j)$ (i.e. $X_i$ is an excedance and $X_j$ a deficiency), *where* $X_j$ *lies below* $X_k$. The pair will contribute an up-step and a down-step to the left of $\ell_k$. However, to the right of $r_k$, the only contribution will be a down-step produced by $X_i$.

Note that there cannot be a deficiency $X_j$ to the left of $X_k$ matched with an excedance to the right of $X_k$, because in this case $X_j$ would have been matched with $X_k$ by the algorithm. In the first three cases, the contribution to both sides of the Dyck path is the same, so that the heights of $r_k$ and $\ell_k$ are equally affected. But since $\pi_k > k$, at least one of the crosses to the left of $X_k$ must be below it, and this must be a matched deficiency as in the fourth case. This implies that the step $r_k$ is at a higher $y$-coordinate than $\ell_k$. Let $h_k$ be the height of $\ell_k$. We now show that $\Psi(\pi)$ has a right-across tunnel at height $h_k$.

Observe that $h_k$ is the number of unmatched crosses to the left of $X_k$, and that the height of $r_k$ is the number of unmatched crosses above $X_k$ (which equals $h_k$) plus the number of excedances above $X_k$ matched with deficiencies below $X_k$. The part of the path between $\ell_k$ and the middle always remains at a height greater than $h_k$. This is because the only possible down-steps in this part can come from matched excedances $X_i$ to the right of $X_k$, but each such $X_i$ is matched with a deficiency $X_j$ to the right of $X_k$ but to the left of $X_i$, which produces an up-step compensating the down-step associated to $X_i$. Similarly, the part of the path between $r_k$ and the middle remains at a height greater than $h_k$. This is because the $h_k$ down-steps to the right of $r_k$ that come from unmatched crosses above $X_k$ do not have a corresponding up-step in the part of the path between $r_k$ and the middle. Hence, $\ell_k$ is the left end of a right-across tunnel, since the right end of this tunnel is to the right of $r_k$, which in turn is closer to the right end of $\Psi(\pi)$ than $\ell_k$ is to its left end (see Figure 2.6).

It can easily be checked that the converse is also true, namely that in every right-across tunnel of $\Psi(\pi)$, the step at its left end corresponds to an unmatched excedance of $\pi$. □

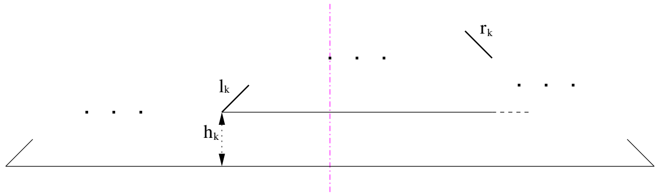


**Figure 2.6** An unmateched excedance produces a right-across tunnel.

As a side note, let us point out that the limiting distribution of the length of the longest increasing subsequence in $\mathcal{S}_n(321)$ has been studied in [26]. From Theorem 2.4, the results in [26] can be translated into results on the limiting distribution of the statistic rank in $\mathcal{S}_n(132)$.

### 2.2.4 The bijection $\psi_{\llcorner}$

We already have proved that $F_{132}(x,q,z) = F_{321}(x,q,z)$. Here we give an expression for the generating function

$$F_{321}(x,q,z) = \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n,$$

which enumerates 321-avoiding permutations with respect to fixed points and excedances. In order to achieve this, we define a bijection $\psi_{\llcorner}$ between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$, which appeared originally in [48, pg. 89] in a slightly different form, and later was used by Richard Stanley in connection to pattern avoidance.

We will give three equivalent definitions of the bijection $\psi_{\llcorner}$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(321)$. For $i \in [n]$, define $a_i = \max\{j : \{1,2,\dots,j\} \subseteq \{\pi_1,\pi_2,\dots,\pi_i\}\}$ ($j$ can be 0, in which case $\{1,2,\dots,j\} = \emptyset$). Now build the Dyck path $\psi_{\llcorner}(\pi)$ by adjoining, for each $i$ from 1 to $n$, one up-step followed by $\max\{a_i - \pi_i + 1, 0\}$ down-steps. For example, for $\pi = 23147586$ we get $a_1 = a_2 = 0$, $a_3 = 3$, $a_4 = a_5 = 4$, $a_6 = a_7 = 5$, $a_8 = 8$, and the corresponding Dyck path is given in Figure 2.7.

Here is an alternative way to define this bijection. Let $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$ be the right-to-left minima of $\pi$, from left to right. For example, the right-to-left minima of 23147586 are $1,4,5,6$. Then, $\psi_{\llcorner}(\pi)$ is precisely the path that starts with $i_1$ up-steps, then has, for each $j$ from 2 to $k$, $\pi_{i_j} - \pi_{i_{j-1}}$

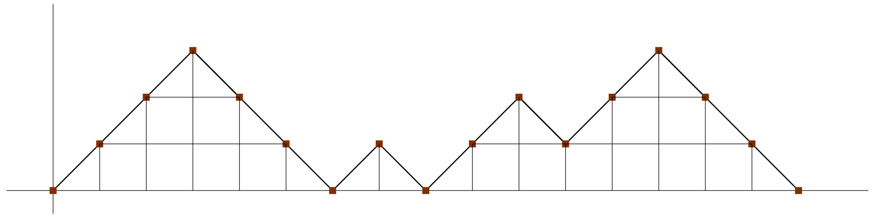

**Figure 2.7** The Dyck path $\psi_{\llcorner}(23147586)$.

down-steps followed by $i_j - i_{j-1}$ up-steps, and finally ends with $n+1-\pi_{i_k}$ down-steps.

The third way to define $\psi_{\llcorner}$ is the easiest one to visualize, and the one that gives us a better intuition for how the bijection works. Consider the array of crosses $\mathrm{arr}(\pi)$ as defined in Section 1.1.3. By definition, excedances correspond to crosses strictly to the right of the main diagonal of the array. It is known (see e.g. [74]) that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Therefore, the elements that are not excedances are precisely the right-to-left minima of $\pi$. Consider the path with *east* and *south* steps along the edges of the squares of $\mathrm{arr}(\pi)$ that goes from the upper-left corner to the lower-right corner of the array, leaving all the crosses to the right and remaining always as close to the main diagonal as possible. Let $U$ be such path. Then $\psi_{\llcorner}(\pi)$ can be obtained from $U$ just by reading an up-step for every south step of $U$, and a down-step for every east step of $U$. Figure 2.8 shows a picture of this bijection, again for $\pi = 23147586$.

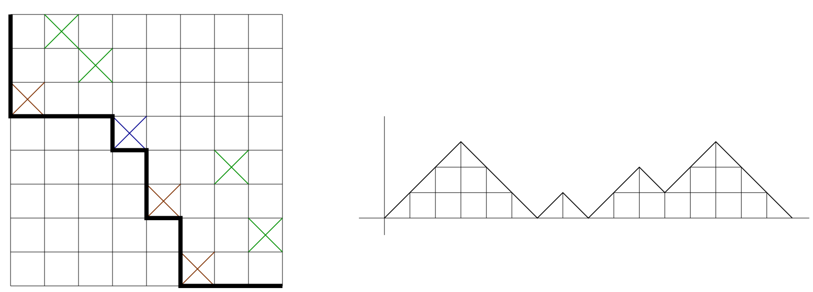

**Figure 2.8** The bijection $\psi_{\llcorner}$.

**Proposition 2.8** *The bijection* $\psi_{\llcorner} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ *satisfies*

(1) $\mathrm{fp}(\pi) = h(\psi_{\llcorner}(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{dr}(\psi_{\llcorner}(\pi))$,

*for all* $\pi \in \mathcal{S}_n(321)$.

*Proof.* To see this, just observe that fixed points of $\pi$ correspond to crosses on the main diagonal of the array, which produce hills in the path. On the other hand, for each cross corresponding to an excedance, the south step of $U$ on the same row as the cross gives an up-step in $\psi_{\llcorner}(\pi)$ which is followed by another up-step, thus forming a double rise. □

Therefore, counting 321-avoiding permutations according to the number of fixed points and excedances is equivalent to counting Dyck paths according to the number of hills and double rises. More precisely,

$$F_{321}(x,q,z) = \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathrm{dr}(D)} z^{|D|}.$$

We can give an equation for $F_{321}$ using the symbolic method summarized in Section 1.3. A recursive definition for the class $\mathcal{D}$ is given by the fact that every non-empty Dyck path $D$ can be decomposed in a unique way as $D = \mathbf{u}A\mathbf{d}B$, where $A, B \in \mathcal{D}$. Clearly if $A$ is empty, $h(D) = h(B) + 1$ and $\mathrm{dr}(D) = \mathrm{dr}(B)$, and otherwise $h(D) = h(B)$ and $\mathrm{dr}(D) = \mathrm{dr}(A) + \mathrm{dr}(B) + 1$. Hence, we obtain the following equation for $F_{321}$:

$$F_{321}(x,q,z) = 1 + z(x + q(F_{321}(1,q,z) - 1))F_{321}(x,q,z). \tag{2.1}$$

Substituting first $x = 1$, we obtain that

$$F_{321}(1,q,z) = \frac{1 + (q-1)z - \sqrt{1 - 2(1+q)z + (1-q)^2 z^2}}{2qt}.$$

Now, solving (2.1) for $F_{321}(x,q,z)$ gives the following result.

**Theorem 2.9**

$$F_{132}(x,q,z) = F_{213}(x,q,z) = F_{321}(x,q,z) = \\ = \frac{2}{1 + (1+q-2x)z + \sqrt{1 - 2(1+q)z + (1-q)^2 z^2}}.$$

To conclude this section, we want to remark that applying this bijection one can also obtain the GF that enumerates 321-avoiding permutations with respect to fixed points, excedances and descents. It follows easily from the description of $\psi_\llcorner$ that the number des($\pi$) of descents of a 321-avoiding permutation $\pi$ equals the number of occurrences of **uud** in the Dyck word of $\psi_\llcorner(\pi)$. Using the same decomposition as before, we obtain the following result.

**Theorem 2.10**

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)}z^n = \frac{2}{1+(1+q-2x)z+\sqrt{1-2(1+q)z+((1+q)^2-4qp)z^2}}.$$

The analogous generalization of Theorem 2.9 which enumerates 132-avoiding permutations with respect to these three statistics is given in Theorem 4.10.

## 2.3 b) 123

For this case we have not been able to find a satisfactory expression for $F_{123}(x,q,z)$. We can nevertheless give summation formulas for the number of permutations in $\mathcal{S}_n(123)$ with a given number of fixed points. The first trivial observation is that if $\pi$ avoids 123, then it can have at most two fixed points. If $\pi_i = i$, we say that $i$ is a *big fixed point* of $\pi$ if $i \geq \frac{n+1}{2}$, and that it is a *small fixed point* if $i < \frac{n+1}{2}$.

We already mentioned that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Using the fact that $\pi$ avoids 123 if and only if $\bar{\pi}$ avoids 321, we obtain a characterization of 123-avoiding permutations as those with the following property: the elements $\pi_i$ such that $\pi_i < n+1-i$ form a decreasing subsequence, and so do the remaining elements. In particular, since no two fixed points can be in the same decreasing subsequence, this implies that $\pi$ can have at most one big fixed point and one small fixed point.

Recall the bijection $\psi_\llcorner : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ that we defined in Section 2.2.4. Composing it with the complementation operation sending $\pi \in \mathcal{S}_n(123)$ to $\bar{\pi} \in \mathcal{S}_n(321)$, we obtain a bijection between $\mathcal{S}_n(123)$ and $\mathcal{D}_n$, which we denote by $\psi_\lrcorner$. Figure 2.9 shows an example when $\pi = (9,6,10,4,8,7,3,5,2,1)$.

Note that the peaks of the path are determined by the crosses of elements $\pi_i$ such that $\pi_i \geq n+1-i$, which form a decreasing subsequence. Now it

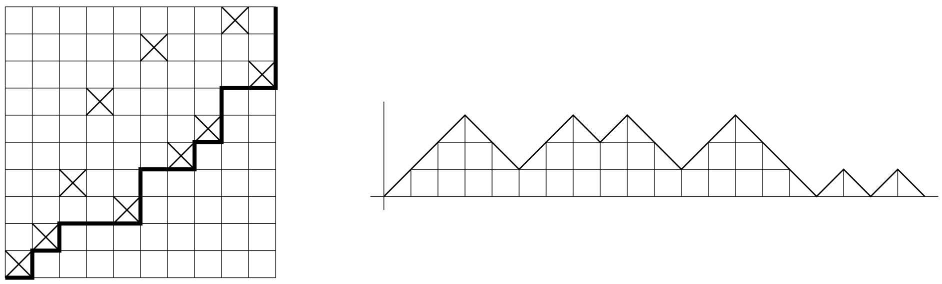

**Figure 2.9** The bijection $\psi_\lrcorner$.

is easy to determine how many permutations have a big (resp. small) fixed point.

**Proposition 2.11** *Let* $n \geq 1$. *We have*

(1) $|\{\pi \in \mathcal{S}_n(123) : \pi \textit{ has a big fixed point}\}| = \mathbf{C}_{n-1}$,

(2) $|\{\pi \in \mathcal{S}_n(123) : \pi \textit{ has a small fixed point}\}| = \begin{cases} \mathbf{C}_{n-1} & \textit{if } n \textit{ is even,} \\ \mathbf{C}_{n-1} - \mathbf{C}^2_{\frac{n-1}{2}} & \\ & \textit{if } n \textit{ is odd.} \end{cases}$

*Proof.* (1) It is clear from the definition of $\psi_\lrcorner$ that $\pi$ has a big fixed point if and only if $\psi_\lrcorner(\pi)$ has a peak in the middle. Now, we can easily define a bijection from the subset of elements of $\mathcal{D}_n$ with a peak in the middle and $\mathcal{D}_{n-1}$, by removing the two middle steps **ud**.

(2) Clearly, $\pi \in \mathcal{S}_n(123)$ if and only if $\widehat{\pi} \in \mathcal{S}_n(123)$ (recall the definition from Section 1.1.3). This involution switches big and small fixed points, except for the possible big fixed point in position $\frac{n+1}{2}$, which remains unchanged. Applying now $\psi_\lrcorner$, a small fixed point of $\pi$ is transformed into a peak in the middle of $\psi_\lrcorner(\widehat{\pi})$ of height at least two (indeed, a hill would correspond to the big fixed point $\frac{n+1}{2}$). Knowing that the number of paths in $\mathcal{D}_n$ with a peak in the middle is $\mathbf{C}_{n-1}$, we just have to subtract those where this middle peak has height 1. If $n$ is even, paths in $\mathcal{D}_n$ cannot have a hill in the middle. If $n$ is odd, such paths have the form $A\mathbf{ud}B$, where $A, B \in \mathcal{D}_{\frac{n-1}{2}}$, so the formula follows. □

For $k \geq 0$, let $s_n^k(123) := |\{\pi \in \mathcal{S}_n(123) : \mathrm{fp}(\pi) = k\}|$. We have mentioned that $s_n^k(123) = 0$ for $k \geq 3$. The following corollary reduces the problem of studying the distribution of fixed points in $\mathcal{S}_n(123)$ to that of determining $s_n^2(123)$.

**Corollary 2.12** *Let $n \geq 1$. We have*

$$(1)\ \ s_n^1(123) = \begin{cases} 2(\mathbf{C}_{n-1} - s_n^2(123)) & \text{if } n \text{ even,} \\ 2(\mathbf{C}_{n-1} - s_n^2(123)) - \mathbf{C}^2_{\frac{n-1}{2}} & \text{if } n \text{ odd.} \end{cases}$$

$$(2)\ \ s_n^0(123) = \begin{cases} \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) & \text{if } n \text{ even,} \\ \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) + \mathbf{C}^2_{\frac{n-1}{2}} & \text{if } n \text{ odd.} \end{cases}$$

*Proof.* (1) By inclusion-exclusion,

$$\begin{aligned} s_n^1(123) = &|\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a big fixed point}\}| \\ &+|\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a small fixed point}\}| - 2s_n^2(123). \end{aligned}$$

Now we apply Proposition 2.11.

(2) Clearly, $s_n^0(123) = \mathbf{C}_n - s_n^1(123) - s_n^2(123)$. □

The next theorem, together with the previous corollary, gives a formula for the distribution of fixed points in 123-avoiding permutations.

**Theorem 2.13**

$$\begin{aligned} s_n^2(123) = \sum_{i=1}^{n-1} \sum_{r,s=1}^{i} \Bigg[ &\left( \binom{2i-r-1}{i-1} - \binom{2i-r-1}{i} \right) \\ &\cdot \left( \binom{2i-s-1}{i-1} - \binom{2i-s-1}{i} \right) \\ &\cdot \sum_{\substack{h=1 \\ n-h \text{ even}}}^{n} \sum_{k=0}^{n-2i} f(k,r,h,n-2i+r) f(n-2i-k,s,h,n-2i+s) \Bigg], \end{aligned}$$

*where*

$$f(k,r,h,\ell) = \begin{cases} \binom{\frac{\ell+h-r}{2}-1}{k}\binom{\frac{\ell-h+r}{2}-1}{k-1} - \binom{\frac{\ell-h-r}{2}-1}{k}\binom{\frac{\ell+h+r}{2}-1}{k-1} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0 \text{ and } \ell = h - r, \\ 0 & \text{otherwise,} \end{cases}$$

*with the convention $\binom{a}{b} := 0$ if $a < 0$.*

*Proof.* Recall that $s_n^2(123)$ counts permutations with both a big and a small fixed point. We have seen already that $\psi_\lrcorner$ maps a big fixed point of the permutation into a peak in the middle of the Dyck path. Now we look at how a small fixed point of the permutation is transformed by $\psi_\lrcorner$. We claim that $\pi \in \mathcal{S}_n(123)$ has a small fixed point if and only if $D = \psi_\lrcorner(\pi)$ satisfies the following condition (which we call condition C1): there exists $i$ such that the $i$-th and $(i+1)$-st up-steps of $D$ are consecutive, the $i$-th and $(i+1)$-st down-steps from the end are consecutive, and there are exactly $n+1-2i$ peaks of $D$ between them. To see this, assume that $i$ is a small fixed point of $\pi$ (see Figure 2.10). Then, the path from the upper-right to the lower-left corner of the array of $\pi$, used to define $\psi_\lrcorner(\pi)$, has two consecutive vertical steps in rows $i$ and $i+1$, and two consecutive horizontal steps in columns $i$ and $i+1$. Besides, there are $n+1-2i$ crosses below and to the right of cross $(i,i)$, each one of which produces a peak in the Dyck path $\psi_\lrcorner(\pi)$. Reciprocally, it can be checked that if $\psi_\lrcorner(\pi)$ satisfies condition C1 then $\pi$ has a small fixed point.

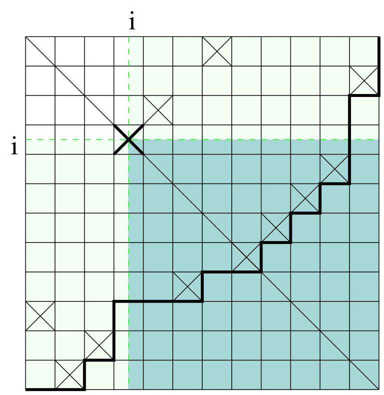


**Figure 2.10** A small fixed point $i$ has $n+1-2i$ crosses below and to the right.

All we have to do is count how many paths $D \in \mathcal{D}_n$ with a peak in the middle satisfy condition C1. For such a Dyck path $D$, define the following parameters: let $i$ be the value such that condition C1 holds, let $h = \nu(D)$ be the height of $D$ in the middle, $r$ the height at which the $i$-th up-step ends, and $s$ the height at which the $i$-th down-step from the end begins. In the example of Figure 2.11, $n = 12$, $i = 4$, $h = 4$, $r = 3$, and $s = 1$.

Fix $n$, $i$, $h$, $r$ and $s$. We will count the number of Dyck paths $D$ with

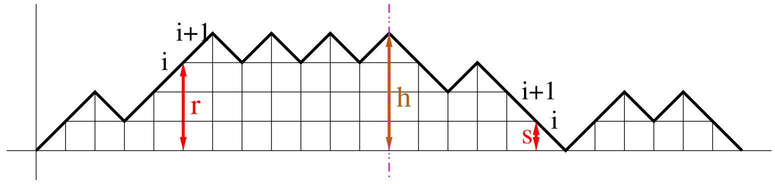


**Figure 2.11** The parameters $i$, $h$, $r$ and $s$ in a Dyck path.

these given parameters. We can write $D = A\mathbf{uu}B_1B_2\mathbf{dd}C$, where the distinguished **u**'s are the $i$-th and $(i+1)$-st up-steps, the two **d**'s are the $i$-th and $(i+1)$-st down-steps from the end, and the middle of $D$ is between $B_1$ and $B_2$. Then $A$ is a path from $(0,0)$ to $(2i-r-1, r-1)$ not going below $y=0$. It is easy to see that there are $\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i}$ such paths $A$. By symmetry, there are $\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i}$ possibilities for $C$.

Now we count the possibilities for $B_1$ and $B_2$. It can be checked that $f(k,r,h,\ell)$ counts the number of paths from $(0,r)$ to $(\ell,h)$ having exactly $k$ peaks, starting and ending with an up-step, and never going below $y=0$. The fragment $\mathbf{u}B_1$ is a path from $(2i-r,r)$ to $(n,h)$ not going below $y=0$, and ending with an up-step (since $D$ has a peak in the middle). If we fix $k$ as the number of peaks of this fragment, then there are $f(k,r,h,n-2i+r)$ such paths $\mathbf{u}B_1$. Similarly, there are $f(n-2i-k,s,h,n-2i-s)$ possibilities for $B_2\mathbf{d}$ with $n-2i-k$ peaks.

Summing over all possible values of $k$, $h$, $r$, $s$ and $i$ we obtain the expression in the theorem. □

Using Corollary 2.12, we can prove that among the derangements of length $n$, the number of 123-avoiding ones is at least the number of 132-avoiding ones. This inequality was conjectured by Miklós Bóna and Olivier Guibert.

**Theorem 2.14 ([16])** *For all $n \geq 4$, $s_n^0(132) < s_n^0(123)$.*

*Proof.* For $n \leq 12$ the result can be checked by exhaustive enumeration of all derangements by computer. Let us assume that $n \geq 13$.

From part (2) of Corollary 2.12, we have that

$$s_n^0(123) \geq \mathbf{C}_n - 2\mathbf{C}_{n-1}.$$

It is known [78] that $s_n^0(132) = \mathbf{F}_n$, the $n$-th *Fine number*. Therefore, the theorem will be proved if we show that

$$\mathbf{F}_n < \mathbf{C}_n - 2\mathbf{C}_{n-1} \qquad (2.2)$$

for $n \geq 13$. Using the identity $\mathbf{F}_n = \frac{1}{2}\sum_{i=0}^{n-2}(\frac{-1}{2})^i \mathbf{C}_{n-i}$, we get the inequality $\mathbf{F}_n < \frac{1}{2}\mathbf{C}_n - \frac{1}{4}\mathbf{C}_{n-1} + \frac{1}{8}\mathbf{C}_{n-2}$, which reduces (2.2) to showing that $\mathbf{C}_n > \frac{7}{2}\mathbf{C}_{n-1} + \frac{1}{4}\mathbf{C}_{n-2}$. This inequality certainly holds asymptotically, because $\mathbf{C}_n$ grows like $\frac{1}{\sqrt{\pi}} n^{-\frac{3}{2}} 4^n$ as $n$ tends to infinity, and it is not hard to see that in fact it holds for all $n \geq 13$. □

## 2.4 c, c') $231 \sim_f 312$

Using the bijection

$$\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}), \end{array}$$

Lemma 2.2 implies that

$$F_{312}(x, q, z) = \sum_{D \in \mathcal{D}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

To enumerate tunnels of depth 0, we will separate them according to their height. For every $h \geq 0$, a tunnel at height $h$ must have length $2(h+1)$ in order to have depth 0. It is important to notice that if a tunnel of depth 0 of $D$ corresponds to a decomposition $D = A\mathbf{u}B\mathbf{d}C$, then $D$ has no tunnels of depth 0 in the part given by $B$. In other words, the projections on the $x$-axis of all the tunnels of depth 0 of a given Dyck path are disjoint. This observation allows us to give a continued fraction expression for $F_{312}(x, 1, z)$.

**Theorem 2.15** *$F_{312}(x, 1, z)$ is given by the following continued fraction.*

$$F_{312}(x,1,z) = \cfrac{1}{1-(x-1)z-\cfrac{z}{1-(x-1)z^2-\cfrac{z}{1-2(x-1)z^3-\cfrac{z}{1-5(x-1)z^4-\cfrac{z}{\ddots}}}}},$$

*where at the $n$-th level, the coefficient of $(x-1)z^{n+1}$ is the Catalan number $\mathbf{C}_n$.*

*Proof.* For every $h \geq 0$, let $\mathrm{td}_0^h(D)$ be the number of tunnels of $D$ of height $h$ and length $2(h+1)$. Note that $\mathrm{td}_0(D) = \sum_{h \geq 0} \mathrm{td}_0^h(D)$. We will show now that for every $h \geq 1$, the generating function for Dyck paths where $x$ marks the statistic $\mathrm{td}_0^0 + \cdots + \mathrm{td}_0^{h-1}$ is given by the continued fraction of the theorem truncated at level $h$, with the $(h+1)$-st level replaced with $\mathbf{C}(z)$.

A Dyck path $D$ can be written uniquely as a sequence of elevated Dyck paths, that is, as $D = \mathbf{u}A_1\mathbf{d} \cdots \mathbf{u}A_r\mathbf{d}$, where each $A_i \in \mathcal{D}$. In terms of the GF $\mathbf{C}(z) = \sum_{D\in\mathcal{D}} z^{|D|}$, this translates into the equation $\mathbf{C}(z) = \frac{1}{1-z\mathbf{C}(z)}$. A tunnel of height 0 and length 2 (i.e., a hill) appears in $D$ for each empty $A_i$. Therefore, the GF enumerating hills is

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)} z^{|D|} = \frac{1}{1 - z[x - 1 + \mathbf{C}(z)]}, \tag{2.3}$$

since an empty $A_i$ has to be counted as $x$, not as 1.

Let us enumerate simultaneously hills (as above), and tunnels of height 1 and length 4. The GF (2.3) can be written as

$$\frac{1}{1 - z\left[x - 1 + \dfrac{1}{1 - z\mathbf{C}(z)}\right]}.$$

Combinatorially, this corresponds to expressing each $A_i$ as a sequence $\mathbf{u}B_1\mathbf{d} \cdots \mathbf{u}B_s\mathbf{d}$, where each $B_j \in \mathcal{D}$. Notice that since each $\mathbf{u}B_j\mathbf{d}$ starts at height 1, a tunnel of height 1 and length 4 is created by each $B_j = \mathbf{ud}$ in the decomposition. Thus, if we want $x$ to mark also these tunnels, such a $B_j$ has to be counted as $xz$, not $z$. The corresponding GF is

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)} z^{|D|} = \frac{1}{1 - z\left[x - 1 + \dfrac{1}{1 - z[(x-1)z + \mathbf{C}(z)]}\right]}.$$

Now it is clear how iterating this process indefinitely we obtain the continued fraction of the theorem. From the GF where $x$ marks $\mathrm{td}_0^0 + \cdots + \mathrm{td}_0^{h-1}$, we can obtain the one where $x$ marks $\mathrm{td}_0^0 + \cdots + \mathrm{td}_0^h$ by replacing the $\mathbf{C}(z)$ at the lowest level with

$$\frac{1}{1 - z[(x-1)\mathbf{C}_h z^h + \mathbf{C}(z)]},$$

to account for tunnels of height $h$ and length $2(h+1)$, which in the decomposition correspond to elevated Dyck paths at height $h$. □

The same technique can be used to enumerate excedances in 312-avoiding permutations, which correspond to tunnels of negative depth in the Dyck path. Recall that

$$\mathbf{C}_{<i}(z) = \sum_{j=0}^{i-1} \mathbf{C}_j z^j$$

denotes the series for the Catalan numbers truncated at degree $i$.

**Theorem 2.16** *$F_{312}(x,q,z)$ is given by the following continued fraction.*

$$F_{312}(x,q,z) = \cfrac{1}{1 - zK_0 + \cfrac{z}{1 - zK_1 + \cfrac{z}{1 - zK_2 + \cfrac{z}{1 - zK_3 + \cfrac{z}{\ddots}}}}},$$

*where $K_n = (x-1)\mathbf{C}_n q^n z^n + (q-1)\mathbf{C}_{<n}(qz)$ for $n \geq 0$.*

Note that the first values of $K_n$ are

$K_0 = x - 1,$
$K_1 = (x-1)qz + q - 1,$
$K_2 = 2(x-1)q^2z^2 + (q-1)(1+qz),$
$K_3 = 5(x-1)q^3z^3 + (q-1)(1+qz+2q^2z^2).$

*Proof.* We use the same decomposition as above, now keeping track of tunnels of negative depth as well. For every $h \geq 0$, let $\mathrm{td}^h_{<0}(D)$ be the number of tunnels of $D$ of height $h$ and length less than $2(h+1)$. Note that $\mathrm{td}_{<0}(D) = \sum_{h\geq 0} \mathrm{td}^h_{<0}(D)$. To follow the same structure as in the previous proof, counting tunnels height by height, it will be convenient that at the $h$-th step of the iteration, $q$ marks not only tunnels of negative depth up to height $h$ but also all the tunnels at higher levels. Denote by $\mathrm{alltun}^{>h}(D)$ the number of tunnels of $D$ of height strictly greater than $h$.

We will show now that for every $h \geq 1$, the generating function for Dyck paths where $x$ marks the statistic $\mathrm{td}^0_0 + \cdots + \mathrm{td}^{h-1}_0$ and $q$ marks $\mathrm{td}^0_{<0} + \cdots + \mathrm{td}^{h-1}_{<0} + \mathrm{alltun}^{>h-1}$ is given by the continued fraction of the theorem truncated at level $h$, with the $(h+1)$-st level replaced with $\mathbf{C}(qz)$.

The analogous to equation (2.3) is now

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}^0_0(D)} q^{\mathrm{td}^0_{<0}(D)+\mathrm{alltun}^{>0}(D)} z^{|D|} = \frac{1}{1 - z[x - 1 + \mathbf{C}(qz)]}. \qquad (2.4)$$

Indeed, decomposing $D$ as $\mathbf{u}A_1\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, $q$ counts all the tunnels that appear in any $A_i$, and whenever an $A_i$ is empty we must mark it as $x$.

Let us enumerate now tunnels of depth 0 and negative depth at both height 0 and height 1. Modifying (2.4) so that $q$ no longer counts tunnels at height 1, we get

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)} q^{\mathrm{td}_{<0}^0(D)+\mathrm{alltun}^{>1}(D)} z^{|D|} = \frac{1}{1-z\left[x-1+\dfrac{1}{1-z\mathbf{C}(qz)}\right]}, \quad (2.5)$$

which corresponds to writing each $A_i$ as $A_i = \mathbf{u}B_1\mathbf{d}\cdots\mathbf{u}B_s\mathbf{d}$, and having $q$ count all tunnels in each $B_j$. Now, in order for $x$ to mark tunnels of depth 0 at height 1, each $B_j = \mathbf{ud}$, that in (2.5) is counted as $qz$, has to be now counted as $xqz$ instead. Similarly, to have $q$ mark tunnels of negative depth at height 1, we must count each empty $B_j$ as $q$, not as 1. This gives us the following GF:

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)} q^{\mathrm{td}_{<0}^0(D)+\mathrm{td}_{<0}^1(D)+\mathrm{alltun}^{>1}(D)} z^{|D|}$$

$$= \frac{1}{1-z\left[x-1+\dfrac{1}{1-z[(x-1)qz+q-1+\mathbf{C}(qz)]}\right]}.$$

Iterating this process level by level indefinitely we obtain the continued fraction of the theorem. At each step, from the GF where $x$ marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^{h-1}$, and $q$ marks $\mathrm{td}_{<0}^0+\cdots+\mathrm{td}_{<0}^{h-1}+\mathrm{alltun}^{>h-1}$, we can obtain the one where $x$ marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^h$ and $q$ marks $\mathrm{td}_{<0}^0+\cdots+\mathrm{td}_{<0}^h+\mathrm{alltun}^{>h}$ by replacing the $\mathbf{C}(qz)$ at the lowest level with

$$\frac{1}{1-z[(x-1)\mathbf{C}_h q^h z^h+(q-1)\mathbf{C}_{<h}(qz)+\mathbf{C}(qz)]}. \quad (2.6)$$

This change makes $x$ account for tunnels of depth 0 at height $h$, which in the decomposition correspond to the $\mathbf{C}_h$ possible elevated Dyck paths of length $2(h+1)$ when they occur at height $h$. It also makes $q$ count tunnels of negative depth at height $h$, which in the decomposition correspond to elevated Dyck paths at height $h$ of length less than $2(h+1)$. The GF for these ones becomes $q\mathbf{C}_{<h}(qz)$ instead of $\mathbf{C}_{<h}(qz)$, since for every $j<h$, an elevated path $\mathbf{u}C\mathbf{d}$ with $C\in\mathcal{D}_j$ contributes to one extra tunnel of negative depth at height $h$, aside from the $j$ tunnels of height more than $h$ that it contains. □

For 231-avoiding permutations we get the following GF.

**Corollary 2.17** *$F_{231}(x,q,z)$ is given by the following continued fraction.*

$$F_{231}(x,q,z)=\cfrac{1}{1-zK'_0+\cfrac{qz}{1-zK'_1+\cfrac{qz}{1-zK'_2+\cfrac{qz}{1-zK'_3+\cfrac{qz}{\ddots}}}}},$$

*where* $K'_n=(x-q)\mathbf{C}_n z^n+(1-q)\mathbf{C}_{<n}(z)$.

The first values of $K'_n$ are

$K'_0=x-q,$
$K'_1=(x-q)z+1-q,$
$K'_2=2(x-q)z^2+(1-q)(1+z),$
$K'_3=5(x-q)z^3+(1-q)(1+z+2z^2).$

*Proof.* By Lemma 1.2, we have that $F_{231}(x,q,z)=F_{312}(x/q,1/q,qz)$, so the expression follows from Theorem 2.16. □

# 3

# Simultaneous avoidance

After having studied in the previous section permutations avoiding one pattern of length 3, the next step is to consider permutations avoiding several patterns at the same time. In this chapter we study the distribution of the statistics 'number of fixed points' and 'number of excedances' on permutations avoiding simultaneously two or more patterns. A systematic enumeration (with no statistics) of permutations avoiding any subset of patterns of length 3 was done in [84]. Here we give refinements of these results, by enumerating the same permutations with respect to the statistics fp and exc.

The main technique that we use are bijections between pattern-avoiding permutations and certain kinds of Dyck paths with some restrictions, in such a way that the statistics in permutations that we study correspond to statistics on Dyck paths that are easy to enumerate.

