Chapter 2

Quantitative estimates on the normal form

2.1 Overview of the chapter

Suppose that we transform a function \( f = \sum_{l \geq 1} f_l \in \mathcal{E} \) through the map \( T_G \) in the definition 1.7 on page 21 to obtain \( T_G f = \sum_{k \geq 1} F_k \) and we are interested in bounding –using some suitable norm to be defined later–, the sum \( \sum_{k \geq r} F_k \). Since \( F_s, s = 1, 2, \ldots \), are obtained as a combination of Poisson brackets of \( f_1, \ldots, f_k, G_3, \ldots, G_{s+1} \) of type,

\[
  F_1 = f_1, \\
  F_2 = f_2 + \{f_1, G_3\}, \\
  F_3 = f_3 + \{f_2, G_3\} + \{f_1, G_4\} + \frac{1}{2}\{\{f_1, G_3\}, G_3\}, \\
  \vdots
\]

Then, assuming certain bounds for the components \( G_j, j \geq 3 \) of the generating function \( G = G_3 + G_4 + \cdots + G_r \) (see (2.6.30) in proposition 2.19) one may attempt –with the help of the more specialized recursive formulas of the Giorgilli-Galgani algorithm–, to obtain estimates for the remainder as the sum \( R^{(r)} = F_{r+1} + F_{r+2} + \ldots \). This is done in section 2.6, using proposition 2.16 stated in section 2.5. This proposition is based on lemma 2.13, completed with lemma A.12 of appendix A. Particularly, we mention that is in lemma 2.13 where the assumptions on the size of \( G_j, 3 \leq j \leq r \) are (through the bounds (2.5.11)) introduced. Next, in the section 2.6.1, we optimize the derived estimates with respect to the degree \( r \) up to which the normal form is carried out. In this way, an optimal \( r_{opt} \) order is obtained as a function of the distance to the critical periodic orbit. The results concerning optimization of the order are formalized in proposition 2.19. Finally, in the last section of the chapter, section 2.7, we let \( f = H \), where \( H \) will be the complexified Hamiltonian (1.6.2). Then it is checked that the early assumed bounds for the terms of the generated function \( G = G_3 + \cdots + G_r \) are fulfilled when these are the solutions of the homological equations (1.7.57). Hence, the optimal normalization order apply, in particular, to the complex Hamiltonian (1.6.2) and the remainder of the reduced Hamiltonian satisfy the same bounds of the proposition. Theorem 2.29 summarizes these results and also, gives an order-independent estimate for the size of the normal form, which will be useful in the forthcoming chapter 3.

Though, in the paragraph above, we have, essentially, outlined the plot of the present chapter. However, we have not mentioned that, besides a preliminary section, with some remarks and
notation (see below), there are two sections (2.3, 2.4) devoted to derive bounds for the solutions of the homological equations, more precisely, to bound the solutions $G_j$, $Z_j$ of the homological equations (1.7.57) in terms of the norm of their corresponding r. h. s., $F_j$. The main result (used later in section 2.7) is lemma 2.7, given at the end of section 2.4.

### 2.2 Preliminaries

We want to point here that, when analogous normal form computations are applied to a semisimple elliptic equilibrium point of a Hamiltonian (see example B.23), homological equations lead to a linear algebraic diagonal system in the coefficients $g_{l,m}$ (of the polynomial $G_s$), whose solutions are $g_{l,m} = \frac{F_{l,m}}{i(\omega I - m)}$, with $|l| + |m| = s$, $l \neq m$, and $F_{l,m}$ the coefficients of $F_s$, (the right hand side of the homological equations). It is usual to introduce some conditions on the frequencies $\omega^* = (\omega_1, \ldots, \omega_n)$. For example, as in Giorgilli et al. (1989), assume that $|\langle \omega, \nu \rangle| \geq \alpha_r$ for those $\nu \notin \mathcal{R}$ (being $\mathcal{R}$ the resonance module considered) and such that, $|\nu| \leq r$. In $\mathcal{A}_s$—the space of homogeneous polynomials of degree $s$ as defined in example B.23—, one introduces the norm,

$$
\|f\| = \sum_{|l| + |m| = s} |f_{l,m}|,
$$

and then the term $G_s$ is easily bounded by $\|G_s\| \leq \frac{1}{\alpha_r} \|F_s\|$.

When the homological equations are not diagonal, as actually happens, the computations are more involved. Moreover, we shall work with functions which are no longer polynomials (see below). Then, it is worth introducing before the appropriate norms and some notation.

### Notation

Let $\mathcal{E}$ denote now the space of the analytic functions $f(\theta_1, I_1, q, p)$, and $2\pi$-periodic with respect $\theta_1$, defined on $D(\rho, R)$, with some $\rho < \rho^*$ and $R < R^*$—see (1.6.3)–. Later on though, this domain will be more precisely defined (section 2.5.1). Therefore a function $f \in \mathcal{E}$ admits an expansion in Taylor series of type (1.7.1), again,

$$
f = \sum_{(l,m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_2^+ \times \mathbb{Z}_2^+} f_{l,m,n}(\theta_1) I_1^l q^m p^n, \tag{2.2.1}
$$

(with $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$). In their turn, the coefficients $f_{l,m,n}(\theta_1)$, expand in Fourier series,

$$
f_{l,m,n}(\theta_1) = \sum_{k \in \mathbb{Z}} f_{k,l,m,n} \exp(ik\theta_1). \tag{2.2.2}
$$

We use the developments (2.2.1) and (2.2.2) to introduce in $\mathcal{E}$ the following norms,

$$
|f_{l,m,n}|_\rho = \sum_{k \in \mathbb{Z}} |f_{k,l,m,n}| \exp(|k|\rho), \tag{2.2.3}
$$

$$
|f|_{\rho,R} = \sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathbb{Z}_2^+} \sum_{n \in \mathbb{Z}_2^+} |f_{l,m,n}|_\rho R^{2l+|m|+|n|}, \tag{2.2.4}
$$

with $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
with the definition of degree given by (1.7.3), section 1.7 of the previous chapter, i. e. counting twice the contribution of the degree in $I$). Some basic properties of these norms are given in the appendix A. If the sums defining these norms are convergent, then

$$\sup_{\|\theta_1\| \leq \rho} |f_i, m, n(\theta_1)| \leq |f_i, m, n|_{\rho}, \quad \sup_{D(\rho, R)} |f| \leq |f|_{\rho, R},$$

i. e., they are bounds for the supremum norms of $f_i, m, n(\theta_1)$, on the complex strip of width $\rho > 0$, and for $f$ on $D(\rho, R)$ (see (1.6.3)). The use of these norms will simplify many of the bounds, specially those of the small divisors (see lemma A.1 of appendix A). Alternatively one could consider the supremum norm and use the bounds in Rüssmann (1975).

We have already used the absolute norm of a vector $x \in \mathbb{R}^n (\mathbb{C}^n)$ $|x|_1 := \sum_{i=1}^{n} |x_i|$; $x \in \mathbb{R}^n (\mathbb{C}^n)$. Given a $n \times n$ real (complex) matrix $A = (a_{i,j})_{1 \leq i, j \leq n}$, the sum

$$|A|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{i,j}| \quad (2.2.5)$$

defines a compatible matrix norm, since it can be shown that $|A x|_1 \leq |A|_1 |x|_1$ (see Fröberg, 1973, chapter 3). We shall use these norms to bound the solutions of the linear system (1.7.19) as the first step to get estimates of the generating function.

Moreover, and for shortness, we shall denote by $\mathcal{E}^s_\mathcal{M}$ the space of homogeneous polynomials of even degree $s$, constituted only by $\mathcal{M}$-type monomials, i. e.,

$$\mathcal{E}^s_\mathcal{M} := \oplus_{l+M=s/2} \mathcal{E}^s_{0,l,M,M}, \quad (2.2.6)$$

(see section 1.7.1).

### 2.3 Bounds for the solutions of the homological equations

Let us return to the matrix formulation of the homological equations given in section 1.7.1 of chapter 1 by the $(M+1) \times (N+1)$ linear system (1.7.21), where $E_\nu$, $\nu = 1, 2, \ldots, N + 1$ were the $(N+1) \times (N+1)$ diagonal matrices $\text{diag}[\nu, \ldots, \nu]$, $D_N = \Omega I_{N+1} - P_N$, with $P_N$ the $(N+1) \times (N+1)$ nilpotent matrix (1.7.22), $P_{N+1}$ the identity matrix of the same order and $\Omega = \Omega_{k, M, N}$, defined as $\Omega_{k, M, N} = i \omega_1 k + i \omega_2 (M - N)$.

We can split each term of degree $s$ of the generating function $G$, $G_s$, as the sum $G_s = G^{(1)}_s + G^{(2)}_s$, with $G^{(2)}_s \in \mathcal{E}^s_\mathcal{M}$ i. e., $G^{(2)}_s$ contains only $\mathcal{M}$-type monomials (see definition 1.11)–

and $G^{(1)}_s \in \mathcal{E}^s_s \setminus \mathcal{E}^s_\mathcal{M}$ (note that $G^{(2)}_s = 0$ for $s$ even). This last one gives rise to linear systems with $\Omega \neq 0$.

Our aim in the present section, is to derive estimates on the norm of $G^{(1)}_s$ so $\Omega \neq 0$ will be assumed throughout, whilst the search for bounds on $G^{(2)}_s$ will be relegated to section 2.4.

The first step is, then, to get bounds for the solutions of (1.7.21). This will provide bounds on the coefficients of the expansion of $G^{(1)}_s$. We are going to see that those solutions can be formally expressed as (if necessary, see section 1.7.1, page 25, to review the notation)

$$g_M = D_N^{-1} f_M,$$

$$g_{M-\nu} = \sum_{j=0}^{\nu-1} (-1)^{\nu+j}(M - \nu + 1)_{\nu-j} D_N^{-\nu+j-1} f_{M-j} + D_N^{-1} f_{M-\nu}, \quad (2.3.1)$$
\( \nu = 1, \ldots, M \) and where the Pochhammer symbols
\[
(\alpha)_\nu := \alpha(\alpha + 1) \cdots (\alpha + \nu - 1), \quad (\alpha)_0 := 1,
\]
are used. Indeed, since the solutions of (1.7.21) can be first obtained in the following recursive form,
\[
g_M = D_N^{-1} f_M, \\
g_{M-\nu} = -(M - \nu + 1)D_N^{-1} g_{M-\nu+1} + D_N^{-1} f_{M-\nu},
\]
so taking \( \nu = 1 \), \( g_{M-1} \) turns out to be \( g_{M-1} = -MD_N^{-1} g_M + D_N^{-1} f_{M-1} \) and substituting the first of (2.3.1),
\[
g_{M-1} = -MD_N^{-2} f_M + D_N^{-1} f_{M-1},
\]
which coincides with the second of (2.3.1) for \( \nu = 1 \). Now, suppose this same relation works also for \( \nu \), with \( 1 < \nu \leq M \). Then for \( \nu + 1 \):
\[
g_{M-\nu-1} = D_N^{-1} f_{M-\nu-1} - (M - \nu)D_N^{-1} g_{M-\nu}
\]
\[
= D_N^{-1} f_{M-\nu-1} - (M - \nu)D_N^{-2} f_{M-\nu}
\]
\[
+ \sum_{j=0}^{\nu-1} (-1)^{\nu+j+1}(M - \nu)(M - \nu + 1)_{\nu-j}D_N^{\nu+j-2} f_{M-j}.
\]
Using that,
\[
(M - \nu)(M - \nu + 1)_{\nu-j} = (M - \nu)(M - \nu + 1) \cdots (M - j)
\]
\[= (M - \nu)_{\nu-j+1}, \]
for \( j = 1, 2, \ldots, \nu + 1 \), the last formula for \( g_{M-\nu-1} \) can be arranged to,
\[
g_{M-\nu-1} = D_N^{-1} f_{M-\nu-1} + \sum_{j=0}^{\nu} (-1)^{\nu+j+1}(M - \nu)_{\nu+1-j}D_N^{\nu-1+j-1} f_{M-j}.
\]
This ends the induction and hence (2.3.1) is fully justified. Moreover, if as in section 1.7.1, we denote the matrix of (1.7.21) by \( A \), it is immediately seen in view of the just obtained solutions for \( g \), that its inverse, \( A^{-1} \), can be written blockwise as
\[
A^{-1} = \\
\begin{pmatrix}
\binom{M}{\nu} p_1 \\
\binom{M-1}{\nu} p_2 \\
\binom{M-2}{\nu} p_3 \\
\vdots \\
\binom{M-\nu+1}{\nu} p_{\nu+1} \\
\binom{M-\nu}{\nu} p_{\nu} \\
\binom{M-\nu-1}{\nu} p_{\nu-1} \\
\vdots \\
\binom{M}{0} p_M \\
\binom{M-1}{0} p_{M-1} \\
\binom{M-2}{0} p_{M-2} \\
\vdots \\
\binom{M-\nu+1}{0} p_{M-\nu+1} \\
\binom{M-\nu}{0} p_{M-\nu} \\
\binom{M-\nu-1}{0} p_{M-\nu-1} \\
\vdots \\
\binom{0}{0} p_1
\end{pmatrix}.
\]
2.3. Bounds for the solutions of the homological equations

where, for convenience, we have introduced \( D_\nu = (-1)^{\nu+1}(\nu - 1)! D_{N}^{-\nu} \), and used that \((M)_\nu = \nu! \left( \frac{M + \nu - 1}{\nu} \right) \). To determine the powers \( D_{N}^{-\nu} \) (\( \nu = 1, 2, \ldots, M + 1 \)), of the inverse matrix \( D_{N}^{-1} \), we proceed from the definition of \( D_{N} \), so

\[
D_{N}^{-\nu} = (\Omega \cdot I_{N+1} - P_{N})^{-\nu},
\]

or, equivalently, \( D_{N}^{-\nu} = \frac{1}{\nu!} \left( I_{N+1} - \frac{1}{\nu!} P_{N} \right)^{-\nu} \) and after binomial expansion

\[
D_{N}^{-\nu} = \sum_{j=0}^{N} \frac{(-1)^j}{\nu!^j} \left( \begin{array}{c} -\nu \\ j \end{array} \right) P_{N}^j
\]

but \( \left( \begin{array}{c} -\nu \\ 1 \end{array} \right) = -\frac{\nu}{\nu!}, \left( \begin{array}{c} -\nu \\ 2 \end{array} \right) = \frac{\nu(\nu+1)}{2\nu!} \), and, in general, for \( \nu = 1, \ldots, M + 1 \),

\[
\left( \begin{array}{c} -\nu \\ N \end{array} \right) = (-1)^N \frac{\nu(\nu+1) \cdots (\nu + N - 1)}{N!} = \frac{(-1)^N}{N!} (\nu)_N. \quad (2.3.4)
\]

Direct computation of the powers \( P_{N}^j \) yields

\[
P_{N}^j = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
j! \binom{N}{j} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & j! \binom{N-1}{j} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & j! \binom{N-2}{j} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & j! & 0 & \cdots & 0
\end{pmatrix}
\]

Here, the coefficient \( j! \binom{N}{j} \) which produces the first row different from zero is placed at the \((j+1)\)-th row, with \( j \) ranging from \( j = 0 \) (given the \((N+1)\times(N+1)\) identity matrix) to \( j = N \). Thus, defining

\[
a_j(\nu, \Omega) = \frac{(\nu)_j}{\nu!}, \quad j = 0, 1, \ldots, N;
\]

and by substitution in our previous expansion of \( D_{N}^{-\nu} \), one obtains an explicit expression for these matrices

\[
D_{N}^{-\nu} = \begin{pmatrix}
(\binom{N}{0}) a_0 \\
(\binom{1}{0}) a_1 & (\binom{N-1}{0}) a_0 \\
(\binom{2}{0}) a_2 & (\binom{N-1}{1}) a_1 & (\binom{N-2}{0}) a_0 \\
\vdots & \vdots & \ddots & \ddots \\
(\binom{N}{j}) a_j & (\binom{N-1}{j-1}) a_{j-1} & (\binom{N-2}{j-2}) a_{j-2} & \cdots & (\binom{N}{0}) a_0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
(\binom{N}{N}) a_N & (\binom{N-1}{N-1}) a_{N-1} & (\binom{N-2}{N-2}) a_{N-2} & \cdots & (\binom{0}{0}) a_0
\end{pmatrix}. \quad (2.3.5)
\]
Remark 2.1. Before continuing, we stress here that, actually, all the entities involved in the linear system (1.7.19) do depend, through \( \Omega \), on \( k \in \mathbb{Z} \), and \( M, N \) non-negative integers with \( M + N + 2l = s \) with \( l = 0, 1, 2, \ldots, \lfloor s/2 \rfloor \) fixed. From the expression of the blocks \( D_{N}^{-1} \) just found, where powers of \( \Omega \) are present, it follows, in view of (2.3.3), that the same applies to the inverse matrix \( \Lambda^{-1} \). To avoid, as far as possible, the use of an excessive number of indices, we shall not write them out explicitly, but ask the reader to keep these dependences in mind. 

The next lemma furnishes an estimate on the norm of \( \Lambda^{-1} \).

**Lemma 2.2.** For the corresponding \( k, M \) and \( N \) (see last remark),

\[
|A^{-1}|_1 \leq \left( 1 + \frac{1}{|\Omega|} \right)^{M+N} \frac{(M+N)!}{|\Omega|}. \tag{2.3.6}
\]

**Proof.** Directly from the structure of the matrix (2.3.3), we have that

\[
|A^{-1}|_1 = \sum_{\nu=1}^{M+1} \binom{M}{\nu-1} |D_\nu|_1,
\]

and with the definition of \( D_\nu \) given before,

\[
|A^{-1}|_1 = \sum_{\nu=1}^{M+1} (\nu-1)! \binom{M}{\nu-1} |D_\nu^{-\nu}|_1.
\]

Then from (2.3.5), it is clear that the norm of \( D_\nu^{-\nu} \) equals to,

\[
|D_\nu^{-\nu}|_1 = \sum_{j=0}^{N} \binom{N}{j} \frac{(\nu)_j}{|\Omega|^{\nu+j}}.
\]

This allows to bound \( |A^{-1}|_1 \) in accordance with

\[
|A^{-1}|_1 = \sum_{\nu=1}^{M+1} (\nu-1)! \binom{M}{\nu-1} \sum_{j=0}^{N} \binom{N}{j} \frac{(\nu)_j}{|\Omega|^{\nu+j}}
\]

\[
\leq \sum_{\nu=0}^{N} \binom{M}{\nu} \frac{(\nu-1)!}{|\Omega|^{\nu+1}} \frac{1}{|\Omega|^N}
\]

\[
= \left( 1 + \frac{1}{|\Omega|} \right)^N \sum_{\nu=0}^{M} \binom{M}{\nu} \frac{(\nu+1)_N}{|\Omega|^{\nu+1}}
\]

\[
= \left( 1 + \frac{1}{|\Omega|} \right)^N \frac{(M+N)!}{|\Omega|} \sum_{\nu=0}^{M} \binom{M}{\nu} \frac{1}{|\Omega|^{\nu}},
\]

where, in the last step the relation:

\[
\nu! (\nu+1)_N = \nu! (\nu+1)(\nu+2) \cdots (\nu+N) = (\nu+N)! \leq (M+N)!
\]

for \( \nu = 0, 1, 2, \ldots, M \) has been applied.

