[1]

SOME REMARKS ON CARTAN-EILENBERG CATEGORIES

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Abstract. In this note we collect some remarks and examples on Cartan-Eilenberg categories.

Contents

1. Introduction

Cartan-Eilenberg categories were introduced in [GNPR] as an approach to homotopical algebra based on two classes of morphisms in a category $\mathcal{C}$, $\mathcal{S} \subseteq \mathcal{W}$, which in the classical case of categories of complexes are the homotopy equivalences and the quasi-isomorphisms (see §2). We

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applied the Cartan-Eilenberg formalism in three different situations: to obtain general criteria to derive functors, to contextualize Sullivan's minimal models within homotopical algebra, and to prove a far general acyclic models theorem.

We had the opportunity to present the main concepts and results on Cartan-Eilenberg categories at the Advanced School on Homotopy Theory and Algebraic Geometry held in Seville in September 2009, where we presented also some examples and results not appearing in the original paper. In this note we collect some of the remarks and examples presented at the Seville School, those not involved with derived functors. According to the main applications of the Cartan-Eilenberg structures, the results in this note my be packed in three groups:

- Sections 3 and 4 are devoted to Cartan-Eilenberg structures on categories of non-negative complexes of a certain category $\mathcal{A}$, $\mathbf{C}_+(\mathcal{A})$. We give two examples of abelian categories $\mathcal{A}$ without enough projective objects such that $\mathbf{C}_+(\mathcal{A})$ is not a Caratn-Eilenberg category: the Freyd abelian category and the category of quasi-coherent modules on the projective line over a field. In §4 we prove that if $\mathcal{E}$ is an exact category with enough projectives, then $\mathbf{C}_+(\mathcal{E})$ is a Cartan-Eilenberg category. As a consequence there is a Cartan-Eilenberg structure in the category of complexes of filtered objects of an abelian category $\mathcal{A}$. This example is a first step towards the applications of this formalism to the category of filtered differential graded algebras over a filed of characteristic zero and its minimal models, which has been developed by Joana Cirici and will be presented elsewhere.

- Section 5 is devoted to the interpretation of two known results as examples of Sullivan categories, that is, as Cartan-Eilenberg categories with enough minimal models. The first result is about finite topological paces, while the second one reviews Schlessinger's fundamental theorem for deformation functors as a Sullivan structure on the category of such functors.

- In the final section we derive the classical Burdick-Conner-Floyd theorem on the uniqueness of ordinary cohomology as a chain homology after our acyclic models theorem.

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2. Cartan-Eilenberg categories

In this brief section we recall the definition of left Cartan-Eilenberg category and left Sullivan category. We refer to [GNPR] for more details on this subject.

2.1. **Cofibrant objects.** Let $\mathcal{C}$ be a category and $\mathcal{S} \subseteq \mathcal{W}$ two classes of morphisms of $\mathcal{C}$, which we call strong and weak equivalences, respectively. We recall from [GNPR] that an object M

of $\mathcal{C}$ is *cofibrant* if for each weak equivalence $w : X \longrightarrow Y \in \mathcal{W}$, the map

$$\begin{array}{rcl} w_* : \mathcal{C}[\mathcal{S}^{-1}](M,X) & \longrightarrow & \mathcal{C}[\mathcal{S}^{-1}](M,Y), \\ g & \mapsto & w \circ g \end{array}$$

is bijective, where $\mathcal{C}[\mathcal{S}^{-1}]$ is the localization of $\mathcal{C}$ by $\mathcal{S}$.

That is to say, cofibrant objects are defined by a lifting property in $\mathcal{C}[\mathcal{S}^{-1}]$ with respect to weak equivalences: for any solid diagram as

$$\begin{tikzcd} & X \arrow[d, "w"] \\ M \arrow[ur, dotted, "g"] \arrow[r, "f"] & Y \end{tikzcd}$$

with $w \in \mathcal{W}$ and f a morphism of $\mathcal{C}[\mathcal{S}^{-1}]$, there exists a unique morphism g of $\mathcal{C}[\mathcal{S}^{-1}]$ making the triangle commutative.

Definition 2.1.1. A category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a *left Cartan-Eilenberg category* if for each object X of $\mathcal{C}$ there is a cofibrant object M and a morphism $\varepsilon : M \longrightarrow X$ in $\mathcal{C}[\mathcal{S}^{-1}]$ which is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.

We say that (M, ε) is a left cofibrant $\mathcal{S}$-model of X. We also say that a left Cartan-Eilenberg category is a category with strong and weak equivalences with sufficiently many cofibrant objects.