In Section 3.1 we solve completely the case of permutations avoiding simultaneously any two patterns of length 3, giving generating functions counting the number of fixed points and the number of excedances. For some particular instances we can generalize the results, allowing one pattern of the pair to have arbitrary length. In Section 3.2 we give the analogous generating functions for permutations avoiding simultaneously any three patterns of length 3 or more. Section 3.3 is concerned with the study of the distribution of these statistics in involutions avoiding any subset of patterns of length 3.

The bijection $\varphi$ defined in Section 2.1 will be one of our main tools in this chapter, together with its properties given in Lemma 2.2. We will also use repeatedly the array representation of a permutation $\pi$ as described in Section 1.1.3, as well as the operations $\bar{\pi}$, $\widehat{\pi}$, and the lemmas proved in that section.

## 3.1 Double restrictions

In this section we consider simultaneous avoidance of any two patterns of length 3. Using Lemma 1.2, the pairs of patterns fall into the following equivalence classes.

**a)** $\{123, 132\} \approx \{123, 213\}$
**b)** $\{231, 321\} \quad \sim_f \quad$ **b')** $\{312, 321\}$
**c)** $\{132, 213\}$
**d)** $\{231, 312\}$
**e)** $\{132, 231\} \approx \{213, 231\} \quad \sim_f \quad$ **e')** $\{132, 312\} \approx \{213, 312\}$
**f)** $\{132, 321\} \approx \{213, 321\}$
**g)** $\{123, 231\} \quad \sim_f \quad$ **g')** $\{123, 312\}$
**h)** $\{123, 321\}$

In [84] it is shown that the number of permutations in $\mathcal{S}_n$ avoiding any of the pairs in the classes **a)**, **b)**, **b')**, **c)**, **d)**, **e)**, and **e')** is $2^{n-1}$, and that for the pairs in **f)**, **g)** and **g')**, the number of permutations avoiding any of them is $\binom{n}{2} + 1$. The case **h)** is trivial because this pair is avoided only by permutations of length at most 4.

In terms of generating functions, this means that when we substitute $x = q = 1$ in $F_\Sigma(x, q, z)$, where $\Sigma$ is any of the pairs in the classes from **a)** to **e')**, we get

$$F_\Sigma(1, 1, z) = \sum_{n \geq 0} 2^{n-1} z^n = \frac{1-z}{1-2z}.$$

If $\Sigma$ is a pair from the classes **f)**, **g)**, **g')**, we get

$$F_\Sigma(1, 1, z) = \sum_{n \geq 0} (\binom{n}{2} + 1) z^n = \frac{1 - 2z + 2z^2}{(1-z)^3}.$$

### 3.1.1 a) $\{123, 132\} \approx \{123, 213\}$

**Proposition 3.1**

$F_{\{123,132\}}(x, q, z) = F_{\{123,213\}}(x, q, z)$
$= \dfrac{1 + xz + (x^2 - 4q)z^2 + (-3xq + q + q^2)z^3 + (xq + xq^2 - 3x^2 q + 3q^2)z^4}{(1 - qz^2)(1 - 4qz^2)}.$

*Proof.* Consider the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Section 2.1. Part (4) of Proposition 2.1 says that the height of the Dyck path $\varphi(\pi)$ is the length of the longest increasing subsequence of $\pi$. In particular, $\pi \in \mathcal{S}_n(12\cdots(k+1), 132)$ if and only if $\varphi(\pi)$ has height at most $k$. Thus, by Lemma 2.2, $F_{\{123,132\}}(x, q, z)$ can be written in terms of Dyck paths as

$$\sum_{D\in\mathcal{D}^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}. \tag{3.1}$$

Let us first find the univariate GF for paths of height at most 2 (with no statistics). Clearly, the GF for Dyck paths of height at most 1 is $\frac{1}{1-z}$, since such paths are just sequences of hills. A path $D$ of height at most 2 can be written uniquely as $D = \mathbf{u}A_1\mathbf{du}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each $A_i$ is a path of height at most 1. The GF for each $\mathbf{u}A_i\mathbf{d}$ is $\frac{z}{1-z}$. Thus,

$$\sum_{D\in\mathcal{D}^{\leq 2}} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = \sum_{n\geq 0} 2^{n-1} z^n.$$

In the rest of this proof, we assume that all Dyck paths that appear have height at most 2 unless otherwise stated. To compute (3.1), we will separate paths according to their height at the middle. Consider first paths whose height at the middle is 0. Splitting such a path at its midpoint we obtain a pair of paths of the same length. Thus, the corresponding GF is

$$\sum_{m\geq 0} 2^{m-1} z^m \cdot 2^{m-1} q^m z^m = \frac{1-3qz^2}{1-4qz^2}, \tag{3.2}$$

since the number of right tunnels of such a path is the semilength of its right half.

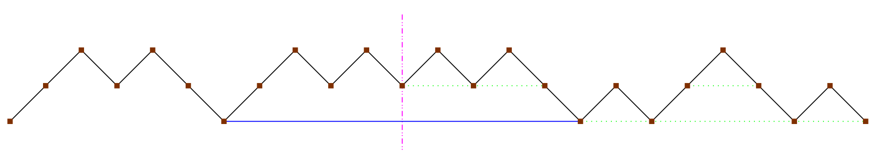

**Figure 3.1** A path of height 2 with a centered tunnel.

Now we consider paths whose height at the middle is 1. It is easy to check that without the variables $x$ and $q$, the GF for such paths is

$$\frac{z}{1-4z^2}. \tag{3.3}$$

Let us first look at paths of this kind that have a centered tunnel. They must be of the form $D = A\mathbf{u}B\mathbf{d}C$ where $A, C \in \mathcal{D}^{\leq 2}$ have the same length, and $B$ is a sequence of an even number of hills. Thus, their GF is

$$xz \cdot \frac{1}{1-qz^2} \cdot \frac{1-3qz^2}{1-4qz^2}, \tag{3.4}$$

where $x$ marks the centered tunnel, $\frac{1}{1-qz^2}$ corresponds to the sequence of hills $B$, half of which give right tunnels, and the last fraction comes from the pair $AC$, which is counted as in (3.2). From (3.3) and (3.4) it follows that the univariate GF (with just variable $z$) for paths with height at the middle 1, not having a centered tunnel, is

$$\frac{z}{1-4z^2} - \frac{z(1-3z^2)}{(1-z^2)(1-4z^2)} = \frac{2z^3}{(1-z^2)(1-4z^2)}.$$

By symmetry, in half of these paths, the tunnel of height 0 that goes across the middle is a right tunnel. Thus, the multivariate GF for all paths with height 1 at the middle is

$$\frac{xz(1-3qz^2)}{(1-qz^2)(1-4qz^2)} + \frac{(q+1)qz^3}{(1-qz^2)(1-4qz^2)}. \tag{3.5}$$

Here the right summand corresponds to paths with no centered tunnel: the term $(q+1)$ distinguishes whether the tunnel that goes across the middle is a right tunnel or not, and the other $q$'s mark tunnels completely contained in the right half.

Paths with height 2 at the middle are easy to enumerate now. Indeed, they must have a peak **ud** in the middle, whose removal induces a bijection between these paths and paths with height 1 at the middle. This bijection preserves the number of right tunnels, and decreases the length and the number of centered tunnels by one. Thus, the GF for paths with height 2 at the middle is $xz$ times expression (3.5). Adding up this GF for paths with height 2 at the middle, to the expressions (3.2) and (3.5) for paths whose height at the middle is 0 and 1 respectively, we obtain the desired expression for $F_{\{123,132\}}(x,q,z)$. □

Let us see how the same technique used in this proof can be generalized to enumerate fixed points in $\mathcal{S}_n(132, 12\cdots(k+1))$ for an arbitrary $k \geq 0$.

**Theorem 3.2** *For $k \geq 0$, let*

$$J_k(x,z) := F_{\{132,12\cdots(k+1)\}}(x,1,z) = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(132,12\cdots(k+1))} x^{\mathrm{fp}(\pi)} z^n.$$

*Then the $J_k$'s satisfy the recurrence*

$$J_k(x,z) = \sum_{\ell=0}^{k} I_{k,\ell}(z)(1+(x-1)zJ_{\ell-1}(x,z)), \tag{3.6}$$

*where $J_{-1}(x,z) := 0$, and $I_{k,\ell}(z)$ is defined as*

$$I_{k,\ell}(z) := \sum_{n\geq 0} g_{k,\ell}^2(n) z^n, \text{ where } \sum_{n\geq 0} g_{k,\ell}(n)z^n = \frac{U_\ell(\frac{1}{2z})}{zU_{k+1}(\frac{1}{2z})},$$

*where $U_m$ are the Chebyshev polynomials of the second kind, defined in Section 1.3.4.*

Before proving this theorem, let us show how to apply it to obtain the GFs $J_k$ for the first few values of $k$. For $k=1$, we have $I_{1,0}(z) = \frac{z}{1-z^2}$, $I_{1,1}(z) = \frac{1}{1-z^2}$, so

$$J_1(x,z) = \frac{1+xz}{1-z^2}.$$

For $k=2$, we get $I_{2,0}(z) = \frac{z^2}{1-4z^2}$, $I_{2,1}(z) = \frac{z}{1-4z^2}$, $I_{2,2}(z) = 1+\frac{z^2}{1-4z^2}$, thus

$$J_2(x,z) = \frac{1+xz+(x^2-4)z^2+(2-3x)z^3+(3+2x-3x^2)z^4}{(1-z^2)(1-4z^2)},$$

which is the expression of Proposition 3.1 for $q=1$.
For $k=3$, we obtain $I_{3,0}(z) = \frac{z^3+z^5}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,1}(z) = \frac{z^2+z^4}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,2}(z) = \frac{z(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,2}(z) = 1+\frac{z^2(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, so
$J_3(x,z) = [1+xz+(x^2-12)z^2+(x^3-11x+2)z^3+(-10x^2+4x+45)z^4+(-10x^3+4x^2+37x-10)z^5 +(25x^2-22x-52)z^6+(25x^3-22x^2-41x+16)z^7+(-12x^2+16x+16)z^8+(-12x^3+16x^2 +12x-8)z^9] \,/\, [(1-z^2)^2(1-4z^2)(1-7z^2+z^4)]$.

*Proof.* As mentioned in the previous proof, $\varphi$ induces a bijection between $\mathcal{S}_n(132, 12\cdots(k+1))$ and $\mathcal{D}^{\leq k}$, the set of Dyck paths of height at most $k$. Thus, by Lemma 2.2,

$$J_k(x,z) = \sum_{D\in\mathcal{D}^{\leq k}} x^{\mathrm{ct}(D)} z^{|D|}.$$

In order to find a recursion for this GF, we are going to apply a trick that consists in consider Dyck paths where some centered tunnels are marked.

That is, we will count pairs $(D,S)$ where $D \in \mathcal{D}^{\leq k}$ and $S$ is a subset of $\mathrm{CT}(D)$, the set of centered tunnels of $D$ ($S$ is the set of marked tunnels). In other words, we are considering Dyck paths where some centered tunnels (namely those in $S$) are marked. Each such pair is given weight $(x-1)^{|S|}z^{|D|}$, so that for a fixed $D$, the sum of weights of all pairs $(D,S)$ will be

$$\sum_{S\subseteq \mathrm{CT}(D)} (x-1)^{|S|}z^{|D|} = ((x-1)+1)^{|\mathrm{CT}(D)|}z^{|D|} = x^{\mathrm{ct}(D)}z^{|D|},$$

which is precisely the weight that $D$ has in $J_k(x,z)$.

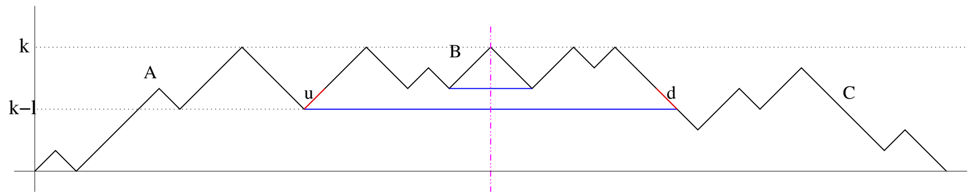


**Figure 3.2** A path of height $k$ with two marked centered tunnels.

If $D \in \mathcal{D}^{\leq k}$ has some marked centered tunnel, consider the decomposition $D = A\mathbf{u}B\mathbf{d}C$ given by the longest marked tunnel (i.e., all the other marked tunnels are inside the part $B$ of the path). Let $\ell$ be the distance between this tunnel and the line $y = k$ (see Figure 3.2). Equivalently, $A$ ends at height $k-\ell$, the same height where $C$ begins. Then, $B$ is an arbitrary Dyck path of height at most $\ell-1$ with possibly some marked centered tunnels, so its corresponding GF is $J_{\ell-1}(x,z)$ (with the convention $J_{-1}(x,z) := 0$, since for $\ell = 0$ there is no such $B$). Giving weight $(x-1)$ to the tunnel that determines our decomposition, we have that the part $\mathbf{u}B\mathbf{d}$ of the path contributes $(x-1)zM_{\ell-1}(x,z)$ to the GF.

Now we look at the GF for the part $A$ of the path. Let $g_{k,\ell}(n)$ be the number of paths from $(0,0)$ to $(n,k-\ell)$ staying always between $y = 0$ and $y = k$. A path of this type can be decomposed uniquely as $A = E_k\mathbf{u}E_{k-1}\mathbf{u}\cdots\mathbf{u}E_{\ell+1}\mathbf{u}E_\ell$, where each $E_i \in \mathcal{D}^{\leq i}$. The GF of Dyck paths of height at most $i$ is

$$J_i(1,z) = \frac{U_i(\frac{1}{2\sqrt{z}})}{\sqrt{z}U_{i+1}(\frac{1}{2\sqrt{z}})},$$

as shown for example in [59]. Let $w = \sqrt{z}$, which is the weight of a single step of a path, and let $\widetilde{R}_{k,\ell}(w) := \sum_{n\geq 0} g_{k,\ell}(n)w^n$. From the above decomposition of $A$,

$$\widetilde{R}_{k,\ell}(w) = J_k(1,w^2)wJ_{k-1}(1,w^2)w\cdots wJ_\ell(1,w^2) = \frac{U_\ell(\frac{1}{2w})}{wU_{k+1}(\frac{1}{2w})}.$$

The part $C$ of the path $D$, flipped over a vertical line, can be regarded as a path with the same endpoints as $A$, since it must have the same length and end at the same height $k-\ell$. Thus, the GF for pairs $(A,C)$ of paths of the same length from height 0 to height $k-\ell$ and not going above $y=k$ is $\sum_{n\geq 0} g_{k,\ell}^2(n)z^n = I_{k,\ell}(z)$.

Hence, the GF for paths $D\in\mathcal{D}^{\leq k}$ having the longest marked centered tunnel at height $k-\ell$ is $I_{k,\ell}(z)(x-1)zM_{\ell-1}(x,z)$.

If $D$ has no marked tunnel, decompose it as $D=AC$ where $A$ and $C$ have the same length. Letting $k-\ell$ be again the height where $A$ ends and $C$ begins, the situation is the same as above but without any contribution coming from the central part of $D$. The parameter $\ell$ can take any value between 0 and $k$. Thus, summing over all possible decompositions of $D$, we get

$$J_k(x,z) = \sum_{\ell=0}^{k} I_{k,\ell}(z)(1+(x-1)zJ_{\ell-1}(x,z)).$$

□

### 3.1.2 b, b') $\{231, 321\}\sim_f\{312, 321\}$

**Proposition 3.3**

$$F_{\{312,321\}}(x,q,z) = \frac{1-qz}{1-(x+q)z+(x-1)qz^2}. \tag{3.7}$$

*Proof.* The length of the longest decreasing subsequence of $\pi$ equals the height of the Dyck path $\varphi(\bar{\pi})$. In particular, we have a bijection

$$\begin{array}{ccc} \mathcal{S}_n(312,321) & \longleftrightarrow & \mathcal{D}_n^{\leq 2} \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$$

Thus, by Lemma 2.2,

$$F_{\{312,321\}}(x,q,z) = \sum_{D\in\mathcal{D}^{\leq 2}} x^{\mathrm{td}_0(D)}q^{\mathrm{td}_{<0}(D)}z^{|D|}.$$

But the only tunnels of depth 0 that a Dyck path of height at most 2 can have are hills, and the only tunnels of negative depth that it can have are peaks at height 2. A path $D \in \mathcal{D}^{\leq 2}$ can be written uniquely as $D = \mathbf{u}A_1\mathbf{d}\mathbf{u}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each $A_i$ is a (possibly empty) sequence of hills. An empty $A_i$ creates a tunnel of depth 0 in $D$, so it contributes as $x$. An $A_i$ of length $2j > 0$ contributes as $q^j z^j$, since it creates $j$ peaks at height 2 in $D$. Thus,

$$F_{\{312,321\}}(x,q,z) = \frac{1}{1 - z\left(x + \dfrac{qz}{1-qz}\right)},$$

which is equivalent to (3.7). □

**Corollary 3.4**

$$F_{\{231,321\}}(x,q,z) = \frac{1-z}{1-(x+1)z+(x-q)z^2}.$$

*Proof.* By Lemma 1.2, $F_{\{231,321\}}(x,q,z) = F_{\{312,321\}}(x/q, 1/q, qz)$. □

As in the previous section, these results can be generalized to the case when instead of the pattern 321 we have a decreasing pattern $(k+1)k\cdots 21$ of arbitrary length. For $i, h \geq 0$, let $\mathbf{C}_i^{\leq h}$ be the number of Dyck paths of length $2i$ and height at most $h$. As mentioned before, it is known that

$$\sum_{i \geq 0} \mathbf{C}_i^{\leq h} z^i = \frac{U_h(\frac{1}{2\sqrt{z}})}{\sqrt{z}\,U_{h+1}(\frac{1}{2\sqrt{z}})},$$

where $U_m$ are the Chebyshev polynomials of the second kind. Let

$$\mathbf{C}_{<i}^{\leq h}(z) := \sum_{j=0}^{i-1} \mathbf{C}_j^{\leq h} z^j.$$

The following theorem deals with fixed points and excedances in the set $\mathcal{S}_n(312, (k+1)k\cdots 1)$ for any $k \geq 0$.

**Theorem 3.5** *Let $\mathbf{C}_i^{\leq h} = |\mathcal{D}_i^{\leq h}|$ and $\mathbf{C}_{<i}^{\leq h}(z)$ be defined as above. Then, for $k \geq 0$,*

$$F_{\{312,(k+1)k\cdots 1\}}(x,q,z) = A_0^k(x,q,z),$$

*where* $A_i^k$ *is recursively defined by*

$$A_i^k(x,q,z) = \begin{cases} \dfrac{1}{1 - z[(x-1)\mathbf{C}_i^{\leq k-i-1} q^i z^i + (q-1)\mathbf{C}_{<i}^{\leq k-i-1}(qz) + A_{i+1}^k(x,q,z)]} & \text{if } i < k, \\ 1 & \text{if } i = k. \end{cases}$$

For example, for $k = 2$ we obtain Proposition 3.3, and for $k = 3$ we get

$$F_{\{312,4321\}}(x,q,z) = \cfrac{1}{1 - z\left[x - 1 + \cfrac{1}{1 - z\left[(x-1)qz + q - 1 + \frac{1}{1-qz}\right]}\right]} =$$

$$\frac{1 - 2qz + (q^2 - xq)z^2 + (xq^2 - q^2)z^3}{1 - (x+2q)z + (xq + q^2 - q)z^2 + (x^2q - xq)z^3 + (-x^2q^2 + 2xq^2 - q^2)z^4}.$$

*Proof.* It is analogous to the proof of Theorem 2.16, with the only difference that here we consider only those paths that do not go above the line $y = k$. □

Making the appropriate substitutions in the statement of Theorem 3.5, we obtain an expression for the generating function $F_{\{231,(k+1)k\cdots 1\}}(x,q,z) = F_{\{312,(k+1)k\cdots 1\}}(x/q, 1/q, qz)$.

### 3.1.3 c) $\{132, 213\}$

**Proposition 3.6**

$$F_{\{132,213\}}(x,q,z) = \frac{1 - (1+q)z - 2qz^2 + 4q(1+q)z^3 - (xq^2 + xq + 5q^2)z^4 + 2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}.$$

*Proof.* We use again the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$. From its description given in Section 2.1, it is not hard to see that a permutation $\pi \in \mathcal{S}_n(132)$ avoids 213 if and only if all the valleys of the corresponding Dyck path $\varphi(\pi)$ have their lowest point on the $x$-axis. A path with such property can be described equivalently as a *sequence of pyramids*. Denote by $\mathcal{P}yr_n \subseteq \mathcal{D}_n$ the set of sequences of pyramids of length $2n$, and let $\mathcal{P}yr := \bigcup_{n\geq 0} \mathcal{P}yr_n$. We have just seen that $\varphi$ restricts to a bijection between $\mathcal{S}_n(132, 213)$ and $\mathcal{P}yr_n$. By Lemma 2.2, we can write $F_{\{132,213\}}(x, q, z)$ as

$$\sum_{D\in\mathcal{P}yr} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

Since for each $n \geq 1$ there is exactly one pyramid of length $2n$, the univariate GF of sequences of pyramids is just $\sum_{D\in\mathcal{P}yr} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = 1 + \sum_{n\geq 1} 2^{n-1} z^n$.

Let us first consider elements of $\mathcal{P}yr$ that have height 0 in the middle (equivalently, the two central steps are **du**). Each one of their halves is a sequence of pyramids, both of the same length. They have no centered tunnels, and the number of right tunnels is given by the semilength of the right half. Thus, their multivariate GF is

$$1 + \sum_{m\geq 1} 4^{m-1} q^m z^{2m} = 1 + \frac{qz^2}{1-4qz^2}. \tag{3.8}$$

Now we count elements of $\mathcal{P}yr$ whose two central steps are **ud**. They are obtained uniquely by inserting a pyramid of arbitrary length in the middle of a path with height 0 at the middle. The tunnels created by the inserted pyramid are all centered tunnels, so the corresponding GF is

$$\frac{xz}{1-xz}\left(1 + \frac{qz^2}{1-4qz^2}\right). \tag{3.9}$$

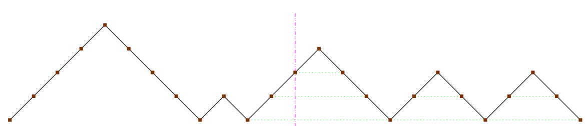

**Figure 3.3** A sequence of pyramids.

It remains to count the elements of $\mathcal{P}yr$ that in the middle have neither a peak nor a valley. From a non-empty sequence of pyramids with height 0 in the middle, if we increase the size of the leftmost pyramid of the right half by an arbitrary number of steps, we obtain a sequence of pyramids whose two central steps are **uu**. Reciprocally, by this procedure every such sequence of pyramids can be obtained in a unique way from a sequence of pyramids with height 0 in the middle. Thus, the GF for the elements of $\mathcal{P}yr$ whose two central steps are **uu** is

$$\frac{qz}{1-qz} \cdot \frac{qz^2}{1-4qz^2}. \tag{3.10}$$

By symmetry, the GF for the elements of $\mathcal{P}yr$ whose two central steps are **dd** is

$$\frac{z}{1-z} \cdot \frac{qz^2}{1-4qz^2}, \tag{3.11}$$

where the difference with respect to 3.10 is that now the pyramid across the middle does not create right tunnels. Adding up (3.8), (3.9), (3.10) and (3.11) we get the desired GF. □

### 3.1.4 d) $\{231, 312\}$

**Proposition 3.7**

$$F_{\{231,312\}}(x, q, z) = \frac{1 - qz^2}{1 - xz - 2qz^2}.$$

*Proof.* We have shown in the proof of Proposition 3.6 that $\varphi$ induces a bijection between $\mathcal{S}_n(132, 213)$ and $\mathcal{P}yr_n$, the set of sequences of pyramids of length $2n$. Composing it with the complementation operation, we get a bijection $\pi \mapsto \varphi(\bar{\pi})$ between $\mathcal{S}_n(231, 312)$ and $\mathcal{P}yr_n$. Together with Lemma 2.2, this allows us to express $F_{\{231,312\}}(x, q, z)$ as

$$\sum_{D \in \mathcal{P}yr} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

All that remains is to observe how many tunnels of zero and negative depth are created by a pyramid according to its size. A pyramid of odd semilength $2m + 1$ creates one tunnel of depth 0 and $m$ tunnels of negative depth. A pyramid of even semilength $2m$ creates only $m$ tunnels of negative depth. Thus, we have that

$$F_{\{231,312\}}(x, q, z) = \frac{1}{1 - \dfrac{xz}{1 - qz^2} - \dfrac{qz^2}{1 - qz^2}},$$

which equals the expression above. □

### 3.1.5 e, e') $\{132, 231\} \approx \{213, 231\} \sim_f \{132, 312\} \approx \{213, 312\}$

**Proposition 3.8**

$$F_{\{132,231\}}(x, q, z) = F_{\{213,231\}}(x, q, z) = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}.$$

*Proof.* As usual, we use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$. Now we are interested in how the condition that $\pi$ avoids 231 is reflected in the Dyck path $\varphi(\pi)$. It is easy to see from the description of $\varphi$ and $\varphi^{-1}$ in Section 2.1 that $\pi$ is 231-avoiding if and only if $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step (equivalently, all the non-isolated up-steps occur at the beginning of the path). Let $\mathcal{E}_n \subseteq \mathcal{D}_n$ be the set of Dyck paths with this condition, and let $\mathcal{E} := \bigcup_{n\geq 0} \mathcal{E}_n$. Then, $\varphi$ induces a bijection between $\mathcal{S}_n(132, 231)$ and $\mathcal{E}_n$. By Lemma 2.2, $F_{\{132,231\}}(x, q, z)$ can be written as

$$\sum_{D\in\mathcal{E}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

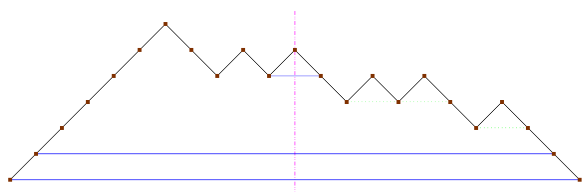

**Figure 3.4** A path in $\mathcal{E}$ with a peak in the middle and two bottom tunnels.

If $D \in \mathcal{E}$, centered tunnels of $D$ can appear only in the following two places. There can be a centered tunnel produced by a peak in the middle of $D$. All the other centered tunnels of $D$ must have their endpoints in the initial ascending run and the final descending run of $D$ (that is, in their corresponding decomposition $D = A\mathbf{u}B\mathbf{d}C$, $A$ is a sequence of up-steps and $C$ is a sequence of down-steps). For convenience we call this second kind of tunnels *bottom* tunnels. All the right tunnels of $D$ come from peaks on the right half.

It is an exercise to check that the number of paths in $\mathcal{E}_n$ having a peak in the middle and $r$ peaks on the right half is $\binom{n-r-1}{r}2^{r-1}$ if $r \geq 1$, and 1 if $r = 0$. Similarly, the number of paths in $\mathcal{E}_n$ with no peak in the middle and $r$ peaks on the right half is $\binom{n-r}{r}2^{r-1}$ if $r \geq 1$, and 0 if $r = 0$. Let us ignore for the moment the bottom tunnels. For peaks in the middle and right tunnels we have the following GF.

$$Q(x,q,z) := \sum_{D\in\mathcal{E}\setminus\mathcal{E}_0} x^{\#\{\text{peaks in the middle of D}\}} q^{\mathrm{rt}(D)} z^{|D|}$$
$$= \sum_{n\geq 1}\left[\sum_{r=1}^{\lfloor n/2\rfloor}\binom{n-r}{r}2^{r-1}q^r + x\left(1+\sum_{r=1}^{\lfloor (n-1)/2\rfloor}\binom{n-r-1}{r}2^{r-1}q^r\right)\right]z^n$$
$$= \frac{xz+(q-x)z^2-xqz^3}{(1-z)(1-z-2qz^2)}. \qquad (3.12)$$

Now, to take into account all centered tunnels, we use that every $D\in\mathcal{E}$ can be written uniquely as $D=\mathbf{u}^k D'\mathbf{d}^k$, where $k\geq 0$ and $D'\in\mathcal{E}$ has no bottom tunnels. The GF for elements of $\mathcal{E}$ that do have bottom tunnels, where $x$ marks peaks in the middle, is $xz+zQ(x,q,z)$ (the term $xz$ is the contribution of the path $\mathbf{ud}$). Hence, the sought GF where $x$ marks all centered tunnels is

$$\begin{aligned} F_{\{132,231\}}(x,q,z) &= \frac{1}{1-xz}[1+Q(x,q,z)-xz-zQ(x,q,z)] \\ &= 1+\frac{1-z}{1-xz}Q(x,q,z), \end{aligned}$$

which together with (3.12) implies the proposition. □

**Corollary 3.9**

$$F_{\{132,312\}}(x,q,z) = F_{\{213,312\}}(x,q,z) = \frac{1-qz-qz^2+xqz^3}{(1-xz)(1-qz-2qz^2)}.$$

*Proof.* Lemma 1.2 gives us $F_{\{132,312\}}(x,q,z) = F_{\{132,231\}}(x/q,1/q,qz)$. □

### 3.1.6 f) $\{132,321\}\approx\{213,321\}$

**Proposition 3.10**

$$F_{\{132,321\}}(x,q,z) = F_{\{213,321\}}(x,q,z) = \frac{1-(1+q)z+2qz^2}{(1-z)(1-xz)(1-qz)}.$$

*Proof.* We saw in part (6) of Proposition 2.1 that the number of peaks of the Dyck path $\varphi(\pi)$ equals the length of the longest decreasing subsequence of $\pi$. In particular, $\pi$ is 321-avoiding if and only if $\varphi(\pi)$ has at most two peaks.

By Lemma 2.2, $F_{\{132,321\}}(x,q,z) = \sum x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}$, where the sum is over Dyck paths $D$ with at most two peaks. Clearly, such a path can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and $D'$ is either empty or a pair of adjacent pyramids (see Figure 3.5). Therefore,

$$F_{\{132,321\}}(x,q,z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z} \cdot \frac{qz}{1-qz}\right),$$

since centered tunnels are produced by the steps outside $D'$, and right tunnels are created by the right pyramid of $D'$. □

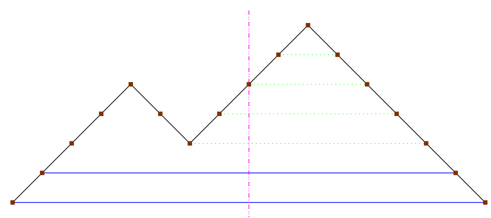

**Figure 3.5** A path with two peaks.

This case can be generalized to the situation when instead of 321 we have a decreasing pattern of arbitrary length. Observe that by Lemma 1.2, $F_{\{132,(k+1)k\cdots21\}}(x,q,z) = F_{\{213,(k+1)k\cdots21\}}(x,q,z)$ for all $k$.

**Theorem 3.11**

$$\sum_{k\geq 0} F_{\{132,(k+1)k\cdots21\}}(x,q,z)\, p^k$$
$$= \frac{2(1+xz(p-1))}{(1-p)[1+(1+q-2x)z - qz^2(p-1)^2 + \sqrt{f_1(q,z)}]},$$

*where* $f_1(q,z) = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2-2q(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$.

*Proof.* We use again the fact from Proposition 2.1 that the number of peaks of $\varphi(\pi)$ equals the length of the longest decreasing subsequence of $\pi$. Thus, $\varphi$ induces a bijection between $\mathcal{S}_n(132,(k+1)k\cdots21)$ and the subset of $\mathcal{D}_n$ of paths with at most $k$ peaks. This implies that we can express $\sum_{k\geq 0} F_{\{132,(k+1)k\cdots21\}}(x,q,z)\, p^k$ as

$$\frac{1}{1-p}\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} p^{\#\{\text{peaks of } D\}} z^{|D|}.$$

The result now follows from Theorem 4.10 and the expression for the generating function $\sum x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} p^{\#\{\text{peaks of } D\}} z^{|D|}$ that we will give in its proof in the next chapter. □

### 3.1.7 g, g') $\{123, 231\} \sim_f \{123, 312\}$

**Proposition 3.12**

$$F_{\{123,312\}}(x, q, z) = \frac{1 + xz + (x^2 - 2q)z^2 + (-x^2q + xq^2 + 3q^2)z^4 + 3q^3z^5 - q^3z^6 - 4q^4z^7 - 2xq^4z^8}{(1 - qz^2)^3(1 - q^2z^3)}.$$

*Proof.* We saw in the proof of Proposition 3.10 that $\varphi$ induces a bijection between $\mathcal{S}_n(132, 321)$ and the set of paths in $\mathcal{D}_n$ with at most two peaks. Composing it with the complementation operation, we get a bijection $\pi \mapsto \varphi(\bar{\pi})$ between $\mathcal{S}_n(123, 312)$ and such set of Dyck paths. Using Lemma 2.2, we can write $F_{\{123,312\}}(x, q, z) = \sum x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}$, where the sum is over Dyck paths $D$ with at most two peaks. Again, such a $D$ can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and $D'$ is either empty or a pair of adjacent pyramids, i.e., $D' = \mathbf{u}^i \mathbf{d}^i \mathbf{u}^j \mathbf{d}^j$ with $i, j \geq 1$. The idea is to consider cases depending on the relations among $i$, $j$ and $k$.

To enumerate Dyck paths with at most two peaks with respect to $\mathrm{td}_0$ and $\mathrm{td}_{<0}$, it is important to look at where the tunnels of depth 0 and depth 1 occur. For convenience in this proof, we call such tunnels *frontier tunnels*, since they determine where tunnels of negative depth are: above them all tunnels have negative depth, and below them tunnels have positive depth. There are four possibilities according to where the frontier tunnels of $D$ occur in the decomposition above:

(1) outside $D'$,

(2) inside one of the pyramids of $D'$,

(3) inside both pyramids of $D'$,

(4) $D$ has no frontier tunnel.

Figure 3.6 shows an example of each of the four cases. The frontier tunnels (whose depth is 0 in this example) are drawn with a solid line, while the dotted lines are the tunnels of negative depth.