Finally, using again the binomial formula \( \sum_{\nu=0}^{M} \binom{M}{\nu} \frac{1}{|\Omega|^\nu} = (1 + \frac{1}{|\Omega|})^M \), we get the estimate (2.3.6).
2.3. Bounds for the solutions of the homological equations

Now, we look at the denominators $|\Omega|$, with $\Omega_{k,M,N} = ik\omega_1 + i\omega_2(M - N)$ and realize that, even though $\omega_1$ and $\omega_2$ are not commensurable, values of the integers $k$ and the difference $M - N$ can be chosen which make $|\Omega_{k,M,N}|$ smaller than any previously fixed quantity.

Thus, to control the size of these small divisors, the frequencies $\omega_1$, $\omega_2$ are asked to satisfy the following Diophantine conditions,

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}$$  \hspace{1cm} (2.3.7)

for $\tau > 1$ and for a certain $\gamma > 0$. Here, $k \in \mathbb{Z}^2 \setminus \{0\}$ and $\omega^* = (\omega_1, \omega_2)$.

With the definitions,

$$\Xi(\tau, \gamma) := \{ \omega \in \mathbb{R}^n : |\langle k, \omega \rangle| \geq \gamma|k|^{-\tau} \text{ for all } k \in \mathbb{Z}^n \setminus \{0\} \}$$  \hspace{1cm} (2.3.8)

and,

$$\Xi(\tau) := \bigcup_{\gamma > 0} \Xi(\tau, \gamma),$$  \hspace{1cm} (2.3.9)

the following theorem (see Lochack and Meunier, 1988, Appendix 4) is well known.

Theorem 2.3. According to the values of $\tau$, there are three different cases,

(i) $0 < \tau < n - 1$; then $\Xi(\tau) = \emptyset$.

(ii) $\tau = n - 1$; $\Xi(\tau)$ has Lebesgue measure zero and Hausdorff measure $n$.

(iii) $\tau > n - 1$; $\mathbb{R}^n \setminus \Xi(\tau)$ has zero Lebesgue measure. More precisely, if $B_R$ is a ball in $\mathbb{R}^n$ of radius $R$, then,

$$\text{meas } \{ \mathbb{R}^n \setminus \Xi(\tau, \gamma) \cap B_R \} \leq C_\tau \gamma R^{n-1},$$

where $C_\tau$ is a constant for a fixed $\tau$.

In our case $n = 2$, so taking $\tau > 1$, the corresponding set $\Xi(\tau)$ of “Diophantine” frequencies has full Lebesgue measure.

From the estimates on $|A^{-1}|_1$ of lemma 2.2 and the Diophantine conditions just imposed, it is now possible to give more explicit bounds of $G^{(1)}_s$ in a smaller domain. Given $0 < \delta < \rho$ and $\chi = e^{-\delta}$, we have

$$|G^{(1)}_s|_{\rho-\delta, R_X} = \sum_{k \in \mathbb{Z}} \sum_{l=0}^{[s/2]} \sum_{M+N} \sum_{m=0}^{M} \sum_{n=0}^{N} |g_{k,l,m,M-m,N-n,n}| R^s e^{-\delta(|k|+s)} e^{\rho|k|}$$  \hspace{1cm} (2.3.10)

(where the sum does not contain terms with $k = 0$ and $M = N$ at the same time). But,

$$|g_{k,l,M,N}|_1 = \sum_{m=0}^{M} \sum_{n=0}^{N} |g_{k,l,m,M-m,N-n,n}|$$  \hspace{1cm} (2.3.11)

is the solution of (1.7.19), with the subscripts $k, l, M, N$ written explicitly, so $|g_{k,l,M,N}|_1 \leq |A^{-1}_{k,l,M,N}|_1 |f_{k,l,M,N}|_1$ (by the consistency property of the matrix norm). Hence, substitution in (2.3.10), gives the bounds,

$$|G^{(1)}_s|_{\rho-\delta, R_X} \leq \sum_{k \in \mathbb{Z}} \sum_{l=0}^{[s/2]} \sum_{M=0}^{s-2l} |A^{-1}_{k,l,M,s-2l-M}|_1 |f_{k,l,M,s-2l-M}|_1 R^s e^{-\delta(|k|+s)} e^{\rho|k|}.$$  \hspace{1cm} (2.3.12)
Introducing now the quantity,

\[ \tilde{\alpha}_s := \sup_{k \in \mathbb{Z}, 0 \leq l \leq [s/2]} \left| A_{k,l,M,s-2l-M}^{-1} \right|_1 e^{-\delta(|k|+s)}, \tag{2.3.13} \]

(2.3.12) reads,

\[
\left| G^{(1)}_s \right|_{\rho-\delta,Rx} \leq \tilde{\alpha}_s \sum_{k \in \mathbb{Z}} \sum_{l=0}^{[s/2]} \sum_{M=0}^{s-2l} \left| f_{k,l,M,s-2l-M} \right|_1 R^s e^{\delta|k|} \leq \tilde{\alpha}_s \left| F_s^{(1)} \right|_{\rho,R}, \tag{2.3.14} \]

where Poisson series, \( F_s^{(1)} \in \mathcal{E}_s \), stands for the right hand side of the homological equations (1.7.11), without \( \mathcal{M} \)-type monomials (see definition 1.11) to conform with our early assumption \( \Omega_{k,M,N} \neq 0 \) for non-negative whole numbers, \( M, N \), with \( 2l + M + N = s \).

Thus, we need reasonable bounds of the factor \( \tilde{\alpha}_s \). Using the Diophantine conditions (2.3.7),

\[ |\Omega| = |k\omega_1 + \omega_2(M - N)| \geq \frac{\gamma}{(|k| + |M - N|)^\tau} \]

in the bounds for \( |A^{-1}|_1 \) on lemma 2.2

\[ |A_{k,l,M,N}^{-1}|_1 \leq \left( 1 + \frac{(|k| + |M - N|)^\tau}{\gamma} \right)^{M+N} \frac{(|k| + |M - N|)^\tau}{\gamma} (M+N)! e^{-\delta(|k|+M+N)} \left( 1 + \frac{(|k| + |M - N|)^\tau}{\gamma} \right)^{s+1} e^{-\delta(|k|+|M-N|)/\gamma}. \tag{2.3.15} \]

From here we can derive estimates for the \( \tilde{\alpha}_s \) constants defined in (2.3.13), explicitly,

\[ \tilde{\alpha}_s \leq \left( 1 + \frac{(|k| + |M - N|)^\tau}{\gamma} \right)^{M+N} \frac{(|k| + |M - N|)^\tau}{\gamma} (M+N)! e^{-\delta(|k|+M+N)} \left( 1 + \frac{(|k| + |M - N|)^\tau}{\gamma} \right)^{s+1} e^{-\delta(|k|+|M-N|)/\gamma}. \tag{2.3.16} \]

Here, we have take into account that:

\[ |M - N| \leq M + N, \quad e^{-\delta(|k|+M+N)} < e^{-\delta(|k|+|M-N|)}; \]

moreover \( M + N = s - 2l \leq s \Rightarrow (M+N)! \leq s!, \) and \( (|k| + |M - N|)^\tau/\gamma \leq 1 + (|k| + |M - N|)^\tau/\gamma \).

Consider now the function,

\[ h(x) := (1 + \frac{x^\tau}{\gamma})^{s+1} e^{-\delta x}, \]

defined for \( x \geq 0 \). If \( 0 < \delta < 1 \) is small enough, its derivative,

\[ h'(x) = \frac{e^{-\delta x}}{\gamma} \left( 1 + \frac{x^\tau}{\gamma} \right)^s g(x), \]

\[ (1) \text{ We take } x = |k| + |M - N| \text{ and further consider } x \text{ as a continuous variable, but } |k| + |M - N| \geq 1, \text{ since } M, N, k \text{ are integers and both } k \text{ and the difference } M - N \text{ can not be zero simultaneously. Hence, it suffices to define the function } h(x) \text{ for } x \geq 1. \text{ Nevertheless, it is easier to discuss the position of its maximum if both, } h(x) \text{ and the auxiliary function } g(x), \text{ are considered in the whole nonnegative semiaxis.} \]
with

\[ g(x) := (s + 1)\tau x^{\tau-1} - \delta x^{\gamma} - \delta, \]

has a zero, \( \hat{x} \), in the interval \( J = (x_1, x_2) \), where

\[ x_1 = \max \left\{ 1, \left( \frac{\tau(1-s+1)}{\delta} \right)^{1/(1-s+1)} \right\} \quad \text{and} \quad x_2 = \frac{\tau(s + 1)}{\delta}, \]

provided \( g(1) > 0 \). Then \( \hat{x} \) corresponds to a point of (absolute) maximum of \( h(x) \). To show this, we first check that \( g(x_1) > 0 \), \( g(x) \) decreases monotonically for \( x > x_1 \) and \( g(x_2) = -\delta < 0, \hat{x} \in J \). It should be clear from the expression of the derivative of \( h(x) \) above, that \( h'(x) > 0 \), for \( x \in (1, \hat{x}) \) and \( h'(x) < 0 \), for \( x > \hat{x} \). Thus \( h(x) \leq h(\hat{x}) \), whenever \( x \geq 1 \). Finally, note that the condition \( g(1) > 0 \) is equivalent to the requirement

\[ \frac{(s + 1)\tau}{\delta} - \gamma > 1, \tag{2.3.17} \]

for \( s \geq 3 \) and \( \tau > 1, \gamma > 0 \) fixed. Indeed, this can be achieved taking \( 0 < \delta < 1 \) sufficiently small. In fact, as \( s \geq 3 \), it will be enough to take, \( \delta < \min\left\{ \frac{4\pi}{1+\gamma}, 1 \right\} \). This considerations will be contemplated again in section 2.5.1, to conveniently adjust the size of the domain.

Moreover, \( g(\hat{x}) = 0 \) implies

\[ 1 + \frac{\hat{x}^{\tau}}{\gamma} = \frac{(s + 1)\tau}{\delta\gamma} \hat{x}^{\tau-1}, \]

and together with the obvious inequality \( e^{-\delta \hat{x}} < e^{-(\tau-1)(s+1)} \), they produce the following bound for \( h(x) \)

\[ h(x) \leq h(\hat{x}) \leq \left( \frac{e}{\gamma} \right)^{s+1} \left( \frac{\tau(s + 1)}{\delta} \right)^{\tau(s+1)} e^{-\tau(s+1)}, \tag{2.3.18} \]

valid for all \( x \geq 1 \). The lemma below summarizes the arguments given in this section and shows the estimates on the generating function, \( G^{(1)}_s \), to be used along the rest of the chapter.

**Lemma 2.4.** With the notation above, and with \( 0 < \delta < \rho, \chi = e^{-\delta} \), and whenever the frequencies \( \omega_1, \omega_2 \) fulfill the Diophantine condition (2.3.7) for some \( \gamma > 0, \tau > 1 \), the piece of the \( s \)-th degree term of the function free of \( M \)-type monomials, \( G^{(1)}_s \), is bounded by,

\[ \left| G^{(1)}_s \right|_{\rho-\delta,R \chi} \leq \alpha_s \left| F^{(1)}_s \right|_{\rho,R}, \tag{2.3.19} \]

with the coefficients \( \alpha_s \) defined as,

\[ \alpha_s := \sqrt{2\pi e^2} \left( \frac{s + 1}{\gamma} \right)^{s+1} \left( \frac{\tau(s + 1)}{\delta e} \right)^{\tau(s+1)} \], \quad \text{if } s \geq 3. \tag{2.3.20} \]

**Proof.** (2.3.16), with the bounds (2.3.18) on \( h \) above, give the following estimates for \( \tilde{\alpha}_s \)

\[ \tilde{\alpha}_s \leq s! \left( \frac{e}{\gamma} \right)^{s+1} \left( \frac{\tau(s + 1)}{\delta e} \right)^{\tau(s+1)}, \]

and with the use of the Stirling’s formula (see Puig Adam, 1939), \( s! = \sqrt{2\pi e^2} \frac{s^{s+\frac{1}{2}} e^{-s + \frac{\xi}{12s}}}{\Gamma(s+1)} \), for \( s > 0 \) and \( 0 < \xi < 1 \). In our case \( s \geq 3 \), so \( \frac{\xi}{12s} \leq 1 \) and then \( e^{\frac{\xi}{12s}} < e \). Furthermore, \( s^{s+\frac{1}{2}} < (s+1)^{s+1} \), so \( s! < \sqrt{2\pi e^2} (s+1)^{s+1} e^{-\tau(s+1)} \). Finally, substitution of this bound for \( s! \) in the last inequality leads to \( \tilde{\alpha}_s \leq \alpha_s \) with \( \alpha_s \) as defined in (2.3.20). \( \square \)

---

\( ^2 \)One checks out immediately that, under the assumption \( g(1) > 0 \), there is another zero of \( g(x) \) in the interval \( (0, \min\{1, x_1\}) \), and no more ones could exist though, because \( g(x) \) decreases for \( x > x_1 \).
Chapter 2. Quantitative estimates on the normal form

2.4 Study of the resonant terms

As it has been repeatedly pointed, up to now, we have only dealt with the solutions of the “nonresonant” (in the sense already explained) part of the homological equations. Time has come, then, to revisit section 1.7.2 and face up to the problem of finding bounds for the solutions of the reduced homological equations (1.7.25).

Additional notation and definitions

To proceed beyond, it is worth revisiting the different variables we use in the representation of polynomials. Originally, we considered polynomials $P \in \mathcal{E}_M$, with $s$ even—see definition (2.2.6)—, in the action $I_1$, the positions $q_1, q_2$ and their conjugate momenta $p_1, p_2$. Later, in section 1.7.2, the coordinates were grouped in the special variables $\eta^* = (\eta_1, \eta_2, \eta_3, \eta_4)$, related with the former ones through,

$$\eta^*(q, p) = (q_1 p_1, q_2 p_2, q_2 p_1).$$

Furthermore, even another set of variables, $\xi^* = (\xi_1, \xi_2, \xi_3, \xi_4)$, was introduced by means of a linear change $\eta(\xi)$,

$$\eta(\xi) = \begin{pmatrix} -i & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

and its inverse $\xi(\eta)$,

$$\xi(\eta) = \begin{pmatrix} i/2 & 0 & i/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}.$$
see equation (1.7.43b). Here, an additional subscript, \( l \), has been added to account for the degree of the action \( I_1 \).

**Definition 2.5.** Given a polynomial \( P = \sum_m a_m z^m \), defined on the complex domain, \( D(\tilde{R}) = \{ z \in \mathbb{C}^5 : |z_j| \leq \tilde{R}, j = 1, 2, 3, 4, 5 \} \), we introduce the auxiliary norm \( \| \cdot \|_{\tilde{R}} \):

\[
\| P \|_{\tilde{R}} := \sum_m |a_m| \tilde{R}^{|m|}. \tag{2.4.6}
\]

**Lemma 2.6.** The norm defined by (2.4.6) satisfies,

(i) If \( P \in \mathcal{E}_s^M \), then \( \| \tilde{P} \|_{\mathcal{R}^2} = |P|_{\rho, R} \) (where \( | \cdot |_{\rho, R} \) is the norm defined by (2.2.4)).

(ii) \( \| PQ \|_{\tilde{R}} \leq \| P \|_{\tilde{R}} \| Q \|_{\tilde{R}} \)

(iii) \( \| \tilde{P}(\eta) \|_{\tilde{R}} \leq \| \tilde{P}(\xi) \|_{\tilde{R}} \leq \| \tilde{P}(\xi) \|_{2\tilde{R}} \).

**Proof.** The first item follows immediately from the definition, while the second and the third ones can be thought of as special cases for polynomials of item (ii), lemma A.2, and of lemma A.4 (applied to particular transformations \( \xi(\eta) \), \( \eta(\xi) \)) respectively, taking into account that,

\[
\max_{i=1,2,3,4} \{ \| \xi_i(\eta) \|_{\tilde{R}} \} \leq \tilde{R}, \quad \text{and} \quad \max_{i=1,2,3,4} \{ \| \eta_i(\xi) \|_{\tilde{R}} \} \leq 2\tilde{R}.
\]

With the definition 2.5, the norm \( \| \cdot \|_{\mathcal{R}^2} \), with \( \chi = \exp(-\delta) \), of \( \tilde{G}_s^{(2)} \) in (2.4.5) is

\[
\| \tilde{G}_s^{(2)} \|_{\mathcal{R}^2} = \left( \sum_{l+\nu+m+n=s/2, \nu \geq 0} \left| f_{l,\nu,m,n} \right| + \sum_{l+\nu+m+n=s/2, \nu \geq 0} \left| g_{l,\nu,m,n} \right| \right) \tilde{R}^s \exp(-s\delta). \tag{2.4.7}
\]

But in view of (1.7.47), the second of the sums in the r. h. s. of (2.4.7) is bounded without effort. Indeed, if we express the independent term of the homological equations restricted to the space \( \mathcal{E}_s^M \), in the variables \( \xi_1, \xi_2, \xi_3 \) and \( \xi_4 \),

\[
\tilde{F}_s^{(2)} = \sum_{l+\nu+m+n=s/2, \nu \geq 0} \hat{f}_{l,\nu,m,n} l_1^i \epsilon_2^\nu \epsilon_1^m \epsilon_3^n + \sum_{l+\nu+m+n=s/2, \nu \geq 1} \hat{g}_{l,\nu,m,n} l_1^i \epsilon_4^\nu \epsilon_1^m \epsilon_3^n \tag{2.4.8}
\]

–where all the coefficients \( \hat{f}_{l,\nu,m,n} \), \( \hat{g}_{l,\nu,m,n} \) are real, as in (1.7.43a)–, and we recall that the coefficients \( \hat{f}_{l,\nu,m,0} \) correspond to resonant monomials, so one can define:

\[
\tilde{Z}_s = \sum_{l+\nu+m+n=s/2, \nu \geq 0} \hat{f}_{l,\nu,m,0} l_1^i \epsilon_2^\nu \epsilon_1^m. \tag{2.4.9}
\]
and hence, according to the norm in definition 2.5 and the relations (2.4.4a)-(2.4.4c), the functions $Z_s(q, I_1, p)$, $s = 3, \ldots, r$, holding the resonant terms, can be bounded, applying lemma 2.6, as:

$$
|Z_s|_{p,R} = \|\tilde{Z}_s\|_{R^2} = \|\tilde{Z}_s\|_{R^2}
\leq \|\tilde{F}_s\|_{R^2} = 2^{s/2}\|\tilde{F}_s\|_{R^2/2} \leq 2^{s/2}\|\tilde{F}_s\|_{R^2} = 2^{s/2}|F_s|_{p,R}. \quad (2.4.10)
$$

Then, we get the estimates,

$$
\sum_{l=0}^{s/2-l-1} |g_{l,v,m,n}| = \sum_{l=0}^{s/2-l-1} \sum_{m \neq 0} \frac{|\tilde{g}_{l,v+1,m,n-1}|}{n} \leq \sum_{l=0}^{s/2-l-1} \sum_{m \neq 0} |\tilde{g}_{l,v+1,m,n-1}| \quad (2.4.11)
$$

(we recall that the arbitrary coefficients $g_{l,v,m,0}$ were taken equal to 0). However, it is a more involved task to obtain suitable bounds for the first sum in (2.4.7). We have split such a task into two basic steps.