As $\mathcal{S} \subseteq \mathcal{W}$, there is a natural functor $j : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$. If $\mathcal{C}_{cof}$ denotes the full subcategory of cofibrant objects of $\mathcal{C}$, j induces a functor on $j : \mathcal{C}_{cof}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$.

For any subcategory $\mathcal{M}$ of $\mathcal{C}_{cof}$ we denote by $\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}]$ the full subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$ whose objects are of $\mathcal{M}$; we call this category the relative localization of $\mathcal{M}$ by $\mathcal{S}$ in $\mathcal{C}$ (see [GNPR], where it is proved that in some good situations the relative localization coincides with $\mathcal{M}[\mathcal{S}^{-1}]$).

Cartan-Eilenberg categories admit the following characterization in terms of relative localizations (see [GNPR], Theorem 2.3.2).

Proposition 2.1.2. *A category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category if and only if the functor*

$$j : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of categories. □

2.2. **Sullivan categories.** In some left Cartan-Eilenberg categories there is a distinguished subcategory $\mathcal{M}$ of $\mathcal{C}_{cof}$ which gives sufficiently many cofibrant models. This is often the case with minimal models, as for example in Sullivan's theory of commutative differential graded algebras over a field of characteristic zero.

Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. Recall that a cofibrant object M of $\mathcal{C}$ is a *minimal object* if

$$End_{\mathcal{C}}(M) \cap \mathcal{W} = Aut_{\mathcal{C}}(M),$$

that is, any weak equivalence $w : M \longrightarrow M$ of $\mathcal{C}$ is an isomorphism. We say that $(\mathcal{C}, \mathcal{W}, \mathcal{S})$ is a *left Sullivan category* if there are sufficiently many minimal $\mathcal{S}$-models.

Let $\mathcal{C}_{min}$ the full subcategory of $\mathcal{C}$ of minimal objects. Remark that $C_{min}[\mathcal{S}^{-1}] = \mathcal{C}_{min}$, so in general we will have $C_{min}[\mathcal{S}^{-1}] \neq C_{min}[\mathcal{S}^{-1}, \mathcal{C}]$. Left Sulivan categories are the left Cartan-Eilenberg categories with $C_{min}[\mathcal{S}^{-1}, \mathcal{C}] = \mathcal{C}[\mathcal{W}^{-1}]$.

3. Two (non) examples

Let $\mathcal{A}$ be an abelian category and consider the category of (non necessarily bounded) chain complexes $\mathbf{C}(\mathcal{A})$ together with the class of homotopy equivalences $\mathcal{S}$ and the class of quasi-isomorphisms $\mathcal{W}$. If $\mathcal{A}$ is a Grothendieck category with enough projective objects, then the triple $(\mathbf{C}(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category; even more, it has a cofibrantly generated model structure for which all objects are fibrant, see [H] 2.3.11 and [GNPR], Theorem 4.1.2. For the similar result for the category of positive chain complexes $\mathbf{C}_+(\mathcal{A})$ it suffices to assume that $\mathcal{A}$ is an abelian category with enough projectives.

The hypothesis of having sufficiently many projective objects seems natural in order to have cofibrant models for all chain complexes of $\mathcal{A}$. In this section we present two examples of abelian categories without enough projectives such that the corresponding chain categories are not Cartan-Eilenberg; the negative answer is based on different reasons: in the first example the localized category is not locally small, while in the second one it is the geometry of the situation that permits to fulfill the details.

3.1. **Freyd's example.**

This example corresponds to an abelian category introduced by Freyd [F] and recently bringed up in relation with Brown's representability theorem by Casacuberta and Neeman in [CN].

Let I be the class of all ordinals and take $R = \mathbb{Z}[I]$, the polynomial ring freely generated by I. This is a big ring, whose underlying set is not of our set category.

Let $\mathcal{A}$ be the abelian category of (small) R-modules, that is, its objects are abelian groups A together with commuting endomorphisms

$$\varphi_i : A \longrightarrow A, \quad i \in I;$$

and its morphisms are the group homomorphisms compatible with the I-action.