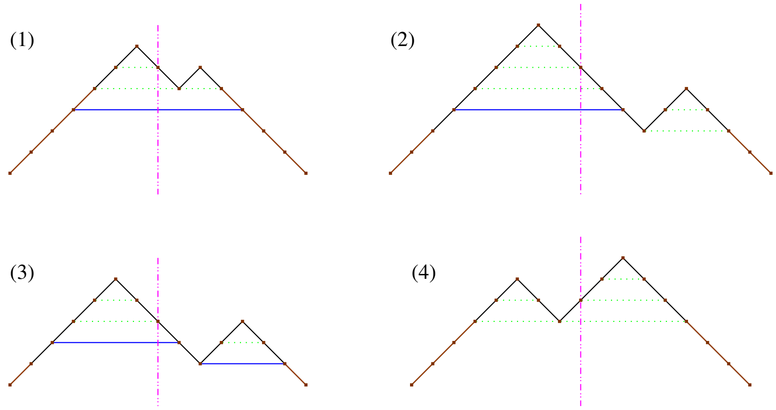


**Figure 3.6** Four possible locations of the frontier tunnels.

Note that in case (4) the tunnels of negative depth are exactly those in $D'$. We show as an example how to find the GF in case (1). In this case, the frontier tunnel $T$ gives a decomposition $D = A\mathbf{u}B\mathbf{d}C$ where $A = \mathbf{u}^m$, $C = \mathbf{d}^m$, $m \geq 0$, and $B$ is a Dyck path with at most two peaks, of semilength $|B| = m$ if $\text{depth}(T) = 0$, and $|B| = m+1$ if $\text{depth}(T) = 1$. It follows from Proposition 3.10 that the GF for Dyck paths with at most two peaks is $\frac{1-2z+2z^2}{(1-z)^3}$. In the situation where $\text{depth}(T) = 0$, we have that $|D| = 2|B|+1$ and $\text{td}_{<0}(D) = |B|$. Thus, the corresponding GF is

$$xz \cdot \frac{1-2qz^2+2q^2z^4}{(1-qz^2)^3}.$$

Similarly, in the situation where $\text{depth}(T) = 1$, we have that $|D| = 2|B|$ and $\text{td}_{<0}(D) = |B|$, thus the corresponding GF is

$$\frac{1-2qz^2+2q^2z^4}{(1-qz^2)^3}.$$

The other cases are similar. Adding up the GFs obtained in each case, we get the desired expression for $F_{\{123,312\}}(x,q,z)$. □

**Corollary 3.13**

$$F_{\{123,231\}}(x,q,z) = \frac{1+xz+(x^2-2q)z^2+(-x^2q+xq+3q^2)z^4+3q^2z^5-q^3z^6-4q^3z^7-2xq^3z^8}{(1-qz^2)^3(1-qz^3)}.$$

*Proof.* By Lemma 1.2, we have $F_{\{123,231\}}(x,q,z) = F_{\{123,312\}}(x/q,1/q,qz)$. □

### 3.1.8 h) $\{123, 321\}$

**Proposition 3.14**

$$F_{\{123,321\}}(x, q, z) = 1 + xz + (x^2 + q)z^2 + (2xq + q^2 + q)z^3 + 4q^2z^4.$$

*Proof.* By a well-known result of Erdös and Szekeres, any permutation of length at least 5 contains an occurrence of either 123 or 321. This reduces the problem to counting fixed points and excedances in permutations of length at most 4, which is trivial. □

## 3.2 Triple restrictions

Here we consider simultaneous avoidance of any three patterns of length 3. Applying Lemma 1.2, the triplets of patterns fall into the following equivalence classes.

**a)** $\{123, 132, 213\}$
**b)** $\{231, 312, 321\}$
**c)** $\{123, 132, 231\} \approx \{123, 213, 231\} \sim_f$ **c')** $\{123, 132, 312\} \approx \{123, 213, 312\}$
**d)** $\{132, 231, 321\} \approx \{213, 231, 321\} \sim_f$ **d')** $\{132, 312, 321\} \approx \{213, 312, 321\}$
**e)** $\{132, 213, 231\} \sim_f$ **e')** $\{132, 213, 312\}$
**f)** $\{132, 231, 312\} \approx \{213, 231, 312\}$
**g)** $\{123, 231, 312\}$
**h)** $\{132, 213, 321\}$
**i)** $\{123, 132, 321\} \approx \{123, 213, 321\}$
**j)** $\{123, 231, 321\} \sim_f$ **j')** $\{123, 312, 321\}$

It is known [84] that the number of permutations in $\mathcal{S}_n$ avoiding the triplets in the classes **a)** and **b)** is the Fibonacci number $F_{n+1}$. The number of permutations avoiding any of the triplets in the classes **c)**, **c')**, **d)**, **d')**, **e)**, **e')**, **f)**, **g)** and **h)** is $n$. The cases of the triplets **i)**, **j)** and **j')** are trivial, because they are avoided only by permutations of length at most 4.

In terms of generating functions, when we substitute $x = q = 1$ in $F_\Sigma(x, q, z)$ where $\Sigma$ is a triplet from one of the classes between **a)** and **g)**, we get

$$F_\Sigma(1, 1, z) = \sum_{n \geq 0} F_{n+1} z^n = \frac{1}{1 - z - z^2}.$$

If $\Sigma$ is any triplet from the classes between **c)** and **h)**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} nz^n = \frac{1-z+z^2}{(1-z)^2}.$$

The following theorem gives all the generating functions of permutations avoiding any triplet of patterns of length 3.

**Theorem 3.15** **a)**

$$F_{\{123,132,213\}}(x,q,z) = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

**b)**

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}$$

**c)**

$$F_{\{123,132,231\}}(x,q,z) = F_{\{123,213,231\}}(x,q,z)$$
$$= \frac{1+xz+(x^2-q)z^2+qz^3+(-x^2q+xq+q^2)z^4}{(1-qz^2)^2}$$

**c')**

$$F_{\{123,132,312\}}(x,q,z) = F_{\{123,213,312\}}(x,q,z)$$
$$= \frac{1+xz+(x^2-q)z^2+q^2z^3+(-x^2q+xq^2+q^2)z^4}{(1-qz^2)^2}$$

**d)**

$$F_{\{132,231,321\}}(x,q,z) = F_{\{213,231,321\}}(x,q,z) = \frac{1-z+qz^2}{(1-z)(1-xz)}$$

**d')**

$$F_{\{132,312,321\}}(x,q,z) = F_{\{213,312,321\}}(x,q,z) = \frac{1-qz+qz^2}{(1-xz)(1-qz)}$$

**e)**

$$F_{\{132,213,231\}}(x,q,z)$$
$$= \frac{1-z-qz^2+2qz^3+(-x^2q+q^2-xq)z^4+(x^2q-2q^2)z^5+xq^2z^6}{(1-z)(1-xz)(1-qz^2)^2}$$

**e')**

$$F_{\{132,213,312\}}(x,q,z) = \frac{1 - qz - qz^2 + 2q^2z^3 + (-x^2q - xq^2 + q^2)z^4 + (x^2q^2 - 2q^3)z^5 + xq^3z^6}{(1-xz)(1-qz)(1-qz^2)^2}$$

**f)**

$$F_{\{132,231,312\}}(x,q,z) = F_{\{213,231,312\}}(x,q,z) = \frac{1 + xqz^3}{(1-xz)(1-qz^2)}$$

**g)**

$$F_{\{123,231,312\}}(x,q,z) = \frac{1 + xz + (x^2 - q)z^2 + xqz^3 + q^2z^4}{(1-qz^2)^2}$$

**h)**

$$F_{\{132,213,321\}}(x,q,z) = \frac{1 - (1+q)z + 2qz^2 - xqz^3}{(1-z)(1-xz)(1-qz)}$$

**i)**

$$F_{\{123,132,321\}}(x,q,z) = F_{\{123,213,321\}}(x,q,z) = 1 + xz + (x^2 + q)z^2 + (xq + q^2 + q)z^3 + q^2z^4$$

**j)**

$$F_{\{123,231,321\}}(x,q,z) = 1 + xz + (x^2 + q)z^2 + (2xq + q)z^3 + q^2z^4$$

**j')**

$$F_{\{123,312,321\}}(x,q,z) = 1 + xz + (x^2 + q)z^2 + (2xq + q^2)z^3 + q^2z^4$$

*Proof.* Throughout this proof we will use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Section 2.1.

**a)** As in the proof of Proposition 3.1, we have that $\pi \in \mathcal{S}_n(132)$ avoids 123 if and only if the Dyck path $\varphi(\pi)$ has height at most 2. Similarly, from the proof of Proposition 3.6, $\pi$ avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids. Thus, $\varphi$ induces a bijection between $\mathcal{S}_n(123, 132, 213)$

and $\mathcal{P}yr_n^{\leq 2} := \mathcal{P}yr^{\leq 2} \cap \mathcal{D}_n$, where $\mathcal{P}yr^{\leq 2}$ denotes the set of sequences of pyramids of height at most 2. By Lemma 2.2,

$$F_{\{123,132,213\}}(x,q,z) = \sum_{D \in \mathcal{P}yr^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

To count centered and right tunnels, we distinguish cases according to which steps are the middle steps of $D$. A path in $\mathcal{P}yr^{\leq 2}$ of height 0 at the middle can be split in two elements of $\mathcal{P}yr^{\leq 2}$ of equal length, only the right one producing right tunnels. Since the number of $D \in \mathcal{P}yr_n^{\leq 2}$ is $F_{n+1}$, the GF for paths of height 0 at the middle is

$$\sum_{n\geq 0} F_{m+1}^2 q^m z^{2m} = \frac{1-qz^2}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

Multiplying this expression by $xz$ (resp. by $x^2z^2$) we obtain the GF for paths in $\mathcal{P}yr^{\leq 2}$ having in the middle a centered pyramid of height 1 (resp. of height 2).

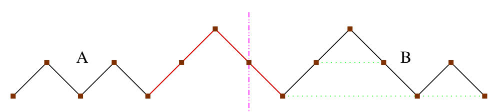


**Figure 3.7** A sequence of pyramids of height at most 2.

Paths $D \in \mathcal{P}yr^{\leq 2}$ whose two middle steps are **dd** can be written as $D = A\mathbf{uudd}B$, where $A, B \in \mathcal{P}yr^{\leq 2}$ and $|B| = |A| + 1$ (see Figure 3.7). Thus, the corresponding GF is

$$\sum_{n\geq 1} F_m F_{m+1} q^m z^{2m+1} = \frac{qz^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

By symmetry, multiplying this expression by $q$ we get the GF for paths whose two middle steps are **uu**.

Adding up all the cases, we get the desired GF

$$F_{\{123,132,213\}}(x,q,z) = \frac{(1+xz+x^2z^2)(1-qz^2)}{(1+qz^2)(1-3qz^2+q^2z^4)} + \frac{(q+q^2)z^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

**b)** Using the same reasoning as in **a)**, we have that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(231,312,321)$ and $\mathcal{P}yr_n^{\leq 2}$. Now, Lemma 2.2 implies that

$$F_{\{231,312,321\}}(x,q,z) = \sum_{D \in \mathcal{P}yr^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

Each pyramid of height 1 produces a tunnel of depth 0, and each pyramid of height 2 creates a tunnel of negative depth. Therefore,

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}.$$

**c)** We saw in the proof of Proposition 3.8 that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Therefore, $\varphi$ induces a bijection between $\mathcal{S}_n(123,132,312)$, and paths in $\mathcal{D}_n$ with the above condition and height at most 2. Such paths (except the empty one) can be expressed uniquely as $D = \mathbf{u}A\mathbf{d}B$, where $A$ and $B$ are sequences of hills (i.e, they have the form $(\mathbf{ud})^k$ for some $k \geq 0$). Lemma 2.2 reduces the problem to enumerating centered tunnels and right tunnels on these paths.

If $B$ is empty, $D = \mathbf{u}A\mathbf{d}$ has a centered tunnel at height 0. The contribution of paths of this kind to our GF is $\frac{xz}{1-qz^2}$ for $|A|$ even, and $\frac{x^2z^2}{1-qz^2}$ for $|A|$ odd.

Assume now that $|A| < |B|$, so that $A$ is within the left half of $D = \mathbf{u}A\mathbf{d}B$. If the middle of $D$ is at height 0, then $D$ is determined by the length of $A$ and the number of hills in $B$ to the left of the middle. Thus, the contribution of this subset to the GF is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by $xz$ gives the GF for paths whose midpoint is on top of a hill of $B$.

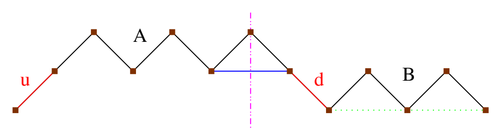


**Figure 3.8** An example with $|A| = 3$ and $|B| = 2$.

It remains the case in which $|A| \geq |B| > 0$. If $|A| - |B|$ is even, the contribution of these paths to the GF is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{1}{1-qz^2},$$

where the last factor counts how larger $A$ is than $B$. If $|A|-|B|$ is odd, the corresponding GF is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{xz}{1-qz^2},$$

since in this case there is a centered tunnel of height 1 inside $A$ (see Figure 3.8).

Summing up all the cases, we get

$$F_{\{123,132,231\}}(x,q,z) = 1 + \frac{xz + x^2z^2}{1-qz^2} + \frac{(1+xz)qz^2}{(1-qz^2)^2} + \frac{qz^3(1+xz)}{(1-qz^2)^2}.$$

**c')** Lemma 1.2 implies that $F_{\{123,132,312\}}(x,q,z) = F_{\{123,132,231\}}(\frac{x}{q}, 1/q, qz)$, so the formula follows from part **c)**.

**d)** As in the proof of Proposition 3.8, we use that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Besides, as in Proposition 3.10, $\pi \in \mathcal{S}_n(132)$ avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. Thus, $\pi \in \mathcal{S}_n(132, 231, 321)$ if and only if $\varphi(\pi) \in \mathcal{D}_n$ has the form $\mathbf{u}^k B \mathbf{d}^k$, where $B$ is either empty or is a pair of pyramids, the second of height 1. Fixed points and excedances of $\pi$ are mapped to centered tunnels and right tunnels of $\varphi(\pi)$ respectively, by Lemma 2.2. Thus, $F_{\{123,132,312\}}(x,q,z)$ equals the GF enumerating centered and right tunnels in these paths.

If $B$ is not empty, the contribution of the first pyramid is $\frac{z}{1-z}$, and the second pyramid contributes $qz$. Centered tunnels come from the steps outside $B$. Hence,

$$F_{\{123,132,312\}}(x,q,z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z} \cdot qz\right).$$

**d')** It follows from part **d)** and Lemma 1.2.

**e)** Let $\pi \in \mathcal{S}_n(132)$. We have seen that the condition that $\pi$ avoids 213 translates into $\varphi(\pi)$ being a sequence of pyramids. The additional restriction of $\pi$ avoiding 231 implies that all but the first pyramid of the sequence $\varphi(\pi)$ must have height 1. Thus, by Lemma 2.2, $F_{\{132,213,231\}}(x,q,z)$ can be obtained enumerating centered and right tunnels in paths of the form $D = AB$, where $A$ is any pyramid and $B$ is a sequence of hills.

The contribution of such paths when $B$ is empty is just $\frac{1}{1-xz}$. Assume now that $B$ is not empty. If $|A| > |B|$, the corresponding contribution is

$$\frac{qz^2}{1-qz^2} \cdot \frac{z}{1-z},$$

where the second factor counts how larger $A$ is than $B$. It remains the case $|A| \leq |B|$, in which $A$ is within the left half of $D$. If the middle of $D$ is at height 0, then $D$ is determined by the length of $A$ and the number of hills in $B$ to the left of the middle. Thus, the contribution of this subset to the GF is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by $xz$ gives the GF for paths whose midpoint is on top of a hill of $B$.

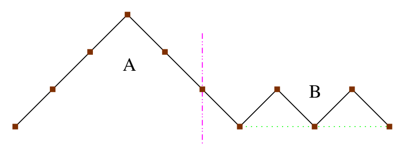


**Figure 3.9** A pyramid followed by a sequence of hills.

Summing all this up, we get

$$F_{\{132,213,231\}}(x,q,z) = \frac{1}{1-xz} + \frac{qz^3}{(1-z)(1-qz^2)} + \frac{(1+xz)qz^2}{(1-qz^2)^2}.$$

**e')** It follows from part **e)** and Lemma 1.2.

**f)** Reasoning as in the proof of **e)**, we see that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(123, 231, 312)$ and the subset of paths in $\mathcal{D}_n$ consisting of a pyramid followed by a sequence of hills. By Lemma 2.2, it is enough to enumerate these paths according to the statistics $\mathrm{td}_0$ and $\mathrm{td}_{<0}$. If the path is nonempty, the first pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size (since then it contains a tunnel of depth 0) and $\frac{qz^2}{1-qz^2}$ if it has even size. The sequence of hills contributes $\frac{1}{1-xz}$. Therefore,

$$F_{\{132,231,312\}}(x,q,z) = 1 + \frac{xz+qz^2}{1-qz^2} \cdot \frac{1}{1-xz}.$$

**g)** Let $\pi \in \mathcal{S}_n(132)$. We have seen that $\pi$ avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids, and that $\pi$ avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. In other words, $\varphi$ induces a bijection between $\mathcal{S}_n(132, 213, 321)$ and the subset of paths in $\mathcal{D}_n$ that are a sequence of at most two pyramids. Composing with the complementation operation, we have that $\pi \in \mathcal{S}_n(123, 231, 312)$ if and only if $\varphi(\bar{\pi})$ is in that subset. Now, Lemma 2.2 implies that $F_{\{123,231,312\}}$ can be obtained enumerating sequences of at most 2 pyramids according to $\mathrm{td}_0$ and $\mathrm{td}_{<0}$. Each pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size and $\frac{qz^2}{1-qz^2}$ if it has even size. Thus,

$$F_{\{123,231,312\}}(x, q, z) = 1 + \frac{xz + qz^2}{1 - qz^2} + \left(\frac{xz + qz^2}{1 - qz^2}\right)^2.$$

**h)** We have shown in the proof of **g)** that $\pi \in \mathcal{S}_n(132, 213, 321)$ if and only if $\varphi(\pi)$ is a sequence of at most two pyramids. Using Lemma 2.2, it is enough to enumerate centered tunnels and right tunnels in such paths. The contribution of paths with exactly two pyramids is

$$\frac{z}{1-z} \cdot \frac{qz}{1-qz},$$

since only the one on the right gives right tunnels. Centered tunnels appear when there is only one pyramid. Thus we obtain

$$F_{\{132,213,321\}}(x, q, z) = \frac{1}{1 - xz} + \frac{qz^2}{(1-z)(1-qz)}.$$

**i, j, j')** These cases are trivial because only permutations of length at most 4 can avoid 123 and 321 simultaneously. □

After having studied all the cases of double and triple restrictions, the next step is to consider restrictions of higher multiplicity. However, for $\Sigma \subseteq \mathcal{S}_3$ with $|\Sigma| \geq 4$, the sets $\mathcal{S}_n(\Sigma)$ are very easy to describe (see for example [84]), and the distribution of fixed points and excedances is trivial. In particular, in these cases we have that $|\mathcal{S}_n(\Sigma)| \in \{0, 1, 2\}$ for all $n$.

## 3.3 Pattern-avoiding involutions

Recall that $\mathcal{I}_n$ denotes the set of involutions of length $n$, i.e., permutations $\pi \in \mathcal{S}_n$ such that $\pi = \pi^{-1}$. In terms of the array representation of $\pi$, this condition is equivalent to $\mathrm{arr}(\pi)$ being symmetric with respect to the main diagonal. In this section we consider the distribution of the statistics fp and exc in involutions avoiding any subset of patterns of length 3.

For any $\pi \in \mathcal{S}_n$, it is clear that $\mathrm{fp}(\pi) + \mathrm{exc}(\pi) + \mathrm{exc}(\pi^{-1}) = n$ (each cross in the array of $\pi$ is either on, to the right of, or to the left of the main diagonal). Thus, if $\pi \in \mathcal{I}_n$, then $\mathrm{exc}(\pi) = \frac{1}{2}(n - \mathrm{fp}(\pi))$, so the number of excedances is determined by the number of fixed points. Therefore, it is enough here to consider only the statistic 'number of fixed points' in pattern-avoiding involutions.

For any set of patterns $\Sigma$, let $\mathcal{I}_n(\Sigma) := \mathcal{I}_n \cap \mathcal{S}_n(\Sigma)$, and let $i_n^k(\Sigma) := |\{\pi \in \mathcal{I}_n(\Sigma) : \mathrm{fp}(\pi) = k\}|$. Define

$$G_\Sigma(x,z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{I}_n(\Sigma)} x^{\mathrm{fp}(\pi)} z^n.$$

By the reasoning above, $\sum_{n\geq 0} \sum_{\pi\in\mathcal{I}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n = G_\Sigma(xq^{-1/2}, zq^{1/2})$. Clearly, $\widehat{\pi}$ is an involution if and only if $\pi$ is an involution. Therefore, from Lemma 1.1 we get the following.

**Lemma 3.16** *Let $\Sigma$ be any set of patterns. We have*

(1) $G_{\widehat{\Sigma}}(x,z) = G_\Sigma(x,z)$,

(2) $G_{\Sigma^{-1}}(x,z) = G_\Sigma(x,z)$.

The property stated in the following lemma is what allows us to apply our techniques for studying statistics on pattern-avoiding permutations to the case of involutions.

**Lemma 3.17** *Let $\pi \in \mathcal{S}_n(132)$ and let $D = \varphi(\pi) \in \mathcal{D}_n$. Then,*

$$\pi \textit{ is an involution} \Longleftrightarrow \varphi(\pi) \textit{ is symmetric.}$$

*Proof.* The array of crosses representing $\pi^{-1}$ is obtained from the one of $\pi$ by reflection over the main diagonal. Therefore, from the description of the bijection $\varphi$ given in Section 2.1, we have that $\varphi(\pi^{-1}) = D^*$. It follows that $\pi$ is an involution if and only if $D = D^*$, which is equivalent to $D$ being a symmetric Dyck path. □

### 3.3.1 Single restrictions

It is known [84] that for $\sigma \in \{123, 132, 213, 321\}$, $|\mathcal{I}_n(\sigma)| = \binom{n}{\lfloor n/2 \rfloor}$, and that for $\sigma \in \{231, 312\}$, $|\mathcal{I}_n(\sigma)| = 2^{n-1}$. From Lemma 3.16 it follows that for all $k \geq 0$, $i_n^k(132) = i_n^k(213)$ and $i_n^k(231) = i_n^k(312)$. In a recent paper [27], Deutsch, Robertson and Saracino prove the following fact:

**Theorem 3.18 ([27])** *The number of* 321*-avoiding involutions* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *equals the number of* 132*-avoiding involutions* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$*, for any* $0 \leq i \leq n$.

Let us show that Theorem 3.18 follows easily from the work in Section 2.2. Recall from Section 1.2.1 that if $D \in \mathcal{D}_n$, $D^*$ denotes the path obtained by reflection of $D$ from a vertical line $x = n$. Now observe that if $\varphi(\pi) = D$, then $\varphi(\pi^{-1}) = D^*$ (see Lemma 3.17). Similarly, if $\Psi(\pi) = D$, then $\Psi(\pi^{-1}) = D^*$ (by the duality of RSK). Therefore, $\pi \in \mathcal{S}_n(321)$ is an involution if and only if so is $\Theta(\pi) \in \mathcal{S}_n(132)$, which implies the result. Furthermore, restricting $\Theta$ to involutions we obtain the following extension of Theorem 3.18:

**Theorem 3.19** *The number of* 321*-avoiding involutions* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ *and* $\mathrm{lis}(\pi) = k$ *equals the number of* 132*-avoiding involutions* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ *and* $\mathrm{rank}(\pi) = n - k$*, for any* $0 \leq i, j, k \leq n$.

By Theorem 3.18 we have that $i_n^k(132) = i_n^k(321)$. Thus, for single restrictions there are three cases to consider.

**Theorem 3.20 ([50, 27])** *Let* $n \geq 1$, $k \geq 0$. *We have*

(1) $i_n^0(123) = i_n^2(123) = \begin{cases} \binom{n-1}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$

$i_n^1(123) = \begin{cases} \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$

$i_n^k(123) = 0$ *if* $k \geq 3$.

(2) $i_n^k(132) = i_n^k(213) = i_n^k(321) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \text{if } n-k \text{ is even,} \\ 0 & \text{if } n-k \text{ is odd.} \end{cases}$

(3) $G_{231}(x,z) = G_{312}(x,z) = \dfrac{1-z^2}{1-xz-2z^2}.$

*Proof.* (1) Clearly a 123-avoiding permutation cannot have more than two fixed points. On the other hand, if $\pi \in \mathcal{I}_n$, we have $\text{fp}(\pi) = n - 2\,\text{exc}(\pi)$, which explains that $i_n^k(123) = 0$ if $n-k$ is odd. This implies that for odd $n$, $\text{fp}(\pi) = 1$ for all $\pi \in \mathcal{I}_n$, so $i_n^1(123) = |\mathcal{I}_n(123)| = \binom{n}{\frac{n-1}{2}}$. For even $n$, all we have to show is that $i_n^0(123) = i_n^2(123)$.

The bijection $\psi_\lrcorner : \mathcal{S}_n(123) \longrightarrow \mathcal{D}_n$ described in Section 2.3 has the property that $\pi \in \mathcal{I}_n(123)$ if and only if $\psi_\lrcorner(\pi)$ is a symmetric Dyck path. If $n$ is even, involutions $\pi \in \mathcal{I}_n$ with $\text{fp}(\pi) = 2$ are mapped to symmetric Dyck paths with a peak in the middle, and those with $\text{fp}(\pi) = 0$ are mapped to symmetric Dyck paths with a valley in the middle. We can establish a bijection between these two sets of Dyck paths just by changing the middle peak **ud** into a middle valley **du** (this can always be done because the height at the middle of a Dyck path of even semilength is always even, so it cannot be 1). This proves that $i_n^0(123) = i_n^2(123)$, and in particular it equals $\frac{1}{2}|\mathcal{I}_n(123)| = \binom{n-1}{\frac{n}{2}}$.

(2) We use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, which by Lemma 3.17 restricts to a bijection between $\mathcal{I}_n(132)$ and $\mathcal{D}s$. Thus, by Lemma 2.2, $G_{132}(x,z)$ can be expressed as $\sum_{D\in\mathcal{D}s} x^{\text{ct}(D)} z^{|D|}$, where the sum is over all symmetric Dyck paths. But the number of centered tunnels of a symmetric Dyck path is just its height at the middle. Therefore, taking only the first half of the path, $i_n^k(132)$ counts the number of paths from $(0,0)$ to $(n,k)$ never going below the $x$-axis, which equals the ballot number given in the theorem.

(3) Consider the bijection $\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$. Then $\pi \in \mathcal{I}_n(312)$ if and only if $\varphi(\bar{\pi})$ is a sequence of pyramids. Together with the proof of Proposition 3.7, this implies (see also [84]) that $\mathcal{I}_n(312) = \mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$. Recall that fixed points of $\pi$ are mapped to tunnels of depth 0 of $\varphi(\bar{\pi})$, which are produced by pyramids of odd size. Thus, as in Proposition 3.7,

$$G_{312}(x,z) = \frac{1}{1 - \frac{xz+z^2}{1-z^2}}.$$

□

## 3.3.2 Multiple restrictions

**Theorem 3.21 a)**

$$G_{\{123,132\}}(x,z) = G_{\{123,213\}}(x,z) = \frac{1 + xz + (x^2-1)z^2}{1-2z^2}$$

**b)**

$$G_{\{231,321\}}(x,z) = G_{\{312,321\}}(x,z) = \frac{1}{1-xz-z^2}$$

**c)**

$$G_{\{132,213\}}(x,z) = \frac{1-z^2}{(1-xz)(1-2z^2)}$$

**d)**

$$G_{\{231,312\}}(x,z) = \frac{1-z^2}{1-xz-2z^2}$$

**e)**

$$G_{\{132,231\}}(x,z) = G_{\{213,231\}}(x,z) = G_{\{132,312\}}(x,z) = G_{\{213,312\}}(x,z) = \frac{1+xz^3}{(1-xz)(1-z^2)}$$

**f)**

$$G_{\{132,321\}}(x,z) = G_{\{213,321\}}(x,z) = \frac{1}{(1-xz)(1-z^2)}$$

**g)**

$$G_{\{123,231\}}(x,z) = G_{\{123,312\}}(x,z) = \frac{1+xz+(x^2-1)z^2+xz^3+z^4}{(1-z^2)^2}$$

**h)**

$$G_{\{123,321\}}(x,z) = 1+xz+(x^2+1)z^2+2xz^3+2z^4$$

*Proof.* All the equalities between $G_\Sigma$ for different $\Sigma$ follow trivially from Lemma 3.16. To find expressions for these GFs, the idea is to use again the same bijections as in Section 3.1, between permutations avoiding two patterns of length 3 and certain subclasses of Dyck paths. The main difference is that here we will have to deal only with symmetric Dyck paths, as a consequence of Lemma 3.17.

**a)** From the proof of Proposition 3.1 and Lemma 3.17, we have that $\varphi$ restricts to a bijection between $\mathcal{I}_n(123,132)$ and symmetric Dyck paths $D \in \mathcal{D}_n$ of height at most 2. By Lemma 2.2, $\varphi$ maps fixed points to centered tunnels, so all we have to do is count elements $D \in \mathcal{D}s$ of height at most 2 according to the number of centered tunnels. Such a $D$ can be uniquely written as $D = ABC$, where $A = C^* \in \mathcal{D}^{\leq 2}$ and $B$ is either empty or has the form $B = \mathbf{u}B_1\mathbf{d}$, where $B_1$ is a sequence of hills. If $|B_1|$ is even (resp. odd), then $D$ has one (resp. two) centered tunnels, so the contribution of $B$ is $1 + \frac{(1+xz)xz}{1-z^2}$. The contribution of $A$ and $C$ is $\frac{1-z^2}{1-2z^2}$. The product of these two quantities gives the expression for $G_{\{123,132\}}(x, z)$.

**b)** As shown above and also in [84], we have that $\mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$. Therefore, $\mathcal{I}_n(231, 321) = \mathcal{S}_n(231, 312, 321)$. This case was treated in Theorem 3.15 **b)**.

**c)** From the proof of Proposition 3.6 and Lemma 3.17, we have that $\varphi$ gives a bijection between $\mathcal{I}_n(132, 213)$ and symmetric sequences of pyramids $D \in \mathcal{P}yr_n$, and that it maps fixed points of the permutation to centered tunnels of the Dyck path. Such a $D$ can be written uniquely as $D = ABC$, where $A = C^* \in \mathcal{P}yr$, and $B$ is either empty or a pyramid. The contribution of $B$ is $\frac{1}{1-xz}$, whereas $A$ and $C$ contribute $\frac{1-z^2}{1-2z^2}$. Multiplying these two expressions we get a formula for $G_{\{132,213\}}(x, z)$.

**d)** Again, $\mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$ implies that $\mathcal{I}_n(231, 312) = \mathcal{S}_n(231, 312)$, which has been considered in Proposition 3.7.

**e)** We have that $\mathcal{I}_n(132, 231) = \mathcal{S}_n(132, 231, 312)$, so the formula follows from Theorem 3.15 **f)**.

**f)** From the proof of Proposition 3.10 and Lemma 3.17, we have that $\varphi$ gives a bijection between $\mathcal{I}_n(132, 321)$ and symmetric paths $D \in \mathcal{D}_n$ with at most two peaks. Counting centered tunnels in such paths is very easy, since they have the form $D = \mathbf{u}^k B \mathbf{d}^k$, where $k \geq 0$ and $B$ is either empty or a pair of identical pyramids. The contribution of $B$ is $\frac{1}{1-z^2}$, whereas the rest contributes $\frac{1}{1-xz}$.

**g)** We have that $\mathcal{I}_n(123, 231) = \mathcal{S}_n(123, 231, 312)$, so the formula follows from Theorem 3.15 **g)**.

**h)** It is trivial since $\mathcal{S}_n(123, 321) = \emptyset$ for $n \geq 5$. □

The case of involutions avoiding simultaneously three or more patterns of length 3 is very easy and does not involve any new idea, so we omit it here.

As a final remark, let us point out that looking at the results of this chapter, one observes that the GFs $F_\Sigma(x, q, z)$ that we have obtained for $\Sigma \subseteq \mathcal{S}_3$ are all rational functions when $|\Sigma| \geq 2$. This is in contrast with the fact that they are not rational when $|\Sigma| = 1$, since in that case $F_\Sigma(1, 1, z) = \frac{1-\sqrt{1-4z}}{2z} = \mathbf{C}(z)$. For the case of involutions, all the GFs $G_\Sigma(x, z)$ for $\Sigma \subseteq \mathcal{S}_3$ are rational except when $\Sigma \in \{\{123\}, \{132\}, \{213\}, \{321\}\}$.

# 4

# A simple and unusual bijection for Dyck paths

In this chapter we introduce a new bijection $\Phi$ from the set of Dyck paths to itself. This bijection has the property that it maps nontrivial statistics that appear in the study of pattern-avoiding permutations into classical statistics on Dyck paths, which have been widely studied in the literature and whose distribution is easy to obtain.

When one tries to enumerate Dyck paths with respect to the number of centered and right tunnels directly, the standard decompositions of Dyck paths do not work. Intuitively, the problem is that unlike hills, peaks, or double rises, which are characteristics of a Dyck path that are defined locally, the notion of tunnel may involve an arbitrarily large number of steps of the path. The bijection $\Phi$ has the advantage that it transforms tunnel-like statistics into locally defined statistics that behave well under the usual decompositions of Dyck paths. As a consequence, several enumeration problems regarding permutation statistics on restricted permutations can be solved more easily considering their counterpart in terms of Dyck paths.

Another important application of $\Phi$ is that it allows us to give a simple bijective proof of Theorem 1.4, which is a weaker version of Theorem 2.3 considering only the number of fixed points. Some results in this chapter are joint work with Emeric Deutsch [33].

In Section 4.1 we present the bijection $\Phi$, and in Section 4.2 we study its properties. In Section 4.3 we give a generalization of $\Phi$, namely a family of bijections depending on an integer parameter $r$, from which the main bijection $\Phi$ is the particular case $r = 0$. These bijections give correspondences involving new statistics on Dyck paths, which generalize ct and rt. We give multivariate generating functions for them. Section 4.4 discusses several applications of these bijections to enumeration of statistics on 321- and 132-avoiding permutations. In particular, we generalize Theorem 1.4, and we find a multivariate generating function for fixed points, excedances and descents in 132-avoiding permutations. Finally, in Section 5.1 we discuss new interpretations of Catalan numbers that follow from our work.

## 4.1 The bijection $\Phi$

In this section we describe a bijection $\Phi$ from $\mathcal{D}_n$ to itself. Let $D \in \mathcal{D}_n$. Each up-step of $D$ has a corresponding down-step together with which it determines a tunnel. Match each such pair of steps. Let $\tau \in \mathcal{S}_{2n}$ be the permutation defined by

$$\tau_i = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd;} \\ 2n+1-\dfrac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

In two-line notation,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2n-2 & 2n-1 & 2n \\ 1 & 2n & 2 & 2n-1 & 3 & 2n-2 & \cdots & n+2 & n & n+1 \end{pmatrix}.$$

Then $\Phi(D)$ is created as follows. For $i$ from 1 to $2n$, consider the $\tau_i$-th step of $D$ (i.e., $D$ is read in zigzag). If its corresponding matching step has not yet been read, define the $i$-th step of $\Phi(D)$ to be an up-step, otherwise let it be a down-step. In the first case, we say that the $\tau_i$-th step of $D$ *opens* a tunnel, in the second we say that it *closes* a tunnel.