**Step 1.** With $s$ and $l$ fixed, we compute explicit solutions of linear algebraic systems of type (1.7.50), now, $\mathcal{N}'_{l,v}f_{l,v} = \tilde{f}_{l,v}$, with

$$
f_{l,v} = \begin{pmatrix}
\hat{f}_{l,v+1,0,s/2-l-v-1} \\
\hat{f}_{l,v+1,1,s/2-l-v-2} \\
\vdots \\
\hat{f}_{l,v+1,s/2-l-v-2} \\
\hat{f}_{l,v+1,s/2-l-v-1}
\end{pmatrix}, \quad \tilde{f}_{l,v} = \begin{pmatrix}
\hat{f}_{l,v,0,s/2-l-v} \\
\hat{f}_{l,v,1,s/2-l-v-1} \\
\vdots \\
\hat{f}_{l,v,s/2-l-v-2} \\
\hat{f}_{l,v,s/2-l-v-1,1}
\end{pmatrix},
\quad (2.4.12)
$$

—i.e., they are vectors with the same structure as in (1.7.49), with a new subscript “$l$” and $s/2-l$ in substitution of $M$. In the same way, the matrix $\mathcal{N}'_{l,v}$ is given by (1.7.51), but with $s/2-l$ instead of $M$ everywhere.

**Step 2.** The components of the vector $f_{l,v}$ are directly bounded in terms of those of $\tilde{f}_{l,v}$. If in addition, one introduces the vectors $f_{l}, \hat{f}_{l} \in \mathbb{R}^d$, $d = (s/2-l)(s/2-l+1)/2$,

$$
f_*^l = (f_*^l,1; f_*^l,2; f_*^l,3; \ldots; f_*^l,s/2-l), \quad (2.4.13a)
\quad \hat{f}_*^l = (\hat{f}_*^l,0; \hat{f}_*^l,1; \hat{f}_*^l,2; \ldots; \hat{f}_*^l,s/2-l-1), \quad (2.4.13b)
$$

then, the sum $|f_l|_1 = \sum |f_{l,v,m,n}|$ in (2.4.7) can be bounded as $|f_l|_1 \leq \frac{1}{2} |\hat{f}_l|_1$.

The first step is accomplished after some direct algebra on the equations (1.7.48a)-(1.7.48c); from there, the solution of the system

$$
\mathcal{N}'_{l,v}f_{l,v} = \tilde{f}_{l,v}, \quad (2.4.14)
$$
is derived straightforward from the general recurrence relation,

\[ f_{l,\nu+1,m,s/2-l-\nu-m-1} = - \frac{1}{s/2-l+\nu-m+1} \hat{f}_{l,\nu,m,s/2-l-\nu-m} \\
+ \frac{1}{s/2-l+\nu-m+1} f_{l,\nu+1,m-2,s/2-l-\nu-m+1}, \]

(2.4.15)

with \( 2 \leq m \leq s/2 - l - \nu - 1 \). The first terms of the solution are,

\[ f_{l,\nu+1,0,s/2-l-\nu-1} = - \frac{1}{s/2-l+\nu+1} \hat{f}_{l,\nu,0,s/2-l-\nu}, \]

\[ f_{l,\nu+1,1,s/2-l-\nu-2} = - \frac{1}{s/2-l+\nu} \hat{f}_{l,\nu,1,s/2-l-\nu-1}, \]

\[ f_{l,\nu+1,2,s/2-l-\nu-3} = - \frac{1}{s/2-l+\nu-1} \hat{f}_{l,\nu,2,s/2-l-\nu-2} + \frac{1}{s/2-l+\nu} \hat{f}_{l,\nu,0,s/2-l-\nu}, \]

\[ f_{l,\nu+1,3,s/2-l-\nu-4} = - \frac{1}{s/2-l+\nu-2} \hat{f}_{l,\nu,3,s/2-l-\nu-3} + \frac{1}{s/2-l+\nu-2} \hat{f}_{l,\nu,1,s/2-l-\nu-1}, \]

\[ f_{l,\nu+1,4,s/2-l-\nu-5} = - \frac{1}{s/2-l+\nu-3} \hat{f}_{l,\nu,4,s/2-l-\nu-4} + \frac{1}{s/2-l+\nu-3} \hat{f}_{l,\nu,2,s/2-l-\nu-2} + \frac{1}{s/2-l+\nu-3} \hat{f}_{l,\nu,0,s/2-l-\nu}, \]

and so on. By induction it can be shown that the general term is,

\[ f_{l,\nu+1,m,s/2-l-\nu-m} = - \frac{1}{s/2-l+\nu-m+1} \hat{f}_{l,\nu,m,s/2-l-\nu-m} \]

\[ + \sum_{\alpha=1}^{\lfloor m/2 \rfloor} (-1)^{\alpha+1} \frac{(s/2-l+\nu-m-1)!!(s/2-l+\nu-m+2\alpha-1)!!}{(s/2-l+\nu-m-1)!!(s/2-l+\nu-m+2\alpha+1)!!} \hat{f}_{l,\nu,m-2\alpha,s/2-l-\nu-m+2\alpha}. \]

(2.4.16)

Indeed, for the formula above has just been checked for \( m = 2, 3, 4 \). Suppose, thus, that it works also for \( 2 < m < s/2 - l - \nu - 1 \). Therefore, substitution in the recurrence relation (2.4.15), written for \( m + 1 \), yields

\[ f_{l,\nu+1,m+1,s/2-l-\nu-m-2} = - \frac{1}{s/2-l+\nu-m} \hat{f}_{l,\nu,m+1,s/2-l-\nu-m-1} \]

\[ + \frac{1}{s/2-l+\nu-m} f_{l,\nu+1,m+2,s/2-l-\nu-m-1} \]

\[ - \sum_{\alpha=1}^{\lfloor (m-1)/2 \rfloor} (-1)^{\alpha+1} \frac{(s/2-l+\nu-m-2)!!(s/2-l+\nu-m+2\alpha)!!}{(s/2-l+\nu-m-2)!!(s/2-l+\nu-m+2\alpha+2)!!} \hat{f}_{l,\nu,m-2\alpha,s/2-l-\nu-m+2\alpha+1} \]

and after the inclusion of the second term in the r. h. s. into the sum, followed by a displacement of the index \( \alpha \) to \( \alpha + 1 \), the last expression arranges to,

\[ f_{l,\nu+1,m+1,s/2-l-\nu-m-2} = - \frac{1}{s/2-l+\nu-m} \hat{f}_{l,\nu,m+1,s/2-l-\nu-m-1} \]

\[ + \sum_{\alpha=1}^{\lfloor (m+1)/2 \rfloor} (-1)^{\alpha+1} \frac{(s/2-l+\nu-m-2)!!(s/2-l+\nu-m+1+2\alpha-1)!!}{(s/2-l+\nu-m-2)!!(s/2-l+\nu-m+1+2\alpha+1)!!} \hat{f}_{l,\nu,m+1-2\alpha,s/2-l-\nu-m+1+2\alpha} \]
which matches (2.4.16) with \( m + 1 \) in the place of \( m \). This completes the induction and hence the first step.

The second step begins with the quest for appropriate bounds on the different components of \( f_{l,\nu} \). For \( m = 0, 1 \) we have directly

\[
|f_{l,\nu+1,0,s/2-l-\nu-1}| = \frac{1}{s/2-l+\nu+1} |\hat{f}_{l,\nu,0,s/2-l-\nu}|,
\]
\[
|f_{l,\nu+1,1,s/2-l-\nu-2}| = \frac{1}{s/2-l+\nu} |\hat{f}_{l,\nu,1,s/2-l-\nu-1}|
\]

while a glance on the general term (2.4.16) of the solution for \( m \geq 2 \) leads to the estimate

\[
|f_{l,\nu+1,m,s/2-l-\nu-m-1}| \leq \frac{1}{s/2-l+\nu-m+1} |\hat{f}_{l,\nu,m,s/2-l-\nu}| + \sum_{\alpha=1}^{[m/2]} \frac{1}{s/2-l+\nu-m+2\alpha+1} |\hat{f}_{l,\nu,m-2\alpha,s/2-l-\nu-m+2\alpha}|.
\]

(2.17)

Let us put \( M = s/2 - l \) and \( j = m - 2\alpha \). Then, note that the coefficient of

\[
|\hat{f}_{l,\nu,m-2\alpha,s/2-l-\nu-m+2\alpha}| = |\hat{f}_{l,\nu,j,M-j-\nu}|,
\]

is \( \frac{1}{M-j+1+\nu} \), which is independent of the index \( \alpha \) in the sum (2.17) above. Therefore:

\[
\sum_{m=0}^{M-1-\nu} |f_{l,\nu+1,m,M-\nu-m-1}| \leq \sum_{j=0}^{M-\nu-1} \frac{N_\#}{M-j+1+\nu} |\hat{f}_{l,\nu,j,M-j-\nu}|,
\]

being

\[
N_\# = \# \left\{ m \text{ such that the term } |\hat{f}_{l,\nu,j,M-j-\nu}| \text{ appear in the sum (2.17)} \right\},
\]

but the first \( m \) giving rise to \( |\hat{f}_{l,\nu,j,M-j-\nu}| \) is \( m = j \), (and \( \alpha = 0 \)), the second is \( m = j + 2 \) (and \( \alpha = 1 \)), the third is \( m = j + 4 \) (\( \alpha = 2 \)) and so on. So it must be \( N_\# = \left\lfloor \frac{M-\nu-j}{2} \right\rfloor + 1 \). Hence:

\[
\frac{N_\#}{M-j+1+\nu} \leq \frac{1}{2} (M-j+\nu+1) \leq \frac{1}{2},
\]

and therefore:

\[
\sum_{\nu=0}^{s/2-l-1} \sum_{m=0}^{s/2-l-\nu-1} |f_{l,\nu+1,m,s/2-l-\nu-m-1}| \leq \frac{1}{2} \sum_{\nu=0}^{s/2-l-1} \sum_{m=0}^{s/2-l-\nu-1} |\hat{f}_{l,\nu,j,s/2-l-j-\nu}|.
\]

Using this in (2.4.7) and taking (2.4.11) into account,

\[
\|C_s^{(2)}\|_{R^2}^2 \leq \left( \sum_{l+s+m+n/2} \frac{1}{l+s+m+n/2} |\hat{f}_{l,\nu,m,n}| + \sum_{l+s+m+n/2} \frac{1}{l+s+m+n/2} |\hat{g}_{l,\nu,m,n}| \right) \frac{1}{2} R^s \exp(-s\delta)
\]

(2.4.18)

\[
\leq \frac{1}{2} \exp(-s\delta) \|\hat{F}_s^{(2)}\|_{R^2}.
\]
2.4. Study of the resonant terms

From the expressions (2.4.2) and (2.4.3) for the linear transformation $\eta(\xi)$ and its corresponding inverse, $\eta^{-1}(\xi)$, it follows that $\|\xi_i(\eta)\|_{\chi^2R^2} = \chi^2R^2$ and $\|\eta_i(\xi)\|_{R^2/2} \leq R^2$, for $i = 1, 2, 3, 4$. Hence, using items (i) and (iii), of lemma 2.6,

$$
|G_s^{(2)}|_{\rho-\delta,R\chi} = \|\tilde{G}_s^{(2)}\|_{R^2\chi^2} \leq \|\tilde{G}_s^{(2)}\|_{R^2\chi^2} \leq \frac{1}{2} \exp(-s\delta)\|\tilde{F}_s^{(2)}\|_{R^2} \\
\leq \frac{1}{2} \exp(-s\delta)\|\tilde{F}_s^{(2)}\|_{2R^2} = \frac{1}{2} 2^{s/2} \exp(-s\delta)\|\tilde{F}_s^{(2)}\|_{R^2} \\
\leq \frac{1}{2} 2^{s/2} \exp(-s\delta)|F_s^{(2)}|_{\rho,R},
$$

which gives an inequality of type,

$$
|G_s^{(2)}|_{\rho-\delta,R\chi} \leq \tilde{\beta}_s|F_s^{(2)}|_{\rho,R},
$$

with $\tilde{\beta}_s := \frac{1}{2} 2^{s/2} \exp(-s\delta)$; but $\tilde{\beta}_s \leq \alpha_s$ for $s \geq 3$ –with $\alpha_s$ given by (2.3.20)–, provided that the coefficient $\gamma$ in the Diophantine conditions (2.3.7) not to be too large (we can suppose by the moment that $\gamma < 1$, see remark 2.10). Moreover, it is:

$$
|G_s|_{\rho-\delta,R\chi} = |G_s^{(1)}|_{\rho-\delta,R\chi} + |G_s^{(2)}|_{\rho-\delta,R\chi},
$$

$$
|F_s|_{\rho,R} = |F_s^{(1)}|_{\rho,R} + |F_s^{(2)}|_{\rho,R},
$$

so then,

$$
|G_s|_{\rho-\delta,R\chi} = |G_s^{(1)}|_{\rho-\delta,R\chi} + |G_s^{(2)}|_{\rho-\delta,R\chi} \\
\leq \alpha_s |F_s^{(1)}|_{\rho,R} + \tilde{\beta}_s |F_s^{(2)}|_{\rho,R} \\
\leq \alpha_s \left( |F_s^{(1)}|_{\rho,R} + |F_s^{(2)}|_{\rho,R} \right) \\
\leq \alpha_s |F_s|_{\rho,R}.
$$

We state this conclusion in a new lemma which extends the preceding lemma 2.4 and gives the effective estimates for the solutions of the $s^{th}$-degree homological equations, $G_s$, to be used in the forthcoming.

**Lemma 2.7.** Consider the homological equations (1.7.11) for the terms of degree $s$, with $s = 3, 4, \ldots$, that is:

$$
L_{H_2}G_s + Z_s = F_s.
$$

With the same notation of lemma 2.4, with $0 < \delta < \rho$, $\chi = e^{-\delta}$, the frequencies $\omega_1, \omega_2$ satisfying the Diophantine condition (2.3.7) for some $0 < \gamma < 1$ and some $\tau > 1$, the terms $G_s$ of the generating function can be estimated –in a slightly reduced domain–, according to

$$
|G_s|_{\rho-\delta,R\chi} \leq \alpha_s |F_s|_{\rho,R},
$$

where $\alpha_s$ is the quantity defined by (2.3.20). Moreover, for the resonant terms $Z_s$, we have:

$$
|Z_s|_{\rho,R} \leq 2^{s/2} |F_s|_{\rho,R}.
$$
From here, the idea is –once the generating function of the canonical change has been determined and bounded–, to use such bounds to estimate the quantities \( f_{l,k} \) in the recursive formula (1.7.6). As their sum gives the components of the transformed function \( T_G f \), say \( F_s \), with \( T_G f = \sum_{s \geq 1} F_s \) (see definition 1.7 in the previous chapter), this process will allow us to bound those components for each degree \( s \). So, if the nonlinear reduction has been carried out up to some finite degree, \( r \) (thus, taking \( G_{r+1} = G_{r+2} = \cdots = 0 \)), to compute the size of the remainder we still need to determine the size of the sum \( \sum_{s > r} F_s \). These questions are investigated in the sections to come.

### 2.5 Bounds on the transformed function

In order to obtain estimates of the \( s \)-degree terms, \( F_s \), in the transformed function \( F = T_G f \), since \( F_s = \sum_{l \geq 1} f_{l,s-1} f_{l,0} = f_l \), some knowledge about the size of \( f_{l,m} \) is first required. From now on and up to the end of this chapter, we shall use the following notation and definitions:

#### 2.5.1 New notation and definitions

We suppose the function \( f = \sum_{l \geq 1} f_l \) is defined and analytic in the complex domain \( D(\rho^*, R^*) \), given by (1.6.3), with \( R^*, \rho^* \) small enough to make \( |f|_{\rho^*, R^*} < +\infty \). Let \( c \) be precisely the norm of \( f \), i.e.,

\[
|f|_{\rho^*, R^*} = c = \left( \frac{R}{R_0} \right)^s .
\]

Next we reduce our initial domain taking \( R_0 \) and \( \rho_0 \) two positive quantities satisfying the inequalities,

\[
R_0 < \min \{ 1, c, R^* \}, \quad \rho_0 < \min \left\{ 1, \rho^*, \frac{32}{e^{\gamma/7}}, \frac{128r}{\gamma+1} \right\} ,
\]

and, still, consider \( \rho, R \) with \( 0 < \rho < \rho_0, 0 < R < R_0 \). Thus, the complex domain is reduced to \( D(\rho, R) \). From the definition of the norm \( | \cdot |_{\rho, R} \) and from (2.5.1) one gets the following analytic bounds on each \( f_s \):

\[
|f_s|_{\rho, R} \leq c \left( \frac{R}{R_0} \right)^s ,
\]

for \( s = 1, 2, \ldots \); they are deduced straightforward since

\[
|f_s|_{\rho, R} = |f_s|_{\rho_0, R_0} \left( \frac{R}{R_0} \right)^s \leq |f_s|_{\rho^*, R^*} \left( \frac{R}{R_0} \right)^s \leq c \left( \frac{R}{R_0} \right)^s .
\]

Clearly, \( |f_s|_{\rho^*, R^*} \leq |f|_{\rho^*, R^*} \), and therefore \( |f_s|_{\rho^*, R^*} \leq |f|_{\rho^*, R^*} \leq c \) (this trick has been used to state the last inequality in the chain above). Nevertheless, we advance that this last domain, \( D(\rho_0, R_0) \), will be later successively shrunken down to a smaller one, in which all the bounds for the functions \( f_{l,\nu} \) should be valid. We define (see corollary A.7 of appendix A):

\[
\rho_2 = \rho, \quad \rho_\nu := \rho_2 - 2 \sum_{\sigma=3}^{\nu} \delta_\sigma ,
\]

\[
R_2 = R, \quad R_\nu := R \exp \left( -2 \sum_{\sigma=3}^{\nu} \delta_\sigma \right) .
\]
In particular, for the sequence \( \{\delta_{\nu}\}_{3 \leq \nu \leq s+2} \) the following determination is chosen: define for some \( r \in \mathbb{N}, r \geq 3 \), and any \( s \geq 1 \) fixed,

\[
\delta := \frac{\rho/16}{r-1},
\]

and then, for the terms \( \delta_{\nu} \), we take \( \delta_{\nu} = \delta_{\nu}^{(s)} \), with:

\[
\delta_{\nu}^{(s)} := \begin{cases} 
\delta, & \text{if } 3 \leq \nu \leq r, \\
\frac{s}{\delta}, & \text{if } r < \nu \leq s+2,
\end{cases}
\]

(so \( \delta_{3}^{(s)} = \delta_{4}^{(s)} = \cdots = \delta_{s+2}^{(s)} \) if \( s \leq r - 2 \)). Hence, for the sum of the terms,

\[
\sum_{\nu=3}^{s+2} \delta_{\nu}^{(s)} = \sum_{\nu=3}^{r} \delta_{\nu}^{(s)} + \sum_{\nu=r+1}^{s+2} \delta_{\nu}^{(s)} = (r-2)\delta + (s-r+2)\frac{\delta}{s} \leq (r-1) \frac{\rho/16}{r-1} = \frac{\rho}{16},
\]

if \( s > r - 2 \). Conversely, if \( s \leq r - 2 \) then

\[
\sum_{\nu=3}^{s+2} \delta_{\nu}^{(s)} = s\delta = s\frac{\rho/16}{r-1} < \frac{\rho}{16}.
\]

Therefore, for all \( 3 \leq \nu \leq s + 2 \),

\[
\sum_{\sigma=3}^{\nu} \delta_{\sigma}^{(s)} \leq \frac{\rho}{16}.
\]

**Remark 2.8.** The superscript \( (s) \) in definition (2.5.6) of \( \delta_{\nu} \) was added only to emphasize their specific dependence on the degree \( s \). We want to stress that for any given \( s_{1} \geq 1 \), the sequence \( \{\delta_{\nu}^{(s_{1})}\}_{3 \leq \nu \leq s_{1}+2} \) will determine the successive reduction of the domains only when we seek bounds for \( F_{s_{1}} \). For a term of different degree, say \( F_{s_{2}} \) with \( s_{2} \neq s_{1} \), one must construct another different sequence, \( \{\delta_{\nu}^{(s_{1})}\}_{3 \leq \nu \leq s_{1}+2} \), letting \( s = s_{2} \) in (2.5.6). Once this precision is understood, the superscript \( (s) \) can be thrown away, as we shall do in what follows to avoid an overload of notation.