Proposition 3.1.1. *The category of complexes $\mathbf{C}(\mathcal{A})$, with the class $\mathcal{S}$ of homotopy equivalences and the class $\mathcal{W}$ of quasi-isomorphisms is not a left (nor right) Cartan-Eilenberg category.*

Proof. Suppose $(\mathbf{C}(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a (left) Cartan-Eilenberg category. The localization of $\mathbf{C}(\mathcal{A})$ with respect to $\mathcal{S}$ is isomorphic to the homotopy category of complexes $\mathbf{K}(\mathcal{A})$, (see for example, [GNPR], Propositiom 1.3.3). So, if $\simeq$ denotes the homotopy relation between complex morphisms, we deduce from Proposition 2.1.2 an equivalence of categories

$$\mathbf{C}(\mathcal{A})_{cof}/\simeq \longrightarrow \mathbf{D}(\mathcal{A}).$$

Obviously the category $\mathbf{C}(\mathcal{A})_{cof}/\simeq$ has small hom sets, so the derived category has also small homs between two objects.

But Freyd observed that $Ext^1_{\mathcal{A}}(\mathbb{Z},\mathbb{Z}) = \mathbf{D}(\mathcal{A})(\mathbb{Z},\mathbb{Z}[1])$ is a proper class (see [CN], Lemma 1.1), so the derived category $\mathbf{D}(\mathcal{A})$ is not locally small, getting a contradiction. □

3.2. **Quasi-coherent sheaves on $\mathbb{P}^1$.** Let k be a field and consider the projective line $\mathbb{P}^1$ over k. It is well known that the abelian category of quasicoherent sheaves on $\mathbb{P}^1$, denoted by $QCoh(\mathbb{P}^1)$, has not enough projective sheaves, see [H] Exercise III.6.2. Elaborating on this example we obtain:

Proposition 3.2.1. *The category $\mathbf{C}_+(QCoh(\mathbb{P}^1))$, with the classes of homotopy equivalences and quasi-isomorphisms, is not a left CE category.*

Proof. Let us assume that $\mathbf{C}_+(QCoh(\mathbb{P}^1))$ is a left Cartan-Eilenberg category, so that any complex of quasi-coherent sheaves has a cofibrant model. In particular, the structural sheaf $\mathcal{O}$ has a cofibrant model

$$\varepsilon : P_* \longrightarrow \mathcal{O}.$$

Take a closed point $x \in \mathbb{P}^1$ with maximal ideal m_x and consider the exact sequence

$$0 \longrightarrow m_x \longrightarrow \mathcal{O} \longrightarrow k(x) \longrightarrow 0.$$

By composition of ε with the surjective morphism $\mathcal{O} \longrightarrow k(x)$, we obtain a morphism $f : P_* \longrightarrow k(x)$, such that $H_0(f) : H_0(P_*) = \mathcal{O} \longrightarrow k(x)$ is surjective.

Let us suppose for a moment that P_0 is a coherent sheaf. Observe that for any line bundle $\mathcal{L}$, the complex $P_* \otimes \mathcal{L}$ is also cofibrant, since the functor $\otimes\mathcal{L}$ additive, exact and invertible. Moreover, for any such sheaf $\mathcal{L}$ we have $k(x) \otimes \mathcal{L} = k(x)$, consequently, after Serre's theorem, we can assume that P_0 is generated by its global sections.

Taking the tensor product of the exact sequence above by the line bundle $\mathcal{O}(-1)$ we obtain an exact sequence

$$0 \longrightarrow m_x(-1) \longrightarrow \mathcal{O}(-1) \longrightarrow k(x) \longrightarrow 0,$$

that we view as a quasi-isomorphism between the complex formed by the first two terms and $k(x)$. Consider the diagram

$$\begin{tikzcd} & (m_x(-1) \longrightarrow \mathcal{O}(-1)) \arrow[d] \\ P_* \arrow[ur, dotted, "g"] \arrow[r, "f"] & k(x) \end{tikzcd}$$

as P_* is cofibrant, there is a morphism g making the diagram commutative up to homotopy. As $\mathcal{O}(-1)$ has no global sections, we see that the global section morphism associated to g is zero, but the fiber at x of P_0 is genertated by global section, so $g_x = 0$, and we deduce that $H_0(g)_x = 0$.

As g and f are homotopic, $H_0(f)_x = H_0(g)_x = 0$, but $H_0(f)$ is surjective so we arrive at a contradiction.

Finally it remains to see that we can assume that P_0 is coherent. As the morphism f is surjective, there is a local section which is 1 at x, and there is a coherent subsheaf G of P_0 which contains

this section (see Exercise II.5.15 of [H]). But then we can assume that G is generated by global sections and proceed as in the above paragraph. $\square$

4. Exact categories with enough projectives

In this section we prove that the category of non-negative chain complexes of an exact category with enough projectives has a natural structure of left Cartan-Eilenberg category.