The bijection $\Phi$ applied to the Dyck paths of semilength at most 3 is shown in Figure 4.1. Figure 4.2 shows $\Phi$ applied to the example of the Dyck path $D =$ **uuduudududdud**.

It is clear from the definition that $\Phi(D)$ is a Dyck path. Indeed, it never goes below the $x$-axis because at any point the number of down-steps drawn so far can never exceed the number of up-steps, since each down-step is drawn when the second step of a matching pair in $D$ is read, and in that case the first step of the pair has already produced an up-step in $\Phi(D)$. Also, $\Phi(D)$ ends in $(2n, 0)$ because each of the matched pairs of $D$ produces an up-step and a down-step in $\Phi(D)$.

To show that $\Phi$ is indeed a bijection, we will describe the inverse map $\Phi^{-1}$. Given $D' \in \mathcal{D}_n$, the following procedure recovers the $D \in \mathcal{D}_n$ such

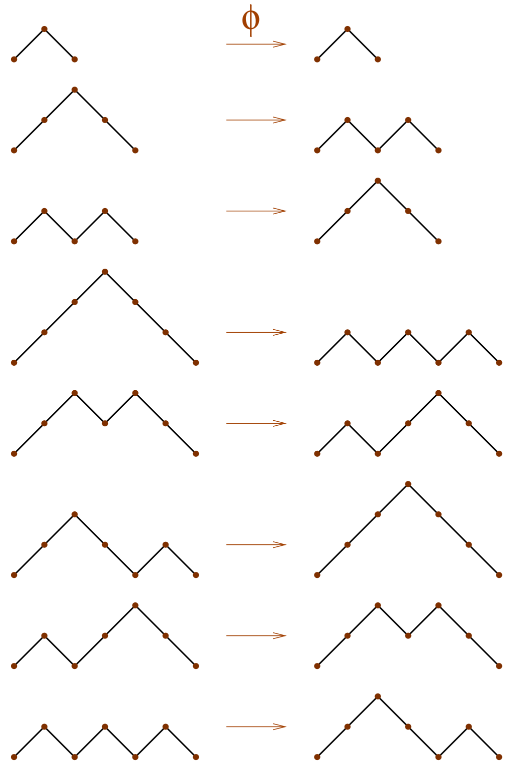


**Figure 4.1** The bijection Φ for paths of length at most 3.

that $\Phi(D) = D'$. Consider the permutation $\tau$ defined above, and let $W = w_1 w_2 \cdots w_{2n}$ be the word obtained from $D'$ as follows. For $i$ from 1 to $2n$, if the $i$-th step of $D'$ is an up-step, let $w_{\tau_i} = o$, otherwise let $w_{\tau_i} = c$. $W$ contains the same information as $D'$, with the advantage that the $o$'s are located in the positions of $D$ in which a tunnel is opened when $D$ is read in zigzag, and the $c$'s are located in the positions where a tunnel is closed. Equivalently, the $o$'s are located in the positions of the left walls of the left and centered tunnels of $D$, and in the positions of the right walls of the right tunnels. For an example see Figure 4.3.

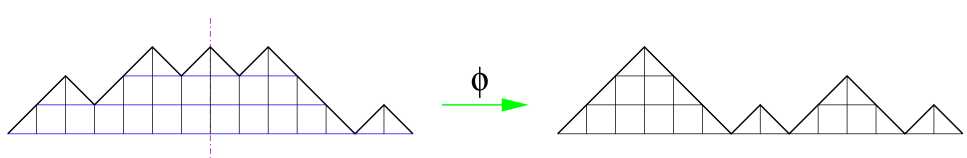


**Figure 4.2** An example of Φ.

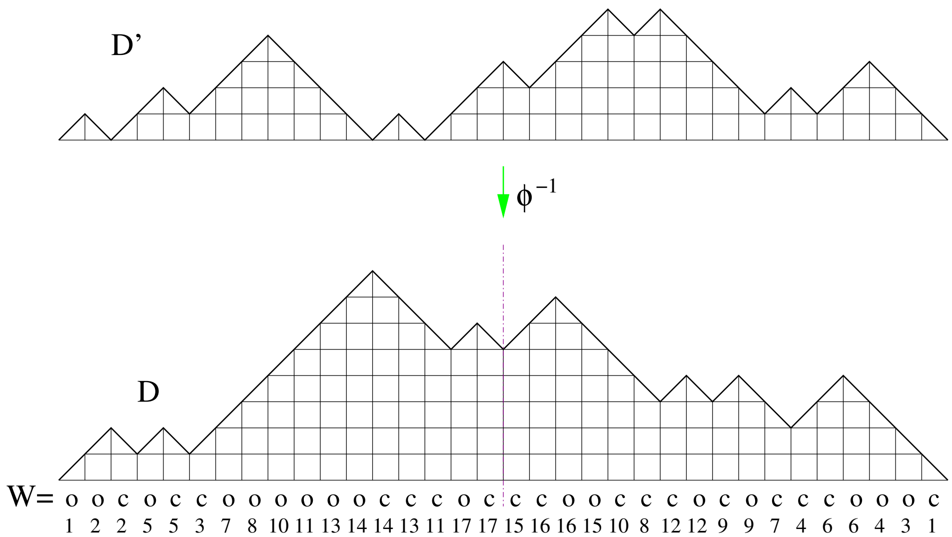


**Figure 4.3** The inverse of $\Phi$.

Now we define a matching between the $o$'s and the $c$'s in $W$, so that each matched pair will give a tunnel in $D$. We will label the $o$'s with $1, 2, \ldots, n$ and similarly the $c$'s, to indicate that an $o$ and a $c$ with the same label are matched. By left (resp. right) half of $W$ we mean the symbols $w_i$ with $i \leq n$ (resp. $i > n$). For $i$ from 1 to $2n$, if $w_{\tau_i} = o$, place in it the smallest label that has not been used yet. If $w_{\tau_i} = c$, match it with the unmatched $o$ in the same half of $W$ as $w_{\tau_i}$ with largest label, if such an $o$ exists. If it does not, match $w_{\tau_i}$ with the unmatched $o$ in the opposite half of $W$ with smallest label. Note that since $D'$ was a Dyck path, at any time the number of $c$'s read so far does not exceed the number of $o$'s, so each $c$ has some $o$ to be paired up with.

Once the symbols in $W$ have been labelled, $D$ can be recovered by reading the labels from left to right, drawing an up-step for each label that is read for the first time, and a down-step for each label that appears the second time. In Figure 4.3 the labelling is shown under $W$.

## 4.2 Properties of $\Phi$

**Lemma 4.1** *Let $D = ABC$ be a decomposition of a Dyck path $D$, where $B$ is a Dyck path, and $A$ and $C$ have the same length. Then $\Phi(ABC) = \Phi(AC)\Phi(B)$. In particular, $\Phi(\mathbf{u}B\mathbf{d}) = \mathbf{ud}\Phi(B)$.*

*Proof.* It follows immediately from the definition of $\Phi$, since the path $D$ is read in zigzag while $\Phi(D)$ is built from left to right. □

**Theorem 4.2** *Let $D$ be any Dyck path, and let $D' = \Phi(D)$. We have the following correspondences:*

(1) $\mathrm{ct}(D) = h(D')$,

(2) $\mathrm{rt}(D) = \mathrm{er}(D')$,

(3) $\mathrm{lt}(D) + \mathrm{ct}(D) = \mathrm{or}(D')$,

(4) $\mathrm{cmt}(D) = \mathrm{ret}(D')$.

*Proof.* First we show (1). Consider a centered tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$. Applying Lemma 4.1 twice, we get $D' = \Phi(A\mathbf{u}B\mathbf{d}C) = \Phi(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi(AC)\mathbf{ud}\Phi(B)$, so we have a hill $\mathbf{ud}$ in $D'$. Reciprocally, any hill in $D'$, say $D' = X\mathbf{ud}Y$, where $X, Y \in \mathcal{D}$, comes from a centered tunnel $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where $Z_1$ and $Z_2$ are respectively the first and second halves of $\Phi^{-1}(X)$.

The proof of (4) is very similar. Recall that $\mathrm{ret}(D')$ equals the number of arches of $D'$. Given a centered multitunnel corresponding to the decomposition $D = ABC$, we have $\Phi(D) = \Phi(AC)\Phi(B)$, so $D'$ has an arch starting at the first step of $\Phi(B)$, which is nonempty.

To show (2), consider a right tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$, where $\mathrm{length}(A) > \mathrm{length}(C)$. Of the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel, $\mathbf{d}$ will be encountered before $\mathbf{u}$ when $D$ is read in zigzag, since $\mathrm{length}(A) > \mathrm{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step in $D'$. Besides, this up-step will be at an even position, since $\mathbf{d}$ was in the right half of $D$. Reciprocally, an even rise of $D'$ corresponds to a step in the right half of $D$ that opens a tunnel when $D$ is read in zigzag, so it is necessarily a right tunnel.

Relation (3) follows from (2) and the fact that the total number of tunnels of $D$ is $\mathrm{lt}(D) + \mathrm{ct}(D) + \mathrm{rt}(D) = n$, and the total number of up-steps of $D'$ is $\mathrm{or}(D') + \mathrm{er}(D') = n$. □

One interesting application of this bijection is that it can be used to enumerate Dyck paths according to the number of centered, left, and right tunnels, and number of centered multitunnels. We are looking for a multivariate generating function for these four statistics, namely

$$\widetilde{R}(x,u,v,w,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)} u^{\mathrm{lt}(D)} v^{\mathrm{rt}(D)} w^{\mathrm{cmt}(D)} z^{|D|}.$$

By Theorem 4.2, this GF can be expressed as

$$\widetilde{R}(x,u,v,w,z) = R(\frac{x}{u},u,v,w,z), \tag{4.1}$$

where

$$R(t,u,v,w,z) = \sum_{D\in\mathcal{D}} t^{h(D)} u^{\mathrm{or}(D)} v^{\mathrm{er}(D)} w^{\mathrm{ret}(D)} z^{|D|}.$$

We can derive an equation for $R$ using again that every nonempty Dyck path $D$ can be decomposed in a unique way as $D = \mathbf{u}A\mathbf{d}B$, where $A, B \in \mathcal{D}$. The number of hills of $\mathbf{u}A\mathbf{d}B$ is $h(B)+1$ if $A$ is empty, and $h(B)$ otherwise. The odd rises of $A$ become even rises of $\mathbf{u}A\mathbf{d}B$, and the even rises of $A$ become odd rises of $\mathbf{u}A\mathbf{d}B$. Thus, we have $\mathrm{er}(\mathbf{u}A\mathbf{d}B) = \mathrm{or}(A) + \mathrm{er}(B)$, and $\mathrm{or}(\mathbf{u}A\mathbf{d}B) = \mathrm{er}(A) + \mathrm{or}(B) + 1$, where the extra odd rise comes from the first step $\mathbf{u}$. We also have $\mathrm{ret}(\mathbf{u}A\mathbf{d}B) = \mathrm{ret}(B) + 1$. Hence, we obtain the following equation for $R$:

$$R(t,u,v,w,z) = 1 + uwz(R(1,v,u,1,z) - 1 + t)R(t,u,v,w,z). \tag{4.2}$$

Denote $R_1 := R(1,u,v,1,z)$, $\widehat{R}_1 := R(1,v,u,1,z)$. Substituting $t = w = 1$ in (4.2), we obtain

$$R_1 = 1 + uz\widehat{R}_1 R_1, \tag{4.3}$$

and interchanging $u$ and $v$,

$$\widehat{R}_1 = 1 + vzR_1\widehat{R}_1. \tag{4.4}$$

Solving (4.3) and (4.4) for $\widehat{R}_1$, gives

$$\widehat{R}_1 = \frac{1+(u-v)z-\sqrt{1-2(v+u)z+(v-u)^2z^2}}{2uz}.$$

Thus, from (4.2),

$$R(t,u,v,w,z) = \frac{1}{1-uwz(\widehat{R}_1-1+t)}$$
$$\frac{2}{2-w+(v+u-2tu)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}. \tag{4.5}$$

Now, switching to $\widetilde{R}$, we obtain the following theorem.

**Theorem 4.3** *The multivariate generating function for Dyck paths according to centered, left, and right tunnels, centered multitunnels, and semilength is*

$$\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}u^{\mathrm{lt}(D)}v^{\mathrm{rt}(D)}w^{\mathrm{cmt}(D)}z^{|D|}$$
$$= \frac{2}{2-w+(v+u-2x)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.$$

As pointed out by Alex Burstein, Lagrange inversion applied to equation (4.2) gives a nice expression for the coefficients of $\widetilde{R}$.

**Corollary 4.4** *Fix integers $c,l,r,m\geq 0$ and let $n=c+l+r$. The number of Dyck paths $D\in\mathcal{D}_n$ with $\mathrm{ct}(D)=c$, $\mathrm{lt}(D)=l$, $\mathrm{rt}(D)=r$ and $\mathrm{cmt}(D)=m$ is given by*

$$\frac{m-c}{n-m}\binom{m}{c}\binom{n-m}{l}\binom{n-m}{r}$$

*if $c<m<c+l<n$, and it is 1 if $c=m=n$.*

*Proof.* The case $c=m=n$ is trivial, since the only path in $\mathcal{D}_n$ with $n$ centered tunnels is $D=\mathbf{u}^n\mathbf{d}^n$. For the rest of the proof we assume that $0\leq c<m<c+l<n$.

We start by applying Lagrange inversion formula (Theorem 1.3) to equation (4.2) for variable $w$, being $f(w)=R(t,u,v,w,z)-1$, $G(w)=uz(\widehat{R}_1-1+t)(w+1)$, $n=m$ and $k=1$ in the theorem. We get that

$$\begin{aligned}[w^m](R(t,u,v,w,z)-1) &= \frac{1}{m}[w^{m-1}](uz(\widehat{R}_1-1+t)(w+1))^m \\ &= u^m z^m(\widehat{R}_1-1+t)^m.\end{aligned}$$

Taking the coefficient of $t^c$,

$$[t^c w^m](R(t,u,v,w,z)-1)=\binom{m}{c}u^m z^m(\widehat{R}_1-1)^{m-c}. \tag{4.6}$$

Isolating $R_1$ in equation (4.3) and substituting it in (4.4) we get

$$\widehat{R}_1 = 1+\frac{vz\widehat{R}_1}{1-uz\widehat{R}_1},$$

which is equivalent to

$$\widehat{R}_1 - 1 = z\widehat{R}_1(u(\widehat{R}_1 - 1) + v).$$

We apply Lagrange inversion formula again, now for variable $z$, with $f(z) = \widehat{R}_1 - 1$, $G(z) = (z+1)(uz+v)$ and $n = s$. This gives us (for $s \neq 0$),

$$[z^s](\widehat{R}_1 - 1)^k = \frac{k}{s}\,[z^{s-k}](z+1)^s(uz+v)^s,$$

so

$$[u^{s-r}v^r z^s](\widehat{R}_1 - 1)^k = \frac{k}{s}\binom{s}{r}[z^{s-k}](z+1)^s z^{s-r} = \frac{k}{s}\binom{s}{r}\binom{s}{r-k}. \qquad (4.7)$$

Now, taking the appropriate coefficients of $u$, $v$ and $z$ in equation (4.6) gives

$$\begin{aligned}[t^c u^{n-r} v^r w^m z^n](R(t,u,v,w,z) - 1) &= \binom{m}{c}[u^{n-m-r}v^r z^{n-m}](\widehat{R}_1 - 1)^{m-c} \\ &= \binom{m}{c}\frac{m-c}{n-m}\binom{n-m}{r}\binom{n-m}{r-m+c},\end{aligned}$$

where the last equality follows from (4.7) with $s = n - m$ and $k = m - c$. Thus, using that $n - r = c + l$ and $n - m - (r - m + c) = l$, we get that for $n \geq 1$,

$$[t^c u^{c+l} v^r w^m z^n]R(t,u,v,w,z) = \frac{m-c}{n-m}\binom{m}{c}\binom{n-m}{r}\binom{n-m}{l}.$$

But by relation (4.1), this coefficient is precisely $[x^c u^l v^r w^m z^n]\widetilde{R}(x,u,v,w,z)$, which is the number of paths $D \in \mathcal{D}_n$ with $\mathrm{ct}(D) = c$, $\mathrm{lt}(D) = l$, $\mathrm{rt}(D) = r$ and $\mathrm{cmt}(D) = m$. □

## 4.3 Generalizations

Here we present a generalization $\Phi_r$ of the bijection $\Phi$, which depends on a nonnegative integer parameter $r$. Given $D \in \mathcal{D}_n$, copy the first $2r$ steps of $D$ into the first $2r$ steps of $\Phi_r(D)$. Now, read the remaining steps of $D$ in zigzag in the following order: $2r+1$, $2n$, $2r+2$, $2n-1$, $2r+3$, $2n-2$, and so on. For each of these steps, if its corresponding matching step in $D$ has not yet been encountered, draw an up-step in $\Phi_r(D)$, otherwise draw a down-step. Note that for $r = 0$ we get the same bijection $\Phi$ as before.

Note that $\Phi_r$ can be defined exactly as $\Phi$ with the difference that instead of $\tau$, the permutation that gives the order in which the steps of $D$ are read is $\tau^{(r)} \in \mathcal{S}_{2n}$, defined as

$$\tau_i^{(r)} = \begin{cases} i & \text{if } i \leq 2r; \\ \dfrac{i+1}{2} + r & \text{if } i > 2r \text{ and } i \text{ is odd;} \\ 2n + 1 - \dfrac{i}{2} - r & \text{if } i > 2r \text{ and } i \text{ is even.} \end{cases}$$

Figure 4.4 shows an example of the bijection $\Phi_r$ for $r = 2$ applied to the path $D =$ **uduuduuduududdddudd**.

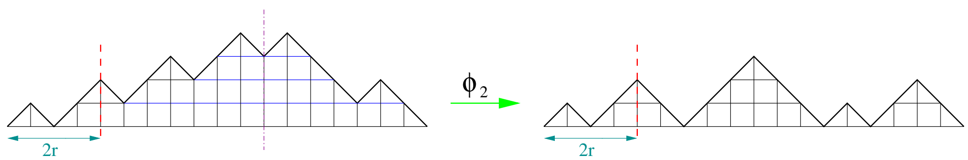


**Figure 4.4** An example of $\Phi_2$.

It is clear from the definition that $\Phi_r(D)$ is a Dyck path. A reasoning similar to the one used for $\Phi$ shows that $\Phi_r$ is indeed a bijection.

The properties of $\Phi$ given in Theorem 4.2 generalize to analogous properties of $\Phi_r$. We will prove them using the next lemma, which follows immediately from the definition of $\Phi_r$.

**Lemma 4.5** *Let $r \geq 0$, and let $D = ABC$ be a decomposition of a Dyck path $D$, where $B$ is a Dyck path, and* $\text{length}(A) = \text{length}(C) + 2r$. *Then* $\Phi_r(ABC) = \Phi_r(AC)\Phi(B)$.

**Theorem 4.6** *Let $r \geq 0$, let $D$ be any Dyck path, and let $D' = \Phi_r(D)$. We have the following correspondences:*

(1) $\#\{$*tunnels of $D$ with midpoint at* $x = n + r\}$
$= \#\{$*hills of $D'$ in* $x > 2r\}$,

(2) $\#\{$*tunnels of $D$ with midpoint in* $x > n + r\}$
$= \#\{$*even rises of $D'$ in* $x > 2r\}$,

(3) $\#\{$*tunnels of $D$ with midpoint in* $x \leq n + r\}$
$= \#\{$*odd rises of $D'$ in* $x > 2r\} + \#\{$*up-steps of $D'$ in* $x \leq 2r\}$,

(4) $\#\{$*multitunnels of $D$ with midpoint at* $x = n + r\}$
$= \#\{$*arches of $D'$ in* $x \geq 2r\}$.

*Proof.* Fist we show (1). A tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$ has its midpoint at $x = n + r$ exactly when $\text{length}(A) = \text{length}(C) + 2r$. Applying Lemmas 4.5 and 4.1, $D' = \Phi_r(A\mathbf{u}B\mathbf{d}C) = \Phi_r(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi_r(AC)\Phi(\mathbf{ud})\Phi(B) = \Phi_r(AC)\mathbf{ud}\Phi(B)$, and $\mathbf{ud}$ is a hill of $D'$ in $x > 2r$, since $\text{length}(\Phi_r(AC)) \geq 2r$. Reciprocally, any hill of $D'$ in $x > 2r$, say $D' = X\mathbf{ud}Y$, where $X, Y \in \mathcal{D}$ and $\text{length}(X) \geq 2r$, comes from a tunnel with midpoint at $x = n+r$, namely $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where $Z_1Z_2 = \Phi_r^{-1}(X)$ and $\text{length}(Z_1) = \text{length}(Z_2) + 2r$.

The proof of (4) is very similar. A multitunnel given by $D = ABC$ has its midpoint at $x = n+r$ exactly when $\text{length}(A) = \text{length}(C)+2r$. In this case, $\Phi_r(D) = \Phi_r(AC)\Phi(B)$ by Lemma 4.5, so $D'$ has an arch starting at the first step of $\Phi(B)$. Notice that this arch is in $x \geq 2r$ because $\text{length}(\Phi_r(AC)) \geq 2r$.

To show (2), consider a tunnel in $D$ with midpoint in $x > n + r$. This is equivalent to saying that it is given by a decomposition $D = A\mathbf{u}B\mathbf{d}C$ with $\text{length}(A) > \text{length}(C) + 2r$. In particular, the tunnel is contained in the halfspace $x \geq 2r$, so the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel are in the part of $D$ that is read in zigzag in the process to obtain $\Phi_r(D)$, and $\mathbf{d}$ will be encountered before $\mathbf{u}$, since $\text{length}(A) - 2r > \text{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step of $D'$ in $x > 2r$. Besides, this up-step will be at an even position, since $\mathbf{d}$ is in $x > n + r$, that is, in the right half of the part of $D$ that is read in zigzag. Reciprocally, an even rise of $D'$ in $x > 2r$ corresponds to a step of $D$ in $x > n + r$ that opens a tunnel when $D$ is read according to $\tau^{(r)}$, so it is necessarily a tunnel with midpoint to the right of $x = n + r$ .

Relation (3) follows from (2) and the fact that the total number of tunnels of $D$ is $\#\{\text{tunnels of } D \text{ with midpoint in } x > n+r\} + \#\{\text{tunnels of } D \text{ with midpoint in } x \leq n + r\} = n$, and the total number of up-steps of $D'$ is $\#\{\text{even rises of } D' \text{ in } x > 2r\} + \#\{\text{odd rises of } D' \text{ in } x > 2r\} + \#\{\text{up-steps of } D' \text{ in } x \leq 2r\} = n$. □

Similarly to how we used the properties of $\Phi$ to prove Theorem 4.3, we can use the properties of $\Phi_r$ to prove a more general theorem. Our goal is to enumerate Dyck paths according to the number of tunnels with midpoint on, to the right of, and to the left of an arbitrary vertical line $x = n + r$, and multitunnels with midpoint on that line. In generating function terms, we are looking for an expression for

$$E(t,u,v,w,y,z) := \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \sum_{D\in\mathcal{D}_n} t^{\alpha}u^{\beta}v^{\gamma}w^{\delta}y^{r}z^{n},$$

where
$\alpha = \#\{\text{tunnels of } D \text{ with midpoint at } x = n+r\}$,
$\beta = \#\{\text{tunnels of } D \text{ with midpoint in } x \leq n+r\}$,
$\gamma = \#\{\text{tunnels of } D \text{ with midpoint in } x > n+r\}$,
$\delta = \#\{\text{multitunnels of } D \text{ with midpoint at } x = n+r\}$.

Note that the variable $y$ marks the position of the vertical line $x = n+r$ with respect to which the tunnels are classified. The following theorem gives an expression for $E$.

**Theorem 4.7** *Let $E$, $R$ and $\mathbf{C}$ be defined as above. Then,*

$$\begin{aligned} E(t,u,v,w,y,z) &= \frac{\mathbf{C}(uyz)R(t,u,v,w,z)}{1-yu^2z^2\mathbf{C}^2(uyz)R(1,u,v,1,z)R(1,v,u,1,z)} \\ &= \frac{2\delta_2(2+(v-u)z+\delta_1)}{[2+(u+v-2tu)wz+w\delta_1]\,[(\delta_1+(v-u)z)\delta_2-4uyz]}, \end{aligned}$$

*where* $\delta_1 := \sqrt{1-2(u+v)z+(u-v)^2z^2}-1$, $\delta_2 := \sqrt{1-4uyz}-1$.

*Proof.* By Theorem 4.6, the generating function $E$ can be expressed as

$$\begin{aligned} &E(t,u,v,w,y,z) = \\ &\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \sum_{D\in\mathcal{D}_n} t^{\#\{\text{hills of } D \text{ in } x>2r\}} \\ &\qquad u^{\#\{\text{odd rises of } D \text{ in } x>2r\}+\#\{\text{up-steps of } D \text{ in } x\leq 2r\}} \\ &\qquad v^{\#\{\text{even rises of } D \text{ in } x>2r\}} w^{\#\{\text{arches of } D \text{ in } x\geq 2r\}} y^r z^n. \end{aligned} \tag{4.8}$$

For each path $D$ in this summation, the $y$-coordinate of its intersection with the vertical line $x = 2r$ has to be even. Fix $h \geq 0$. We will now focus only on the paths $D \in \mathcal{D}$ for which this intersection has $y$-coordinate equal to $2h$. Let $D = AB$, where $A$ and $B$ are the parts of the path respectively to the left and to the right of $x = 2r$. Then, $\#\{\text{hills of } D \text{ in } x > 2r\} = \#\{\text{hills of } B\}$, $\#\{\text{odd rises of } D \text{ in } x > 2r\} = \#\{\text{odd rises of } B\}$, $\#\{\text{up-steps of } D \text{ in } x \leq 2r\} = \#\{\text{up-steps of } A\}$, and $\#\{\text{arches of } D \text{ in } x \geq 2r\} = \#\{\text{arches of } B\}$.

$B$ can be any path starting at height $2h$ and landing on the $x$-axis, never going below it. If $h > 0$, consider the first down-step of $B$ that lands at height $2h-1$. Then $B$ can be decomposed as $B = B_1\mathbf{d}B'$, where $B_1$ is any Dyck path, and $B'$ is any path starting at height $2h-1$ and landing on the $x$-axis, never going below it. Applying this decomposition recursively, $B$ can be written uniquely as $B = B_1\mathbf{d}B_2\mathbf{d}\cdots B_{2h}\mathbf{d}B_{2h+1}$, where the $B_i$'s for $1 \le i \le 2h+1$ are arbitrary Dyck paths. The number of hills and number of arches of $B$ are given by those of $B_{2h+1}$. The odd rises of $B$ are the odd rises of the $B_i$'s with odd subindex plus the even rises of those with even subindex. In a similar way one can describe the even rises of $B$. The semilength of $B$ is the sum of semilengths of the $B_i$'s plus $h$, which comes from the $2h$ additional down-steps. Thus, the generating function for all paths $B$ of this form, where $t$, $u$, $v$, and $z$ mark respectively number of hills, number of odd rises, number of even rises, and semilength, is

$$z^h R^h(1,u,v,1,z) R^h(1,v,u,1,z) R(t,u,v,w,z). \tag{4.9}$$

Similarly, $A$ can be decomposed uniquely as $A = A_1\mathbf{u}A_2\mathbf{u}\cdots A_{2h}\mathbf{u}A_{2h+1}$. The number of up-steps of $A$ is the sum of the number of up-steps of each $A_i$, plus a $2h$ term that comes from the additional up-steps. The generating function for paths $A$ of this form, where $u$ marks the number of up-steps, and $y$ and $z$ mark both the semilength, is

$$z^h y^h u^{2h} \mathbf{C}^{2h+1}(uyz). \tag{4.10}$$

The product of (4.9) and (4.10) gives the generating function for paths $D = AB$ where the height of the intersection point of $D$ with the vertical line between $A$ and $B$ is $2h$, where the variables mark the same statistics as in (4.8). Note that the exponent of $y$ is half the distance between the origin of $D$ and this vertical line. Summing over $h$, we obtain

$$\begin{aligned} &E(t,u,v,w,y,z) \\ &\quad= \sum_{h\ge 0} z^{2h} y^h u^{2h} \mathbf{C}^{2h+1}(uyz) R^h(1,u,v,1,z) R^h(1,v,u,1,z) R(t,u,v,w,z) \\ &\quad= \frac{\mathbf{C}(uyz) R(t,u,v,w,z)}{1 - yu^2z^2\mathbf{C}^2(uyz) R(1,u,v,1,z) R(1,v,u,1,z)}. \end{aligned}$$

The second expression in the statement of the theorem follows from the formula (4.5) that we had for $R$. □

## 4.4 Connection to pattern-avoiding permutations

The bijection $\Phi$ has applications to enumeration of statistics on pattern-avoiding permutations. The first one is that it can be used together with the bijections defined in Chapter 2 to give another bijective proof of Theorem 1.4. Here we show a more general result. We use the bijection $\Phi_r$ to give a combinatorial proof of the following generalization of Theorem 1.4. Note that the particular case $r = 0$ gives a new bijective proof of such theorem.

**Theorem 4.8** *Fix $r, n \geq 0$. For any $\pi \in \mathcal{S}_n$, define $\alpha_r(\pi) = \#\{i : \pi_i = i + r\}$, $\beta_r(\pi) = \#\{i : i > r, \pi_i = i\}$. Then, the number of 321-avoiding permutations $\pi \in \mathcal{S}_n$ with $\beta_r(\pi) = k$ equals the number of 132-avoiding permutations $\pi \in \mathcal{S}_n$ with $\alpha_r(\pi) = k$, for any $0 \leq k \leq n$.*

*Proof.* Recall that the bijection $\psi_{\llcorner} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ defined in Section 2.2.4 satisfies that $\mathrm{fp}(\pi) = h(\psi_{\llcorner}(\pi))$. More precisely, it can be easily checked that $i$ is a fixed point of $\pi$ if and only if $\psi_{\llcorner}(\pi)$ has a hill with $x$-coordinate $2i - 1$. This implies that $\beta_r(\pi) = \#\{\text{hills of } \psi_{\llcorner}(\pi) \text{ in } x > 2r\}$.

The second bijection that we use is $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, defined in Section 2.1. In Proposition 2.1 we showed that $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$. Recall that in the proof of this proposition, we associated a unique tunnel of $D$ to each cross of the array $\mathrm{arr}(\pi)$. An element $i$ with $\pi_i = i + r$ is represented by a cross $(i, i + r)$ in the array. From the description of the association between crosses and tunnels, it follows that such a cross $(i, i + r)$ corresponds to a tunnel of $\varphi(\pi)$ with midpoint $r$ units to the right of the center. That is, an element $i$ with $\pi_i = i+r$ gives a tunnel with midpoint at $x = n+r$. Therefore, we have that $\alpha_r(\pi) = \#\{\text{tunnels of } \varphi(\pi) \text{ with midpoint at } x = n + r\}$.

Now all we need to do is use $\Phi_r$ and property (1) given in Theorem 4.6. From this it follows that the bijection $\psi_{\llcorner}^{-1} \circ \Phi_r \circ \varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{S}_n(321)$ has the property that $\beta_r(\psi_{\llcorner}^{-1} \circ \Phi_r \circ \varphi(\pi)) = \#\{\text{hills of } \Phi_r \circ \varphi(\pi) \text{ in } x > 2r\} = \#\{\text{tunnels of } \varphi(\pi) \text{ with midpoint at } x = n + r\} = \alpha_r(\pi)$. □

While in Section 2.2.4 we describe a simple way to enumerate 321-avoiding permutations with respect to the statistics fp and exc, the analogous enumeration for 132-avoiding permutations is harder to do directly. Here we use the properties of $\Phi$ to give a more direct derivation of the multivariate generating function for 132-avoiding permutations according to the number of fixed points and the number of excedances.

**Corollary 4.9 (of Theorem 4.3)**

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}v^{\mathrm{exc}(\pi)}z^n = \frac{2}{1+(1+v-2x)z+\sqrt{1-2(1+v)z+(1-v)^2z^2}}. \quad (4.11)$$

*Proof.* Proposition 2.1 shows that $\varphi$ maps fixed points to centered tunnels, and excedances to right tunnels, i.e., $\mathrm{fp}(\pi)=\mathrm{ct}(\varphi(\pi))$ and $\mathrm{exc}(\pi)=\mathrm{rt}(\varphi(\pi))$. Thus, the left hand side of (4.11) equals $\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}v^{\mathrm{rt}(D)}z^{|D|}$. The result now is obtained applying Theorem 4.3 for $u=w=1$. □

Comparing this expression (4.11) with the equation obtained in 2.2.4 for $F_{321}(x,q,z)$, we obtain another proof of Theorem 2.3.

As a further application, we can use the bijection $\Phi$ to give the following refinement of Corollary 4.9, which gives an expression for the multivariate generating function for number of fixed points, number of excedances, and number of descents in 132-avoiding permutations. The analogous result for 321-avoiding permutations is given in Theorem 2.10.

**Theorem 4.10** *Let*

$$L(x,q,p,z) := 1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)+1}z^n.$$

*Then*

$$L(x,q,p,z) = \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{f_1(q,z)}}, \quad (4.12)$$

*where* $f_1(q,z)=1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2-2q(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$.

*Proof.* We use again that $\varphi$ maps fixed points to centered tunnels, and excedances to right tunnels. It is shown in Proposition 2.1 that it also maps descents of the permutation to valleys of the corresponding Dyck path. Clearly, the number of valleys of any nonempty Dyck path equals the number of peaks minus one (in the empty path both numbers are 0). Thus, $L$ can be expressed as

$$L(x,q,p,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}.$$

By Theorem 4.2, $\Phi$ maps centered tunnels into hills and right tunnels into even rises. Let us see what peaks are mapped to by $\Phi$. Given a peak $\mathbf{ud}$ in $D \in \mathcal{D}$, $D$ can be written as $D = A\mathbf{ud}C$, where $A$ and $C$ are the parts of the path before and after the peak respectively. This decomposition corresponds to a tunnel of $D$ that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$. Assume first that the peak occurs in the left half (i.e., length$(A) <$ length$(C)$). When $D$ is read in zigzag, the $\mathbf{u}$ opens a tunnel that is closed by the $\mathbf{d}$ two steps later. This produces in $\Phi(D)$ an up-step followed by a down-step two positions ahead, that is, an occurrence of $\mathbf{u}\star\mathbf{d}$ in the Dyck word of $\Phi(D)$, where $\star$ stands for any symbol (either a $\mathbf{u}$ or a $\mathbf{d}$).