**Remark 2.9.** With (2.5.7) and from the definitions (2.5.4a), (2.5.4b), it is clear that,

\[
7\rho/8 \leq \rho_{\nu}, \quad R \exp(-\rho/8) \leq R_{\nu},
\]

for all \( \nu = 3, 4, \ldots, s + 2 \). We can take \( D(7\rho/8, R \exp(-\rho/8)) \) as the common domain where all the estimates in the norm of \( f_{l,\nu} \) will work for all \( \nu = 3, 4, \ldots, s + 2 \), and all \( s \geq 1 \).

**Remark 2.10.** The condition on \( \rho_{0} \) in (2.5.2), together with the definition (2.5.5) for \( \delta \) guarantees,

(C1) \( 1 \leq \alpha_{j} \leq \alpha_{k} \), for \( 3 \leq j \leq k \leq r \), because it makes \( \gamma(\delta e)^{\tau} < 1 \) in (2.2.20). Indeed,

\[
\gamma(\delta \exp(1))^{\tau} = \gamma \left( \frac{\rho \exp(1)}{16(r-1)} \right)^{\tau} \leq \gamma \left( \frac{\rho \exp(1)}{32} \right)^{\tau} < \gamma \left( \frac{1}{32} \right) \gamma \left( \frac{\exp(1)}{\gamma} \right)^{\tau} = \frac{1}{\gamma} = 1.
\]

Note also that with this choice of \( \rho_{0} \) the condition \( \gamma < 1 \) in lemma 2.7 can be dropped.
(C2) With \( \rho < \rho_0 \), the condition (2.3.17) is fulfilled. Really,
\[
\frac{(s+1)\tau}{\delta} - \gamma = \frac{16\frac{(s+1)(r-1)\tau}{\rho}}{\rho} - \gamma \geq \frac{128\tau}{\rho} - \gamma > \frac{128\tau}{\rho_0} - \gamma > \frac{128\tau(\gamma+1)}{128\tau} - \gamma = 1.
\]

Item (C1) states that the \( r - 2 \) quantities \( \alpha_3, \alpha_4, \ldots, \alpha_r \) form a nondecreasing sequence with all its terms greater than 1. This will play—as we shall see later on in section 2.7—, an important role in the process of bounding the terms of the generating function.

To state lemma 2.13, which provides bounds for the terms \( f_{l,\nu} \) (those given in definition (1.7.6)), we need to introduce a pair of definitions—included also in appendix A—, (see lemma A.11).

**Definition 2.11.** Let \( m, n \) be two nonnegative integers, with \( n > 0 \); then
\[
[m, n] := m - n \left\lfloor \frac{m}{n} \right\rfloor,
\]
i.e., the square bracket of the integers \( m \geq 0 \) and \( n > 0 \), \([m, n]\), is the remainder of the integer division of \( m \) by \( n \).

**Definition 2.12.** Consider the sequence of \( r - 1 \) numbers, \( \{\beta_\nu\}_{3 \leq \nu \leq r} \), such that,
\[
1 = \beta_2 \leq \beta_3 \leq \beta_4 \leq \cdots \leq \beta_{r-1} \leq \beta_r.
\]
Given two positive integers \( r, s \), the quantities \( \mathcal{W}(r, s) \) are defined by the products,
\[
\mathcal{W}(r, s) := \left( \prod_{\nu=1}^{r-2} \beta_{\nu+2} \right) \left[ \frac{r-2}{\beta_2} \right] \left( \prod_{\nu=0}^{[s,r-2]} \beta_{\nu+2} \right).
\]

With these two previous definitions in mind, we go on with the enunciation of the just mentioned lemma.

**Lemma 2.13.** For some given \( r \geq 3 \), let \( s \geq 1 \) be a fixed integer, consider the sequence \( \{\delta_\nu\}_{3 \leq \nu \leq s+2} \) defined by (2.5.6)(3), the corresponding sequences \( \{\rho_\nu\}_{3 \leq \nu \leq s+2} \), \( \{R_\nu\}_{3 \leq \nu \leq s+2} \) introduced in (2.5.4a) and (2.5.4b) respectively and \( \{\beta_\nu\}_{2 \leq \nu \leq r} \) a non decreasing sequence with \( \beta_2 = 1 \). Assume that,
\[
|G_3|_{\rho_3+\delta_3, R_3 \exp(\delta_3)} \leq \beta_3 b \left( \frac{R}{R_0} \right)^3,
\]
\[
|G_\nu|_{\rho_\nu+\delta_3, R_\nu \exp(\delta_3)} \leq \beta_3 \beta_4 \cdots \beta_\nu \frac{\alpha^{\nu-3} b}{\delta_3^2} \left( \frac{R}{R_0} \right)^\nu, \quad 3 < \nu \leq r.
\]
(with \( a, b \) positive constants). Then, the following bounds on \( f_{l,\nu} \) apply,
\[
|f_{l,\nu}|_{\rho_{\nu+2}, R_{\nu+2}} \leq \mathcal{W}(r, \nu) \frac{\partial_{\nu} c}{\delta_3^2} \left( \frac{R}{R_0} \right)^{l+\nu}.
\]

(3)With the superindex \((s)\) dropped, (see remark 2.8).
for all \( l, \nu \in \mathbb{N} \) with \( l \geq 1 \), \( 0 \leq \nu \leq s \). Here, the terms \( \vartheta_\nu \) are defined recursively through,

\[
\vartheta_0 = 1, \\
\vartheta_\nu = \frac{\delta_3}{\vartheta_{\nu+2}} \sum_{j=1}^{\min(\nu, r-2)} \frac{j}{\nu} a^{j-1} d \vartheta_{\nu-j}, \quad 1 \leq \nu \leq s
\]

(2.5.13)

with the constants \( d = \frac{17b \exp(1)}{2R_0^2} \) and \( c \) given by (2.5.1); whilst the symbol \( W(r, s) \) stands for the products (2.5.10) introduced in definition 2.12.

**Remark 2.14.** We note that to obtain the bounds for \( f_{l, \nu} \) we have stated a priori estimations for the terms \( G_\nu \) of the generating function, where the constants \( a, b \) will be conveniently exacted later in proposition 2.28.

**Remark 2.15.** Before proceeding with the proof of lemma 2.13, we point out that bounds for sums of Poisson brackets of type \( \sum_{j=1}^{\nu} \frac{j}{\nu} \{ f_{l, \nu-j}, G_{2+j} \} \) will be required, but every term therein can be bounded directly applying corollary A.7, so

\[
\sum_{j=1}^{\nu} \frac{j}{\nu} \{ f_{l, \nu-j}, G_{2+j} \} |_{\rho_{\nu+2}, R_{\nu+2}} \leq
\]

\[
\leq \sum_{j=1}^{\nu} \frac{j}{\nu} 17 \exp(1) | f_{l, \nu-j} | | f_{\nu-j+2, \nu-j+2} | | G_{2+j} | | G_{2+j} | | G_{2+j} | | G_{2+j} | | \exp(\delta_2) |
\]

(2.5.14)

This provides recursive bounds on these sums. Without explicit mention, we shall apply the above formula throughout.

**Proof of lemma 2.13.** The estimate (2.5.12) works for \( \nu = 1 \) and for all \( l \geq 1 \),

\[
| f_{l,1} |_{\rho_3, R_3} = | \{ f_{l,0}, G_3 \} |_{\rho_3, R_3} \leq \frac{17 \exp(1)}{2\delta_3 \delta_3 R^2} | f_{l,0} |_{\rho_2, R_2} | G_3 |_{\rho_3 + \delta_3, R_3} \exp(\delta_3)
\]

\[
\leq \beta_3 \frac{17b \exp(1)}{2\delta_3 \delta_3 R^2} \left( \frac{R}{R_0} \right)^{l+3},
\]

but \( W(r, 1) = \beta_3 \); this, together with the arrangement \( \frac{1}{R^7} \left( \frac{R}{R_0} \right)^{l+3} = \frac{1}{R_0^7} \left( \frac{R}{R_0} \right)^{l+1} \) and further identification of the constant \( d \), will lead to

\[
| f_{l,1} |_{\rho_3, R_3} \leq W(r, 1) \frac{d c}{\delta_3} \left( \frac{R}{R_0} \right)^{l+1},
\]

which agrees with (2.5.12) for \( \nu = 1 \), for according with (2.5.13), is \( \vartheta_1 = d \). Assume thus, the same inequality is verified also for \( 1 \leq \nu \leq k - 1 \); therefore, taking the norm \( | \cdot |_{\rho_{k+2}, R_{k+2}} \), at both sides of \( f_{l,k} = \sum_{j=1}^{k} \frac{j}{k} \{ f_{l,k-j}, G_{2+j} \} \) –see remark 2.15–, and realizing that \( G_{r+1} = \).
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\[ G_{r+2} = \cdots = 0, \] one deduces readily,

\[
|f_{l,k}|_{\rho_{k+2}, R_{k+2}} \leq \frac{17 \exp(1)}{2R^2 \delta_3 \delta_{r+2}} \sum_{j=1}^{\min \{k, r-2\}} \frac{j}{k} |f_{l,k-j}|_{\rho_{k-j+2}, R_{k-j+2}} |G_{2+j}|_{\rho_{j+2}, R_{j+2}} \exp(\delta_{j+2})
\]

\[
\leq \frac{17b \exp(1)}{2R^2 \delta_3 \delta_{r+2}} \sum_{j=1}^{\min \{k, r-2\}} \frac{j}{k} a^{j-1} \vartheta_{k-j} \beta_3 \cdots \beta_{j+2} W(r, k-j) \frac{c}{\delta_3 (k-j) \delta_{j+2}} \left( \frac{R}{R_0} \right)^{l+k-2}
\]

\[
= \mathcal{W}(r, k) \frac{\vartheta_k c}{\delta_3^2} \left( \frac{R}{R_0} \right)^{l+k}
\]

where to obtain the first inequality we apply: \( \delta_{j+2} = \delta_3 \) for \( j \leq r-2 \), and for the third we use that, according with lemma A.11, it must be

\[
\beta_3 \beta_4 \cdots \beta_{j+2} \mathcal{W}(r, k-j) \leq \mathcal{W}(r, k),
\]

for all \( 1 \leq j \leq \min \{k, r-2\} \). Finally, identification of the constant \( d \) in the sum between parenthesis and the whole sum (times the quotient \( \delta_3 / \delta_{r+2} \)) with the term \( \vartheta_k \) close the proof. \( \square \)

The lemma above may be completed with lemma A.12 of the appendix A, which provides bounds for the terms \( \vartheta_s \) and, on the other hand, the bounds (2.5.11) are proved to be valid –for the solutions \( G_3, \ldots, G_r \) of the homological equations (1.7.57)–, in proposition 2.28, at the end of this chapter. This yields estimates for the norm of \( f_{l,s}, l \geq 1, s \geq 1 \) in the somewhat reduced domain, \( \mathcal{D}(7\rho/8, R \exp(-\rho/8)) \). Such result constitutes the matter of the next proposition.

**Proposition 2.16.** Under the same hypothesis of lemma 2.13, the following estimates on \( f_{l,s} \) apply,

\[
|f_{l,s}|_{7\rho/8, R \exp(-\rho/8)} \leq \mathcal{W}(r, s) \frac{cdg^{s-1}}{\delta_r^{2s}} \left( \frac{R}{R_0} \right)^{l+s},
\]

where,

\[
g := \max \{1, ed + 2ae\},
\]

and a subscript \( r \) has been added to remark the dependence of \( \delta \) on \( r \),

\[
\delta_r = \frac{\rho/16}{r - 1}.
\]

These estimates will be the keystone to deal with the bounding of the remainder.

### 2.6 Bounds for the remainder of the transformation

Let us precise what we mean when talking about the “bounding of the remainder”. Up to now, we have been considering the formal transformation \( T_G f = F_1 + F_2 + F_3 + \cdots + F_r + \cdots \). This is an infinite process, which in practical computations is carried out only up to some finite
order (degree) $r$ i.e., the last term computed is $F_r$. So it is natural to define, the \textit{remainder of order} $r$ of the transformation (or of the transformed function) as,

$$\mathcal{R}^{(r)} = F_{r+1} + F_{r+2} + F_{r+3} + \ldots$$  \hfill (2.6.1)

In the previous section it has been justified that one can consider the functions $f_{l,s}$ and the terms $F_1, F_2, \ldots, F_r, \ldots$ defined in the domain, $\mathcal{D}(7\rho/8, R \exp(-\rho/8))$. To avoid an overload of notation we introduce the following convention for the norm,

$$||| \cdot ||| = | \cdot |_{7\rho/8, R \exp(-\rho/8)}.$$  \hfill (2.6.2)

This will not cause any confusion, because the domain will no longer be reduced along this section. Keeping in mind the definition of the Giorgilli-Galgani algorithm, definition 1.7, one realizes easily that,

$$|||F_{r+1}||| \leq |||f_{r+1,0}||| + |||f_{r,1}||| + |||f_{r-1,2}||| + \ldots$$

$$\ldots + |||f_{3,r-2}||| + |||f_{2,r-1}||| + |||f_{1,r}|||,$$

$$|||F_{r+2}||| \leq |||f_{r+2,0}||| + |||f_{r+1,1}||| + |||f_{r,2}||| + \ldots$$

$$\ldots + |||f_{4,r-2}||| + |||f_{3,r-1}||| + |||f_{2,r}||| + |||f_{1,r+1}|||,$$

$$|||F_{r+3}||| \leq |||f_{r+3,0}||| + |||f_{r+2,1}||| + |||f_{r+1,2}||| + \ldots$$

$$\ldots + |||f_{5,r-2}||| + |||f_{4,r-1}||| + |||f_{3,r}||| + |||f_{2,r+1}||| + |||f_{1,r+2}|||,$$

$$\vdots$$

$$|||F_{r+k}||| \leq |||f_{r+k,0}||| + |||f_{r+k-1,1}||| + |||f_{r+k-2,2}||| + \ldots$$

$$\ldots + |||f_{k+2,r-2}||| + |||f_{k+1,r-1}||| + |||f_{k,r}|||$$

$$+ |||f_{k-1,r+1}||| + |||f_{k-2,r+2}||| + \cdots + |||f_{1,r+k-1}|||,$$

$$\vdots$$

Therefore, the sum of the norms $|||F_{r+1}||| + |||F_{r+2}||| + \ldots$, may be grouped in two terms, $S_1$ and $S_2$, defined as,

$$S_1 = |||f_{r+1,0}||| + |||f_{r+2,0}||| + |||f_{r+3,0}||| + \cdots + |||f_{r+k,0}||| + \ldots$$

$$+ |||f_{r,1}||| + |||f_{r+1,1}||| + |||f_{r+2,1}||| + \cdots + |||f_{r+k-1,1}||| + \ldots$$

$$+ |||f_{r-1,2}||| + |||f_{r,2}||| + |||f_{r+1,2}||| + \cdots + |||f_{r+k-2,2}||| + \ldots$$

$$+ |||f_{r-2,3}||| + |||f_{r-1,3}||| + |||f_{r,3}||| + \cdots + |||f_{r+k-3,3}||| + \ldots$$

$$+ \cdots$$

$$+ |||f_{3,r-2}||| + |||f_{4,r-2}||| + |||f_{5,r-2}||| + \cdots + |||f_{k+2,r-2}||| + \ldots,$$  \hfill (2.6.3)
and,

$$S_2 = |||f_{2,r-1}||| + |||f_{3,r-1}||| + |||f_{4,r-1}||| + \cdots + |||f_{k,r-1}||| + \cdots$$

$$+ |||f_{1,r}||| + |||f_{2,r}||| + |||f_{3,r}||| + \cdots + |||f_{k,r}||| + \cdots$$

$$+ |||f_{1,r+1}||| + |||f_{2,r+1}||| + |||f_{3,r+1}||| + \cdots + |||f_{k,r+1}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,2r-4}||| + |||f_{2,2r-4}||| + |||f_{3,2r-4}||| + \cdots + |||f_{k,2r-4}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,2r-3}||| + |||f_{2,2r-3}||| + |||f_{3,2r-3}||| + \cdots + |||f_{k,2r-3}||| + \cdots$$

$$+ |||f_{1,2r-2}||| + |||f_{2,2r-2}||| + |||f_{3,2r-2}||| + \cdots + |||f_{k,2r-2}||| + \cdots$$

$$+ |||f_{1,2r-1}||| + |||f_{2,2r-1}||| + |||f_{3,2r-1}||| + \cdots + |||f_{k,2r-1}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,3r-6}||| + |||f_{2,3r-6}||| + |||f_{3,3r-6}||| + \cdots + |||f_{k,3r-6}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,3r-5}||| + |||f_{2,3r-5}||| + |||f_{3,3r-5}||| + \cdots + |||f_{k,3r-5}||| + \cdots$$

$$+ |||f_{1,3r-4}||| + |||f_{2,3r-4}||| + |||f_{3,3r-4}||| + \cdots + |||f_{k,3r-4}||| + \cdots$$

$$+ |||f_{1,3r-3}||| + |||f_{2,3r-4}||| + |||f_{3,3r-4}||| + \cdots + |||f_{k,3r-4}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,4r-8}||| + |||f_{2,4r-8}||| + |||f_{3,4r-8}||| + \cdots + |||f_{k,4r-8}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,j(r-2)+1}||| + |||f_{2,j(r-2)+1}||| + \cdots + |||f_{k,j(r-2)+1}||| + \cdots$$

$$+ |||f_{1,j(r-2)+2}||| + |||f_{2,j(r-2)+2}||| + \cdots + |||f_{k,j(r-2)+2}||| + \cdots$$

$$+ |||f_{1,j(r-2)+3}||| + |||f_{2,j(r-2)+3}||| + \cdots + |||f_{k,j(r-2)+3}||| + \cdots$$

$$+ \cdots$$

$$+ |||f_{1,(j+1)(r-2)}||| + |||f_{2,(j+1)(r-2)}||| + \cdots + |||f_{k,(j+1)(r-2)}||| + \cdots$$

$$+ \cdots$$

(2.6.4)