If the peak occurs in the right half of $D$ (i.e., length$(A) >$ length$(C)$), the reasoning is analogous, with the difference that the $\mathbf{d}$ opens a tunnel that is closed by the $\mathbf{u}$ two steps ahead. So, such a peak produces also an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$. Reciprocally, we claim that an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ can come only from a peak of $D$ either in the left or in the right half. Indeed, using the notation from the procedure above describing the inverse of $\Phi$, an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ corresponds to either an occurrence of $oc$ in the left half of $W$ or an occurrence of $co$ in the right half of $W$. In both cases, the algorithm given above will match these two letters $c$ and $o$ with each other, so they correspond to an occurrence of $\mathbf{ud}$ in $D$.

If the peak occurs in the middle (i.e., length$(A) =$ length$(C)$), then by Lemma 4.1, $\Phi(A\mathbf{ud}C) = \Phi(AC)\mathbf{ud}$, so it is mapped to an occurrence of $\mathbf{ud}$ at the end of $\Phi(D)$. Clearly we have such an occurrence only when $D$ has a peak in the middle.

Thus, we have shown that peaks in $D$ are mapped by $\Phi$ to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ and occurrences of $\mathbf{ud}$ at the end of $\Phi(D)$, or, equivalently, to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)\mathbf{d}$ (here $\Phi(D)\mathbf{d}$ is a Dyck path followed by a down-step). Denote by $\lambda(D)$ the number of occurrences of $\mathbf{u}\star\mathbf{d}$ in $D\mathbf{d}$. This implies that $L$ can be written as

$$L(x,q,p,z) = \sum_{D \in \mathcal{D}} x^{h(D)} q^{\mathrm{er}(D)} p^{\lambda(D)} z^{|D|}.$$

We are left with a Dyck path enumeration problem, which is solved in the following lemma. Let $J$ be defined in Lemma 4.11. It is easy to see that we have $L(x,q,p,z) = 1 + J(x,1,p,1,q,p,z)$. Making use of (4.13) and (4.14), it follows at once that

$$L(x,q,p,z) = \frac{1 - xz + xpz}{1 - xz - zK_1},$$

where $K_1$ is given by

$$zK_1^2 - [1 - z - qz + q(1-p)^2z^2]K_1 + p^2qz = 0.$$

From these equations we obtain (4.12). □

**Lemma 4.11** *Denote by* $\mathrm{ih}(D)$ *(*$\mathrm{fh}(D)$*) the number of initial (final) hills in* $D$ *(obviously, their only possible values are 0 and 1). Denote by* $\mu(D)$ *the number of occurrences of* $\mathbf{u} \star \mathbf{d}$ *in* $D$*. Then the generating function*

$$J(x,t,s,u,v,y,z) := \sum x^{h(D)} t^{\mathrm{ih}(D)} s^{\mathrm{fh}(D)} u^{\mathrm{or}(D)} v^{\mathrm{er}(D)} y^{\mu(D)} z^{|D|},$$

*where the summation is over all nonempty Dyck paths, is given by*

$$J(x,t,s,u,v,y,z) = \frac{uz[xts + (1 - xu(1-t)(1-s)z)K]}{1 - xuz - uzK}, \tag{4.13}$$

*where* $K$ *is given by*

$$uzK^2 - [1 - (u+v)z + uv(1-y)^2z^2]K + y^2vz = 0. \tag{4.14}$$

*Proof.* Every nonempty Dyck path has one of the following four forms: $\mathbf{ud}$, $\mathbf{ud}B$, $\mathbf{u}A\mathbf{d}$, or $\mathbf{u}A\mathbf{d}B$, where $A$ and $B$ are nonempty Dyck paths. The generating functions of these four pairwise disjoint sets of Dyck paths are

(i) $xtsuz$,

(ii) $xtuzJ(x,1,s,u,v,y,z)$,

(iii) $uzJ(1,y,y,v,u,y,z)$,

(iv) $uzJ(1,y,y,v,u,y,z)J(x,1,s,u,v,y,z)$,

respectively. Only the third factor in (iii) and (iv) needs an explanation: the hills of $A$ are not hills in $\mathbf{u}A\mathbf{d}$; an initial (final) hill in $A$ gives a $\mathbf{uud}$ ($\mathbf{udd}$) in $\mathbf{u}A\mathbf{d}$; an odd (even) rise in $A$ becomes an even (odd) rise in $\mathbf{u}A\mathbf{d}$.

Consequently, the generating function $J$ satisfies the functional equation

$$\begin{aligned} J(x,t,s,u,v,y,z) &= xtsuz + xtuzJ(x,1,s,u,v,y,z) \\ &+ uzJ(1,y,y,v,u,y,z) + uzJ(1,y,y,v,u,y,z)J(x,1,s,u,v,y,z). \end{aligned} \tag{4.15}$$

From equation (4.15) it is clear that, whether interested or not in the statistics 'number of initial (final) hills', we had to introduce them for the sake of the statistic marked by the variable $y$. Also, without any additional effort we could use two separate variables to mark the number of **uud**'s and the number of **udd**'s, and obtain a slightly more general generating function, although we do not need it here.

Denoting $H = J(x, 1, s, u, v, y, z)$, $K = J(1, y, y, v, u, y, z)$, equation (4.15) becomes

$$J = xtsuz + xtuzH + uzK + uzHK. \tag{4.16}$$

Setting here $t = 1$, we obtain

$$H = xsuz + xuzH + uzK + uzHK. \tag{4.17}$$

Solving (4.17) for $H$ and introducing it into (4.16), we obtain (4.13).

It remains to show that $K$ satisfies the quadratic equation (4.14). Setting $x = 1$, $t = y$, $s = y$ in (4.16), and interchanging $u$ and $v$, we get

$$K = y^2vz + yvzM + vz\widehat{K} + vzM\widehat{K}, \tag{4.18}$$

where $M = J(1, 1, y, v, u, y, z)$ and $\widehat{K}$ is $K$ with $u$ and $v$ interchanged, namely $\widehat{K} = J(1, y, y, u, v, y, z)$.

Now in (4.16) we set $x = 1$, $t = 1$, $s = y$, and we interchange $u$ and $v$, to get

$$M = yvz + vzM + vz\widehat{K} + vzM\widehat{K}. \tag{4.19}$$

Eliminating $M$ from (4.18) and (4.19), we obtain

$$vz(2yvz - y^2vz + 1 - vz)\widehat{K} + (vz - 1)K + vzK\widehat{K} + y^2vz = 0. \tag{4.20}$$

Finally, eliminating $\widehat{K}$ from (4.20) and the equation obtained from (4.20) by interchanging $u$ and $v$, we obtain equation (4.14). Note that, as expected, $J$ is symmetric in the variables t and s and linear in each of these two variables. □

From Theorem 4.10 one can see that the first terms of $L(x, q, p, z)$ are

$$1 + xpz + (qp^2 + x^2p)z^2 + (q^2p^2 + qp^2 + xqp^3 + xqp^2 + x^3p)z^3 + \cdots,$$

corresponding to Dyck paths of semilength at most 3 (or equivalently, to 132-avoiding permutations of length at most 3).

# 5

# Other bijections and combinatorial interpretations

In this chapter we describe several combinatorial objects that are enumerated by the Catalan, Fine and Narayana numbers. They arise naturally from the work in the previous chapters, and some of them give new combinatorial interpretations of these numbers. In Section 5.2 we describe three bijections between 321-avoiding permutations and Dyck paths, which show that certain statistics on Dyck paths and on permutations are equidistributed.

In Section 5.3 we consider a class of permutations defined in terms of non-crossing matchings of points around a circle. We study their structure and provide generating functions enumerating them with respect to the statistic 'number of descents', and also with respect to the number of fixed points and the number excedances.

## 5.1 Some interpretations of the Catalan and Fine numbers

Our first new interpretation of the Catalan numbers follows immediately from the results in the previous chapter. Note that any nonempty Dyck path $D \in \mathcal{D}_n$ has a multitunnel that goes from $(0,0)$ to $(2n,0)$. We call this the *basic multitunnel*.

**Proposition 5.1** *Let* $n \geq 0$*. The number of Dyck paths of length* $2n+2$ *with no centered multitunnels other than the basic one is* $\mathbf{C}_n$*.*

*Proof.* We could give a non-bijective argument using generating functions, but now the bijection $\Phi$ yields a simple combinatorial proof. We know from part (4) of Theorem 4.2 that the set of paths $D \in \mathcal{D}_{n+1}$ with $\mathrm{cmt}(D) = 1$ is in bijection with the set of Dyck paths of length $2n+2$ with only one return. But these are precisely elevated Dyck paths of the form $\mathbf{u}A\mathbf{d}$, where $A \in \mathcal{D}_n$. The number of them is $\mathbf{C}_n$. □

**Proposition 5.2** *The following quantities are equal to* $\mathbf{C}_n$*:*

(1) *The total number of fixed points in elements of* $\mathcal{S}_n(321)$*.*

(2) *The total number of fixed points in elements of* $\mathcal{S}_n(132)$*.*

(3) *The total number of centered tunnels in Dyck paths of length* $2n$*.*

*Proof.* By the first part of Theorem 4.2, (3) equals the total number of hills in Dyck paths of length $2n$. To prove that this number is $\mathbf{C}_n$, we define the following bijection between paths in $\mathcal{D}_n$ with a marked hill and the set $\mathcal{D}_n$ itself. Given a path with a distinguished hill $A\mathbf{ud}B \in \mathcal{D}_n$, where $A, B \in \mathcal{D}$, map it to the path $\mathbf{u}A\mathbf{d}B \in \mathcal{D}_n$. This is obviously a bijection, since each $D \in \mathcal{D}_n$ can be expressed uniquely as $D = \mathbf{u}A\mathbf{d}B$, with $A, B \in \mathcal{D}$.

The equality (2)=(3) is a consequence of the first part of Proposition 2.1. Finally, by Proposition 2.8, fixed points in 321-avoiding permutations are in one-to-one correspondence with hills of Dyck paths, which proves (1).

Another argument to compute (1) directly is the following. We are counting the number of pairs $(\pi, i)$, where $\pi \in \mathcal{S}_n(321)$ and $i$ is a fixed point of $\pi$. Given $1 \leq i \leq n$, the number of $\pi \in \mathcal{S}_n(321)$ having $i$ as a fixed point is $\mathbf{C}_{i-1}\mathbf{C}_{n-i}$, since $\pi_i = i$ if and only if $\pi_1\pi_2 \cdots \pi_{i-1} \in \mathcal{S}_{i-1}(321)$ and $(\pi_{i+1} - i)(\pi_{i+2} - i) \cdots (\pi_n - i) \in \mathcal{S}_{n-i}(321)$. Therefore, the total number of such pairs $(\pi, i)$ is $\sum_{i=1}^{n} \mathbf{C}_{i-1}\mathbf{C}_{n-i} = \mathbf{C}_n$. □

In [78] it is proved that the number of permutations $\pi \in \mathcal{S}_n(132)$ (or $\pi \in \mathcal{S}_n(321)$) with no fixed points is the *Fine number* $\mathbf{F}_n$. This sequence is most easily defined by its relation to Catalan numbers:

$$\mathbf{C}_n = 2\,\mathbf{F}_n + \mathbf{F}_{n-1} \quad \text{for } n \geq 2, \quad \text{and } \mathbf{F}_1 = 0,\ \mathbf{F}_2 = 1.$$

Although defined some time ago, Fine numbers have received much attention in recent years (see a survey [28]). Special cases of the bijections in Section 2.2 give simple bijections between these two combinatorial interpretations of Fine numbers and a new one: the set of Dyck paths without centered tunnels. In particular, we obtain a bijective proof of the following result.

**Proposition 5.3** *The number of Dyck paths $D \in \mathcal{D}_n$ without centered tunnels is equal to $\mathbf{F}_n$.*

Yet another combinatorial proof of this fact follows from the bijections in Chapter 4. The bijection $\Phi$ maps Dyck paths without centered tunnels to Dyck paths without hills, which in turn correspond through the bijection $\psi_{\llcorner}$ to 321-avoiding derangements.

For the next interpretation of the Catalan and Fine numbers, consider the directed graph $G$ drawn in Figure 5.1. Its nodes are the infinite set $\{q_{i,j} : i,j \geq 0, i+j \text{ even}\}$. The set of edges is $\{(q_{i,j} \rightarrow q_{i+1,j+1}), (q_{i+1,j+1} \rightarrow q_{i,j}) : i,j \geq 0, i+j \text{ even}\} \cup \{(q_{0,2j-2} \rightarrow q_{0,2j}), (q_{0,2j} \rightarrow q_{0,2j}) : j \geq 2\} \cup \{(q_{2i-2,0} \rightarrow q_{2i,0}), (q_{2i,0} \rightarrow q_{2i,0}) : i \geq 2\} \cup \{(q_{0,0} \rightarrow q_{0,0})\}$.

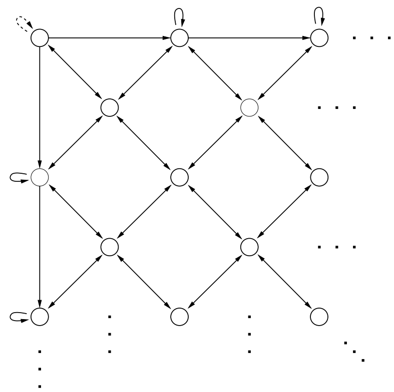

**Figure 5.1** The graph $G$.

**Proposition 5.4** *Let $G$ be the directed graph described above, and let $G'$ be the graph obtained from it by removing the edge $(q_{0,0} \rightarrow q_{0,0})$. Fix $n \geq 0$.*

(1) *The number of paths in $G$ from $q_{0,0}$ to $q_{0,0}$ with $n$ steps is $\mathbf{C}_n$.*

(2) *The number of paths in $G'$ from $q_{0,0}$ to $q_{0,0}$ with $n$ steps is $\mathbf{F}_n$.*

(3) *The number of paths in $G$ from $q_{0,0}$ to $q_{0,0}$ with $n+1$ steps not having $q_{0,0}$ as an interior point is $\mathbf{C}_n$.*

*Proof.* We will construct a bijection between $\mathcal{D}_n$ and the set of paths in $G$ from $q_{0,0}$ to $q_{0,0}$ with $n$ steps. Let $D \in \mathcal{D}_n$. Read the steps of $D$ two by two, starting with the two middle steps $n$ and $n+1$, then $n-1$ and $n+2$, and progressively moving away from the middle, finishing with the pair 1 and $2n$. For $k$ from 1 to $n$, let $D_k$ be the subpath of $D$ consisting of the steps at distance at most $k$ from the middle. Let $a_k$ (resp. $b_k$) be the height difference between the leftmost (resp. rightmost) point of $D_k$ and its lowest point (the height is given by the $y$-coordinate). Equivalently, $a_k$ (resp. $b_k$) is the number of left (resp. right) tunnels of $D$ with exactly one of its two delimiting steps belonging to $D_k$. Define the $k$-th node of our path in $G$ to be $q_{a_k,b_k}$.

Note that $q_{a_0,b_0} = q_{a_n,b_n} = q_{0,0}$, and that for every $k$ there is an edge from $q_{a_k,b_k}$ to $q_{a_{k+1},b_{k+1}}$ in $G$, by the way the numbers $a_k$ and $b_k$ change every time that a pair of steps is read from $D$. It is not hard to see that this defines a bijection between $\mathcal{D}_n$ and paths in $G$ with $n$ steps starting and ending at $q_{0,0}$. Indeed, the numbers $a_k$ and $b_k$ encode enough information to reconstruct the Dyck path. This proves (1).

To show parts (2) and (3), observe that the node $q_{0,0}$ is used whenever $a_k = b_k = 0$, which means that there is a centered multitunnel between the two endpoints of $D_k$. Similarly, the edge $(q_{0,0} \to q_{0,0})$ is used when $a_k = b_k = a_{k+1} = b_{k+1} = 0$, and this condition is equivalent to $D$ having a centered tunnel between the endpoints of $D_{k+1}$. To prove (2), we use the fact from Corollary 5.3 that the number of Dyck paths of length $2n$ with no centered tunnels is $\mathbf{F}_n$. Part (3) follows now from Proposition 5.1, and in fact is also a direct consequence of part (1). □

Combining the bijection just defined with $\Phi$, an alternative and perhaps simpler bijection between $\mathcal{D}_n$ and the set of paths in $G$ from $q_{0,0}$ to $q_{0,0}$ with $n$ steps can be defined. It will be convenient to describe the path in $G$ backwards. Equivalently, we will give a path $P$ in the graph obtained from $G$ by reversing all the edges. Given $D \in \mathcal{D}_n$, read the steps from left to right two at a time, and construct $P$ as follows. Let $q_{i,j}$ be the current node in $P$. If a **uu** is read, add an edge $(q_{i,j} \to q_{i+1,j+1})$ to the path. If a pair **ud** is encountered, add an edge $(q_{i,j} \to q_{i+1,j-1})$ if $j > 0$, or a loop $(q_{i,j} \to q_{i,j})$ otherwise. For each pair **du**, add an edge $(q_{i,j} \to q_{i-1,j+1})$ if $i > 0$, or a loop $(q_{i,j} \to q_{i,j})$ otherwise. Finally, for each pair **dd**, add an edge $(q_{i,j} \to q_{i-1,j-1})$ if $i, j > 0$, $(q_{i,j} \to q_{i-2,j})$ if $j = 0$, or $(q_{i,j} \to q_{i,j-2})$ if $i = 0$. It can be checked that this is a bijection as well. Note that if at a given point of the construction the current node in $P$ is $q_{i,j}$, then the fragment of $D$ that has been read so far ends at height $i + j$.

Our last interpretation of the Catalan numbers is joint work with Emeric Deutsch and Astrid Reifegerste.

**Proposition 5.5** *The following quantities are equal to* $\mathbf{C}_n$*:*

(1) *The number of permutations* $\pi \in \mathcal{S}_{2n+1}(321)$ *such that* $\psi_{\llcorner}(\pi) = \phi_{\llcorner}(\pi)$.

(2) *The number of symmetric parallelogram polyominoes*[1] *of perimeter* $4(2n+1)$ *having exactly one horizontal (equivalently, vertical) boundary segment at each level.*

*Proof.* The equivalency between (1) and (2) is clear when we draw $\psi_{\llcorner}(\pi)$ (as in Figure 2.8) and $\phi_{\llcorner}(\pi)$ (as in Figure 5.3) as lattice paths from the top-left to the bottom-right corner of the array of $\pi$. The two paths form a parallelogram polyomino which is symmetric (and satisfies the conditions of (2)) exactly when $\psi_{\llcorner}(\pi) = \phi_{\llcorner}(\pi)$ as Dyck paths.

Now we show that the permutations in (1) are counted by the Catalan numbers. Let $\pi \in \mathcal{S}_{2n+1}(321)$, let $i_1 < i_2 < \cdots < i_e$ be the positions of the excedances of $\pi$ and let $j_1 < j_2 < \cdots < j_{2n+1-e}$ be the remaining positions. Then, $\psi_{\llcorner}(\pi) = \phi_{\llcorner}(\pi)$ if and only if $e = n$ ($\pi$ has $n$ excedances) and $\pi_{i_k} = j_k + 1$ for all $1 \leq k \leq n$. Each permutation satisfying these conditions is uniquely determined by its excedance set $\{i_1, i_2, \ldots, i_n\}$. Now, these sets are in bijection with Dyck paths of length $2n$: given such a set, construct a Dyck path having up-steps in positions $\{i_1, i_2, \ldots, i_n\}$ and down-steps everywhere else. Figure 5.2 shows an example for $\pi = 4512736$, whose excedance set is $\{1, 2, 5\}$. □

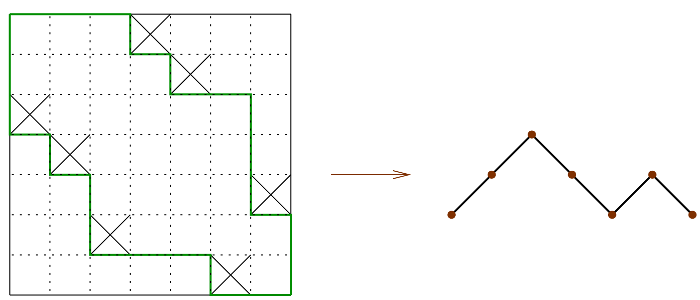

**Figure 5.2** A permutation satisfying $\psi_{\llcorner}(\pi) = \phi_{\llcorner}(\pi)$, its symmetric parallelogram polyomino, and the corresponding Dyck path.

[1] Parallelogram polyominoes are unordered pairs of lattice paths starting at $(0, 0)$, using steps $(1, 0)$ and $(0, -1)$, ending at the same point, and only intersecting at the beginning and at the end.

## 5.2 Some other bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$

Considering the array of crosses associated to a permutation, as we did to define $\psi_{\llcorner}$ in Section 2.2.4, some other known bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ can easily be viewed in a systematic way, as paths with east and south steps from the upper-left corner to the lower-right corner of the $n \times n$ array. For each of these bijections, our canonical example will be $\pi = 23147586$. One such bijection was established by Billey, Jockusch and Stanley in [7, p. 361]. Denote it by $\phi_{\llcorner}$. Consider the path with east and south steps that leaves the crosses corresponding to excedances to the right, and stays always as far from the main diagonal as possible (Figure 5.3). Then $\phi_{\llcorner}(\pi)$ can be obtained from it just by reading an up-step for every east step of this path and a down-step for every south step.

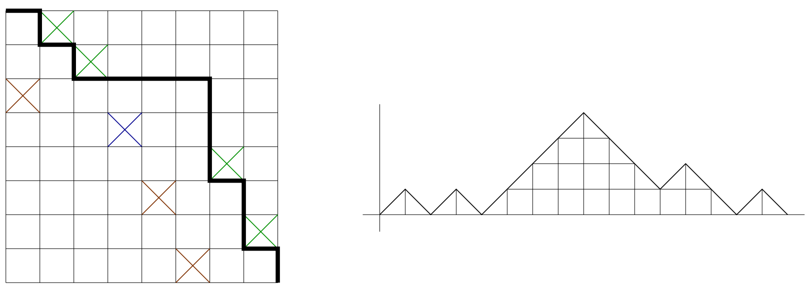

**Figure 5.3** The bijection $\phi_{\llcorner}$.

In [59], Krattenthaler describes a bijection from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. If we omit the last step, consisting of reflecting the path over a vertical line, and compose the bijection with the reversal operation, that maps a permutation $\pi_1\pi_2\cdots\pi_n$ into $\pi_n\cdots\pi_2\pi_1$, we get a bijection from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$. Denote it by $\psi_{\neg}$. In the array representation, $\psi_{\neg}(\pi)$ corresponds (by the same trivial transformation as before) to the path with east and south steps that leaves all the crosses to the left and remains always as close to the main diagonal as possible (Figure 5.4).

Our first bijection is related to this last one by $\psi_{\neg}(\pi) = \psi_{\llcorner}(\pi^{-1})$. In a similar way, we could still define a fourth bijection $\phi_{\neg} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ by $\phi_{\neg}(\pi) := \phi_{\llcorner}(\pi^{-1})$ (Figure 5.5).

Combining these bijections and their inverses, one can get some automorphisms on Dyck paths and on 321-avoiding permutations with interesting properties. Recall from Section 1.2.1 that $\mathrm{va}(D)$ and $p_2(D)$ denote respec-

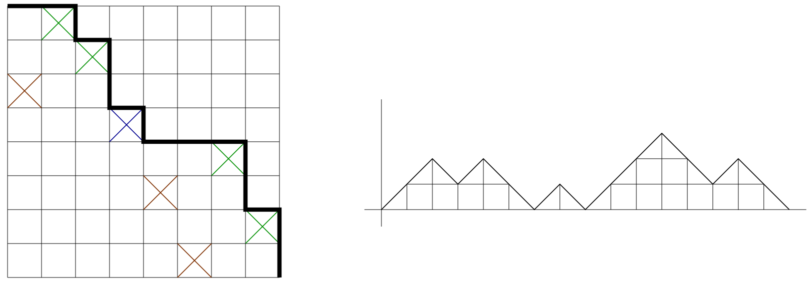

**Figure 5.4** The bijection $\psi_\neg$.

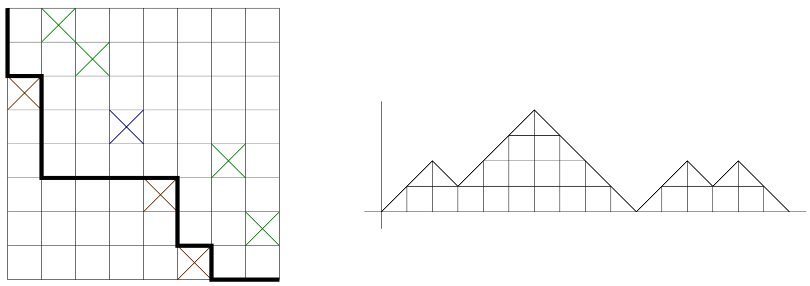

**Figure 5.5** The bijection $\phi_\neg$.

tively the number of valleys and the number of peaks of height at least 2 of $D$. It can be checked that $\psi_\llcorner \circ \phi_\llcorner^{-1}$ is an involution on $\mathcal{D}_n$ with the property that $\mathrm{va}(\psi_\llcorner \circ \phi_\llcorner^{-1}(D)) = \mathrm{dr}(D)$ and $\mathrm{dr}(\psi_\llcorner \circ \phi_\llcorner^{-1}(D)) = \mathrm{va}(D)$. Indeed, this follows from the fact that excedances are mapped to valleys by $\phi_\llcorner$ and to double rises by $\psi_\llcorner$. This bijection gives a new proof of the symmetry of the bivariate distribution of the pair $(\mathrm{va}, \mathrm{dr})$ of statistics on Dyck paths. A different involution with this property was introduced in [24].

Another involution on $\mathcal{D}_n$ is given by $\psi_\llcorner \circ \psi_\urcorner^{-1}$. This one shows the symmetry of the distribution of the pair $(\mathrm{dr}, p_2)$, because $\mathrm{dr}(\psi_\llcorner \circ \psi_\urcorner^{-1}(D)) = p_2(D)$ and $p_2(\psi_\llcorner \circ \psi_\urcorner^{-1}(D)) = \mathrm{dr}(D)$. In addition, it preserves the number of hills, i.e., $h(\psi_\llcorner \circ \psi_\urcorner^{-1}(D)) = h(D)$. To see this, just note that both $\psi_\neg$ and $\psi_\llcorner$ map fixed points to hills, whereas excedances are mapped to peaks of height at least 2 by $\psi_\neg$ and to double rises by $\psi_\llcorner$.

It is well known that the number of permutations in $\mathcal{S}_n$ with $k$ excedances equals the number of permutations in $\mathcal{S}_n$ with $k+1$ weak excedances (recall that $i$ is a *weak excedance* of $\pi$ if $\pi_i \geq i$). We can show combinatorially that the analogous results for 321-avoiding and for 132-avoiding permutations hold as well.

**Proposition 5.6** *Fix $n, k \geq 0$. The following quantities are equal to the* Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$.

(1) *The number of* 321*-avoiding permutations $\pi \in \mathcal{S}_n$ with $k$ excedances.*

(2) *The number of* 321*-avoiding permutations $\pi \in \mathcal{S}_n$ with $k+1$ weak excedances.*

(3) *The number of* 132*-avoiding permutations $\pi \in \mathcal{S}_n$ with $k$ excedances.*

(4) *The number of* 132*-avoiding permutations $\pi \in \mathcal{S}_n$ with $k+1$ weak excedances.*

(5) *The number of* 132*-avoiding permutations $\pi \in \mathcal{S}_n$ with $k$ descents.*

*Proof.* By Proposition 2.8, excedances of $\pi \in \mathcal{S}_n(321)$ correspond to double rises of $\psi_{\mathsf{L}}(\pi)$. It is known that the number of Dyck paths in $\mathcal{D}_n$ with $k$ double rises is given by the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$.

To prove the equality (1)=(2), consider the involution on $\mathcal{S}_n(321)$ that maps $\pi$ to $(\phi_{\mathsf{L}}^{-1}(\psi_{\mathsf{L}}(\pi)))^{-1}$. Excedances of $\pi \in \mathcal{S}_n(321)$ give double rises in $\psi_{\mathsf{L}}(\pi)$. On the other hand, the bijection $\phi_{\mathsf{L}}^{-1}$ maps valleys to excedances. Combining this together, we have that $\text{exc}(\pi) = \text{dr}(\psi_{\mathsf{L}}(\pi)) = n - \text{va}(\psi_{\mathsf{L}}(\pi)) - 1 = n - \text{exc}(\phi_{\mathsf{L}}^{-1}(\psi_{\mathsf{L}}(\pi))) - 1$, where the second equality follows from the fact that each up-step of a Dyck path is either the beginning of a double rise or the beginning of a peak, so the number of peaks plus double rises equals the semilength of the path. Finally, we use that the number of excedances of a permutation in $\mathcal{S}_n$ plus the number of weak excedances of its inverse is $n$.

The equalities (1)=(3) and (2)=(4) are immediate consequences of Theorem 2.3. Finally, to see that (1)=(5), notice that if $\pi \in \mathcal{S}_n(321)$, then $\text{exc}(\pi) = \text{va}(\phi_{\mathsf{L}}(\pi))$. On the other hand, if $\pi \in \mathcal{S}_n(132)$, then $\text{des}(\pi) = \text{va}(\varphi(\pi))$ by Proposition 2.1. Thus, $\varphi^{-1} \circ \phi_{\mathsf{L}} : \mathcal{S}_n(321) \longrightarrow \mathcal{S}_n(132)$ maps excedances to descents. □

## 5.3 Noncrossing permutations

In this section we consider a different class of permutations enumerated by the Catalan numbers, but which is not defined in terms of pattern avoidance (it is not equal to $\mathcal{S}_n(\sigma)$ for any pattern $\sigma$). They were introduced in [53] by Hernando, Hurtado and Noy. We will denote this class $\mathcal{N}_n \subset \mathcal{S}_n$, and call them *noncrossing permutations*. They are defined as those permutations obtained in the following way. Consider $2n$ points around a circle, labelled counterclockwise as $1, 1', 2, 2', \ldots, n, n'$. Now consider a noncrossing perfect matching between the $2n$ points (that is, match pairs of points by drawing a line segment, in such a way that no two segments cross). Notice that each point with a label from the set $\{1, 2, \ldots, n\}$ gets matched with a point with a label in $\{1', 2', \ldots, n'\}$. Thus, to every such matching we can associate the permutation $\pi$ that maps $i$ to $\pi_i = j$ for each matched pair of the form $(i, j')$. Let $\mathcal{N}_n$ be the set of permutations obtained by this procedure. Clearly, $|\mathcal{N}_n| = \mathbf{C}_n$, since the number of noncrossing perfect matchings of $2n$ points on a cycle is the Catalan number $\mathbf{C}_n$.

The class $\mathcal{N}_n$ can be characterized alternatively as the set of permutations $\pi \in \mathcal{S}_n$ for which there do not exist indices $i < j$ such that either of the following conditions holds:

(1) $i < j \leq \pi_i < \pi_j$,

(2) $\pi_i < \pi_j < i < j$,

(3) $i \leq \pi_j < \pi_i < j$,

(4) $\pi_j < i < j \leq \pi_i$.

We now give a recursive description of noncrossing permutations, which will be convenient for enumeration purposes. Given $\pi \in \mathcal{S}_n$, let $\widetilde{\pi}$ the permutation obtained from $\pi$ by replacing $n$ with 1 and increasing all of the remaining entries by one. Let $\widetilde{\mathcal{N}}_n = \{\widetilde{\pi} : \pi \in \mathcal{N}_n\}$. Recall the definition of the reduction $\rho$ from Section 1.1.1.

**Lemma 5.7** *Fix $n \geq 1$, let $\pi \in \mathcal{S}_n$, let $k$ be the index such that $\pi_k = n$, and write $\pi = \tau\, n\, \omega$. Then, $\pi \in \mathcal{N}_n$ if and only if $\tau \in \mathcal{N}_{k-1}$ and $\rho(\omega) \in \widetilde{\mathcal{N}}_{n-k}$ (note that in particular $\tau$ is a permutation of $\{1, 2, \ldots, k-1\}$).*

*Proof.* Consider the matching of $2n$ points that corresponds to $\pi$. Notice that $k$ is matched with $n'$. This splits the rest of the matching into two parts. On one side of the chord connecting $k$ and $n'$ we have a noncrossing matching of $\{1, 1', 2, 2', \ldots, k-1, (k-1)'\}$, which corresponds to the permutation $\tau \in \mathcal{N}_{k-1}$. On the other side there is a noncrossing matching of $\{k', k+1, (k+1)', k+2, \ldots, (n-1)', n\}$, which gives rise to $\omega$. □

It is an immediate consequence of this lemma that any $\pi \in \widetilde{\mathcal{N}}_n$ can be decomposed uniquely as $\pi = \tau\, 1\, \omega$ where $\tau$ is a permutation of $\{2, 3, \cdots, k\}$ such that $\rho(\tau) \in \mathcal{N}_{k-1}$, and $\rho(\omega) \in \widetilde{\mathcal{N}}_{n-k}$.

### 5.3.1 Distribution of descents

The decomposition above allows us to easily enumerate noncrossing permutations with respect to the number of descents. Let

$$N(u,z)=\sum_{n\geq 0}\sum_{\pi\in\mathcal{N}_n} u^{\mathrm{des}(\pi)}z^n.$$

**Proposition 5.8** *The GF for noncrossing permutations with respect to the number of descents is*

$$N(u,z)=\frac{1-(1-u)z-\sqrt{1-2(1+u)z+(1-u)^2z^2}}{2uz}.$$

*Proof.* Let

$$\widetilde{N}(u,z)=\sum_{n\geq 0}\sum_{\pi\in\widetilde{\mathcal{N}}_n} u^{\mathrm{des}(\pi)}z^n.$$

The decomposition in Lemma 5.7 translates into the equation

$$N(u,z)=1+zN(u,z)[u(\widetilde{N}(u,z)-1)+1],$$

since the descents in $\tau n\omega$ are those in $\tau$ plus those in $\omega$, plus an extra descent if $\omega$ is nonempty. A similar reasoning yields the equation

$$\widetilde{N}(u,z)=1+z[u(N(u,z)-1)+1]\widetilde{N}(u,z).$$

From these equations it follows that $N(u,z)=\widetilde{N}(u,z)$, and solving for $N$ we get the desired GF. □

The coefficient of $u^kz^n$ in $N(u,z)$ is the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. This implies together with Proposition 5.6 that the distribution of descents is the same in $\mathcal{N}_n$ as in $\mathcal{S}_n(132)$ (and in fact the same as in $\mathcal{S}_n(213)$, $\mathcal{S}_n(231)$ and $\mathcal{S}_n(312)$ as well). A bijection proving this can be described recursively using the fact that 231-avoiding permutations admit a decomposition which is very similar to that of noncrossing permutations. Indeed, any $\pi\in\mathcal{S}_n(231)$ can be written uniquely as $\pi=\tau\, n\, \omega$ where $\tau\in\mathcal{S}_{k-1}(231)$ and $\rho(\omega)\in\mathcal{S}_{n-k}(231)$, for some $k$.

It follows from the symmetry of the Narayana numbers and it is also easy to see directly that the number of ascents in $\mathcal{N}_n$ has the same distribution as the number of descents.

### 5.3.2 Distribution of fixed points and excedances

Our goal in this subsection is to find an expression for the GF

$$A(x,q,z) := \sum_{n\geq 0}\sum_{\pi\in\mathcal{N}_n} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n.$$

Let us first consider fixed points in noncrossing permutations. An important observation is that from any $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{N}_n$ we can obtain a permutation in $\mathcal{N}_{n+1}$ with a fixed point in position $i$ (for $1 \leq i \leq n+1$) just by inserting a new element $i$ in position $i$ and shifting up all the elements greater or equal than $i$ by one. That is, we build the permutation $\pi'_1\pi'_2\cdots\pi'_{i-1}i\pi'_i\cdots\pi'_n \in \mathcal{N}_{n+1}$, where

$$a' := \begin{cases} a & \text{if } a < i, \\ a+1 & \text{if } a \geq i, \end{cases}$$

for all $a$. Besides, any permutation in $\mathcal{N}_{n+1}$ with a fixed point in position $i$ can be obtained in this way. In terms of the noncrossing matching corresponding to $\pi$, this operation consists in inserting two points connected with an edge between the labels $(i-1)'$ and $i$, labelling them $i$ and $i'$, and shifting up all the labels that come after by one.

By removing all the fixed points in a noncrossing permutation and relabelling the remaining elements accordingly, we obtain a noncrossing derangement. The above reasoning implies that, conversely, every noncrossing permutation can be uniquely obtained by starting from a noncrossing derangement and inserting a certain number of fixed points before each element and at the end. In terms of the GF $A(0,1,z)$ that enumerates noncrossing derangements, this translates into the equation

$$\frac{1}{1-z}A(0,1,\frac{z}{1-z}) = \mathbf{C}(z). \tag{5.1}$$

The substitution of $\frac{z}{1-z}$ for the third variable in $A(0,1,z)$ indicates that every element of the derangement is replaced with a sequence of fixed points followed by the element itself. The factor $\frac{1}{1-z}$ corresponds to the fixed points inserted at the end of the permutation, and the right hand side is the GF for noncrossing permutations, which are counted by the Catalan numbers. The substitution $z = \frac{t}{1+t}$ in (5.1) gives us

$$A(0,1,t) = \frac{1}{1+t}\mathbf{C}(\frac{t}{1+t}) = \frac{1-\sqrt{\dfrac{1-3t}{1+t}}}{2t}.$$

Now it is clear, again by the same argument, that the GF for noncrossing permutations where $x$ marks the number of fixed points is

$$A(x,1,t) = \frac{1}{1-xz}A(0,1,\frac{z}{1-xz}) = \frac{1-\sqrt{\dfrac{1-(3+x)z}{1+(1-x)z}}}{2z}.$$

We will now look at the same problem is a different way, which will allow us to enumerate noncrossing permutations also with respect to the number of excedances.

**Theorem 5.9** *The GF for noncrossing permutations with respect to the number of fixed points and the number of excedances is*

$$A(x,q,z) = \frac{1+(1-x)z-\sqrt{1-2(1+x)z+(x^2+2x+1-4q)z^2}}{2z[1+(q-x)z]}.$$

*Proof.* Recall the bijection between noncrossing matchings of points around a circle (also called chord diagrams) and Dyck paths defined in Section 1.2.2 (see Figure 1.4). Since each noncrossing permutation corresponds to one such matching, when we apply this bijection reading the points around the circle in the order $\{1,1',2,2',\ldots,n,n'\}$, we get a bijection between $\mathcal{N}_n$ and $\mathcal{D}_n$, which we denote $\theta$.

A fixed point in the permutation corresponds to a chord joining points labelled $i$ and $i'$. Through the bijection $\theta$, such chords are mapped to peaks at odd height in the Dyck path. Indeed, it is clear that chords joining two consecutive points in the circle (other than the pair $(n',1)$) are mapped to peaks by $\theta$. However, we only want chords joining a pair $(i,i')$, which appear after an even number of points has been read; hence the corresponding peak has odd height (we call this an *odd peak*).

On the other hand, an excedance in the permutation corresponds to a chord joining a pair $(i,j')$ where $i<j$. Such a chord produces in the Dyck path an up-step in an odd position (i.e., an even rise) when $i$ is read, followed by another up-step when $i'$ is read, since by the noncrossing condition $i'$ cannot be matched with an element with a label less than $i$. Conversely, any even rise followed by another up-step (we call this an *odd double rise*) comes from a chord corresponding to an excedance.

Therefore, our problem is equivalent to the enumeration of Dyck paths with respect to odd peaks and odd double rises. We use the standard decomposition of a nonempty Dyck path as $D=\mathbf{u}A\mathbf{d}B$, with $A,B\in\mathcal{D}$, and then we write $A = \mathbf{u}A_1\mathbf{du}A_2\mathbf{d}\cdots$ as a sequence of elevated paths. The odd double rises of $D$ are those in $B$ plus those in each one of the $A_i$, plus an extra one created by the two $\mathbf{u}$'s preceding $A_1$ if $A$ is nonempty. The odd peaks of $D$ are those in $B$ plus those in each one of the $A_i$, plus an extra one if $A$ is empty. In terms of the GF, we get

$$A(x,q,z) = 1 + z\left(\frac{q}{1 - zA(x,q,z)} + x - q\right)A(x,q,z).$$

The solution to this equation is the expression in the statement of the theorem. □

We have seen that fixed points and excedances are mapped respectively to odd peaks and odd double rises by $\theta$. This means that for any noncrossing permutation $\pi$, we have that $\mathrm{fp}(\pi) + \mathrm{exc}(\pi) = \mathrm{or}(\theta(\pi))$, since every odd rise is the first step of either a double rise or a peak depending on whether the next step is up or down. Therefore, deficiencies of $\pi$ correspond to even rises of the path $\theta(\pi)$. By part (2) of Theorem 4.2 and part (2) of Proposition 2.1, even rises in Dyck paths have the same distribution as excedances in 132-avoiding permutations. Since $\pi$ avoids 132 if and only if so does $\pi^{-1}$, the distribution of deficiencies in 132-avoiding permutations is also the same. We conclude that the statistic 'number of deficiencies' has the same distribution in noncrossing permutations as in 132-avoiding permutations. In particular, the number of permutations in $\mathcal{N}_n$ with $k$ deficiencies is $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ (see Proposition 5.6).

The set $\mathcal{N}_n$ is not closed under inversion $\pi \mapsto \pi^{-1}$. However, it is closed under other interesting operations. One of them is the transformation $\pi \mapsto \widehat{\pi}$, which in terms of the permutation array consists in a reflection along the secondary diagonal. Equivalently, we have that $\pi_i = j$ if and only if $\widehat{\pi}_{n+1-j} = n+1-i$. If $\pi \in \mathcal{N}_n$ is represented by a matching of $2n$ points around a circle, the matching corresponding to $\widehat{\pi} \in \mathcal{N}_n$ is obtained from it by writing the labels backwards, that is, $n'$ is now labelled 1, $n$ becomes $1'$, $(n-1)'$ becomes 2 and so on. Another property of noncrossing permutations is that $\pi \in \mathcal{N}_n$ if and only if $(\pi(1,2,\ldots,n))^{-1} \in \mathcal{N}_n$, where now we are using cycle notation, i.e, $(1,2,\ldots,n)$ is a cycle of length $n$. In terms of the matching, this corresponds to shifting the labels one position in clockwise direction, that is, 1 becomes $n'$, $1'$ becomes 1, 2 is now $1'$, $2'$ becomes 2 and so on. Applying this operation twice, we obtain the permutation $(1,2,\ldots,n)^{-1}\pi(1,2,\ldots,n)$. This shows that $\mathcal{N}_n$ is closed under conjugation by the cycle $(1,2,\ldots,n)$.

# 6

# Consecutive patterns

In this chapter we consider a different notion of pattern avoidance. We introduce the concept of *consecutive patterns*, which we also call *subwords*. The difference with respect to the ordinary patterns studied in the previous chapters is that now, in order to form an occurrence of a consecutive pattern, the corresponding elements in the permutation have to be adjacent, that is, in consecutive positions. In this chapter all the patterns that we consider will be consecutive patterns.

We study the distribution of the number of occurrences of a permutation $\sigma$ as a subword among all permutations in $\mathcal{S}_n$. We solve the problem in several cases depending on the shape of $\sigma$ by obtaining the corresponding bivariate exponential generating functions as solutions of certain linear differential equations with polynomial coefficients. Our method is based on the representation of permutations as increasing binary trees and on symbolic methods. Most results in this chapter are joint work with Marc Noy [35].

Let $m, n$ be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_m$ be two permutations. We say that $\pi$ *contains* $\sigma$ *as a subword* if there exist $m$ *consecutive* elements $\pi_{l+1} \cdots \pi_{l+m}$ such that $\rho(\pi_{l+1} \cdots \pi_{l+m}) = \sigma$, where $\rho$ is the reduction that consists in relabelling the elements with $\{1, \ldots, m\}$ so that they keep the same order relationships they had in $\pi$.

For example, if $\sigma = 4132 \in \mathcal{S}_4$, then $\pi = 6725341 \in \mathcal{S}_7$ contains $\sigma$ as a subword, because $\rho(7253) = 4132$. However, $\pi = 41352 \in \mathcal{S}_5$ does not contain $\sigma$ as a subword (even though it contains it as a subsequence, that is, in non-consecutive positions); in this case we say that $\pi$ *avoids* $\sigma$. Occurrences of a subword can be overlapped, for instance, 5716243 contains two occurrences of $\sigma$, namely 7162 and 6243.

Denote by $A_n(\sigma)$ the set of permutations of $\mathcal{S}_n$ that do not contain $\sigma$ as a subword, and let $\alpha_n(\sigma) = |A_n(\sigma)|$. If we want to exclude several subwords $\sigma, \tau, \dots$ we use the corresponding notations $A_n(\sigma, \tau, \dots)$ and $\alpha_n(\sigma, \tau, \dots)$. Our main purpose is to compute $\alpha_n(\sigma)$ for a given subword $\sigma$. More generally, we are also interested in the distribution of the number of occurrences of $\sigma$ among all permutations in $\mathcal{S}_n$.

Some well-known counting problems in permutations can be stated in terms of forbidden subwords. For instance, occurrences of 12 correspond to ascents and are counted by Eulerian numbers; up-and-down permutations are those in $A_n(123, 321)$; permutations whose longest increasing run is at most $k-1$ correspond to $A_n(12\cdots k)$; and it is not difficult to see that $\alpha_n(123, 132)$ is precisely the number of involutions. Related results for occurrences of subwords of length three can be found in [18] and, more recently, in [56].

The basis of our work is the use of symbolic methods for specifying combinatorial classes, following the approach described in the books by Flajolet and Sedgewick [41, 83] and summarized in Section 1.3.2. The key point is the representation of permutations as binary increasing trees. From this representation, using symbolic methods, we derive differential equations satisfied by the corresponding exponential generating functions. In all cases we have encountered, the differential equations become linear after a suitable substitution. The reader can compare this technique with [40], where a similar approach is taken for analyzing certain geometric configurations, the difference being that here we deal with exponential GFs that are transcendental instead of being algebraic.

A related approach is taken in [39] for counting occurrences of a given subtree in binary search trees. The main difference in our case is however that patterns corresponding to forbidden subwords may be split into two subtrees in several different ways.

The organization of this chapter is as follows. First, we present some preliminaries on the representation of permutations as increasing trees and on asymptotic enumeration. In Section 6.2 we enumerate occurrences of a subword $\sigma$ in two cases. Firstly, when $\sigma$ is totally increasing (or decreasing); and secondly when $\sigma = 12\cdots(a-1)a\,\tau\,(a+1)$, and $\tau$ is any non empty permutation of the elements $\{a+2, a+3, \dots, m+2\}$. Notice that the fact that $\tau$ and $a$ are arbitrary means that our result covers a very large number of subwords $\sigma$. In Section 6.3 we show how these results specialize in the case of subwords of length three and four. We conclude with some remarks.

# 6.1 Preliminaries

## 6.1.1 Tree representation of permutations

Following Stanley [89, Chapter 1], we represent a permutation as an increasing binary tree, that is, a binary tree in which the labels along any path from the root are increasing. This representation allows us to translate combinatorial properties of permutations (such as avoiding a certain subword) into combinatorial properties of trees, which can be handled more conveniently.

Let $\pi = a_1 a_2 \cdots a_n$ be a word on the alphabet of positive integers with no repeated letters. Define a binary tree $T(\pi)$ as follows. If $\pi = \emptyset$, then $T(\pi) = \emptyset$. Otherwise, let $i$ be the least element of $\pi$, so that $\pi$ can be factored uniquely in the form $\pi = \sigma i \tau$. Now define $T(\pi)$ by induction as the tree with root $i$, and having $T(\sigma)$ and $T(\tau)$ as left and right subtrees, respectively. This correspondence gives a bijection between $\mathcal{S}_n$ and the set of *increasing binary trees* on $n$ vertices. In particular, we see that the number of such trees is $n!$.

**Generating functions for increasing trees.** We apply the machinery described in Section 1.3.2 to increasing trees. The following derivation will be used repeatedly.

Let $\mathcal{B}^0$ be the labelled class of (possibly empty) binary increasing trees. It satisfies the recursive definition

$$\mathcal{B}^0 = \{\epsilon\} + (\{z\}^\square \star \mathcal{B}^0 \star \mathcal{B}^0),$$

where $\epsilon$ is the empty tree, $\{z\}$ represents the tree with one single node, and the box indicates that the root contains the smallest label. So, the equation for the EGF is

$$I^0(z) = 1 + \int_0^z I^0(t)^2 dt,$$

which reduces to $I^{0\prime}(z) = I^0(z)^2$ with initial condition $I^0(0) = 1$ (derivatives are always with respect to $z$ in this chapter), admitting the solution $I^0(z) = 1/(1-z)$. Thus, $I^0_n = n!$ as expected.

## 6.1.2 Equivalent subwords

We say that two subwords $\sigma, \tau \in \mathcal{S}_m$ are *equivalent* if the BGF counting occurrences of $\sigma$ and $\tau$ are the same (equivalently, for all $n, k$, the number of permutations in $\mathcal{S}_n$ with $k$ occurrences of $\sigma$ equals the number of those with $k$ occurrences of $\tau$). We write $\sigma \sim \tau$ to denote equivalence.

In Section 1.1.3 we described two simple operations that give equivalent subwords: *reversal*, which transforms $\sigma = \sigma_1 \cdots \sigma_m$ into $\sigma^R = \sigma_m \cdots \sigma_1$, and *complementation*, transforming $\sigma$ into $\bar{\sigma} = (m+1-\sigma_1)\cdots(m+1-\sigma_m)$. The explanation is that a permutation $\pi$ has as many occurrences of $\sigma$ as $\pi^R$ has of $\sigma^R$, and as $\bar{\pi}$ has of $\bar{\sigma}$.

### 6.1.3 Asymptotic enumeration

Let $F(z)$ be a meromorphic function on a domain of the complex plane including the origin, and let $\rho$ be the unique pole of $F(z)$ such that $|\rho|$ is minimum. Then the following asymptotic estimate holds:

$$[z^n]F(z) \sim \gamma \cdot \rho^{-n},$$

where $\gamma$ is the residue of $F$ in $\rho$. If $F(z)$ is defined in the whole complex plane and has no poles, then

$$\lim_{n\to\infty} [z^n]F(z) = 0.$$

See [41, Chapter 4] for a discussion.

## 6.2 Main results

This section contains the main results of this chapter. We obtain the counting BGF of occurrences of a subword $\sigma$ in two cases. First we treat the case of increasing (or decreasing) subwords.

### 6.2.1 Increasing subwords

**Theorem 6.1** *Let $m$ be a positive integer, let $\sigma = 12\cdots(m+1)(m+2) \in \mathcal{S}_{m+2}$, and let $P(u,z)$ be the BGF of permutations where $u$ marks the number of occurrences of the subword $\sigma$. Then, $P(u,z) = 1/\omega(u,z)$, where $\omega$ is the solution of*

$$\omega^{(m+1)} + (1-u)(\omega^{(m)} + \omega^{(m-1)} + \ldots + \omega' + \omega) = 0 \tag{6.1}$$

*with $\omega(0) = 1$, $\omega'(0) = -1$, and $\omega^{(k)}(0) = 0$ for $2 \le k \le m$.*

*Proof.* We use the correspondence of permutations as binary increasing trees. We get for $P(u,z)$ a system of $m+1$ first order differential equations, which will be reduced to a differential equation of order $m+1$ with the substitution $P(u,z) = 1/\omega(u,z)$. Let $\mathcal{P}$ be the class of all permutations, let $\mathcal{K}_i$ be the class of permutations not beginning with $12\cdots(m+2-i)$, and let $K_i(u,z)$ be the BGF of $\mathcal{K}_i$ where $u$ marks occurrences of $\sigma$. With some abuse of notation, we introduce the parameter $u$ in the equations for classes meaning that it will be placed there when we write the corresponding differential equations for the BGF. With this notation, we can write

$$\mathcal{P} = \{\epsilon\} + \{z\}^{\Box} \star \mathcal{P} \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)].$$

This is because occurrences of $\sigma$ in a permutation (seen as a binary increasing tree) can be separated into occurrences on the left subtree and occurrences on the right subtree, taking into account that if the permutation on the right subtree begins with $12\cdots(m+1)$ (that is, belongs to $\mathcal{P} - \mathcal{K}_1$), then there is an additional occurrence of $\sigma$ beginning at the root. The corresponding equation for BGFs is, after differentiating,

$$P' = P(K_1 + u(P - K_1)).$$

The next step is to find an equation for $\mathcal{K}_1$. Note that for a permutation not to begin with a $12\cdots(m+1)$, it is not enough that the permutation on the left subtree does not begin with this subword. We have to exclude also the case in which the left subtree is empty and the right subtree begins with a $12\cdots m$. This gives us

$$\mathcal{K}_1 = \{\epsilon\} + \{z\}^{\Box} \star (\mathcal{K}_1 - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\Box} \star \mathcal{K}_2,$$

which translates to

$$K_1' = (K_1 - 1)(K_1 + u(P - K_1)) + K_2$$

when we differentiate the equation for BGFs. Now it is clear how to find an equation for $\mathcal{K}_2$, in which $\mathcal{K}_3$ will appear, and so on, until we arrive to $\mathcal{K}_m$ (permutations not beginning with 12), which satisfies

$$\mathcal{K}_m = \{\epsilon\} + \{z\}^{\Box} \star (\mathcal{K}_m - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}.$$

All these equations yield a system of $m+1$ differential equations involving the corresponding BGFs. It is convenient to apply the substitution

$$R = uP + (1-u)K_1.$$

Note that $R' = uP' + (1-u)K_1' = uPR + (1-u)((K_1-1)R + K_2) = (uP + (1-u)K_1)R + (1-u)(-R + K_2) = R^2 + (u-1)(R - K_2)$, and we obtain

$$\begin{cases} P' = PR \\ R' = R^2 + (u-1)(R-K_2) \\ {K_2}' = (K_2-1)R + K_3 \\ {K_3}' = (K_3-1)R + K_4 \\ \vdots \\ {K_{m-1}}' = (K_{m-1}-1)R + K_m \\ {K_m}' = (K_m-1)R + 1 \end{cases} \qquad \begin{array}{l} \text{with } P(0) = R(0) = 1, \\ K_i(0) = 1 \quad \text{for all} \quad i. \end{array} \tag{6.2}$$

Now we only have to check that setting $P(u,z) = 1/\omega(u,z)$, then $\omega$ satisfies (6.1). The first equation $P' = PR$ gives $R = -\frac{\omega'}{\omega}$. Substituting this into the second one, we get $K_2 = \frac{\omega'' + (1-u)\omega'}{(u-1)\omega}$. By induction on $i$, we see that

$$K_i = \frac{\omega^{(i)} + (1-u)(\omega^{(i-1)} + \omega^{(i-2)} + \ldots + \omega')}{(u-1)\omega}$$

and $\omega^{(i)}(0) = 0$ for $2 \le i \le m$. Finally, (6.1) is obtained substituting in the last equation of (6.2) the expressions for $K_m$ and $R$ in terms of $\omega$. □

For $u = 0$ the solution of the differential equation can be expressed as a linear combination of exponentials

$$w = \sum_{j=1}^{m+1} c_j e^{\lambda_j z},$$

where the $\lambda_j = \exp(\frac{2\pi i j}{m+2})$ are the non trivial $(m+2)$-th roots of unity, and the indetermined coefficients $c_i$ are the solution of the linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{m+1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_{m+1}^m \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_{m+1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

The matrix $M$ of the previous system is easy to invert, since

$$MM^* = (m+2)I - J,$$

where $A^*$ denotes the conjugate transpose of $A$, and $J$ is the all ones matrix. From this it follows that

$$M^{-1} = \frac{1}{m+2} M^*(I+J),$$

and we can obtain the value of the $c_j$, thus an explicit expression for $P(0,z)$, the GF of permutations avoiding the subword $\sigma = 12\cdots(m+2)$. For example, for $m=4$ one gets

$$c_1 = (1+i)/4, \quad c_2 = 1/2, \quad c_3 = (1-i)/4,$$

which agrees with the solution $w = (\cos z - \sin z + e^{-z})/2$ given in Section 6.3.

### 6.2.2 Other subwords

The next result gives the distribution of subwords of a certain, more general shape.

**Theorem 6.2** *Let $m, a$ be positive integers with $a \leq m$, let $\sigma = 12\cdots(a-1)a\,\tau\,(a+1) \in \mathcal{S}_{m+2}$, where $\tau$ is any permutation of the elements $\{a+2, a+3, \ldots, m+2\}$, and let $P(u,z)$ be the BGF of permutations where $u$ marks the number of occurrences of the subword $\sigma$. Then, $P(u,z) = 1/\omega(u,z)$, where $\omega$ is the solution of*

$$\omega^{(a+1)} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}\omega' = 0$$

*with $\omega(0) = 1$, $\omega'(0) = -1$ and $\omega^{(k)}(0) = 0$ for $2 \leq k \leq a$. In particular, the distribution does not depend on $\tau$.*

*Proof.* Again we find a system of $a+1$ differential equations for $P(u,z)$ that, after the substitution $P(u,z) = 1/\omega(u,z)$, yield a single differential equation of order $a+1$. Let $\mathcal{P}$ be as before the class of all permutations. For $1 \leq i \leq m+1$, we denote $\sigma_{>i} = \rho(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{m+2})$. Note that for $i < a$, $\sigma_{>i}$ has its smallest element in the first position, while for $i \geq a$, the smallest element of $\sigma_{>i}$ is the last one. Now let $\mathcal{K}_i$ be the class of permutations not beginning with any of the following: $\sigma_{>1}, \sigma_{>2}, \ldots, \sigma_{>i}$, and let $K_i(u,z)$ be the BGF of $\mathcal{K}_i$ where $u$ marks occurrences of $\sigma$. With this notation, we can write

$$\mathcal{P} = \{\epsilon\} + \{z\}^{\Box} \star \mathcal{P} \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)].$$

The explanation, as in the previous proof, is that in the decomposition of the increasing tree, new occurrences of $\sigma$, apart from the ones on the left and right subtrees, can appear beginning in the root, when the permutation on the right subtree begins with $\sigma_{>1}$ (that is, belongs to $\mathcal{P} - \mathcal{K}_1$).

For $\mathcal{K}_1$ we derive the equation

$$\mathcal{K}_1 = \{\epsilon\} + \{z\}^{\Box} \star (\mathcal{K}_1 - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\Box} \star [\mathcal{K}_2 + u(\mathcal{P} - \mathcal{K}_1)].$$

Note that the last summand corresponds to the case where the left subtree is empty. Then, the permutation on the right subtree cannot begin with $\sigma_{>2}$, which would produce, together with the root, a permutation beginning with $\sigma_{>1}$. So, after separating on the right subtree the permutations beginning with $\sigma_{>1}$ ($\mathcal{P} - \mathcal{K}_1$), we are left with those beginning with neither $\sigma_{>1}$ nor $\sigma_{>2}$, that is, $\mathcal{K}_2$.

Analogous expressions can be found for $\mathcal{K}_i$, $1 < i < a$:

$$\mathcal{K}_i = \{\epsilon\} + \{z\}^{\Box} \star (\mathcal{K}_i - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\Box} \star [\mathcal{K}_{i+1} + u(\mathcal{P} - \mathcal{K}_1)],$$

which yield the equations

$$K_i' = (K_i - 1)(K_1 + u(P - K_1)) + (K_{i+1} + u(P - K_1)).$$

For $\mathcal{K}_a$ we have

$$\mathcal{K}_a = \{\epsilon\} + \{z\}^{\Box} \star (\mathcal{K}_a - \{\epsilon\} - \{\rho(\tau)\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\Box} \star [\mathcal{K}_a + u(\mathcal{P} - \mathcal{K}_1)].$$

The difference now is that in order to avoid a permutation beginning with $\sigma_{>a} = \rho(\tau\,(a{+}1))$, the left subtree cannot be $\rho(\tau)$. (Remember that the BGF corresponding to $\{\rho(\tau)\}$ is $z^{m-a+1}/(m-a+1)!$ since it is a permutation of size $m - a + 1$ and does not contain the subword $\sigma$.) Another difference is that no new variables appear, since when the left subtree is empty there is no danger of beginning with $\sigma_{>a}$, and so there are no additional restrictions for the right subtree.

From all these we get a system of $a+1$ differential equations. After applying the substitutions $R = uP + (1-u)K_1$ as before, and also $S_i = K_{i-1} - K_i$ for $2 \le i \le a$, we obtain

$$\begin{cases} P' = PR \\ R' = R^2 + (u-1)S_2 \\ {S_2}' = S_2 R + S_3 \\ {S_3}' = S_3 R + S_4 \\ \vdots \\ {S_{a-1}}' = S_{m-1} R + S_a \\ {S_a}' = (S_a + \frac{z^{m-a+1}}{(m-a+1)!})R \end{cases} \qquad \begin{array}{l} \text{with } P(0) = R(0) = 1, \\ S_i(0) = 0 \quad \text{for all } \; i. \end{array} \tag{6.3}$$

Finally, it only remains to set $P(u,z) = 1/\omega(u,z)$ and to check that $\omega$ is the solution of (6.4). Analogously to the previous proof, it can be shown by induction that $S_i = \frac{\omega^{(i)}}{(1-u)\omega}$ for $2 \leq i \leq a$. Then, (6.4) follows from the substitution in the bottom equation of the system. □

Note that the fact that $a \leq m$ ensures that $\tau$ is not empty. The case in which $\tau$ is empty corresponds to the case where $\sigma$ is the increasing permutation, and has already been treated in the previous theorem.

We end this section with a certain condition under which two subwords $\sigma$ and $\tau$ are equivalent, i.e., $\sigma \sim \tau$. We say that a subword $\sigma \in \mathcal{S}_m$ is *non-self-overlapping* if there is no permutation in $\mathcal{S}_{2m-2}$ with more than one occurrence of $\sigma$. (Note that in $\mathcal{S}_{2m-1}$ such a permutation would always exist.) The non-self-overlapping condition implies that for any two occurrences $\pi_{i+1}\cdots\pi_{i+m}$ and $\pi_{j+1}\cdots\pi_{j+m}$ of $\sigma$ in a permutation $\pi$, we have that $|i-j| \geq m-1$, that is, the two occurrences cannot overlap in more than one element. For example, 136254 is a non-self-overlapping subword, but 1324 is not, because the permutation $\pi = 142536$ has two occurrences of 1324.

**Proposition 6.3** *Let $m \geq 3$ and let $\sigma \in \mathcal{S}_m$ be a non-self-overlapping subword such that $\sigma_1 = 1$. Let $a = \sigma_m - 1$, and let $\tau = 12\cdots a(a+2)(a+3)\cdots m(a+1)$. Then we have that $\sigma \sim \tau$.*

Before proving this proposition, note that now Theorem 6.2 can be generalized as follows.

**Corollary 6.4** *Let $m \geq 3$ and let $\sigma \in \mathcal{S}_m$ be a non-self-overlapping subword such that $\sigma_1 = 1$. Let $a = \sigma_m - 1$, and let $P(u,z)$ be the BGF of permutations where $u$ marks the number of occurrences of the subword $\sigma$. Then $P(u,z) = 1/\omega(u,z)$, where $\omega$ is the solution of*

$$\omega^{(a+1)} + (1-u)\frac{z^{m-a-1}}{(m-a-1)!}\omega' = 0 \tag{6.4}$$

*with $\omega(0) = 1$, $\omega'(0) = -1$ and $\omega^{(k)}(0) = 0$ for $2 \leq k \leq a$.*

**Example.** The patterns 136254 and 125364 are non-self-overlapping. Thus, by Proposition 6.3, 136254 $\sim$ 125364 $\sim$ 123564, and the BGF counting occurrences of any of them in permutations is given by $P(u,z) = 1/\omega(u,z)$, where $\omega$ is the solution of

$$\omega'''' + (1-u)\frac{z^2}{2}\omega' = 0,$$

with $\omega(0) = 1$, $\omega'(0) = -1$ and $\omega''(0) = \omega'''(0) = 0$. $P(u, z)$ can be expressed in terms of hypergeometric series.

*Proof of Proposition 6.3.* Assume that $\tau$ and $\sigma$ are different, otherwise there is nothing to prove. Notice that $\tau$ is non-self-overlapping, since it has only one descent, in position $m - 1$. Besides, the only way in which an occurrence of $\sigma$ and an occurrence of $\tau$ can overlap in a permutation $\pi$ is with the occurrence of $\tau$ starting to the left of the occurrence of $\sigma$. Indeed, assume for a contradiction that $\pi$ has an occurrence $\pi_{i+1}\pi_{i+2}\cdots\pi_{i+m}$ of $\sigma$ overlapping with an occurrence $\pi_{j+1}\pi_{j+2}\cdots\pi_{j+m}$ of $\tau$ with $1 \leq j-i \leq m-2$. The fact that $\sigma$ is non-self-overlapping and starts with an ascent implies that $\sigma_{m-1} > \sigma_m$ (otherwise there would be a permutation in $\mathcal{S}_{2m-2}$ with two occurrences of $\sigma$ starting in positions 1 and $m - 1$ respectively). But then, $\pi_{i+m-1} > \pi_{i+m}$, which contradicts the condition $\pi_{j+1} < \pi_{j+2} < \cdots < \pi_{j+m-1}$ needed for it to be an occurrence of $\tau$.

Now we define a bijection $\Gamma : \mathcal{S}_n \longrightarrow \mathcal{S}_n$ such that for every $\pi \in \mathcal{S}_n$, the number of occurrences of $\sigma$ (resp. $\tau$) in $\pi$ equals the number of occurrences of $\tau$ (resp. $\sigma$) in $\Gamma(\pi)$. From our previous observations, the occurrences of $\sigma$ and $\tau$ in $\pi$ are either isolated (not overlapping with any other occurrence of $\sigma$ or $\tau$) or they appear in a pair formed by an occurrence of $\tau$ overlapping with one of $\sigma$, with the occurrence of $\tau$ starting to the left of one of $\sigma$.

To construct $\Gamma(\pi)$, read the permutation $\pi$ from left to right and do the following:

(1) For each pair of overlapping occurrences of $\tau$ and $\sigma$, leave it unchanged.

(2) For each isolated occurrence of $\sigma$, reorder the corresponding elements in $\pi$ so that they form an occurrence of $\tau$ instead. There are now two possibilities. If this modification does not create any new occurrence of $\sigma$, we can jump to the following occurrence. Assume on the contrary that this process created a new $\sigma$. Such an occurrence is necessarily overlapping with the new occurrence of $\tau$, starting the one of $\sigma$ to the right of the one of $\tau$. In this case, reorder the elements of $\pi$ corresponding to this new occurrence of $\sigma$ so that they form an occurrence of $\tau$ instead. Note that this erases the previous occurrence of $\tau$ that we had just created, since $\tau$ is non-self-overlapping. Again, check if any new occurrence of $\sigma$ has been created, and repeat this process until it creates no more occurrences of $\sigma$. After this iteration, the initial isolated $\sigma$ has been transformed into an isolated occurrence of $\tau$. Observe also that this procedure cannot overwrite an existing occurrence of $\sigma$ or $\tau$,

because the new occurrences of $\sigma$ that may be created in the process cannot overlap with occurrences of $\sigma$ or $\tau$ that start more to the right.

(3) For each isolated occurrence of $\tau$, reorder the corresponding elements in $\pi$ so that they form an occurrence of $\sigma$ instead. If this modification does not create any new occurrence of $\tau$, we can jump to the following occurrence. On the other hand, if this created a new occurrence of $\tau$, which is necessarily overlapping with the new occurrence of $\sigma$, starting the one of $\tau$ to the left of the one of $\sigma$, reorder the elements of $\pi$ corresponding to this new occurrence of $\tau$ so that they form an occurrence of $\sigma$ instead. If necessary, repeat the process as before, until no more occurrences of $\tau$ are created. After this iteration, the initial isolated occurrence of $\tau$ has been transformed into an isolated occurrence of $\sigma$. By the same reasoning as above, this procedure cannot overwrite an existing occurrence of $\sigma$ or $\tau$.

It is easy to see that $\Gamma$ is in fact an involution, since steps (2) and (3) are inverses from each other, one transforming isolated occurrences of $\sigma$ into isolated occurrences of $\tau$, and the other doing the opposite. This proves that $\sigma \sim \tau$. □

It seems from experimental computations that a more general version of Proposition 6.3 holds, namely, that any two non-self-overlapping subwords $\sigma, \tau \in \mathcal{S}_m$ with $\sigma_1 = \tau_1$ and $\sigma_m = \tau_m$ satisfy $\sigma \sim \tau$, but we have not been able to prove this fact. However, we can make small variations of Proposition 6.3 to obtain similar results like the following, whose proof is now straightforward.

**Proposition 6.5** *Let $m \geq 3$ and let $\sigma, \tau \in \mathcal{S}_m$ be non-self-overlapping subwords such that no permutation in $\mathcal{S}_{2m-2}$ contains $\sigma$ and $\tau$ simultaneously (i.e., sigma and tau cannot overlap with each other). Then $\sigma \sim \tau$.*

**Example.** The previous proposition shows that $24153 \sim 25143$ and that $351264 \sim 362154$.

## 6.3 Subwords of length at most four

Occurrences of the two subwords 12 and 21 of length two correspond, respectively, to ascents and descents in permutations, giving rise to the well-known Eulerian numbers [23].