Using the proposition 2.16, we can bound the sums

$$|||f_{r+1,0}||| + |||f_{r+2,0}||| + \cdots, \quad |||f_{r,1}||| + |||f_{r+1,1}||| + \cdots, \ldots$$
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entering in the r. h. s. of (2.6.3),

\[\sum_{j \geq r+1} |||f_{j,0}||| \leq c_1 \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right],\]

\[\sum_{j \geq r} |||f_{j,1}||| \leq c_2 \beta_3 \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right] \left( \frac{g}{\delta_r^2} \right),\]

\[\sum_{j \geq r-1} |||f_{j,2}||| \leq c_2 \beta_3 \beta_4 \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right] \left( \frac{g}{\delta_r^2} \right)^2,\]

\[\vdots\]

\[\sum_{j \geq 3} |||f_{j,r-2}||| \leq c_2 \beta_3 \beta_4 \cdots \beta_r \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right] \left( \frac{g}{\delta_r^2} \right)^{r-2},\]

with the constants \(c_1\) and \(c_2\) given by,

\[c_1 = c, \quad c_2 = \frac{cd}{g}.\] (2.6.5)

Adding up the terms in all the partial sums above, we get the following bounds for \(S_1\),

\[S_1 \leq c_1 \left( 1 - \frac{R}{R_0} \right)^{-1} \left( \frac{R}{R_0} \right)^{r+1} + \]

\[+ c_2 \left[ \beta_3 \frac{g}{\delta_r^2} + \beta_3 \beta_4 \left( \frac{g}{\delta_r^2} \right)^2 + \cdots + \beta_3 \beta_4 \cdots \beta_r \left( \frac{g}{\delta_r^2} \right)^{r-2} \right] \left( 1 - \frac{R}{R_0} \right)^{-1} \left( \frac{R}{R_0} \right)^{r+1},\]

and where the geometric series,

\[\left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \left( \frac{R}{R_0} \right)^{r+3} + \cdots = \frac{(R/R_0)^{r+1}}{1 - R/R_0}\]

has been summed \((R/R_0 < 1)\). If we introduce \(\xi\) and \(\xi_s\) as,

\[\xi = \left( \frac{R}{R_0} \right)^{1/2},\]

\[\xi_s = \beta_3 \beta_4 \cdots \beta_s \left( \frac{g}{\delta_r^2} \xi \right)^{s-2}, \quad \text{for } 3 \leq s \leq r\] (2.6.6)

and take into account the nondecreasing character of the sequence \(\{\beta_s\}_{3 \leq s \leq r}\) (with \(\beta_3 \geq 1\), the definition (2.5.5) of \(\delta_r\), and that \(g \geq 1\) (see proposition 2.16), we can derive an easier estimate for \(S_1\). Explicitly,

\[S_1 \leq \frac{c_1 \xi^{2r+2}}{1 - \xi^2} + \frac{c_2 (r-2) \xi_s \xi^{r+4}}{1 - \xi^2}.\] (2.6.7)
For $S_2$, one proceeds in the same way: proposition 2.16 allows to bound separately, every one of the sums $|||f_{k,r-1}||| + |||f_{r-1}||| + \ldots + |||f_{k,r}||| + |||f_{k,r}||| + \ldots + |||f_{k,r}||| + \ldots$, $|||f_{1,r+1}||| + |||f_{k,r}||| + \ldots + |||f_{k,r}||| + \ldots$, \ldots in (2.6.4), i.e.,

$$
\sum_{k \geq 2} |||f_{k,r-1}||| \leq c_2 \beta_3 \beta_3 \beta_r \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{r-1},
$$
$$
\sum_{k \geq 1} |||f_{k,r}||| \leq c_2 \beta_3 \beta_4 \beta_3 \beta_4 \beta_r \left[ \left( \frac{R}{R_0} \right)^{r+1} + \left( \frac{R}{R_0} \right)^{r+2} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{r},
$$
$$
\sum_{k \geq 1} |||f_{k,r+1}||| \leq c_2 \beta_3 \beta_4 \beta_3 \beta_4 \beta_r \left[ \left( \frac{R}{R_0} \right)^{r+2} + \left( \frac{R}{R_0} \right)^{r+3} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{r+1},
$$
$$
\sum_{k \geq 1} |||f_{k,2r-4}||| \leq c_2 \beta_3 \beta_4 \beta_r^2 \left[ \left( \frac{R}{R_0} \right)^{2r-3} + \left( \frac{R}{R_0} \right)^{2r-2} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{2r-4},
$$
$$
\sum_{k \geq 1} |||f_{k,2r-2}||| \leq c_2 \beta_3 \beta_4 \beta_r^2 \left[ \left( \frac{R}{R_0} \right)^{2r-2} + \left( \frac{R}{R_0} \right)^{2r-1} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{2r-3},
$$
$$
\sum_{k \geq 1} |||f_{k,2r-1}||| \leq c_2 \beta_3 \beta_4 \beta_r^2 \left[ \left( \frac{R}{R_0} \right)^{2r-1} + \left( \frac{R}{R_0} \right)^{2r} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{2r-2},
$$
$$
\sum_{k \geq 1} |||f_{k,3r-6}||| \leq c_2 \beta_3 \beta_4 \beta_r^3 \left[ \left( \frac{R}{R_0} \right)^{3r-5} + \left( \frac{R}{R_0} \right)^{3r-4} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{3r-6},
$$

\[ \vdots \]

$$
\sum_{k \geq 1} |||f_{k,(r-2)+1}||| \leq c_2 \beta_3 \beta_3 \beta_4 \beta_r \beta_r^4 \times
\left( \frac{R}{R_0} \right)^{j(r-2)+2} + \left( \frac{R}{R_0} \right)^{j(r-2)+3} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{j(r-2)+1},
$$
$$
\sum_{k \geq 1} |||f_{k,(r-2)+2}||| \leq c_2 \beta_3 \beta_4 \beta_3 \beta_4 \beta_r \beta_r^4 \times
\left( \frac{R}{R_0} \right)^{j(r-2)+3} + \left( \frac{R}{R_0} \right)^{j(r-2)+4} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{j(r-2)+2},
$$
$$
\sum_{k \geq 1} |||f_{k,(r-2)+3}||| \leq c_2 \beta_3 \beta_4 \beta_3 \beta_4 \beta_r \beta_r^4 \times
\left( \frac{R}{R_0} \right)^{j(r-2)+4} + \left( \frac{R}{R_0} \right)^{j(r-2)+5} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{j(r-2)+3},
$$
$$
\sum_{k \geq 1} |||f_{k,(j+1)(r-2)}||| \leq c_2 \beta_3 \beta_4 \beta_r \beta_r^{j+1} \times
\left( \frac{R}{R_0} \right)^{j(r-2)+4} + \left( \frac{R}{R_0} \right)^{j(r-2)+5} + \ldots \right] \left( \frac{g}{\delta^2} \right)^{(j+1)(r-2)},
$$

\[ \vdots \]

and, after addition of the terms on the right and on the left of the \leq symbol (using as before, the expression for the sum of the geometric series \( \left( \frac{R}{R_0} \right)^m + \left( \frac{R}{R_0} \right)^{m+1} + \ldots \) in the square
2.6. Bounds for the remainder of the transformation

brackets), we get—in terms of \( \xi \) and \( \xi_s \) defined by (2.6.6)—,

\[
S_2 \leq \frac{c_2 \xi^2}{1 - \xi^2} (\xi_3 \xi + \xi_4 \xi^2 + \xi_5 \xi^3 + \cdots + \xi_s \xi^{r-2}) \times \\
\times [\xi_r \xi^{r-2} + (\xi_r \xi^{r-2})^2 + (\xi_r \xi^{r-2})^3 + \cdots + (\xi_r \xi^{r-2})^j + \cdots].
\]

So, since \( |||R^{(r)}||| \leq S_1 + S_2 \), we shall take such a sum as a bound of the remainder, i.e.,

\[
|||R^{(r)}||| \leq \frac{c_1 \xi^{2r+2}}{1 - \xi^2} + c_2 \left\{ (r - 2) \xi^4 + \left( \sum_{j=3}^{r} \xi_j \xi^{j-2} \right) \left( \sum_{j=0}^{\infty} (\xi_r \xi^{r-2})^j \right) \right\} \frac{\xi_r \xi^{r-2}}{1 - \xi^2}. \tag{2.6.8}
\]

Therefore, given a fixed order \( r \) up to which the normal form is computed, it follows from the last inequality that the transformation \( \phi^{(r)} \) generated by \( G^{(r)} = \sum_{s=3}^{r} G_s \) i.e., such that \( f \circ \phi^{(r)} = T_{G^{(r)}} f \), will be defined and analytic in some subset of \( D(7\rho/8, R \exp(-\rho/8)) \), provided \( R < R_0 \) is small enough to satisfy,

\[
\xi_r \xi^{r-2} < 1. \tag{2.6.9}
\]

Conversely, for a sufficiently small \( R < R_0 \) (hence a fixed size of the domain), we may wonder what should be the order of the normal form, \( r \), to minimize \( |||R^{(r)}||| \), or at least, some appropriate bound of it. Thus, as in Giorgilli et al. (1989), “one can look for the optimal normalization order \( r_{opt} \) as a function of \( R \), by minimizing the bound (2.6.8) with respect to \( r \)”.

2.6.1 The optimal normalization order

Prior to the computations leading to \( r_{opt} \), it is necessary to concrete the analytical expression of the terms \( \beta_s, s = 3, \ldots, r \). A suitable choice for them (see proposition 2.28) turns out to be,

\[
\beta_s = c_3 \left( \frac{c_2}{\rho} \right)^{r(s+1)} \exp \left( 12\tau \int_{s-1}^{s} x \ln x \, dx \right), \tag{2.6.10}
\]

with the constants,

\[
c_3 = \sqrt{2\pi e^2} (> 1), \quad c_4 = \frac{64\tau}{\gamma_1 e} (\geq 16). \tag{2.6.11}
\]

Another point to remark is that, regardless the more or less accurate selection of the quantities \( \beta_s \), the r.h.s. of (2.6.8) is—as a function of \( r \) and \( \xi \), too intricate to allow analytic optimization with respect to \( r \). However, one can appreciate that the most significative terms of that bound is the product \( \xi_r \xi^{r-2} \). Hence, to determine \( r_{opt} \) it seems reasonable, for \( R > 0 \) fixed, to find the minimum of \( \xi_r \xi^{r-2} \) as a function of \( r \), and then ask \( R \) to be small enough to make \( \xi_r \xi^{r-2} < 1 \). In view of the r.h.s. of (2.6.8), we see that this optimization assures the analyticity of the transformation, provided the sum \( \sum_{j=3}^{r} j \xi_j \xi^{j-2} \) is bounded.

Actually, to simplify even more the calculations, what is minimized is not \( \xi_r \xi^{r-2} \), but the bound purposed by the lemma below.

Lemma 2.17. The product \( \xi_r \xi^{r-1} \) admits the upper bound,

\[
\xi_r \xi^{r-2} \leq \exp \left\{ (r - 2) \ln(c_3 \xi) + c_6 \int_{\frac{r}{2}}^{r} x \ln x \, dx + c_7 \right\}, \tag{2.6.12}
\]
valid for \( r \geq 3 \) and with the constants \( c_5, c_6, c_7 \) given by,

\[
c_5 := c_3 g, \quad c_6 := 16\tau + 4\tau \ln \frac{c_4}{\rho}, \quad c_7 := 4\tau \ln 2 - 2\tau.
\] (2.6.13)

**Proof.** Inequality (2.6.12) is derived straightforward from the definition of \( \xi_r \), the above expression for \( \beta_s \), \( s = 3, \ldots, r \), the aid of the (immediate) auxiliary relations,

\[
\frac{1}{\delta_r} = \frac{16}{\rho} (r - 2) \leq \frac{c_4}{\rho} r, \quad \text{since} \quad c_4 \geq 16;
\] (2.6.14)

\[
2r - 4 + \tau \sum_{j=4}^{r+1} j \leq \tau r^2, \quad \text{for} \quad r \geq 3;
\] (2.6.15)

\[
r^2 \leq 4 \int_2^r x \ln x \, dx, \quad \text{for} \quad r \geq 3.
\] (2.6.16)

and the evaluation of the integral,

\[
\int_2^r x \ln x \, dx = \frac{r^2}{2} \ln r - \frac{1}{4} r^2 + 1 - 2 \ln 2.
\] (2.6.17)

Making the computations explicitly,

\[
\xi_r \xi_{r-2} = \beta_3 \beta_4 \cdots \beta_r \left( \frac{g}{\delta_r} \xi \right)^{r-2} \xi^{r-2}
\leq \left( \frac{c_4}{\rho} \right)^{\tau \sum_{s=3}^{r+1} (s+1)} \left( \frac{1}{\delta_r} \xi \right)^{r-2} \exp \left( 12\tau \sum_{s=3}^r \int_{s-1}^s x \ln x \, dx \right) \times (c_3 g \xi)^{r-2}
\leq \left\{ \frac{1}{\delta_r} \leq \frac{c_4}{\rho} r \right\} \left( \frac{c_4}{\rho} \right)^{\tau \sum_{s=3}^{r+1} (s+1) + 2r - 4} \exp \left( 12\tau \sum_{s=3}^r \int_{s-1}^s x \ln x \, dx \right) \times (c_5 \xi^2)^{r-2}
\leq \left( \frac{c_4}{\rho} r \right)^{\tau r^2} \exp \left( 12\tau \int_2^r x \ln x \, dx \right) \times (c_5 \xi^2)^{r-2}
= \exp \left\{ (r - 2) \ln(c_5 \xi^2) + \tau r^2 \ln \frac{c_4}{\rho} + \tau r \ln r + 12\tau \int_2^r x \ln x \, dx \right\},
\] (2.6.18)

but, in virtue of the inequality (2.6.16) and the integral (2.6.17), we see that the sum

\[
\tau r^2 \ln r + \tau r^2 \ln \frac{c_4}{\rho} = 2\tau \int_2^r x \ln x \, dx + \left( \frac{\tau}{2} + \tau \ln \frac{c_4}{\rho} \right) r^2 + 4\tau \ln 2 - 2\tau
\leq \left( 4\tau + 4\tau \ln \frac{c_4}{\rho} \right) \int_2^r x \ln x \, dx + 4\tau \ln 2 - 2\tau,
\]

and finally, application of the above inequality to (2.6.18) and further identification of the constants \( c_5, c_6 \) and \( c_7 \) lead to (2.6.12). \( \square \)
Let us now consider \( r \) a real continuous variable and the function

\[
h(r) := \exp \left\{ (r - 2) \ln(c_5 \xi^2) + c_6 \int_2^r x \ln x \, dx + c_7 \right\}, \quad (r \geq 3)
\]

– i.e., \( h(r) \) is defined by the the r. h. s. of (2.6.12)–. We denote by \( \hat{r} \) the value of \( r \) minimizing this function. Then, \( \hat{r} \) must be a solution of the equation \( h'(r) = 0 \), which may be expressed as,

\[
e^{\ln r} \ln r = -\frac{1}{c_6} \ln(c_5 \xi^2);
\]

and still, letting \( w = \ln r \) and \( z = -\frac{1}{c_6} \ln(c_5 \xi^2) \), it takes the form,

\[
w e^w = z. \tag{2.6.20}
\]

Provided \( z \geq 0 \), this equation has just one solution, as the function \( w e^w \) increases monotonically from \( 0 \) to \(+\infty\) when \( w \) goes from \( 0 \) to \( \infty \).

The function \( W : \mathbb{C} \rightarrow \mathbb{C} \) such that \( W(z) e^{W(z)} = z \) is a special function known as the Lambert \( W \) function – see Corless et al. (1996), for a concise introduction to the \( W \)-logy. As a complex function \( W(z) \) is multivalued, with infinite number of branches denoted by \( W_k(z) \), \( k \in \mathbb{Z} \). From the division of the complex plane into branches proposed in the paper of Corless et al.:

(i) the branch \( W_0(z) \), called the principal branch, contains the real axis from \(-1/e\) up to \( \infty \); it has a second-order branch point at \( z = -1/e \) which corresponds to \( w = -1 \), with branch cut \( \{ z \in \mathbb{R} : -\infty < z < -1/e \} \). This branch point is shared with \( W_{-1}(z) \), \( W_{1}(z) \).

(ii) \( W_{-1}(z) \), \( W_{1}(z) \) each have a double branch cut: \( \{ z \in \mathbb{R} : -\infty < z < -1/e \} \) and \( \{ z \in \mathbb{R} : -\infty < z < 0 \} \). By convention, the branch cuts are closed on the top and it turns out that this choice for the closure implies that \( W_{-1}(z) \) is real for \( z \in [-1/e, 0) \), so \( W_0 \) and \( W_{-1}(z) \) are the only branches of \( W(z) \) taking real values.

(iii) All other branches \( W_k(z) \), \( k = \pm 2, \pm 3, \ldots \), have only one branch cut, the one matching the real negative axis. Thus, these branches are similar to those of the logarithm.

In our case \( z \) will take real large values, since we consider \( R \) small (\( R/R_0 \ll 1 \)); so from the items above, one deduces that the solutions of (2.6.19) we are looking for will be conveniently expressed by the principal branch \( W_0(z) \). More precisely,

\[
\hat{r} = e^{W_0(\ln(c_5 \xi^2)^{-1/c_6})}, \tag{2.6.21}
\]

and take as the optimal normalizing order, \( r_{\text{opt}} = \lfloor \hat{r} \rfloor \).

Remark 2.18. Though it has not been explicitly pointed out, the choice of the principal branch of \( W(z) \) carries out an implicit assumption on the smallness of \( R \), as it is necessary to impose that \( c_5 \xi^2 < 1 \), or, equivalently:

\[
\frac{R}{R_0} < \frac{1}{c_5},
\]

to have \( z > 0 \) and hence \( w > 0 \), giving rise to solutions \( \hat{r} > 1 \), as desired. \( \Box \)

\(^{4)}\) It can be seen easily that for \(-1/e < z < 0 \), (2.6.20) has two negative solutions, both in the interval \((0, -1)\).