### 6.3.1 Subwords of length three

Among the 6 permutations of three elements, there are only two different classes regarding its distribution as subwords of permutations since, by reversal and complementation, we have that

$$123 \sim 321$$
$$132 \sim 231 \sim 312 \sim 213$$

By the results in the previous section we get:

**Theorem 6.6** *Let $P(u,z)$ and $Q(u,z)$ be the BGFs of permutations where $u$ marks, respectively, the number of occurrences of the subword* 123 *and* 132*. Then*

$$Q(u,z) = \frac{1}{1-\int_0^z e^{(u-1)t^2/2}dt},$$

$$P(u,z) =$$
$$= \frac{2e^{\frac{1}{2}(1-u+\sqrt{(u-1)(u+3)})z}\sqrt{(u-1)(u+3)}}{1+u+\sqrt{(u-1)(u+3)}+e^{\sqrt{(u-1)(u+3)}z}(-1-u+\sqrt{(u-1)(u+3)})},$$

$$P(0,z) = \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z+\frac{\pi}{6})}.$$

*Furthermore, the numbers $\alpha_n(123)$ and $\alpha_n(132)$ of permutations avoiding, respectively, the subwords* 123 *and* 132*, satisfy*

$$\alpha_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n!$$

$$\alpha_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!$$

*where $\rho_1 = 3\sqrt{3}/(2\pi)$, $\gamma_1 = e^{3\sqrt{3}\pi}$, $(\rho_2)^{-1}$ is the unique positive root of $\int_0^z e^{-t^2/2}dt = 1$, and $\gamma_2 = \exp((\rho_2)^{-2}/2)$, the approximate values being*

$$\rho_1 = 0.8269933, \qquad \gamma_1 = 1.8305194$$
$$\rho_2 = 0.7839769, \qquad \gamma_2 = 2.2558142$$

Table 6.1 indicates the number of permutations of length $n$ with $k$ occurrences of the subwords 123 (top) and 132 (bottom).

The asymptotic estimates are obtained as an application of the result quoted in Section 6.1.3. (The computation of $\rho_2$ has been done numerically using

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | 1<br>1 | | | | |
| 2 | 2<br>2 | | | | |
| 3 | 5<br>5 | 1<br>1 | | | |
| 4 | 17<br>16 | 6<br>8 | 1 | | |
| 5 | 70<br>63 | 41<br>54 | 8<br>3 | 1 | |
| 6 | 349<br>296 | 274<br>368 | 86<br>56 | 10 | 1 |

**Table 6.1** Occurrences of subwords of length 3 in permutations.

the computer algebra system MAPLE.) Since $\rho_1 > \rho_2$ in the previous theorem, we see that $\alpha_n(123)$ is asymptotically larger than $\alpha_n(132)$. Looking at Table 6.1 one is led to conjecture that this is always so. Indeed, we have the following result, which is analogous to Theorem 2.14 for the case of consecutive patterns.

**Proposition 6.7** *For every $n \geq 4$, we have*

$$\alpha_n(123) > \alpha_n(132).$$

*Proof.* Given $\sigma \in \mathcal{S}_m$, let $B_n(\sigma) = \mathcal{S}_n - A_n(\sigma)$ be the permutations of $\mathcal{S}_n$ containing $\sigma$. Define a map

$$\gamma : B_n(123) \longrightarrow B_n(132)$$

as follows. If $\pi \in B_n(123)$ contains occurrences of both 123 and 132, then $\gamma(\pi) = \pi$. Otherwise (that is, $\pi$ contains 123's but not 132's), define $\gamma(\pi)$ as the permutation obtained by traversing $\pi$ from left to right and substituting every occurrence of 123 by 132 (transposing the elements in the positions corresponding to 2 and 3).

It only remains to check that $\gamma$ is one to one, and this is because when a 123 is changed to a 132, no new occurrences of 123 appear that did not exist before the substitution. To prove that the inequality is strict for $n \geq 4$, observe that any permutation beginning with 1423 and having no 123 cannot be of the form $\gamma(\pi)$ for any $\pi$. □

### 6.3.2 Subwords of length four

By reversal and complementation, and by the results in the previous section, they fall into seven classes:

I. 1234 $\sim$ 4321
II. 2413 $\sim$ 3142
III. 2143 $\sim$ 3412
IV. 1324 $\sim$ 4231
V. 1423 $\sim$ 3241 $\sim$ 4132 $\sim$ 2314
VI. 1342 $\sim$ 2431 $\sim$ 4213 $\sim$ 3124 $\sim$ 1432 $\sim$ 2341 $\sim$ 4123 $\sim$ 3214
VII. 1243 $\sim$ 3421 $\sim$ 4312 $\sim$ 2134

The results in the previous section give the BGFs for the distribution of occurrences of subwords in classes I, VI and VII.

**Theorem 6.8** *In each of the following cases, let $P(u,z)$ be the BGF of permutations where $u$ marks the number of occurrences of the corresponding subword.*

*Case 1342.* $P(u,z) = \dfrac{1}{1-\int_0^z e^{(u-1)t^3/6}dt}.$

*Case 1234.* $P(u,z) = 1/\omega$, *where $\omega$ is the solution of*

$$\omega''' + (1-u)(\omega'' + \omega' + \omega) = 0$$

*with $\omega(0) = 1$, $\omega'(0) = -1$, $\omega''(0) = 0$. For $u = 0$, the solution is*

$$P(0,z) = \frac{2}{\cos z - \sin z + e^{-z}}$$

*Case 1243.* $P(u,z) = 1/\omega$, *where $\omega$ is the solution of*

$$\omega''' + (1-u)z\omega' = 0$$

*with $\omega(0) = 1$, $\omega'(0) = -1$, $\omega''(0) = 0$.*

*Furthermore, the numbers $\alpha_n(1342), \alpha_n(1234)$ and $\alpha_n(1243)$ satisfy*

$$\begin{aligned} \alpha_n(1342) &\sim \gamma_1 \cdot (\rho_1)^n \cdot n! \\ \alpha_n(1234) &\sim \gamma_2 \cdot (\rho_2)^n \cdot n! \\ \alpha_n(1243) &\sim \gamma_3 \cdot (\rho_3)^n \cdot n! \end{aligned}$$

*where $(\rho_1)^{-1}$ is the smallest postitive solution of $\int_0^z e^{(u-1)t^3/6}dt = 1$, $(\rho_2)^{-1}$ is the smallest positive solution of $\cos z - \sin z + e^{-z} = 0$, and $\rho_3$ is the* *solution of a certain equation involving Airy functions. The approximate values are*

$$\rho_1 = 0.954611, \qquad \gamma_1 = 1.8305194$$
$$\rho_2 = 0.963005, \qquad \gamma_2 = 2.2558142$$
$$\rho_3 = 0.952891, \qquad \gamma_3 = 1.6043282$$

| $n\backslash k$ | | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| | I | 23 | 1 | | | |
| | II | 23 | 1 | | | |
| | III | 23 | 1 | | | |
| 4 | IV | 23 | 1 | | | |
| | V | 23 | 1 | | | |
| | VI | 23 | 1 | | | |
| | VII | 23 | 1 | | | |
| | I | 111 | 8 | 1 | | |
| | II | 110 | 10 | | | |
| | III | 110 | 10 | | | |
| 5 | IV | 110 | 10 | | | |
| | V | 110 | 10 | | | |
| | VI | 110 | 10 | | | |
| | VII | 110 | 10 | | | |
| | I | 642 | 67 | 10 | 1 | |
| | II | 632 | 86 | 2 | | |
| | III | 631 | 88 | 1 | | |
| 6 | IV | 632 | 86 | 2 | | |
| | V | 631 | 88 | 1 | | |
| | VI | 630 | 90 | | | |
| | VII | 630 | 90 | | | |
| | I | 4326 | 602 | 99 | 12 | 1 |
| | II | 4237 | 766 | 37 | | |
| | III | 4223 | 794 | 23 | | |
| 7 | IV | 4229 | 782 | 29 | | |
| | V | 4218 | 804 | 18 | | |
| | VI | 4210 | 820 | 10 | | |
| | VII | 4204 | 832 | 4 | | |

**Table 6.2** Occurrences of subwords of length 4 in permutations.

In the last case, the equation $\omega''' + (1-u)z\omega' = 0$, for $u = 0$ and $v = w'$, can be solved in terms of Bessel functions (note that the equation for $v$

is actually a slight variant of the Airy equation). The computations have been performed with the help of Maple. Table 6.2 shows the number of occurrences in each of the seven classes for $n \leq 7$, showing that no two of them have the same distribution. (Entries II, III, IV, V in Table 6.2 have been computed directly.)

As in the case of length 3, one might expect that again for any two permutations $\sigma, \tau \in \mathcal{S}_4$, if $\alpha_{n_0}(\sigma) > \alpha_{n_0}(\tau)$, then $\alpha_n(\sigma) > \alpha_n(\tau)$ for all $n > n_0$. However, inequality $\alpha_n(1324) \geq \alpha_n(2143)$ holds for $n \leq 11$ but does not hold for $n = 12$. Indeed,

$$\alpha_{11}(1324) = 27959880 > 27954521 = \alpha_{11}(2143),$$

but

$$\alpha_{12}(1324) = 320706444 < 320752991 = \alpha_{12}(2143).$$

These results have been obtained by exhaustive computation, as we do not know the associated EGFs. There is however one relation among classes that we have been able to establish

**Proposition 6.9** *For every $n \geq 7$, we have*

$$\alpha_n(1342) > \alpha_n(1243).$$

*Proof.* As in the proof of Proposition 6.7, let $B_n(\sigma) = \mathcal{S}_n - A_n(\sigma)$ be the set of permutations containing $\sigma$. Define a map

$$\gamma : B_n(1342) \longrightarrow B_n(1243)$$

as follows. Let $\gamma(\pi) = \pi$ if $\pi$ contains both 1342 and 1243. Otherwise, replace, from left to right, all occurrences of 1342 by 1243. Note that when we replace an occurrence 1342 by 1243, we never create new occurrences of 1342 or 1243. (This is not true in general for other patterns, so that the corresponding $\gamma$ is not a bijection.)

It is clear that $\gamma(B_n(1342) \cap B_n(1243)) \cap \gamma(B_n(1342) \cap A_n(1243)) = \emptyset$, because in the second case there are no 1342 left. Let $\pi \neq \eta \in B_n(1342) \cap A_n(1243)$. Now suppose that $\gamma(\pi) = \gamma(\eta)$. Let $i$ be the smallest index so that $\pi_i \neq \eta_i$. Either $\pi_i$ or $\eta_i$ must be moved by $\gamma$.

Now observe that if, say, $\pi_i$ is changed, it cannot be transposed with any of the preceding elements, so it must be the '3' of a 1342 in $\pi$, and thus is interchanged with the '2' in position $i + 2$. But now, after replacing this

1342 by 1243, the '3' can no longer be moved, because it is neither the '2' nor the '3' of any other 1342. If $\eta_i$ is also moved by $\gamma$, this reasoning implies that $\gamma(\pi)_{i+2} = \pi_i \neq \eta_i = \gamma(\eta)_{i+2}$, which is impossible since $\gamma(\pi) = \gamma(\eta)$. So $\eta_i$ is not moved by $\gamma$, but then positions $i-1$, $i$, $i+1$ and $i+2$ of $\gamma(\pi)$ are $\pi_{i-1}\pi_{i+2}\pi_{i+1}\pi_i = \eta_{i-1}\eta_i\eta_{i+1}\eta_{i+2}$, because this positions of $\gamma(\eta)$ cannot have been moved by $\gamma$. This is a 1243, which contradicts the fact that $\eta \in A_n(1243)$. □

## 6.4 Multiple subwords

Instead of occurrences of a single subword one may consider several subwords. For the case of length three, many of the possible combinations are treated in [56]. For example, the class $A_n(123, 321)$ avoiding 123 *and* 321 is clearly that of *up-and-down* permutations. According to a classical result (see [23]), the corresponding EGF is

$$2\left(\tan z + \frac{1}{\cos z}\right) - 1 - z.$$

The class $A_n(213, 312)$ is that of permutations $\pi$ having no *valleys*, that is, positions $i$ such that $\pi_{i-1} > \pi_i < \pi_{i+1}$ (not to be confused with valleys of a Dyck path). The BGF for permutations where $u$ marks valleys is easily shown to be

$$\frac{\sqrt{1-u}}{\sqrt{1-u} - \tanh(z\sqrt{1-u})}.$$

And the class $A_n(123, 132)$ is equinumerous with the class of involutions. This can be easily explained using the classical correspondence of Foata [89, Section 1.3]. Indeed, given the decomposition of a permutation into disjoint cycles, with their smallest element first and ordering the cycles in decreasing order of the smallest elements, all cycles of length greater or equal than three begin with either a 123 or a 132. Conversely, if the described decomposition in cycles contains one of these subwords, all three elements are necessarily in the same cycle, and thus the permutation with that cyclic decomposition cannot be an involution.

We can also find a multivariate generating function $Q(u, v, z)$ where $u$ marks occurrences of 123 and $v$ marks occurrences of 132. Using similar arguments as before, it can be seen that $Q(u, v, z)$ is the solution of

$$\begin{cases} R' = R^2 + [(u-1) + (v-1)z]R - (u-1) \\ Q' = QR \end{cases} \quad \text{with } R(0) = Q(0) = 1. \tag{6.5}$$

One of the cases not solved in [56] is the simultaneous avoidance of the subwords 123 and 231. We can find a multivariate generating function $P(u, v, z)$ where the coefficient of $u^k v^l z^n/n!$ is the number of elements of $\mathcal{S}_n$ with $k$ occurrences of 123 and $l$ occurrences of 231.

Let $\mathcal{P}$ be as before the class of all permutations, and let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be subclasses of $\mathcal{P}$ defined as follows: $\mathcal{K}$ are the permutations not beginning with 12, $\mathcal{L}$ are those not ending with 12, and $\mathcal{M} = \mathcal{K} \cap \mathcal{L}$ are the ones that neither begin nor end with 12. Let $P(u, v, z)$, $K(u, v, z)$, $L(u, v, z)$, $M(u, v, z)$ be respectively the generating functions of these four classes, where $u$ marks occurrences of 123 and $v$ marks occurrences of 231.

We get the following relations for these classes.

$$\begin{aligned}
\mathcal{P} &= \{\epsilon\} + \{z\}^{\square} \star [\mathcal{L} + v(\mathcal{P} - \mathcal{L})] \star [\mathcal{K} + u(\mathcal{P} - \mathcal{K})] \\
\mathcal{K} &= \{\epsilon\} + \{z\} + \{z\}^{\square} \star [\mathcal{M} - \{\epsilon\} + v(\mathcal{K} - \mathcal{M})] \star [\mathcal{K} + u(\mathcal{P} - \mathcal{K})] \\
\mathcal{L} &= \{\epsilon\} + \{z\}^{\square} \star [\mathcal{L} + v(\mathcal{P} - \mathcal{L})] \star [\mathcal{M} - \{z\} + u(\mathcal{L} - \mathcal{M})] \\
\mathcal{M} &= \{\epsilon\} + \{z\} \\
&\quad + \{z\}^{\square} \star [\mathcal{M} - \{\epsilon\} + v(\mathcal{K} - \mathcal{M})] \star [\mathcal{M} - \{z\} + u(\mathcal{L} - \mathcal{M})]
\end{aligned}$$

The idea is the same as in previous proofs. It is based on the fact that a left subtree ending in 12 produces an occurrence of 231 along with the root, and a right subtree beginning with 12 produces a 123 with the root. Therefore, this situations must be marked with $v$ and $u$ respectively. For example, in the third relation, in order for the permutation not to end with 12, the right subtree must be an element of $\mathcal{L} - \{z\}$, and if it belongs to $\mathcal{L} - \mathcal{M}$ (i.e., begins with 12) then it must be marked with a $u$.

This gives the following system of differential equations for the generating functions ($u$ and $v$ are considered as parameters).

$$\begin{cases}
P' = [L + v(P - L)][K + u(P - K)] \\
K' = 1 + [M + v(K - M) - 1][K + u(P - K)] \\
L' = [L + v(P - L)][M + u(L - M) - z] \\
M' = 1 + [M + v(K - M) - 1][M + u(L - M) - z]
\end{cases} \tag{6.6}$$
$$\text{with } P(0) = K(0) = L(0) = M(0) = 1.$$

In particular, if we are interested in the EGF $A(z) = P(0, 0, z)$ whose coefficients are the number of permutations avoiding 123 and 231 simultaneously, we obtain the following result.

**Theorem 6.10** *The EGF $A(z)$ of permutations avoiding both 123 and 231 as subwords is the solution the following system of equations, where derivatives are with respect to $z$:*

$$\begin{cases} A' = CB \\ B' = 1 + (D + z - 1)B \\ C' = CD \\ D' = (D + z - 1)D \end{cases} \quad \text{with } A(0) = B(0) = C(0) = D(0) = 1.$$

An involved explicit form for $A(z)$ can be found in terms of integrals containing the error function, but it seems not suitable for obtaining asymptotic results. One can however obtain from the above system as many terms of $A(z)$ as desired, and we get that the corresponding counting sequence begins with

$$1, 2, 4, 11, 39, 161, 784, 4368, 27260, 189540, 1448860, 12076408,$$
$$109102564, 1061259548, \ldots$$

## 6.5 Concluding remarks

Let us make a few comments concerning the distribution of the number of occurrences of a subword. For $\sigma \in \mathcal{S}_m$ and $n \geq m$, let $X_{\sigma,n}$ be the random variable defined on $\mathcal{S}_n$ equal to the number of occurrences of $\sigma$. It is easy to see that its expectation is $E(X_{\sigma,n}) = \frac{n-m+1}{m!}$ and that its variance is $Var(X_{\sigma,n}) = c_\sigma n^2$ for some constant $c_\sigma$. It follows that the distribution is asymptotically concentrated around the expected value. In fact, using the reasoning in [29], it can be shown that $X_{\sigma,n}$ is asymptotically normal.

The asymptotic behavior of the numbers $\alpha_n(\sigma)$ as $n$ goes to infinity is considered in the next chapter.

# 7

# Asymptotic enumeration of permutations avoiding generalized patterns

In this chapter we discuss a generalization of the notion of pattern avoidance. The concept of *generalized pattern* includes both the classical definition from Section 1.1.1 used in Chapters 2, 3 and 4, and the notion of consecutive patterns described in Chapter 6. In Section 7.1 we introduce the definitions and we give the exponential generating functions for permutations avoiding a special kind of generalized patterns.

In Section 7.2 we study the asymptotic behavior as $n$ goes to infinity of the number of permutations in $\mathcal{S}_n$ avoiding a generalized pattern. We separate the patterns in different cases. In some of them we can describe their asymptotic behavior, but in other cases the behavior in the limit is unknown.

## 7.1 Generalized patterns

In [4], Babson and Steingrímsson introduced the notion of *generalized patterns*, which allows the requirement that two adjacent letters in a pattern must be adjacent in the permutation. A generalized pattern is written as a sequence where two adjacent elements may or may not be separated by a dash. In this context, we write a classical pattern with dashes between any two adjacent letters of the pattern (for example, 1423 as 1-4-2-3). If we omit the dash between two letters, we mean that for it to be an occurrence in a permutation $\pi$, the corresponding elements of $\pi$ have to be adjacent. For example, in an occurrence of the pattern 12-3-4 in a permutation $\pi$, the entries in $\pi$ that correspond to 1 and 2 are adjacent. The permutation

$\pi = 3542617$ has only one occurrence of the pattern 12-3-4, namely the subsequence 3567, whereas $\pi$ has two occurrences of the pattern 1-2-3-4, namely the subsequences 3567 and 3467.

In Chapter 6 we studied some cases of avoidance of patterns where all letters have to occur in consecutive positions. Claesson [21] presented a complete solution for the number of permutations avoiding any single 3-letter generalized pattern with exactly one adjacent pair of letters. Claesson and Mansour [22] (see also [64]) did the same for any pair of such patterns. On the other hand, Kitaev [56] investigated simultaneous avoidance of two or more 3-letter generalized patterns without internal dashes.

Throughout this chapter, all the patterns that appear will represent generalized patterns. Therefore, a pattern without dashes will denote a consecutive pattern like the ones in Chapter 6. If we want to consider a classical pattern in the sense of Section 1.1.1, we will represent it with dashes between any two elements, namely, as $\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_m$.

If $\sigma$ is a generalized pattern, $\mathcal{S}_n(\sigma)$ denotes the set of permutations in $\mathcal{S}_n$ that have no occurrences of $\sigma$ in the sense described above. Note that if $\sigma$ is a consecutive pattern, then $\mathcal{S}_n(\sigma)$ is the set that was denoted $A_n(\sigma)$ in Chapter 6. For a generalized pattern $\sigma$, let $\alpha_n(\sigma) = |\mathcal{S}_n(\sigma)|$, and let

$$A_\sigma(z) = \sum_{n\geq 0} \alpha_n(\sigma)\frac{z^n}{n!}$$

be the exponential generating function counting permutations that avoid $\sigma$.

### 7.1.1 Patterns of the form 1-$\sigma$

In this section we study a very particular class of generalized patterns, namely those that start with 1-, followed by a consecutive pattern.

**Proposition 7.1** *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ be a consecutive pattern, and let 1-$\sigma$ denote the generalized pattern $1\text{-}(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_k+1)$. Then,*

$$A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\,dt\right).$$

*Proof.* For any permutation $\pi$, if $m_1 > m_2 > \cdots > m_r$ are the values of its left-to-right minima, we can write $\pi = m_1w_1m_2w_2\cdots m_rw_r$, where each $w_i$ is a (possibly empty) subword of $\pi$, each of whose elements is greater than

$m_i$. We claim that $\pi$ avoids 1-$\sigma$ if and only if each of the blocks $w_i$ (more precisely, its reduction $\rho(w_i)$) avoids $\sigma$ as a subword. Indeed, it is clear that if one of the blocks $w_i$ contains the subword $\sigma$, then $m_i$ together with the occurrence of $\sigma$ forms an occurrence of 1-$\sigma$. Reciprocally, if $\pi$ contains 1-$\sigma$, then the elements of $\pi$ corresponding to $\sigma$ have to be adjacent, and none of them can be a left-to-right minimum (since the element corresponding to '1' has to be to their left), therefore they must be all inside the same block $w_i$ for some $i$.

If we denote by $\mathcal{A}$ the class of permutations avoiding $\sigma$ as a subword, then, in the notation of Table 1.2, the class of permutations avoiding 1-$\sigma$ can be expressed as

$$\Pi(\{z\}^{\square} \star \mathcal{A}),$$

where $\{z\}^{\square} \star \mathcal{A}$ corresponds to a block $m_i w_i$, with the box indicating that the left-to-right minimum has the smallest label. The set construction arises from the fact given a collection of blocks $m_i w_i$, there is a unique way to order them, namely with the left-to-right minima in decreasing order. The expression $A_{1\text{-}\sigma}(z) = \exp(\int_0^z A_\sigma(t)\, dt)$ follows now from this construction. □

**Example.** The only permutation avoiding the subword $\sigma = 12$ (resp. $\sigma = 21$) is the decreasing (resp. increasing) one. Therefore, by Proposition 7.1,

$$A_{1\text{-}23}(z) = A_{1\text{-}32}(z) = \exp\left(\int_0^z e^t dt\right) = e^{e^z - 1},$$

the EGF for Bell numbers, which agrees with the result in [21].

**Example.** For the subwords 132, 231, 312 and 213, we gave in Theorem 6.6 the corresponding generating functions counting their occurrences in permutations. Now, by Proposition 7.1, we get the following expression:

$$A_{1\text{-}243}(z) = A_{1\text{-}342}(z) = A_{1\text{-}423}(z) = A_{1\text{-}324}(z) = \exp\left(\frac{1}{1 - \int_0^z e^{-t^2/2} dt}\right).$$

**Example.** The EGF for permutations avoiding the subwords 123 and 321 was also given in Theorem 6.6. Proposition 7.1 implies now that

$$A_{1\text{-}234}(z) = A_{1\text{-}432}(z) = \exp\left(\frac{\sqrt{3}}{2} \frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2} z + \frac{\pi}{6})}\right).$$

Together with the results of Theorems 6.1 and 6.2, Proposition 7.1 gives expressions for the EGFs $A_{1\text{-}\sigma}(z)$ where $\sigma$ has one of the following forms:

$\sigma = 123\cdots k$,
$\sigma = k(k-1)\cdots 21$,
$\sigma = 12\cdots a\,\tau\,(a+1)$,
$\sigma = (a+1)\,\tau\,a(a-1)\cdots 21$,
$\sigma = k(k-1)\cdots(k+1-a)\,\tau'\,(k-a)$,
$\sigma = (k-a)\,\tau'\,(k+1-a)(k+2-a)\cdots k$,

where $k, a$ are positive integers with $a \leq k-2$, $\tau$ is any permutation of $\{a+2, a+3, \cdots, k\}$ and $\tau'$ is any permutation of $\{1, 2, \cdots, k-a-1\}$.

## 7.2 Asymptotic enumeration

Here we discuss the behavior of the numbers $\alpha_n(\sigma)$ as $n$ goes to infinity, for a given generalized pattern $\sigma$. We use the symbol $\sim$ to indicate that two sequences of numbers have the same asymptotic behavior (i.e., we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$), and we use the symbol $\ll$ to indicate that a sequence is asymptotically smaller than another one (i.e., we write $a_n \ll b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$).

Let us first consider the case of consecutive patterns.

**Theorem 7.2** *Let $k \geq 3$ and let $\sigma \in \mathcal{S}_k$ be a consecutive pattern.*

(1) *There exist constants $0 < c, d < 1$ such that*

$$c^n n! \leq \alpha_n(\sigma) \leq d^n n!.$$

(2) *There exists a constant $0 < w \leq 1$ such that*

$$\lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} = w.$$

Note that $c$, $d$ and $w$ depend only on $\sigma$.

*Proof.* The key observation is that, for any subword $\sigma$,

$$\alpha_{m+n}(\sigma) \leq \alpha_m(\sigma)\alpha_n(\sigma)\binom{m+n}{n}. \tag{7.1}$$

To see this, just observe that a $\sigma$-avoiding permutation of length $m+n$ induces two juxtaposed $\sigma$-avoiding permutations of lengths $m$ and $n$.

By induction on $n \geq k$ one gets

$$\alpha_{m+n}(\sigma) \leq d^m m!\, d^n n! \binom{m+n}{n} = d^{m+n}(m+n)!$$

for some positive $d < 1$.

For the lower bound, let $\tau = \rho(\sigma_1\sigma_2\sigma_3)$ be the reduction of the first three elements of $\sigma$. Clearly $\mathcal{S}_n(\tau) \subseteq \mathcal{S}_n(\sigma)$ for all $n$, since an occurrence of $\sigma$ in a permutation produces also an occurrence of $\tau$, hence $\alpha_n(\tau) \leq \alpha_n(\sigma)$. But the fact that $\sigma \in \mathcal{S}_3$ implies that $\alpha_n(\sigma)$ equals either $\alpha_n(123)$ or $\alpha_n(132)$. In any case, by Proposition 6.7 and Theorem 6.6, we have that

$$\alpha_n(\sigma) \geq \alpha_n(132) \geq c^n n!$$

for some $c > 0$.

To prove part (2), we can express (7.1) as

$$\frac{\alpha_{m+n}(\sigma)}{(m+n)!} \leq \frac{\alpha_m(\sigma)}{m!}\frac{\alpha_n(\sigma)}{n!}$$

and apply *Fekete's lemma* (see [91, Lemma 11.6] or [37]) to the function $n!/\alpha_n(\sigma)$ to conclude that $\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}$ exists. Calling it $w$, then by part (1) we have that $w \leq 1$ and $w \geq \lim_{n\to\infty}\left(\frac{\alpha_n(132)}{n!}\right)^{1/n} = 0.7839769.$ □

In order to study the asymptotic behavior of $\alpha_n(\sigma)$ for a generalized pattern $\sigma$ we separate the problem into the following cases. Assume from now on that $k \geq 3$ and that $\sigma$ is a generalized pattern of length $k$.

**Case 1** *The pattern $\sigma$ has dashes between any two adjacent elements, i.e., $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$.*

These are just the classical patterns, which have been widely studied for the last decade. The asymptotic behavior of the number of permutations avoiding them is given by the Stanley-Wilf conjecture, which has been recently proved by Marcus and Tardos [70], after several authors had given partial results over the last few years [2, 3, 14, 57].

**Theorem 7.3 (*Stanley-Wilf conjecture*, proved in [70])** *For every classical pattern $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$, there is a constant $\lambda$ (which only depends on $\sigma$) such that*

$$\alpha_n(\sigma) < \lambda^n$$

*for all $n \geq 1$.*

On the other hand, it is clear that $\alpha_n(\sigma) > \alpha_n(\rho(\sigma_1\text{-}\sigma_2\text{-}\sigma_3)) = \mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}} 4^n$. As pointed out by Arratia [3], Theorem 7.3 is equivalent to the statement that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)}$ exists. The value of this limit has been computed for several classical patterns: it is clearly 4 for patterns of length 3, it is known [72] to be $(k-1)^2$ for $\sigma = 1\text{-}2\text{-}\cdots\text{-}k$, it is shown in [11] that for $\sigma = 1\text{-}3\text{-}4\text{-}2$ this limit is 8, and it has recently been proved [15] to be nonrational for certain patterns.

**Case 2** *The pattern $\sigma$ has three consecutive elements without a dash between them, i.e., $\sigma = \cdots \sigma_i\sigma_{i+1}\sigma_{i+2}\cdots$.*

**Proposition 7.4** *Let $\sigma$ be a generalized pattern having three consecutive elements without a dash. Then there exist constants $0 < c, d < 1$ such that*

$$c^n n! \leq \alpha_n(\sigma) \leq d^n n!.$$

*Proof.* For the upper bound, notice that if a permutation contains the consecutive pattern $\sigma_1\sigma_2\sigma_3\cdots\sigma_k$ obtained by removing all the dashes in $\sigma$, then it also contains $\sigma$. Therefore, $\alpha_n(\sigma) \leq \alpha_n(\sigma_1\sigma_2\sigma_3\cdots\sigma_k)$ for all $n$, and now the upper bound follows from part (1) of Theorem 7.2.

For the lower bound, we use that $\alpha_n(\sigma) \geq \alpha_n(\rho(\sigma_i\sigma_{i+1}\sigma_{i+2})) \geq \alpha_n(132) \geq c^n n!$. □

**Case 3** *The pattern $\sigma$ has pairs of adjacent elements without a dash between them, but not three consecutive elements without dashes.*

This case includes all the patterns not considered in Cases 1 and 2. The asymptotic behavior of $\alpha_n(\sigma)$ for these patterns is not known in general. For patterns of length 3 we have the following result due to Claesson [21]. Let $\mathbf{B}_n$ denote the $n$-th *Bell number*, which counts the number of partitions of an $n$-element set.

**Proposition 7.5 ([21])** *Let $\sigma$ be a generalized pattern of length 3 with one dash.*

(1) *If* $\sigma \in \{1\text{-}23, 3\text{-}21, 32\text{-}1, 12\text{-}3, 1\text{-}32, 23\text{-}1, 3\text{-}12, 21\text{-}3\}$, *then* $\alpha_n(\sigma) = \mathbf{B}_n$.

(2) *If* $\sigma \in \{2\text{-}13, 2\text{-}31, 31\text{-}2, 13\text{-}2\}$, *then* $\alpha_n(\sigma) = \mathbf{C}_n$.

It is known that the asymptotic behavior of the Catalan numbers is given by $\mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}} 4^n$. For the Bell numbers, we have the formula

$$\mathbf{B}_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{n+1/2} e^{\lambda(n)-n-1},$$

where $\lambda(n)$ is defined by $\lambda(n)\ln(\lambda(n)) = n$. Another useful description of the asymptotic behavior of $\mathbf{B}_n$ is the following:

$$\frac{\ln \mathbf{B}_n}{n} = \ln n - \ln\ln n + O\left(\frac{\ln\ln n}{\ln n}\right).$$

This shows in particular that $\delta^n \ll \mathbf{B}_n \ll c^n n!$ for any constants $\delta, c > 0$.

For patterns $\sigma$ of length at least 4 that have pairs of adjacent elements without a dash, but not adjacent triplets without dashes, not much is known in general about the number of permutations avoiding them. The asymptotic behavior of $\alpha_n(\sigma)$ could be anywhere between $\delta^n$ for some constant $\delta > 0$ (obviously, if $\sigma$ contains one of the patterns in part (1) of Proposition 7.5, then this lower bound can be improved to $\mathbf{B}_n$) and $d^n n!$ for some constant $0 < d < 1$. In the rest of this chapter we discuss a few partial results in this direction.

The next statement about permutations of the form 1-$\sigma$, is an easy consequence of Proposition 7.1.

**Corollary 7.6** *Let $\sigma$ be a consecutive pattern, and let 1-$\sigma$ be defined as in Proposition 7.1. Then,*

$$\lim_{n\to\infty} \left(\frac{\alpha_n(1\text{-}\sigma)}{n!}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

*Proof.* By Proposition 7.1 we know that $A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\,dt\right)$. Since the exponential is an analytic function, we obtain that $A_{1\text{-}\sigma}(z)$ has the same radius of convergence as $A_\sigma(z)$, from where the result follows. □

### 7.2.1 The pattern 12-34

The next proposition gives an upper and a lower bound for the numbers $\alpha_n(\text{12-34})$. Given two formal power series $F(z) = \sum_{n\geq 0} f_n$ and $G(z) = \sum_{n\geq 0} g_n$, we use the notation $F(z) < G(z)$ to indicate that $f_n < g_n$ for all $n$, and $F(z) \ll G(z)$ to indicate that $f_n \ll g_n$ for all $n$.

**Proposition 7.7** *For $k \geq 1$, let*

$$\begin{aligned}
h_k &= 1 + \frac{1}{2} + \cdots \frac{1}{k},\\
b_k(z) &= \sum_{i=0}^{k} \binom{k}{i}^2 [z + 2(h_{k-i} - h_i)]\, e^{iz},\\
c_k(z) &= \frac{e^{(k+1)z}}{k+1} - \sum_{i=0}^{k} \binom{k}{i}\binom{k+1}{i} \left[z + 2(h_{k-i} - h_i) + \frac{1}{k+1-i}\right] e^{iz},\\
S(z) &= \sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z).
\end{aligned}$$

*Then*

$$e^{S(z)} < A_{\text{12-34}}(z) < e^{S(z)+e^z+z-1}.$$

If we write $e^{S(z)} = \sum l_n \frac{z^n}{n!}$ and $e^{S(z)+e^z+z-1} = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the graph in Figure 7.1 shows the values of $\sqrt[n]{\alpha_n(\text{12-34})/n!}$ for $n \leq 13$, bounded between the values $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 120$. The two horizontal dotted lines are at height 0.7839769 and 0.8269933, which are $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ for $\sigma = 132$ and $\sigma = 123$ respectively, given by Theorem 6.6. From this figure it seems plausible that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\text{12-34})/n!} = 0$, although we have not succeeded in proving this.

Note that the lower bound, together with the fact that $S(z) \gg e^z - 1$ (which follows from the definition), shows that $A_{\text{12-34}}(z) > e^{S(z)} \gg e^{e^z-1}$, which means that $\alpha_n(\text{12-34}) \gg \mathbf{B}_n$, that is, the number of 12-34-avoiding permutations is asymptotically larger than the Bell numbers.

*Proof.* Let $\pi$ be a permutation that avoids 12-34. This means that it has no two ascents such that the second one starts at a higher value than where the first one ends. We can write $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where $a_1$ and the element preceding it form the first ascent of $\pi$, $a_2$ and the element preceding it form the first ascent such that $a_2 < a_1$, $a_3$ and the element preceding it

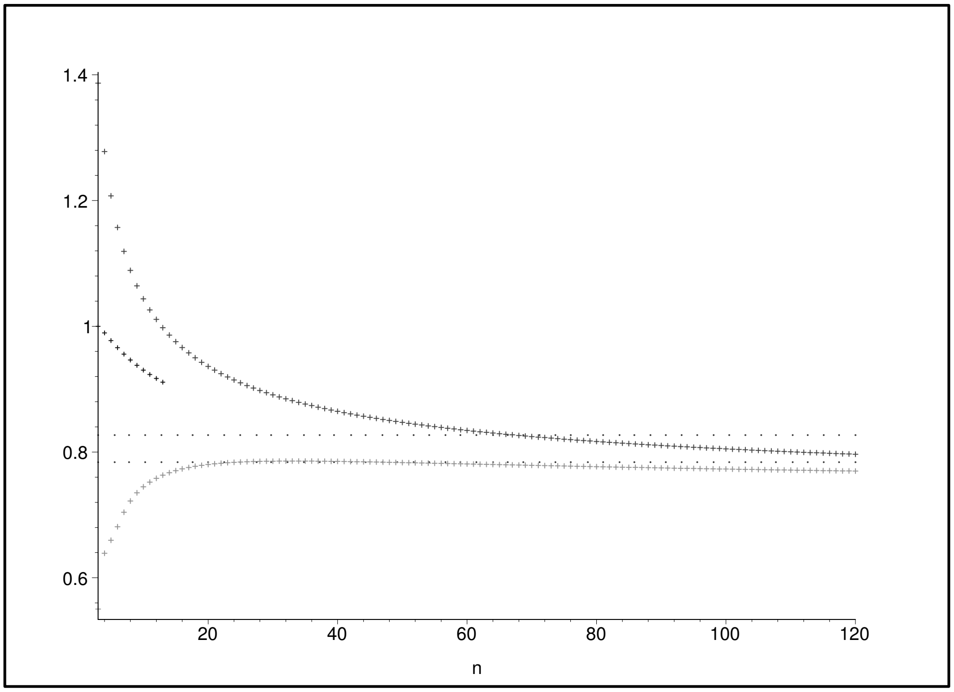


**Figure 7.1** The first values of $\sqrt[n]{\alpha_n(12\text{-}34)/n!}$ between the lower and the upper bound given by Proposition 7.7.

form the first ascent such that $a_3 < a_2$, and so on. By definition, $B_0$ is a non-empty decreasing subword whose last element is less than $a_1$, and each $B_i$ with $i \geq 1$ can be written uniquely as a sequence $B_i = w_{i,0}w_{i,1}w_{i,2}\cdots w_{i,r_i}$ for some $r_i \geq 1$ ($r_i$ can be 0 if $w_{i,0}$ is nonempty) with the following properties:

(1) each $w_{i,j}$ is a decreasing word,

(2) for $j \geq 1$, $w_{i,j}$ is nonempty and its first element is bigger than $a_i$,

(3) the last element of each $w_{i,j}$ is less than $a_i$,

(4) the last element of $B_i$ is less than $a_{i+1}$.

These properties ensure that $\pi$ avoids 12-34 (since no $B_i$ has an ascent above $a_i$), and that the decomposition is unique.

Ideally we would like to use this decomposition to find a generating function for the numbers $\alpha_n(12\text{-}34)$. Unfortunately, the structure of the decomposition is a bit too complicated to find an exact formula. Instead, we will add and remove restrictions to simplify this description, which allows us to give lower and upper bounds respectively.

To find an upper bound, we will count permutations of the form $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where the $B_i$ and $a_i$ satisfy the properties above, except for the requirement that the last element of each $B_i$ has to be less than $a_{i+1}$. Omitting this requirement we are overcounting permutations, and thus we get an upper bound. The first step now is to find the EGF for a block $K_i$ of the form $a_i B_i$, where $B_i$ satisfies properties (1), (2) and (3) from above.

Let us first assume that $w_{i,0}$ is empty, that is, $B_i = w_{i,1} w_{i,2} \cdots w_{i,r_i}$. We compute the EGF for $K_i = a_i B_i$ where $r_i$ is fixed by induction on $r_i$. If $r_i = 0$, then we have that $K_i = a_i$, so the EGF is $b_0(z) := z$. If $r_i = 1$, then $K_i = a_i w_{i,1}$, where $w_{i,1}$ is a decreasing word starting above $a_i$ and ending below it. The EFG for $w_{i,1}$ is $e^z$. Now, to incorporate the condition that the largest and the smallest labels of $K_i$ lie in $w_{i,1}$, we use a generalization of the boxed product construction described in Section 1.3.2. A double derivative is now needed to mark the two special elements. We get that the EGF for such a block is

$$\int_0^z \int_0^y t \left( \frac{d^2}{dt^2} e^t \right) dt\, dy = \int_0^z \int_0^y t e^t dt\, dy = (z-2)e^z + z + 2 = b_1(z).$$

Let now be $r_i = 2$. The case in which both the largest and the smallest label of $K_i = a_i w_{i,1} w_{i,2}$ are contained in $w_{i,2}$ corresponds to the EGF

$$\int_0^z \int_0^y b_1(t) \left( \frac{d^2}{dt^2} e^t \right) dt\, dy. \tag{7.2}$$

If we write each $w_{i,j}$ as $w_{i,j}^+ w_{i,j}^-$, separating the elements above and below $a_i$ ($w_{i,j}^+$ and $w_{i,j}^-$ respectively), then the largest element of $K_i$ can be either in $w_{i,1}^+$ or in $w_{i,2}^+$, and the smallest element of $K_i$ can be either in $w_{i,1}^-$ or in $w_{i,2}^-$. Thus, all the possibilities are obtained from the case counted by the EGF (7.2) by permuting the upper and lower parts of the $w_{i,1}$ and $w_{i,2}$ in the 4 different ways. It follows that the EGF for $K_i$ when $r_i = 2$ is

$$4 \int_0^z \int_0^y b_1(t)\, e^t dt\, dy = (z-3)e^{2z} + 4ze^z + z + 3 = b_2(z).$$

In general, if $b_{k-1}(z)$ is the EGF for the case $r_i = k-1$, then the EGF for the case $r_i = k$ is given by

$$b_k(z) = k^2 \int_0^z \int_0^y b_{k-1}(t)\, e^t dt\, dy.$$

It is straightforward to check that the functions $b_k(z)$ defined in the statement of the proposition satisfy this recurrence.

The case where $w_{i,0}$ is nonempty can be treated similarly. Now we have $B_i = w_{0,i}w_{i,1}w_{i,2}\cdots w_{i,r_i}$. If $r_i = 0$, the EGF for $a_i w_{0,i}$ is $c_0(z) := e^z - 1 - z$ (since the block has at least 2 elements). If $r_i = 1$, then a block of the form $a_i w_{0,i} w_{i,1}$ can be obtained from the case where the largest and the smallest element are in $w_{i,1}$ by permuting $w_{0,i}$ and $w_{i,1}^-$ if necessary. This yields the EGF

$$2\int_0^z \int_0^y c_0(t)\left(\frac{d^2}{dt^2}e^t\right)dt\,dy = \frac{e^{2z}}{2} + 2(1-z)e^z - z - \frac{5}{2} = c_1(z).$$

In general, for nonempty $w_{i,0}$, if $c_{k-1}(z)$ is the EGF for the case $r_i = k-1$, then the EGF for the case $r_i = k$ is given by

$$c_k(z) = k(k+1)\int_0^z \int_0^y c_{k-1}(t)\,e^t dt\,dy.$$

This is the recurrence satisfied by the functions $c_k(z)$ defined in the statement of the proposition.

The generating function for a set of blocks $K_i = a_i B_i$ of the form just described is

$$\exp\left(\sum_{k\geq 0} b_k(z) + \sum_{k\geq 0} c_k(z)\right) = \exp(S(z) + z + e^z - 1 - z).$$

From such a set there is a unique way to form a sequence $a_1B_1a_2B_2a_3B_3\cdots$ where $a_1 > a_2 > a_3 > \cdots$. Finally, we multiply by $e^z$ to take into account the initial decreasing segment $B_0$ of the permutation $\pi = B_0a_1B_1a_2B_2a_3B_3\cdots$, again ignoring the condition that its last element should be smaller than $a_1$. This gives the upper bound $e^z\exp(S(z) + e^z - 1) = \exp(S(z) + e^z + z - 1)$.

Now we use a similar reasoning to obtain a lower bound. We have seen that $b_k(z)$ counts blocks of the form $a_iw_{i,1}w_{i,2}\cdots w_{i,k}$, where each $w_{i,j}$ is a decreasing word starting above $a_i$ and ending below it. If $k \geq 1$, using the notation $w_{i,k} = w_{i,k}^+ w_{i,k}^-$ to separate the elements that are bigger than $a_i$ from those that are smaller, we can move the last part of the block to the beginning and write $L_i = w_{i,k}^- a_i w_{i,1} w_{i,2}\cdots w_{i,k}^+$. Similarly, a block of the form $a_iw_{i,0}w_{i,1}w_{i,2}\cdots w_{i,k}$ like the ones counted by $c_k(z)$ with $k \geq 1$ can be reordered as $L_i' = w_{i,k}^- a_i w_{i,0} w_{i,1} w_{i,2} \cdots w_{i,k}^+$. The EGF that counts sets of pieces of the forms given by $L_i$ and $L'_i$ is

$$\exp\left(\sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z)\right) = \exp(S(z)).$$

Ordering the pieces of such a set by decreasing order of the $a_i$, the sequence that they form by juxtaposition is a 12-34-avoiding permutation. Besides, no such permutation is obtained in more than one way by this construction. However, notice that not every 12-34-avoiding permutation is produced by this process, hence this construction gives only a lower bound. □

The decomposition of 12-34-avoiding permutations given in this proof can be generalized to permutations avoiding a pattern of the form 12-$\sigma$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ is a consecutive pattern, 12-$\sigma$ denotes the generalized pattern 12-$(\sigma_1+2)(\sigma_2+2)\cdots(\sigma_k+2)$.

Any permutation $\pi$ that avoids 12-$\sigma$ can be uniquely decomposed as $\pi = B_0a_1B_1a_2B_2a_3B_3\cdots$, where $a_1$ and the element preceding it form the first ascent of $\pi$, $a_2$ and the element preceding it form the first ascent such that $a_2 < a_1$, $a_3$ and the element preceding it form the first ascent such that $a_3 < a_2$, and so on. Then, by definition, $B_0$ is a non-empty decreasing subword whose last element is less than $a_1$, and each $B_i$ with $i \geq 1$ can be written uniquely as a sequence $B_i = w_{i,0}U_{i,1}w_{i,1}U_{i,2}w_{i,2}\cdots U_{i,r_i}w_{i,r_i}$ for some $r_i \geq 1$ ($r_i$ can be 0 if $w_{i,0}$ is nonempty) with the following properties:

(1) each $w_{i,j}$ is a decreasing word all of whose elements are less than $a_i$,

(2) each $U_{i,j}$ is a nonempty permutation avoiding $\sigma$ as a subword, all of whose elements are greater than $a_i$,

(3) $w_{i,j}$ is nonempty for $j \geq 1$,

(4) the last element of $B_i$ is less than $a_{i+1}$.

From this decomposition the following result follows immediately.

**Proposition 7.8** *If $\sigma \sim \tau$ are two consecutive patterns, then* 12-$\sigma \sim$ 12-$\tau$*.*

The structure of 21-$\sigma$-avoiding permutations (defined analogously) can be described using the same ideas, and it is not hard to see that the following result holds as well.

**Proposition 7.9** *If $\sigma$ is a consecutive pattern, then* 12-$\sigma \sim$ 21-$\sigma$*.*

### 7.2.2 The pattern 1-23-4

Similarly to what we did for the pattern 12-34, analyzing the structure of permutations avoiding 1-23-4 we can give lower and upper bounds for the numbers $\alpha_n$(1-23-4). Let $\mathbf{C}^{\exp} := \sum_{n\geq 0} \mathbf{C}_n \frac{z^n}{n!}$ be the EGF for the Catalan numbers.

**Proposition 7.10** *We have that*

$$\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} < A_{1\text{-}23\text{-}4}(z) < \mathbf{C}^{\exp}(e^z-1).$$

Writing $\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} = \sum l_n \frac{z^n}{n!}$ and $\mathbf{C}^{\exp}(e^z-1) = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the values of $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 90$ are plotted in Figure 7.2, bounding the values of $\sqrt[n]{\alpha_n(1\text{-}23\text{-}4)/n!}$ for $n \leq 11$.

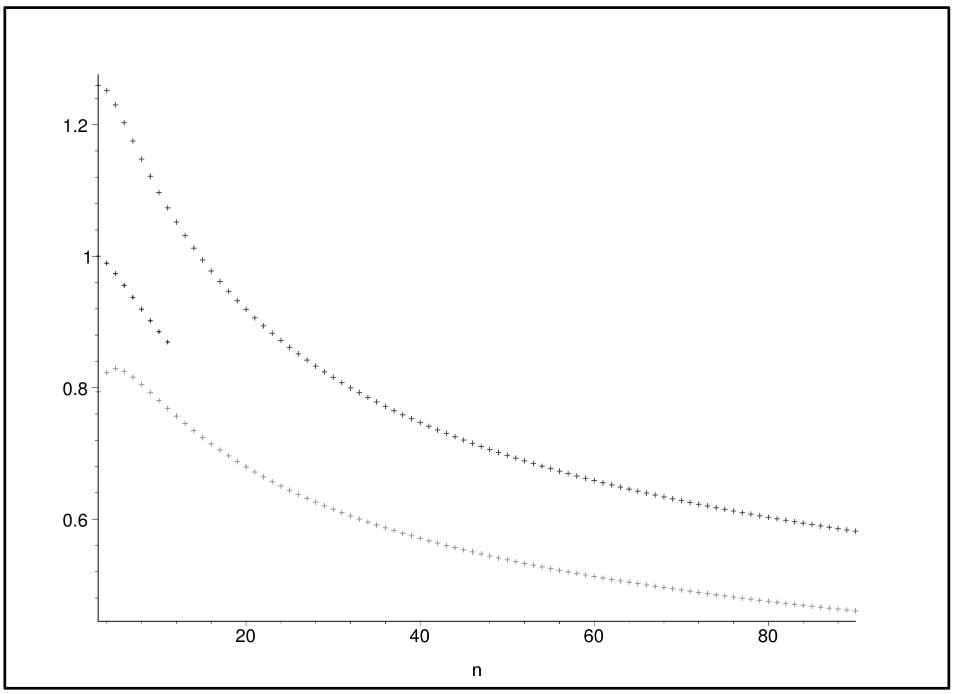


**Figure 7.2** The first values of $\sqrt[n]{\alpha_n(1\text{-}23\text{-}4)/n!}$ between the lower and the upper bound given by Proposition 7.10.

Note that the lower bound implies that $\alpha_n(1\text{-}23\text{-}4) \gg \mathbf{B}_n$, since $e^{2e^z-2} \gg e^{e^z-1}$.

*Proof.* Let $\pi$ be a permutation that avoids 1-23-4. Let $a_1 > a_2 > a_3 > \cdots > a_r$ be the left-to-right minima of $\pi$, and let $b_1 > b_2 > b_3 > \cdots > b_s$ be its right-to-left maxima. Then, marking the positions of the left-to-right minima and right-to-left maxima, we can write $\pi = c_1 w_1 c_2 w_2 \cdots c_{r+s-1} w_{r+s-1} c_{r+s}$, where $c_i \in \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s\}$ for all $i$ (in fact the number of $c_i$'s could be less than $r+s$ if some element is simultaneously a left-to-right minimum and a right-to-left maximum). Note that $c_1 = a_1$ and $c_{r+s} = b_s$. Now, the condition that $\pi$ avoids 1-23-4 is equivalent to the fact that each $w_i$ is a (possibly empty) decreasing word. Indeed, if there was an ascent inside one of the $w_i$, then together with the closest left-to-right minimum to the left of $w_i$ and the closest right-to-left maximum to the right of $w_i$, it would form an occurrence of 1-23-4. On the other hand, it is clear that if all $w_i$ are decreasing, then no such occurrence can exist.

We use this decomposition to obtain upper and lower bounds for $\alpha_n(1\text{-}23\text{-}4)$. Let us first show the lower bound. For that we count only a special type of 1-23-4-avoiding permutations, namely the ones where all the left-to-right minima come before all the right-to-left maxima. Such a $\pi$ can be written as $\pi = a_1 w_1 a_2 w_2 \cdots a_r w_r b_1 w_{r+1} b_2 w_{r+2} \cdots w_{r+s-1} b_s$, where for $1 \le i \le r$ the elements of the decreasing words $w_i$ have values between $a_i$ and $b_1$, and for $r \le i \le r+s-1$ the elements of $w_i$ have values between $a_r$ and $b_{i+1}$. The EGF for the part $a_1 w_1 a_2 w_2 \cdots a_{r-1} w_{r-1}$ is $e^{e^z-1}$, since it is an arbitrary 1-23-avoiding permutation (see the example following Proposition 7.1). Similarly, the EGF for the part $w_{r+1} b_2 w_{r+2} \cdots w_{r+s-1} b_s$ is also $e^{e^z-1}$ (it can be seen as a set of blocks of the form $w_{r+i} b_{i+1}$, each one contributing $e^z - 1$, arranged by decreasing order of the $b_i$'s). The decreasing word $w_r$ contributes $e^z$. Now, to get the EGF for the whole permutation $a_1 w_1 a_2 w_2 \cdots a_r w_r b_1 w_{r+1} b_2 w_{r+2} \cdots w_{r+s-1} b_s$ we use the boxed product construction to require that the biggest element of the block is $b_1$ and the smallest one is $a_r$. The EGF that we obtain is

$$\int_0^z \int_0^y e^{e^t-1} \left(\frac{d}{dt} t\right) e^t \left(\frac{d}{dt} t\right) e^{e^t-1} \, dt \, dy = \frac{1}{2} \int_0^z (e^{2e^y-2} - 1) \, dy,$$

which gives a lower bound for the coefficients of $A_{1\text{-}23\text{-}4}(z)$.

To find the upper bound, consider first permutations of the form $\pi = c_1 w_1 c_2 w_2 \cdots c_{r+s-1} w_{r+s-1} c_{r+s}$ where all the $w_i$ are empty. Such permutations, where every element is either a left-to-right minimum or a right-to-left maximum, are precisely those avoiding 1-2-3, which are counted by the Catalan numbers. Thus, the EGF for such permutations is $\mathbf{C}^{\exp}(z)$.

The next step is to insert a decreasing word $w_i$ after each $c_i$. If $c_i$ is a left-to-right minimum, we require that the elements of $w_i$ are bigger than $c_i$, so the EGF for the block $c_i w_i$ is $e^z - 1$. We omit the requirement that the elements of $w_i$ have to be smaller than the nearest right-to-left maximum to the right of $w_i$; this is why we only get an upper bound. Similarly, if $c_j$ is a right-to-left maximum, we require that the elements of $w_j$ are smaller than $c_j$, so the EGF for the block $c_j w_j$ is also $e^z - 1$. We also omit the requirement that after the last right-to-left maximum there is no decreasing word. Replacing each $c_i$ for a block $c_i w_i$ as just described translates in terms of generating functions into substituting $e^z - 1$ for the variable $z$ in $\mathbf{C}^{\exp}(z)$. This gives the upper bound of the statement. □

The upper bound given in the above proposition yields the following corollary.

**Corollary 7.11** *We have that*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\text{1-23-4})}{n!}\right)^{1/n} = 0.$$

*Proof.* The power series $\mathbf{C}^{\exp}(z)$ can be bounded by

$$\mathbf{C}^{\exp}(z) < \sum_{n\geq 0} 4^n \frac{z^n}{n!} = e^{4z},$$

which converges for all $z$. Therefore, so does $\mathbf{C}^{\exp}(e^z - 1)$, which is an upper bound for $A_{\text{1-23-4}}(z)$. The result follows now from the observations in Section 6.1.3. □

The decomposition of 1-23-4-avoiding permutations given in the proof of the above proposition can be generalized to permutations avoiding a pattern of the form 1-$\sigma$-$k$, defined as follows. If $\sigma = \sigma_1\sigma_2\cdots\sigma_{k-2} \in \mathcal{S}_{k-2}$ is a consecutive pattern, let 1-$\sigma$-$k$ denote the generalized pattern 1-$(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_{k-2}+1)$-$k$.

Any permutation $\pi$ that avoids 1-$\sigma$-$k$ can be uniquely decomposed as $\pi = c_1 w_1 c_2 w_2 \cdots c_{m-1} w_{m-1} c_m$, where the $c_i$ are all the left-to-right minima and right-to-left maxima of $\pi$, and each $w_i$ is a permutation that avoids $\sigma$ as a subword, all of whose elements are bigger than the closest left-to-right minimum to its left and smaller than the closest right-to-left maximum to its right.

Using exactly the same reasoning as in the proof of Proposition 7.10, we obtain the following lower and upper bounds for the numbers $\alpha_n(1\text{-}\sigma\text{-}k)$.

**Proposition 7.12** *Let $\sigma \in \mathcal{S}_{k-2}$ be a consecutive pattern, and let $1\text{-}\sigma\text{-}k$ be defined as above. Then,*

$$\int_0^z \int_0^v e^{2\int_0^y A_\sigma(t)dt+y}\, dy\, dv < A_{1\text{-}\sigma\text{-}k}(z) < \mathbf{C}^{\exp}\left(\int_0^z A_\sigma(t)\, dt\right).$$

**Corollary 7.13** *With the same definitions as in the above proposition,*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{-}\sigma\text{-}k)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

*Proof.* The upper and lower bounds for $A_{1\text{-}\sigma\text{-}k}(z)$ given in Proposition 7.12 are analytic functions of $A_\sigma(z)$, since essentially they only involve exponentials and integrals. Therefore, $A_{1\text{-}\sigma\text{-}k}(z)$ and $A_\sigma(z)$ have the same radius of convergence, hence the limits above coincide. □

Finally, the following proposition is an immediate consequence of the structure of permutations avoiding $1\text{-}\sigma\text{-}k$ discussed above. In particular, it implies that 1-23-4 $\sim$ 1-32-4.

**Proposition 7.14** *If $\sigma \sim \tau$ are two consecutive patterns in $\mathcal{S}_{k-2}$, then $1\text{-}\sigma\text{-}k \sim 1\text{-}\tau\text{-}k$.*

### 7.2.3 Other patterns

We have proved that $\alpha_n(1\text{-}23\text{-}4) \gg \mathbf{B}_n$ and that $\alpha_n(1\text{-}23\text{-}4) \ll c^n n!$ for any constant $c > 0$. For the pattern 12-34 we showed that the analogue to the first statement holds as well, and the second one seems to be true from numerical computations. It remains as an open problem to give more general results concerning the asymptotical behavior of the numbers $\alpha_n(\sigma)$ where $\sigma$ is a generalized pattern.

In Figure 7.3 we have plotted the initial values (connected by lines) of the sequences $\sqrt[n]{\alpha_n(\sigma)/n!}$ for other cases that appear to have some interest. The

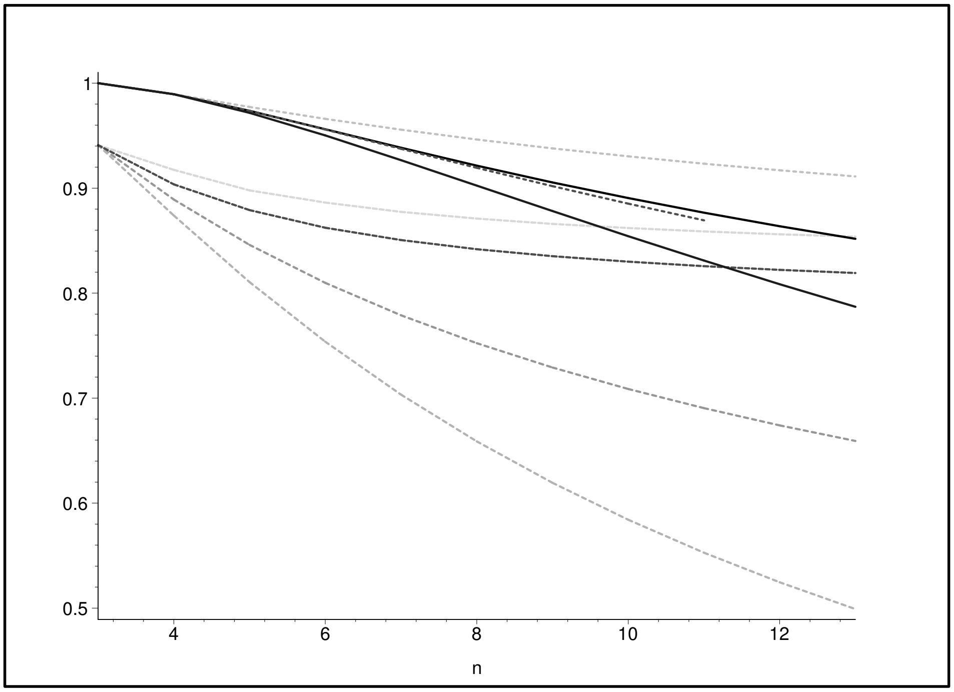


**Figure 7.3** The first values of $\sqrt[n]{\alpha_n(\sigma)/n!}$ for several generalized patterns $\sigma$.

two dotted lines at the bottom of the graph correspond to the sequences $\sqrt[n]{\mathbf{C}_n/n!}$ and $\sqrt[n]{\mathbf{B}_n/n!}$, which are known to tend to 0 as $n$ goes to infinity. The two dashed lines that start at the same point (around 0.941) and tend to a constant correspond to the sequences $\sqrt[n]{\alpha_n(132)/n!}$ and $\sqrt[n]{\alpha_n(123)/n!}$, for which their limits are known to be 0.7839769 and 0.8269933 respectively. Among the lines starting at 1, the two dotted ones correspond to the patterns 1-23-4 (the lower line) and 12-34 (the upper line) discussed in the previous subsections.

Of the two solid lines, the one below corresponds to the pattern 3-14-2. This pattern has a special interest because all of its subpatterns of length 3 are among those in part (2) of Proposition 7.5. Since it does not contain any of the patterns in part (1), we cannot say that $\alpha_n(\text{3-14-2}) \geq \mathbf{B}_n$ for all $n$. In fact, comparting the slopes in Figure 7.3 it seems quite plausible that $\alpha_n(\text{3-14-2})$ is asymptotically smaller than $\mathbf{B}_n$, and proving this is an interesting open question. The other solid line in the graph corresponds to the pattern 13-24, for which we do not know the asymptotic behavior either.

# Conclusions

This thesis is focused on the enumeration of pattern-avoiding permutations with respect to certain statistics, and on the enumeration of permutations avoiding generalized patterns. We now review the results we have obtained and discuss lines of future research.

In Chapter 2 we have classified patterns of length 3 according to the distribution of the statistics 'number of fixed points' and 'number of excedances' in permutations avoiding them. We have introduced bijections between pattern-avoiding permutations and Dyck paths that have played an important role throughout the thesis. They were presented in a graphical way which made it easier to study their properties. The main result of the chapter was that the joint distribution of this pair of statistics is the same in 321-avoiding as in 132-avoiding permutations. This generalizes a recent theorem of Robertson, Saracino and Zeilberger. We gave a bijection between these two sets of permutations that preserves both the number of fixed points and the number of excedances, thus giving a combinatorial proof of the result. Our bijection is a composition of two bijections into Dyck paths, and the result follows from a new analysis of these bijections. The Robinson-Schensted-Knuth correspondence is a part of one of them, and from it stemmed the difficulty of the analysis. The key idea was to introduce a new class of statistics on Dyck paths, based on the concept of *tunnel*, which we introduced in Chapter 1. For the patterns 132, 213 and 321 we gave generating functions with variables enumerating the two statistics mentioned above, and in some cases additional statistics as well. For the patterns 231 and 312 we expressed the corresponding generating functions as continued fractions. For the case of the pattern 123 we could only give partial results regarding the number of fixed points, and we used them to prove a recent conjecture of Bóna and

Guibert.

It would be interesting to try to find a generating function for fixed points and excedances in 123-avoiding permutations, the only case of patterns of length 3 that remains unsolved. Even for the enumeration of fixed points in these permutations, we expect that a simpler expression than the one in Theorem 2.13 can be given.

In Chapter 3 we have studied permutations avoiding simultaneously two or more patterns of length 3, enumerating them with respect to the number of fixed points and the number of excedances [32]. By means of the bijections between restricted permutations and Dyck paths described in Chapter 2, additional restrictions on permutations correspond to certain conditions on the paths, and thus the problem was reduced to enumerating such paths with respect to the statistics that fixed points and excedances are mapped to by these bijections. We solved all the cases of avoidance of two or more patterns by giving the corresponding multivariate generating functions, which are rational and have relatively simple expressions. In some instances we were able to give a generalization to the case where one of the patterns had arbitrary length. Then we enumerated involutions avoiding any subset of patterns of length 3 with respect to the same two parameters. The main technique consists in using bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that our statistics in permutations correspond to statistics on Dyck paths which are easier to enumerate.

An interesting extension of this work would be to study the distribution of statistics in permutations avoiding longer patterns. The enumeration of such permutations is itself a very difficult problem, and not even the case of length 4 is completely solved (see [11, 45, 47] for single patterns, [13, 60, 94, 95] for pairs of patterns of length 4, and [20, 59, 66, 67, 68] for other pairs). For the case of patterns of length 4, we have checked by computer that the only cases in which the number of derangements in $\mathcal{S}_n(\sigma)$ is the same for different patterns $\sigma \in \mathcal{S}_4$ are those in which there exists a trivial bijection (such as $\pi \mapsto \pi^{-1}$ or $\pi \mapsto \widehat{\pi}$) proving this fact. Therefore, Theorems 1.4 and 2.3 do not seem to have an analogue for patterns of length 4. Still, there would be some interest in finding generating functions to enumerate permutations avoiding patterns of length 4 or more with respect to statistics such as the number of fixed points and the number of excedances. For permutations avoiding a pattern of length 4 there are 13 different equivalence classes with respect to the distribution of the statistic fp.

It is possible that Theorem 2.3 admits generalizations to other permutation statistics, or some variations. We have not succeeded in finding any other case of equidistribution of a statistic for different patterns having such an interesting and nontrivial proof. An example of a much simpler result is that the statistic 'number of descents' has the same distribution in $\mathcal{S}_n(132)$, $\mathcal{S}_n(213)$, $\mathcal{S}_n(231)$ and $\mathcal{S}_n(312)$.

Another further direction of research would consist in describing the cycle structure of pattern-avoiding permutations. Using the same bijective techniques from Chapters 2 and 3, one can easily derive generating functions for the *augmented cycle index* of permutations in $\mathcal{S}_n(231, 312)$, in $\mathcal{S}_n(231, 321)$ and in $\mathcal{S}_n(132, 321)$. However, it is not clear whether for permutations avoiding other subsets of patterns of length 3, the distribution of the cycle type has a simple description.

One might wonder if the fact that the number of fixed points has the same distribution both in 321- and in 132-avoiding permutations admits a generalization concerning the cycle structure in $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$. We have for example that the cycle structure of 321-avoiding involutions and 132-avoiding involutions is the same. However, it is not true that the cycle structure of permutations in $\mathcal{S}_6(321)$ is the same as that of permutations in $\mathcal{S}_6(132)$, as shown in [27].

In Chapter 4 we have given the simplest known bijection between 321- and 132-avoiding permutations that preserves the number of fixed points. The main ingredient is a new unusual bijection from the set of Dyck paths to itself. We also presented a generalization of it, which gave additional correspondences of statistics, as well as applications to the enumeration of Dyck paths and restricted permutations with respect to several statistics.

In Chapter 5 we have presented some new interpretations of the Catalan and Fine numbers, and a few natural bijections between 321-avoiding permutations and Dyck paths. Then we have considered a class of permutations obtained from noncrossing matchings of $2n$ points around a circle. They are counted by the Catalan numbers, but are not defined in terms of pattern avoidance. We found the ordinary generating functions with variables marking the number of descents and the number of fixed points and excedances.

In Chapter 6 we have introduced a variation to the notion of pattern avoidance, with the additional requirement that the elements forming the pattern have to occur in consecutive positions in the permutation. We studied the number $\alpha_n(\sigma)$ of permutations avoiding $\sigma$ as a subword and, more generally, the number of occurrences of $\sigma$ in permutations of length $n$. For the case of the increasing pattern of any length and of another pattern of a fairly general shape, using bijections between permutations and binary increasing trees, we were able to solve the problem by obtaining the corresponding bivariate exponential generating functions as inverses of solutions of linear differential equations. This provides a complete solution for all consecutive patterns of length 3, and for three out of the seven classes of patterns of length 4. We also considered some cases of simultaneous avoidance of subwords.

Besides extending our results to other subwords not covered by our analysis, there remain some interesting problems. For instance, it appears from our computations that the increasing subword $12\cdots m$ is always dominating, in the sense that $\alpha_n(12\cdots m) > \alpha_n(\sigma)$ for any $\sigma \in \mathcal{S}_m$ and $n$ large enough. We conjecture that this is always the case.

Finally, in Chapter 7 we have discussed the notion of generalized patterns, which generalizes the definitions of both classical and consecutive patterns. For patterns of the form 1-$\sigma$ with no dashes in $\sigma$ we have obtained the exponential generating function in terms of that for $\sigma$-avoiding permutations. Next we have studied the asymptotic behavior of the number of permutations in $\mathcal{S}_n$ avoiding a fixed generalized pattern as $n$ goes to infinity. For the case of classical patterns, a description of this asymptotic behavior had been given by the recently proven Stanley-Wilf conjecture [70]. We consider the analogous problem for generalized patterns, solving it for consecutive patterns and in some other cases. For a few additional generalized patterns such as 12-34 and 1-23-4 we have shown lower and upper bounds on the number of permutations avoiding them, which gives an estimate of their asymptotic behavior.

An interesting problem would be to present a complete classification of all generalized patterns $\sigma$ according to the asymptotic behavior of the numbers $\alpha_n(\sigma)$ of permutations avoiding them as $n$ goes to infinity. This would conclude the work initiated in this last chapter.

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