\(^{5)}\) \( W \)-logy: the science of the Lambert \( W \) function.
In what follows, we assume that the nonlinear normalization process has been carried out up to the –in the foregoing explained sense–, optimal order $r_{\text{opt}}$. Then,

$$
\xi_{r_{\text{opt}}} \xi_{r_{\text{opt}}-2} \leq \exp \left\{ \left( r_{\text{opt}} - 2 \right) \ln(\xi c_5^2) + c_6 \int_2^{r_{\text{opt}}} x \ln x \, dx + c_7 \right\},
$$

(2.6.22)

and introducing $0 < \chi = \frac{r_{\text{opt}} - 1}{r_{\text{opt}}} < 1$, we have,

$$
c_6 \int_2^{r_{\text{opt}}} x \ln x \, dx + c_7 = \frac{c_6}{2} (1 - \chi) r_{\text{opt}} \ln r_{\text{opt}} + \frac{c_6}{2} r_{\text{opt}} (1 - \chi) \ln(1 - \chi)
$$

$$
= \frac{c_6}{4} r_{\text{opt}}^2 + c_6 (1 - 2 \ln 2) + c_7
$$

$$
= \frac{1}{2} (1 - \chi) r_{\text{opt}} \ln(c_5 \xi^2) + \frac{c_6}{2} r_{\text{opt}} (1 - \chi) \ln(1 - \chi)
$$

$$
- \frac{c_6}{4} r_{\text{opt}}^2 + c_6 (1 - 2 \ln 2) + c_7,
$$

but $\frac{c_6}{2} r_{\text{opt}} (1 - \chi) \ln(1 - \chi) < 0$, $-\frac{c_6}{4} r_{\text{opt}} < 0$ (we recall that all the constants are positive) and also explicit computation gives

$$
c_6 (1 - 2 \ln 2) + c_7 = (14 \tau + 4 \tau \ln \frac{c_4}{\rho})(1 - 2 \ln 2) < 0.
$$

So according with (2.6.22) $\xi_{r_{\text{opt}}} \xi_{r_{\text{opt}}-2}$ can be still bounded by,

$$
\xi_{r_{\text{opt}}} \xi_{r_{\text{opt}}-2} \leq \exp \left\{ \left[ \frac{r_{\text{opt}}}{2} (1 + \chi) - 2 \right] \ln(c_5 \xi^2) \right\} \leq \left( c_5 \frac{R}{R_0} \right)^{\frac{r_{\text{opt}}}{2} - 2} < 1.
$$

(2.6.23)

This last inequality is true, provided $\frac{R}{R_0} \leq \frac{1}{c_5}$ (as pointed in remark 2.18), and small enough to make $r_{\text{opt}}(R) > 4$. Also, both conditions will from now on assumed.

It is still necessary to check the bounded character of $\sum_{j=3}^{r_{\text{opt}}} \xi_j \xi_j^{-2}$. In fact, its terms decay faster than the corresponding ones in $\sum_{j=0}^{\frac{1}{\beta}} \frac{1}{j!}$. To show this last point, let us compute, with $3 \leq j \leq r_{\text{opt}} - 1$, the quotient

$$
\frac{\xi_{j+1} \xi_{j-1}}{\xi_j \xi_{j-2}} = \beta_{j+1} \frac{g_2^{\frac{r_{\text{opt}}}{\rho}}}{\beta_{r_{\text{opt}}}} \xi^2
$$

$$
\leq c_3 \left( \frac{c_4}{\rho} r_{\text{opt}} \right)^{r(j+2)+2} \exp \left( 12 \tau \int_j^{j+1} x \ln x \, dx \right) \xi^2.
$$

By explicit cast of the quadrature one derives, after some arrangements:

$$
12 \tau \int_j^{j+1} x \ln x \, dx \leq \ln(j + 1)^{12r j + 6 \tau} - 3 \tau.
$$

Now, taking into account:

- Here, we suppose $\frac{c_4}{\rho} > 1$ –and hence $\ln \frac{c_4}{\rho} > 0$–, but since $c_4 \geq 16$, this is guaranteed if now (and in the sequel) we take $\rho < 1$.

- Note that $\sum_{j=3}^{r_{\text{opt}}} \xi_j \xi_j^{-2}$ should be bounded independently of $r_{\text{opt}}$, since $r_{\text{opt}}(R) \to \infty$ when $R \to 0$. 

(i) the value of the constants \(c_3\) and \(c_6\), i. e.,

\[
c_3 = \sqrt{2\pi e^2}(< e^3), \quad c_6 = 16\tau + 4\ln \frac{\tau}{\rho},
\]

(ii) \(j + 1 \leq r_{\text{opt}} \leq \lfloor \tilde{r} \rfloor \leq \tilde{r}\),

(iii) and that \(\tilde{r}\) is the (unique) solution of the equation (2.6.19), so \(\xi^2 = \frac{1}{c_5} \tilde{r}^{-c_6\tilde{r}}\),

one has, after some trivial computations:

\[
\frac{\xi_{j+1}\xi_{j-1}}{\xi_j^2} \leq \left( \frac{c_4}{\rho} \right)^{(j+2)\tau^2} e^{13\tau j + 8\tau + 2\xi^2} \leq \frac{1}{c_5} \exp \left\{ (\tau(j + 2) + 2 - 4\tau\tilde{r} \ln \tilde{r}) \ln \frac{c_4}{\rho} \right\} \times \exp \left\{ (-16\tau\tilde{r} + 13\tau j + 8\tau + 2) \ln \tilde{r} \right\},
\]

(2.6.24)

but again, using item (ii) above, it is straightforward to check the inequalities,

\[
\exp \left\{ (\tau(j + 2) + 2 - 4\tilde{r} \ln \tilde{r}) \ln \frac{c_4}{\rho} \right\} \leq 1,
\]

and

\[
\exp \left\{ (-16\tilde{r} + 13\tau j + 8\tau + 2) \ln \tilde{r} \right\} \leq \exp \{-3\tau j \ln(j + 1)\} = (j + 1)^{-3\tau j},
\]

so, their product times \(1/c_5\) in (2.6.24) leads to the bound,

\[
\frac{\xi_{j+1}\xi_{j-1}}{\xi_j^2} = \beta_{j+1} \times \frac{g}{\delta_{r_{\text{opt}}}^2} \times \frac{R}{R_0} \leq \frac{1}{c_5} \frac{1}{(j + 1)^{3\tau j}},
\]

(2.6.25)

for all \(3 \leq j \leq r_{\text{opt}} - 1\); but the r. h. s. of this last inequality cannot be greater that \(\frac{1}{(j+1)^{3\tau j}}\) (we recall that \(c_5, \tau > 1\)). This proves our assertion on the decay of the terms in \(\sum_{j \geq 3}^* \xi_j \xi_{j-2}^2\).

Using such (i. e., \(1/(j + 1)\)) rough bound for the quotient above, one finds that the sum may be estimated according with

\[
\sum_{j=3}^{r_{\text{opt}}} \xi_j \xi_{j-2} \leq \left( 1 + 3! c_3^3 \sum_{j \geq 4}^* \frac{1}{j!} \left( \frac{c_4}{c_5} \right)^j \right) \tilde{r}^3 \xi \leq 3! c_3^3 \tilde{r}^3 c_6^3 \xi_3 \xi, \quad (2.6.26)
\]

and it is thus “controlled” (for \(R\) small enough), since

\[
\xi_3 \xi = \beta_3 \frac{g}{\delta_{r_{\text{opt}}}^2} \xi_2 = c_3 \left( \frac{c_4}{\rho} \right)^{4\tau} \frac{1}{\delta_{r_{\text{opt}}}^2} \exp \left( 12\tau \int_2^3 x \ln x dx \right) \frac{R}{R_0} \leq \tilde{c}_3 \left( \frac{c_4}{\rho} \right)^{4\tau} \tilde{r}^{4\tau + 2} \frac{R}{R_0} \leq \tilde{c}_4 \left( \frac{c_4}{\rho} \right)^{4\tau + 2} \tilde{r}^{4\tau + 2 - c_6\tilde{r}},
\]

(2.6.27)

where the constants \(\tilde{c}_3, \tilde{c}_4\) are independent of \(R\). Here, we have used that \(r_{\text{opt}} = \lfloor \tilde{r} \rfloor \leq \tilde{r}\) and that \(\tilde{r}\) is a solution of the equation \(\ln \tilde{r}^2 = \ln (c_5 \frac{R_0}{R})^{-1/c_6}\) (see (2.6.19)), so \(\frac{R}{R_0} = \frac{1}{c_5} \tilde{r}^{-c_6\tilde{r}}\). But \(\tilde{r}(R)\) tends to infinity when \(R \to 0\) and \(c_6 > 0\). Then it is clear, from the rightmost term of the
expression above that, taking $R < R_0$ small enough one can make $\xi_3 \xi$ smaller than any prefixed constant, for example $\xi_3 \xi < 1$.

With all these elements, we can write down an effective bound for the remainder. Before the statement of the corresponding proposition (which summarizes the foregoing discussion), we introduce for the domains, the abbreviations $D_0$ and $D_1$ as,

\[
D_0 := D(\rho_0, R_0),
\]

\[
D_1 := D(7\rho/8, R \exp(-\rho/8)),
\]

with $\rho_0$ and $R_0$ given by (2.5.2).

**Proposition 2.19.** Consider the generating function $G^{(r)} = \sum_{s=3}^r G_s$, defined in the domain $D_1$ and such that

\[
\|\|G_3\|\| \leq \beta_3 b \left( \frac{R}{R_0} \right)^3,
\]

\[
\|\|G_s\|\| \leq \beta_3 \beta_4 \cdots \beta_s \frac{a^{s-3} b}{\delta_r^{2(s-3)}} \left( \frac{R}{R_0} \right)^s, \quad 3 < s \leq r,
\]

with the terms,

\[
\beta_s = c_3 \left( \frac{c_4}{\rho} \right)^{\tau(s+1)} \exp \left( 12\tau \int_{s-1}^s x \ln x \, dx \right), \quad s = 3, \ldots, r
\]

and the constants,

\[
c_3 > 1, \quad c_4 \geq 16.
\]

Furthermore, let $f$ be a complex function defined and analytic in $D_0$. Then, for $0 < \rho < \rho_0$ and for $0 < R < R_0$ sufficiently small:

(i) The canonical transformation $\phi^{G^{(r)}}$ given by $f \circ \phi^{G^{(r)}} = T^{G^{(r)}} f$, is defined and analytic in the domain $D_1$.

(ii) Let

\[
T^{G^{(r)}} f = F_1 + F_2 + F_3 + \cdots + F_r + \mathcal{R}^{(r)},
\]

be the expansion of the transformed function of $f$, where

\[
\mathcal{R}^{(r)} = F_{r+1} + F_{r+2} + F_{r+3} + \cdots
\]

Then, there exists an “optimal” normalizing order $r_{\text{opt}} = \lceil \tilde{r} \rceil$, depending on $R$ through,

\[
\tilde{r}(R) = e^{W_0 \left( \ln \left( c_5 \frac{R}{R_0} \right)^{-1/c_6} \right)},
\]

(\text{where $W_0$ denotes the principal branch of the Lambert W function}), such that,

\[
\|\|\mathcal{R}^{r_{\text{opt}}(R)}\|\| \leq c_8 \left( 1 - \frac{R}{R_0} \right)^{-1} \left( \frac{c_5 R}{R_0} \right)^{\frac{r_{\text{opt}(R)}}{2} - 1},
\]

and $c_5$, $c_6$, $c_8$ are positive constants which depend upon $\rho$, $\tau$, $\gamma$ and on $R_0$ but not on $R$. 

(iii) The remainder $\mathfrak{R}(r_{\text{opt}})$ goes to zero with $R/R_0$ faster than any analytic order in $R/R_0$. More precisely,

$$\mathfrak{R}(r_{\text{opt}}) = o\left(\left(\frac{R}{R_0}\right)^n\right), \quad (R/R_0 \to 0)$$

(2.6.35)

for any given positive integer $n$.

**Proof.** The first item and (2.6.33) in (ii) has been already proved. The bound (2.6.34) is proved straightforward from (2.6.8) and the bound (2.6.26). Finally, the assertion of (iii) is derived from

$$|||\mathfrak{R}(r_{\text{opt}})||| \leq c_8 c_5 \left(1 - \frac{R}{R_0}\right)^{-1} \left(\frac{c_5}{R_0}\right)^{r_{\text{opt}}(R) - n-1}. \quad (R/R_0 \to 0)$$

Thus, (2.6.35) follows immediately if one knows that $W_0(z)$ tends to (positive) infinity when $z \to +\infty$ (see remark below), so $\lim_{R/R_0 \to 0} r_{\text{opt}} = +\infty$ and the r. h. s. of the last inequality—for all arbitrary but fixed $n \in \mathbb{N}$—, goes to 0 as $R/R_0$ does.

**Remark 2.20.** In the semisimple case (i. e., when the eigenvalues of the monodromy matrix around the periodic orbit are pairwise different), it has been proved (see Jorba and Villanueva, 1997a), that the remainder is exponentially small with $R$, with bounds of type\(^{(8)}\).

$$|||\mathfrak{R}(r_{\text{opt}})||| \leq \text{constant} \cdot \exp\left(-\text{constant} \left(\frac{1}{R}\right)^{\frac{1}{1+\tau}}\right).$$

In this sense, the results shown in the proposition above are worse, as can immediately be deduced from the behavior of $W_0(z)$ at infinity\(^{(9)}\):

$$W(z) = \ln(z) - \ln \ln(z) + O\left(\frac{\ln \ln z}{\ln z}\right),$$

(see de Bruijn, 1958), for $z \in \mathbb{R}$ with $z \gg 1$. Then, it is clear that, the r. h. s., of (2.6.34) do decay with $R$ slower than exponentially.

\(\blacklozenge\)

### 2.7 On the bounds for $G_s$

It should be clear that the theses of proposition 2.19 are tied to the conditions (2.6.30) –derived from lemma 2.13—, with the given expression for the quantities $\beta_s$ in (2.6.31). In this section, we

\(^{(8)}\)The paper of Jorba and Villanueva discuss seminormal forms are around elliptic lower dimensional invariant tori, so their results can be applied in the particular case of nonresonant elliptic periodic orbits.

\(^{(9)}\)In §2.4 of de Bruijn’s book, it is shown that, for $z$ real and large enough, the solutions of equation (2.6.20) admit the following (convergent) development:

$$w = \ln z - \ln \ln z + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} c_{k,m}(\ln \ln z)^{m+1}(\ln z)^{-k-m-1},$$

Corless et al. (1996)–using two points not noted in de Bruijn’s proof—, extended this development to give the asymptotics for all nonprincipal branches of $W(z)$ both at (complex) infinity and at zero.
consider the finite order (degree) generating function, \( G^{(r)} = G_3 + \cdots + G_r \) set up in chapter 1. There, from an analytic linearly reduced Hamiltonian, \( H : \mathcal{D}_0 \to \mathbb{C}, H = H_2 + H_3 + \cdots \), with
\[
H_2(q, I_1, p) = \omega_1 I_1 + i\omega_2 (q_1 p_1 + q_2 p_2) + q_2 p_1,
\]
(2.7.1)
\[
H_s(\theta_1, q, I_1, p) = \sum_{k \in \mathbb{Z}} \sum_{m,n \in \mathbb{N}, n_1 = s} h_{k,m,n} I_1^m q^n \exp(ik\theta_1), \quad (s \geq 3),
\]
(2.7.2)

\( G^{(r)} \) was formally constructed such that the generated canonical change, \( \phi^{G^{(r)}} \) transforms \( H \) to give,
\[
H \circ \phi^{G^{(r)}}(\theta_1, q, I_1, p) = \sum_{s=2}^{r} Z_s(q, I_1, p) + \mathcal{R}^{(r)}(\theta_1, q, I_1, p),
\]
(theorem 1.24). The main target of the present section, is just to discuss whether proposition 2.19 can be applied to such \( G^{(r)} \). If so, this will guarantee the analyticity of the reduced Hamiltonian in some domain \( \mathcal{D}_2 \subset \mathcal{D}_1 \), for \( R \) small enough and justify that, at least locally, the remainder \( \mathcal{R}^{(r)} \) can be dealt as a small perturbation of the normal form \( Z^{(r)} = \sum_{s=2}^{r} Z_s \). Hence we must prove that the hypotheses on the size of \( G_s, s = 3, 4, \ldots \), asked in proposition 2.19 hold.

In view of the equation (2.4.20), for the bounds of the solutions of the homological equations, we realize that for the purpose described above, it is worth finding out bounds of the terms \( F_s \), but these are given in a recursive way by the formulas (1.7.58) in prop 1.21.

**Remark 2.21.** We shall work assuming for the Hamiltonian \( H \) the same conditions than for the function \( f \) in section 2.5.1. In particular, \( H = \sum_{s \geq 2} H_s \) is defined and analytic in \( \mathcal{D}(\rho^*, R^*) \); \( |H|_{\rho^*, R^*} = c \) and hence,
\[
|H_s|_{\rho, R} \leq c \left( \frac{R}{R_0} \right)^s,
\]
where \( 0 < \rho < \rho_0, 0 < R < R_0 \), with \( \rho_0, R_0 \) satisfying (2.5.2).

Moreover by (2.4.10), the terms \( Z_s(q, I_1, p) \), \( 3 \leq s \leq r \) can be bounded in \( \mathcal{D}(\rho_{s-1}, R_{s-1}) \), as:
\[
|Z_s|_{\rho_{s-1}, R_{s-1}} \leq 2^{s/2} |F_s|_{\rho_{s-1}, R_{s-1}}.
\]
(2.7.3)

**Proposition 2.22.** Let \( \delta \) be given by (2.5.5) and define, for \( 2 \leq k \leq r \),
\[
\rho_2 := \rho, \quad \rho_k := \rho_2 - 2(k-2)\delta, \quad R_2 := R, \quad R_k := R_2 \exp(-2(k-2)\delta).
\]
(2.7.4)
\[
\rho_{k+1, R_{k+1}} \leq \theta_{l,k} \frac{\alpha_3 \cdots \alpha_{k+1} C_{k+1}}{\delta^{2k} R_0^2} \left( \frac{R}{R_0} \right)^{2+l+k}, \quad (1 \leq k \leq r-2),
\]
(2.7.6)
\[
|F_{k+2}|_{\rho_{k+1}, R_{k+1}} \leq \eta_k \frac{\alpha_3 \cdots \alpha_{k+1} C_k}{\delta^{2(k-1)} R_0^{2(k-1)}} \left( \frac{R}{R_0} \right)^{k+2}, \quad (2 \leq k \leq r-2);
\]
(2.7.7)
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with $F_3 = H_3$ and the terms $\eta_1, \ldots, \eta_{r-2}$; $\theta_{l,0}, \theta_{l,1}, \ldots, \theta_{l,r-l-2}$ ($0 \leq l \leq r - 2$) are defined recursively by the relations

$$
\theta_{l,k} = \frac{\gamma_2}{k} \sum_{j=1}^{k} j \eta_j \theta_{l,k-j} \quad (k \geq 1),
$$

(2.7.8)

$$
\eta_k = \frac{\gamma_2}{k} \sum_{j=1}^{k-1} j 2^{(k-j+2)/2} \eta_j \eta_{k-j} + \frac{1}{k} \sum_{j=1}^{k} j \theta_{j,k-j} \quad (k \geq 2),
$$

(2.7.9)

with

$$
\gamma_2 = \frac{17e}{2}, \quad \eta_1 = 1, \quad \theta_{l,0} = 1 \quad (\text{for all } l \geq 0),
$$

(2.7.10)

From the inequality (2.4.20) and the bounds for $F_k$, $k = 4, \ldots, r$ in the proposition above, the next corollary follows straightforward.

**Corollary 2.23.** The terms $G_k$ of order $k$, $k = 3, \ldots, r$ of the generating functions are bounded according to,

$$
|G_{k+2}|_{\rho_{k+2}\delta, R_{k+2}\exp(\delta)} \leq \eta_k \frac{\alpha_3 \cdots \alpha_{k+2} c^k}{\delta^{2(k-1)} R_0^{2(k-1)}} \left( \frac{R}{R_0} \right)^{k+2},
$$

(2.7.11)

for $1 \leq k \leq r - 2$.

**Proof of proposition 2.22.** We use the reduction algorithm described in section 1.7.3. For $k = 1$, and the definition of $H_{l,k}$ we have,

$$
|H_{2+l,1}|_{\rho_3, R_3} = |L_{G_3} H_{2+l,0}|_{\rho_3, R_3} = |\{ H_{2+l}, G_3 \}|_{\rho_3, R_3}.
$$

Again, using corollary A.7, with $\delta_\nu = \delta$ for all $\nu$ to bound the Poisson bracket throughout.

$$
|H_{2+l,1}|_{\rho_3, R_3} \leq \frac{17e}{2\delta^2 R^2} |H_{2+l}|_{\rho, R} |G_3|_{\rho_3, R_3} \exp(\delta)
$$

$$
\leq \frac{17e}{2} \frac{\alpha_3 c^2}{\delta^2 R_0^2} \left( \frac{R}{R_0} \right)^{2+l+1}
$$

$$
= \theta_{l,1} \frac{\alpha_3 c^2}{\delta^2 R_0^2} \left( \frac{R}{R_0} \right)^{2+l+1}
$$

since, $|G_3|_{\rho_3, R_3} \exp(\delta) = |G_3|_{\rho-\delta, R \exp(-\delta)} \leq \alpha_3 |F_3|_{\rho, R} = \alpha_3 |H_3|_{\rho, R} \leq \alpha_3 c \left( \frac{R}{R_0} \right)^3$, and by (2.7.8) $\theta_{l,1} = \gamma_2$. In the same way, for $k = 2$,

$$
|F_4|_{\rho_3, R_3} \leq \frac{1}{2} |\{ Z_3, G_3 \}|_{\rho_3, R_3} + \frac{1}{2} |H_{3,1}|_{\rho_3, R_3} + |H_4|_{\rho_3, R_3}
$$

$$
\leq \frac{17e}{2\delta^2 R^2} |Z_3|_{\rho_3, R_3} \exp(2\delta) |G_3|_{\rho_3, R_3} \exp(\delta) + \frac{1}{2} |H_{3,1}|_{\rho_3, R_3} + |H_4|_{\rho, R}
$$

$$
\leq \frac{17e}{2\delta^2 R^2} 2^{3/2} \alpha_3 |F_3|^2_{\rho, R} + \frac{1}{2} \theta_{l,1} \frac{\alpha_3 c^2}{\delta^2 R_0^2} \left( \frac{R}{R_0} \right)^4 + c \left( \frac{R}{R_0} \right)^4
$$

$$
\leq \left( \frac{\gamma_2}{2} 2^{3/2} \eta_1 \eta_1 + \frac{1}{2} \theta_{l,1} + 1 \right) \frac{\alpha_3 c^2}{\delta^2 R_0^2} \left( \frac{R}{R_0} \right)^4,
$$


where, in the last step we use that, by \((2.5.2)\), \(R_0 < c\), \(R_0 < 1\) then, \(c^2/R_0^2 \geq cR_0^2 \geq c\); together with the obvious inequality \(\alpha^2 > 1\). But from \((2.7.9)\), \(\eta_2 = \frac{2}{2^2} 2^{3/2} \eta_1 \eta_1 + \frac{1}{2} \theta_{1,1} + \theta_{2,0}\), and hence,

\[
|F_4|_{\rho_3, R_3} \leq \eta^2 \frac{\alpha_3 c^2}{\delta^2 R_0^2} \left( \frac{R}{R_0} \right)^4.
\]

**Remark 2.24.** We recall here that \(Z_s = 0\) for \(s\) odd. However, we shall ignore this fact and use the bounds:

\[
|Z_s|_{\rho_{s-1}, R_{s-1}} \leq 2^{s/2} |F_s|_{\rho_{s-1}, R_{s-1}}, \quad s = 3, \ldots, r,
\]

throughout (see \((2.7.3)\)).

Similarly, \(H_{2+l,2} = \frac{1}{2} \{H_{2+l,1}, G_3\} + \{H_{2+l,0}, G_4\}\) so,

\[
|H_{2+l,2}|_{\rho_4, R_4} \leq \frac{17e/2}{2 R_0^2} |H_{2+l,1}|_{\rho_3, R_3} |G_3|_{\rho_{s+\delta}, R_{s+\delta}} \exp(\delta) + \frac{17e}{2 R_0^2} |H_{1+2,0}|_{\rho, R} |G_4|_{\rho_{s+\delta}, R_{s+\delta}} \exp(\delta)
\]

\[
\leq \frac{\gamma_2}{2} (\theta_{1,1} \eta_1 + 2 \theta_{1,0} \eta_2) \frac{\alpha_3 \alpha_4 c^3}{\delta^4 R_0^4} \left( \frac{R}{R_0} \right)^{l+4}
\]

\[
= \theta_{l,2} \frac{\alpha_3 \alpha_4 c^3}{\delta^4 R_0^4} \left( \frac{R}{R_0} \right)^{l+4}.
\]

Assume now that \((2.7.6)\) and \((2.7.7)\) work for all \(\nu, \ 2 < \nu < k\) (and \(k \leq r - 2\)). Therefore \((2.7.11)\) should be valid also for \(2 < \nu < k\). Then, for \(\nu = k\) and \(1 \leq j \leq k\),

\[
|\{Z_{k-j+2}, G_{2+j}\}|_{\rho_{s+1}, R_{s+1}} \leq \frac{17e}{2 R_0^2} |Z_{k+2-j}|_{\rho_{k-j+1}, R_{k-j+1}} |G_{2+j}|_{\rho_{s+\delta}, R_{s+\delta}} \exp(\delta)
\]

\[
\leq \frac{17e}{2 R_0^2} 2^{(k+2-j)/2} |F_{k-j+2}|_{\rho_{k-j+1}, R_{k-j+1}} |G_{2+j}|_{\rho_{s+\delta}, R_{s+\delta}} \exp(\delta)
\]

\[
\leq \frac{17e}{2 R_0^2} 2^{(k+2-j)/2} \eta_{k-j} \frac{\alpha_3 \cdots \alpha_{k-j+1} c^{k-j}}{\delta^{2(k-j-1)} R_0^{2(k-j-1)}} \left( \frac{R}{R_0} \right)^{k-j+2}
\]

\[
\times \eta_j \frac{\alpha_3 \cdots \alpha_{j+2 c^j}}{\delta^{2(j-1)} R_0^{2(j-1)}} \left( \frac{R}{R_0} \right)^{j+2}.
\]

Using the non decreasing character of the sequence \(\{\alpha_j\}_{3 \leq j \leq r}\), \(\alpha_{i+2} \leq \alpha_{k-j+i+1}\), for all \(i = 1, \ldots, j\), we have

\[
\alpha_3 \cdots \alpha_{k-j+1} \alpha_3 \cdots \alpha_{j+2} = \left( \prod_{i=3}^{k-j+1} \alpha_i \right) \left( \prod_{i=1}^{j} \alpha_{i+2} \right)
\]

\[
\leq \left( \prod_{i=3}^{k-j+1} \alpha_i \right) \left( \prod_{i=1}^{j} \alpha_{k-j+i+1} \right)
\]

\[
= \alpha_3 \cdots \alpha_{k+1}.
\]
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Thus, the Poisson bracket above can be bounded as,

$$\left|\{Z_{k-j+2}, G_{2+j}\}\right|_{\rho_{k+1}, R_{k+1}} \leq \frac{17e}{2} 2^{(k+2-j)/2} \eta_{k-j} \alpha_j \alpha_{k+1} c^k \left( \frac{R}{R_0} \right)^{k+2}. \quad (2.7.12)$$

Next, using the induction hypotheses, we can estimate the sum

$$\sum_{j=1}^{k} j |H_{2+j,k-j}|_{\rho_{k+1}, R_{k+1}} \leq \frac{1}{k} \sum_{j=1}^{k} j |H_{2+j,k-j}|_{\rho_{k-j+2}, R_{k-j+2}}$$

$$\leq \frac{1}{k} \sum_{j=1}^{k-1} j \theta_{j,k-j} \frac{\alpha_3 \cdots \alpha_{k-j+2} c^{k-j+1}}{\delta^{2(k-j)} R_0^{2(k-j)}} \left( \frac{R}{R_0} \right)^{k+2} + |H_{2+k,0}|_{\rho,R}, \quad (2.7.13)$$

and for $|H_{2+k,0}|_{\rho,R}$, it is immediate that

$$|H_{2+k,0}|_{\rho,R} \leq c \left( \frac{R}{R_0} \right)^{k+2} \leq \frac{\alpha_3 \cdots \alpha_{k+1} c^k}{\delta^{2(k-1)} R_0^{2(k-1)}} \left( \frac{R}{R_0} \right)^{k+2}, \quad (2.7.14)$$

where it has been used that,

(1) as $c/R_0 > 1$, with $R_0 < 1$ it is,

$$\frac{c^{k-j+1}}{R_0^{2(k-j)}} = \left( \frac{c}{R_0} \right)^{k-j+1} \frac{1}{R_0^{k-j-1}} \leq \left( \frac{c}{R_0} \right)^{k} \frac{1}{R_0^{k-j-1}} \leq \left( \frac{c}{R_0} \right)^{k} \frac{1}{R_0^{k-2}} = \frac{c^k}{R_0^{2(k-1)}}.$$

(2) $\alpha_3 \cdots \alpha_{k-j+2} \leq \alpha_3 \cdots \alpha_{k+1}$,

for all $1 \leq j \leq k$. Putting together (2.7.12), (2.7.13), (2.7.14),

$$|F_{k+2}|_{\rho_{k+1}, R_{k+1}} \leq \sum_{j=1}^{k} \frac{1}{k} |\{Z_{k+2-j}, G_{2+j}\}|_{\rho_{k+1}, R_{k+1}} + \sum_{j=1}^{k} \frac{1}{k} |H_{2+j,k-j}|_{\rho_{k+1}, R_{k+1}}$$

$$\leq \left( \frac{72}{k} \sum_{j=1}^{k-1} j \gamma_2^{2(k-j+2)} \eta_j \eta_{k-j} \frac{\alpha_3 \cdots \alpha_{k+1} c^k}{\delta^{2(k-1)} R_0^{2(k-1)}} \left( \frac{R}{R_0} \right)^{k+2} \right.$$

$$\left. = \eta_k \alpha_3 \cdots \alpha_{k+1} c^k \frac{R}{R_0} \right)^{k+2} \left( \frac{R}{R_0} \right)^{k+2}.$$
Analogously, making use of the induction hypothesis and the properties (1), (2) above,

\[
|H_{2+l,k}|_{\rho_{k+2}, R_{k+2}} \leq \sum_{j=1}^{k} \left( \frac{j}{k} \right) \sum_{j=1}^{k} \frac{17e}{2R^2\delta^2} \left| G_{2+j} \right|_{\rho_{j+2}\delta, R_{j+2}} \exp(\delta) \left| H_{2+l,k-j} \right|_{\rho_{j-k+2}, R_{j-k+2}} \\
\leq \sum_{j=1}^{k-1} \frac{17e}{2R^2\delta^2} \left| \alpha_3 \cdots \alpha_{j+2}c^j \right| \frac{\alpha_3 \cdots \alpha_{k+2}c^k}{\delta^2(k-j)} \frac{R^2}{R_0} \times \\
\times \theta_{l,k-j} \frac{\alpha_3 \cdots \alpha_{k+2}c^k}{\delta^2(k-j)} \frac{R^2}{R_0} \frac{R}{R_0}^{2l+k-j} \\
+ \frac{17e}{2R^2\delta^2} \eta_k \frac{\alpha_3 \cdots \alpha_{k+2}c^k}{\delta^2(k-j)} \frac{R^2}{R_0} \frac{R}{R_0}^{4l+k} \\
\leq \left( \frac{\gamma_2}{k} \right) \sum_{j=1}^{k} j \theta_{l,k-j} \eta_j \frac{\alpha_3 \cdots \alpha_{k+2}c^k}{\delta^2k} \frac{R^2}{R_0} \frac{R}{R_0}^{2l+k} \\
= \theta_{l,k} \frac{\alpha_3 \cdots \alpha_{k+2}c^k}{\delta^2k} \frac{R^2}{R_0} \frac{R}{R_0}^{2l+k},
\]

(assuming again that \( \alpha_3 \cdots \alpha_{k+2}c^k \leq \alpha_3 \cdots \alpha_{k+j+2}c^j \), since \( \alpha_{i+2} \leq \alpha_{k-j+i+2} \), for all \( i = 1, \ldots, j \) and for all \( 2 \leq k \leq r - 2 \). This ends the induction and closes the proof of the proposition. \( \square \)

Our purpose is to obtain bounds for the terms of the generating function, \( G_s, s = 3, \ldots, r \). Corollary 2.23, just provides estimates of type (2.5.11), (with \( \delta_3 = \cdots = \delta_r = \delta \)). But therein, a factor \( \eta_k \) defined recursively in (2.7.9) appears. Hence, it is worth obtaining estimates for these coefficients. To this end, one defines the quantities,

\[
a_{l,k} = \frac{4}{k} \sum_{j=1}^{k} j b_j a_{l-k-j}, \quad \text{for } l, k \geq 0, \quad (2.7.15)\\
b_k = \frac{4}{k} \sum_{j=1}^{k-1} j b_j b_{l-j} + \frac{1}{k} a_{l,k-1}, \quad \text{for } k \geq 2, \quad (2.7.16)
\]

with \( b_1 = 1 = a_{l,0} \), for \( l \geq 0 \). Some relevant properties of these coefficients are described by the next lemma.

**Lemma 2.25.** The coefficients \( a_{l,k}, b_k \) defined by (2.7.15) and (2.7.16) respectively, satisfy the following properties:

(i) \( b_s = a_{1,s-1}, \) for \( s \geq 1 \).

(ii) \( a_{1,k} \geq 4^{k+1}k a_{1,k-1}, \) for all \( k \geq 1 \).

(iii) \( a_{l,s} = a_{l-1,s}, \) for all \( l \geq 1 \) and \( s \geq 0 \).
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(iv) $a_{1,k-1} \geq \sum_{j=2}^{k} ja_{j,k-j}$, for $k \geq 1$.

(v) $b_k \leq 4^{k-1} k!$, for all $k \geq 2$.

Proof. (i) $b_1 = a_{1,0} = 1$ by definition. Assume it is verified from 2 up to $s - 1$. Therefore,

$$a_{1,s-1} = \frac{4}{s-1} \sum_{j=1}^{s-1} ja_{1,j-1}a_{1,s-j-1},$$

and

$$b_s = \frac{4}{s} \sum_{j=1}^{s-1} ja_{1,j-1}a_{1,s-j-1} + \frac{1}{s-1} a_{1,s-1} = \frac{s-1}{s} a_{1,s-1} + \frac{1}{s} a_{1,s-1} = a_{1,s-1},$$

since $\frac{4}{s-1} \sum_{j=1}^{s-1} ja_{1,j-1}a_{1,s-j-1} = a_{1,s-1}$, in accordance with (2.7.15) and the induction hypothesis. This proves the first item.

(ii) follows straightforward from the definition of $a_{l,k}$ and from (i). Setting $l = 1$ in (2.7.15),

$$a_{1,k} = \frac{4}{k} \sum_{j=1}^{k} j b_j a_{1,k-j} \geq \frac{4}{k} (b_1 a_{1,k-1} + k b_k a_{1,0}) = \frac{4}{k} \frac{1}{k} a_{1,k-1}$$

(all the quantities $b_j$ and $a_{1,k-j}$ are positive, $b_1 = a_{1,0} = 1$ by definition and $b_k = a_{1,k-1}$ by (i)).

(iii) For $s = 0$ it is satisfied for all $l \geq 1$: $a_{l,0} = a_{l-1,0} = 1$ by definition. If it should work also for all $1 \leq k \leq s - 1$ and $l \geq 1$; then, working with (2.7.15),

$$a_{l,s} = \frac{4}{s} \sum_{j=1}^{s} j b_j a_{l,s-j} = \frac{4}{s} \sum_{j=1}^{s} j b_j a_{l-1,s-j} = a_{l-1,s},$$

because, by the induction hypothesis $a_{l,s-1} = a_{l-1,s-1}$, $a_{l,s-2} = a_{l-1,s-2}$, \ldots, $a_{l,1} = a_{l-1,1}$ and, trivially, $a_{l,0} = a_{l-1,0}$.

(iv) The inequality works for $k = 2$, since, setting $k = 2$ in (ii),

$$a_{1,1} \geq \frac{4}{1} \frac{1}{1} a_{1,0} = 8 \geq \sum_{j=2}^{2} j a_{j,2-j} = 2a_{2,0} = 2.$$ 

Assume the inequality is also valid from 2 up to $k$. Then,

$$a_{1,k-1} + a_{1,k-1} \geq \sum_{j=2}^{k} j a_{j,k-j} + a_{1,k-1},$$

which implies,

$$2a_{1,k-1} \geq \sum_{j=1}^{k} j a_{j,k-j};$$
and applying now the results of the second and third items:

\[
a_{1,k} \geq 4^{k+1} \frac{k+1}{k} a_{1,k-1} \geq 2^{k+1} \frac{k+1}{k} \sum_{j=1}^{k} j a_{j,k-j} = 2^{k+1} \frac{k+1}{k} \sum_{j=1}^{k} j a_{j+1,k-j}
\]

\[
= 2^{k+1} \frac{k+1}{k} \sum_{s=2}^{k+1} (s-1)a_{s,k+1-s},
\]

(where a shift \(s = j + 1\) in the summation index has been introduced in the last step). But it is easy to check that \(2^{k+1} \frac{k+1}{k} (s-1) \geq s\), when \(s \geq 2\). This ends the induction and proves item (iv).

(v) is satisfied trivially for \(k = 1\), as \(b_1 = 1\). Let us suppose it works for \(k\) and inquire what happens for \(k + 1\). By (i), \(b_{k+1} = a_{1,k}\) and then, by definition (2.7.15):

\[
b_{k+1} = 4^{k} \frac{k}{k} \sum_{j=1}^{k} j b_{j} a_{1,k-j} = 4^{k} \frac{k}{k} \sum_{j=1}^{k} j b_{j} b_{k-j+1}
\]

\[
\leq 4^{k} \frac{k}{k} \sum_{j=1}^{k} j 4^{j-1} 4^{k-j} (k-j+1)! = 4^{k} \sum_{j=1}^{k} j! (k-j+1)!,
\]

but it happens that the sum in the las term is not greater than \((k+1)!\). Again, we proceed by induction: it is checked for \(k = 1\):

\[
\sum_{j=1}^{1} j! (k-j+1)! = 1 = (1+1)! = 2.
\]

Let us assume this works for \(k\) and check its validity for \(k + 1\):

\[
\sum_{j=1}^{k+1} j j! (k-j+2)! = \frac{k+1}{k} \sum_{j=1}^{k} j j! (k-j+2)! + (k+1)!
\]

\[
\leq \sum_{j=1}^{k} j j! (k-j+1)! (k-j+2) + (k+1)!
\]

\[
\leq (k+1) \sum_{j=1}^{k} j j! (k-j+1)! + (k+1)!
\]

\[
\leq (k+1)! (k+1)! + (k+1)!
\]

\[
= (k+2)!,
\]

and where, besides the induction hypothesis at the third inequality; we have used the fact that \(k-j+2 \leq k+1\) for all \(j \geq 1\) (and hence for all summation indices at the second term of the r. h. s. of the expression above). This last argument completes the proof. \(\square\)

Next, from \(a_{l,k}\) and \(b_{k}\) we introduce \(\tilde{\theta}_{l,k}\) and \(\tilde{\eta}_{k}\) by the definitions:

\[
\tilde{\theta}_{l,k} = \gamma_{2} k^{2} a_{l,k}, \quad (2.7.17)
\]

\[
\tilde{\eta}_{k} = \gamma_{2}^{k-1} k^{2} b_{k}, \quad (2.7.18)
\]
with $\gamma_2 \geq 1$ (in our case $\gamma_2 = 17e/2$), and the formulas (2.7.15), (2.7.16) for $a_{l,k}$ and $b_k$ respectively, give rise to recursive inequalities for $\tilde{\theta}_{l,k}$ and $\tilde{\eta}_k$. Immediately,

$$\tilde{\theta}_{l,k} = \gamma_2^{k-2} a_{l,k}$$

$$= \frac{4\gamma_2}{k} \sum_{j=1}^{k} j \gamma_2^{-1} j^{2j} b_j \gamma_2^{-k} \gamma_2^{-2} 2^{2j(k-j)} \geq \frac{4\gamma_2}{k} \sum_{j=1}^{k} j \tilde{\eta}_j \tilde{\theta}_{l,k-j} \geq \frac{\gamma_2}{2} \sum_{j=1}^{k} j \tilde{\eta}_j \tilde{\theta}_{l,k-j}.$$  

Here, apart from the definitions of $\tilde{\theta}_{l,k-j}$ and $\tilde{\eta}_j$, it has been used also that $2^{2j(k-j)} \geq 1$, for all $1 \leq j \leq k$. On the other hand,

$$\tilde{\eta}_k = \gamma_2^{k-1} 2 \gamma_2 b_k$$

$$= \frac{4\gamma_2}{k} \sum_{j=1}^{k-1} j \gamma_2^{-1} j^{2j} b_j \gamma_2^{-k} \gamma_2^{-2} 2^{2j(k-j)} + \frac{1}{k} \gamma_2^{k-1} 2 \gamma_2 b_k$$

$$\geq \frac{4\gamma_2}{k} \sum_{j=1}^{k} j \gamma_2^{(k-j+2)/2} \gamma_2 \tilde{\eta}_k-j + \frac{1}{k} \sum_{j=1}^{k} j \tilde{\theta}_{j,k-j}.$$  

**Remark 2.26.** $2a_{1,k-1} \geq \sum_{j=1}^{k} ja_{j,k-j}$ (see the proof of lemma 2.25), and $4j(k-j) \geq k-j + 2 \Leftrightarrow (4j-1)(k-j) \geq 2$, for all $1 \leq j \leq k-1$ and for all $k \geq 2$, so $2^{2j(k-j)} \geq 2^{(k-j+2)/2}$, for the same values of $j$ and $k$. From here:

$$\gamma_2^{k-2} 2 a_{1,k-1} \geq \sum_{j=1}^{k} j \gamma_2^{k-j} 2^{(k-j)} a_{j,k-j} = \sum_{j=1}^{k} j \tilde{\theta}_{j,k-j},$$

since $k^2-1 = (k-j+1)^2-1 = (k-j)^2 + j^2 + 2j(k-j) - 1 \geq (k-j)^2$, for all $1 \leq j \leq k$ and for all $k \geq 1$ and $\gamma_2^{k-1} \geq \gamma_2^{k-j}$ (because $\gamma_2 = 17e/2 \geq 1$ and $k-1 \geq k-j$ for all $1 \leq j \leq k$). In particular, it is true that

$$\tilde{\theta}_{l,k} \geq \gamma_2 \sum_{j=1}^{k} j \tilde{\eta}_j \tilde{\theta}_{l,k-j}, \tag{2.7.19}$$

$$\tilde{\eta}_k \geq \frac{\gamma_2}{k} \sum_{j=1}^{k-1} j \gamma_2^{(k-j+2)/2} \tilde{\eta}_k-j + \frac{1}{k} \sum_{j=1}^{k} j \tilde{\theta}_{j,k-j}. \tag{2.7.20}$$

As $\tilde{\theta}_{l,0} = 1 = \theta_{l,0}$ for all $l \geq 0$, and $\tilde{\eta}_1 = 2 \geq \eta_1 = 1$, induction shows that $\tilde{\theta}_{l,k} \geq \theta_{l,k}$, for all $l,k \geq 0$ and $\tilde{\eta}_k \geq \eta_k$, for all $k \geq 1$. Indeed, let us suppose it is true for $1 \leq \nu \leq k-1$ and for all $l \geq 0$ and show that therefore it is accomplished also for $k$. Directly form (2.7.20):

$$\tilde{\eta}_k \geq \gamma_2 \frac{1}{k} \sum_{j=1}^{k-1} j \gamma_2^{(k-j+2)/2} \eta_{j,k-j} + \frac{1}{k} \sum_{j=1}^{k} j \tilde{\theta}_{j,k-j}$$

$$= \eta_k,$$

(applying the induction hypothesis term by term), so with the same arguments,

$$\tilde{\theta}_{l,k} \geq \gamma_2 \frac{1}{k} \sum_{j=1}^{k} j \tilde{\eta}_j \tilde{\theta}_{l,k-j} = \theta_{l,k}.$$  

Hence, $\eta_k \leq \tilde{\eta}_k = \gamma_2^{k-1} 2 \gamma_2 b_k$ and using item (v) of lemma 2.25, we state
Lemma 2.27. For all \( k \geq 1 \), the terms \( \eta_k \) defined by the recursive formulas of proposition 2.22, are bounded by

\[
\eta_k \leq \gamma_k^{2^{-1}} 2^k k!,
\]

for all \( k \geq 1 \).

With the above estimates on the coefficients \( \eta_k \), if we define

\[
\tilde{\beta}_2 = 1, \\
\tilde{\beta}_k = 4^{k-2}(k-2)\alpha_k, \quad k = 3, \ldots, r;
\]

with \( \alpha_k \) given by (2.3.20), then

\[
\tilde{\beta}_s = c_3 4^{s-2}(s-2) \left( \frac{s + 1}{\gamma_1} \right)^{s+1} \left( \frac{\tau(s+1)}{\delta e} \right)^{\tau(s+1)},
\]

with \( \gamma_1 = \min\{1, \gamma\} \),

(2.7.21)

and using \( (s + 1)^{(\tau + 2)(s + 1)} \leq \exp \left( 12\tau \int_{s-1}^{s} x \ln x \, dx \right) \) for \( s \geq 3 \), we realize that

\[
\tilde{\beta}_s \leq \beta_s, \quad s = 3, \ldots, r,
\]

with \( \beta_s \) defined by (2.6.31), and the constants

\[
c_3 = \sqrt{2\pi e^2}, \quad c_4 = \frac{64\tau}{\gamma_1 e}, \quad \gamma_1 = \min\{1, \gamma\},
\]

(2.7.22)

(2.7.23)

(2.7.24)

(2.7.25)

(2.7.26)

Proposition 2.28. The terms \( G_3, \ldots, G_r \) of the generating function \( G = \sum_{s=3}^{r} G_s \) satisfy the bounds of lemma 2.13, i. e.,

\[
|G_3|_{\rho_3 + \delta_3, R_3 \exp(\delta_3)} \leq \beta_3 b \left( \frac{R}{R_0} \right)^3,
\]

\[
|G_\nu|_{\rho_\nu + \delta_\nu, R_\nu \exp(\delta_\nu)} \leq \beta_3 \beta_4 \cdots \beta_\nu \frac{a^{\nu-3}b}{\delta_3^{2(\nu-3)}} \left( \frac{R}{R_0} \right)^\nu, \quad 3 < \nu \leq r,
\]

where the quantities \( \{\beta_s\}_{2 \leq s \leq r} \) are given by

\[
\beta_2 = 1, \\
\beta_s = c_3 \left( \frac{c_4}{\rho} \right)^{\tau(s+1)} \exp \left( 12\tau \int_{s-1}^{s} x \ln x \, dx \right), \quad s = 3, \ldots, r;
\]

with

\[
c_3 = \sqrt{2\pi e^2}, \quad c_4 = \frac{64\tau}{\gamma_1 e}, \quad \gamma_1 = \min\{1, \gamma\},
\]

(2.7.21)

(2.7.22)

(2.7.23)

(2.7.24)

(2.7.25)

(2.7.26)
Proof. For $G_3$:

$$|G_3|_{\rho_3 + \delta_3, R_3 \exp(\delta_3)} \leq \alpha_3 c \left( \frac{R}{R_0} \right)^3 = 2\alpha_3 c \left( \frac{R}{R_0} \right)^3 \leq 4\alpha_3 c \left( \frac{R}{R_0} \right)^3 = \beta_3 b \left( \frac{R}{R_0} \right)^3 \leq \beta_3 b \left( \frac{R}{R_0} \right)^3.$$ 

And, for $G_\nu$, $3 < \nu \leq r$, using (2.7.11) and (2.7.21),

$$|G_\nu|_{\rho_\nu + \delta_\nu, R_\nu \exp(\delta_\nu)} \leq \frac{\alpha_3 \cdots \alpha_\nu \nu^{-2}}{\delta_3^{2(\nu-3)} R_\nu^{2(\nu-3)}} \frac{\gamma_2 \nu^{-3} 4^{\nu-3} (\nu-2)! 2^{(\nu-2)^2}}{\nu} \frac{R}{R_0} \nu^{2} \frac{R}{R_0} \nu^{3} \left( \frac{R}{R_0} \right)^\nu = \frac{\beta_3 \cdots \beta_\nu}{\delta_3^{2(\nu-3)}} R_{\nu}^{\nu-3} \left( \frac{R}{R_0} \right)^\nu \leq \frac{\beta_3 \cdots \beta_\nu}{\delta_3^{2(\nu-3)}} R_{\nu}^{\nu-3} \left( \frac{R}{R_0} \right)^\nu,$$

where we have made use of the relations:

$$2^{2\nu-6} \gamma_2 (\nu-2)^{2} = \frac{1}{2} 2^{\nu-3} \nu^{-2} 2 (\nu-2)^{2} = \frac{1}{2} 2^{\nu-3} \nu^{-2} (\nu-1),$$

$$4^1 4^2 \cdots 4^{\nu-2} = 4^{1+2+\cdots+\nu-2} = 4^{(\nu-2)(\nu-1)/2} = 2^{(\nu-1)(\nu-2)},$$

to pass from the first to the second term on the r. h. s. in the inequality above.

Therefore, the results of this section justify the use of such bounds for the components $G_\nu$ of the generating function. The theorem below summarizes the quantitative study of the reduced (or normalized, up to some optimal order $r_{opt}$) Hamiltonian, i. e.:

$$H(\theta_1, q, I_1, p) = Z^{(r_{opt})}(q, I_1, p) + \mathcal{R}^{(r_{opt})}(\theta_1, q, I_1, p), \quad (2.7.27)$$

(see theorem 1.24 in the previous chapter). Where $Z^{(r_{opt})} = \sum_{s=2}^{r_{opt}} Z_s$ was the normal form itself while $\mathcal{R}^{(r_{opt})}(\theta_1, q, I_1, p)$ is now the remainder of the reduced Hamiltonian. In other words, now we fix the “generic” function $f$ to be the linear normalized and complexified Hamiltonian (see (1.6.2) in section 1.6 of chapter 1).

**Theorem 2.29.** With the hypothesis and assumptions in theorem 1.24, if the normalization of the linear reduced (and complexified) Hamiltonian defined in $\mathcal{D}(p_0, R_0)$ is carried out up to the order $r_{opt} = \lceil \tilde{r}(R) \rceil$, with $0 < R < R_0$ small enough and $\tilde{r}(R)$ given by formula (2.6.21) (proposition 2.19); then, the normalized Hamiltonian (2.7.27) is defined in $\mathcal{D}(7\rho/8, R \exp(-\rho/8))$ (with $0 < \rho < \rho_0$) and the following bounds for the Remainder

$$|||\mathcal{R}^{(r_{opt})}||| \leq c_8 \left( 1 - \frac{R}{R_0} \right)^{-1} \left( \frac{R}{R_0} \right)^{r_{opt}(R)/2}, \quad (2.7.28)$$
and for the sum $Z_{\ge 6}^{(r_{\text{opt}})} \equiv \sum_{s=6}^{r_{\text{opt}}} Z_s$,
\[ ||| Z_{\ge 6}^{(r_{\text{opt}})} ||| \le c_9 \left( \frac{R}{R_0} \right)^6, \tag{2.7.29} \]
hold for $0 < R < R_0$ small enough. Here $c_5, c_8, c_9$ are constants depending on $\rho, \tau, \gamma, R_0$ but not on $R$ and $||| \cdot |||$ is the norm stated by (2.6.2), i. e.: $||| \cdot ||| \equiv | \cdot |_{7\rho/8, R \exp(-\rho/8)}$.

Proof. All this theorem, but the last bound (2.7.29) for the sum $Z_6 + \cdots + Z_{r_{\text{opt}}}$, can be derived from the proposition 2.19, letting $f = H$, the linear normalized and complexified Hamiltonian (1.6.2). To derive (2.7.29), we shall consider the estimate:
\[ ||| Z_6 ||| \le \tilde{c}_6 \left( \frac{R}{R_0} \right)^6, \tag{2.7.30} \]
with $\tilde{c}_6$ independent on $R$. Here, we recall that $Z_6$ is an homogeneous polynomial of degree six (we mean, adapted degree, as defined by (1.7.3) in section 1.7 of chapter 1) and the previous terms in the normal form do not change when this is carried out up to some higher order). Moreover, using (2.7.3) and (2.7.7) together with the inequality (2.7.30) given above, it is clear that the sum $Z_{\ge 6}^{(r_{\text{opt}})}$ can be bounded as,
\[ ||| Z_{\ge 6}^{(r_{\text{opt}})} ||| \le \left\{ \tilde{c}_6 + \sum_{j=7}^{r_{\text{opt}}} \tilde{Z}_j \right\} \left( \frac{R}{R_0} \right)^6, \tag{2.7.31} \]
being
\[ \tilde{Z}_j = 2^{\frac{j+1}{2}} c^{j-1} \tilde{\eta}_j^{-1} \frac{\alpha_3 \alpha_4 \cdots \alpha_j}{\delta^{(j-2)}_{r_{\text{opt}}} R_0^{j-2}} \left( \frac{R}{R_0} \right)^{j-5}, \quad j = 7, \ldots, r_{\text{opt}}-1, \]
with $\tilde{\eta}_j = \gamma_j^{j-1} 2^j 4^{j-1} j! \ge \eta_j$ (see lemma 2.27). But the ratio between any term in and its previous one in the sum on the right hand side are computed to be:
\[ \frac{\tilde{Z}_{j+1}}{\tilde{Z}_j} = 8\sqrt{2} c \gamma_j^2 \frac{j}{j-1} \frac{1}{4^{j-1} \alpha_{j+1}} \frac{1}{\delta^{(j+1)}_{r_{\text{opt}}} R_0^j} \left( \frac{R}{R_0} \right), \quad \text{for } j = 7, \ldots, r_{\text{opt}}-1. \]
But $(j-1) 4^{j-1} \alpha_{j+1} = \tilde{\beta}_{j+1}$ (see (2.7.22)) and according with (2.7.23), it is $\tilde{\beta}_s \le \beta_s$, for $s = 3, \ldots, r_{\text{opt}}$. Therefore the quotient above may bounded as,
\[ \frac{\tilde{Z}_{j+1}}{\tilde{Z}_j} \le 16 \sqrt{2} \frac{c \gamma_j^2}{g R_0^2} \frac{\beta_{j+1}}{\delta^{(j+1)}_{r_{\text{opt}}} R_0} \left( \frac{R}{R_0} \right) \le \frac{4 \sqrt{2}}{e} \frac{\beta_{j+1}}{\delta^{(j+1)}_{r_{\text{opt}}} R_0} \left( \frac{R}{R_0} \right), \quad \text{for } j = 7, \ldots, r_{\text{opt}}-2, \tag{2.7.32} \]
where, in the last step, we have used that by its definition –equation (2.5.16)– in proposition 2.16, is $g := \max\{1, ed + 2ae\}$, so $g \ge 2ae = \frac{4ae c_2}{R_0^2}$ (we recall that $a = \frac{2c_2}{R_0^2}$ is one of the coefficients exacted in proposition 2.28, see (2.7.26)). Now, by comparison of the last term in the r. h. s. of (2.7.32) with (2.6.25) one may see that the following bounds for the quotients
\[ \frac{\tilde{Z}_{j+1}}{\tilde{Z}_j} \le \frac{4 e c_5}{(j+1)3^2}, \quad 7 \le j \le r_{\text{opt}} - 1 \]
hold if $0 < R < R_0$ is sufficiently small. In particular, as $c_5$ is independent of $R$, this will imply that the sum $\sum_{j=7}^{r_{opt}-1} \tilde{Z}_j$, appearing between the braces on the r. h. s. in (2.7.31), may be bounded by a constant also independent of $R$ if

$$\tilde{Z}_7 = 2^4 c^6 \frac{\eta_7^2}{\delta_{r_{opt}}^2 R_0} \times \frac{R}{R_0}$$

doess. But, with the same arguments than those employed to “control” $\xi_3 \xi$ in section 2.6.1 (see (2.6.27)), one may check that, for $R < R_0$ small enough, $\tilde{Z}_7$ can be made smaller than any prefixed positive constant. Hence, in (2.7.31),

$$\tilde{c}_6 + \sum_{j=7}^{r_{opt}-1} \tilde{Z}_j \leq c_9$$

with $c_9 > 0$ independent of $R$, for $0 < R < R_0$ small enough. This ends the proof of the last statement and thus of the theorem. \qed