4.1. **Exact categories.** Let us recall the definition of exact category: Let $\mathcal{E}$ be an additive category and a class of composable morphisms

$$A' \xrightarrow{i} A \xrightarrow{p} A'' \qquad (\sharp)$$

such that i is the kernel of p and p is the cokernel of i. We say that i is an admissible mono ($\rightarrowtail$) and that p is an admissible epi ($\twoheadrightarrow$).

Suppose that this class of morphisms is closed under isomorphism. With this structure, $\mathcal{E}$ is an *exact category* if it satisfies the following properties:

- (E0) for any A in $\mathcal{E}$, id_A is an admissible mono (resp. admissible epi),
- (E1) the class of admissible monos (resp. admissible epis) is closed by composition,
- (E2) the pushout of an admissible mono always exists and is an admissible mono (resp. the pullback of an admissible epi exists and is an admissible epi), that is, we can complete the solid diagrams

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dotted] \\ A' \arrow[r, tail, dotted] & B' \end{tikzcd} \qquad\qquad \begin{tikzcd} A' \arrow[r, two heads, dotted] \arrow[d, dotted] & B' \arrow[d] \\ A \arrow[r, two heads] & B \end{tikzcd}$$

For an exact category $\mathcal{E}$, the sequences ($\sharp$) are called the exact sequences of $\mathcal{E}$.

Examples 4.1.1. 1. If $\mathcal{A}$ is an additive category, the split exact sequences

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0,$$

define an exact category structure on $\mathcal{A}$.

2. If $\mathcal{A}$ is an abelian category, the short exact sequences of $\mathcal{A}$ define an exact structure.

3. We will end this section with the example of filtered objects of an abelian category.

4.2. **Category of complexes.** Let $\mathcal{E}$ an exact category. As it is an additive category, we can consider the category of non-negative chain complexes $\mathbf{C}_+(\mathcal{E})$ and its homotopy category $\mathbf{K}_+(\mathcal{E})$. We now recall the definition of the derived category $\mathbf{D}_+(\mathcal{E})$, (see [K]).

A complex A_* of $\mathcal{E}$ is *acyclic* if the differentials factorize as

$$\begin{tikzcd} A_{n+1} \arrow[rr, "f"] \arrow[dr, "e"'] & & A_n \\ & Z_{n+1} \arrow[ur, "m"'] & \end{tikzcd},$$

such that the sequences

$$Z_{n+1} \rightarrowtail A_n \twoheadrightarrow Z_n$$

are exact. A morphism of complexes $f : A_* \longrightarrow B_*$ of $\mathcal{E}$ is a *quasi-isomorphism* if its cone $c(f)$ is an acyclic complex.

Denote by $\mathbf{Ac}_+(\mathcal{E})$ the full subcategory of $\mathbf{K}_+(\mathcal{E})$ formed by the acyclic complexes. This subcategory is triangulated (cf. [B]), so we can define the derived category of non-negative complexes of $\mathcal{E}$ as the Verdier quotient

$$\mathbf{D}_+(\mathcal{E}) = \mathbf{K}_+(\mathcal{E})/\mathbf{Ac}_+(\mathcal{E}).$$

For an arbitrary exact category $\mathcal{E}$, the acyclic complexes do not form a thick subcategory of $\mathbf{K}_+(\mathcal{E})$; moreover, a null-homotopic complex is not necessarily acyclic. These possible inconveniences disappear if we assume that $\mathcal{E}$ is idempotent complete, (cf. [K], [B]).

4.3. **Projective objects.** An object P in an exact category $\mathcal{E}$ is projective if it has the usual lifting property with respect to admissible epimorphisms: for any admissible epi $A \twoheadrightarrow A''$ the induced map

$$Hom(P, A) \longrightarrow Hom(P, A'')$$

is surjective. We denote by $\mathcal{P}$ the full subcategory of $\mathcal{E}$ of projective objects.

We say that an exact category $\mathcal{E}$ has enough projective objects if for any object A there is a projective object P and an admissible epimorphism $P \twoheadrightarrow A$.

4.4. **Cartan-Eilenberg structure of $\mathbf{C}_+(\mathcal{E})$.** Let $\mathcal{E}$ be an idempotent complete exact category with enough projectives. Denote by $\mathcal{S}$ the class of homotopy equivalences and by $\mathcal{W}$ the class of quasi-isomorphisms. Since $\mathcal{E}$ is idempotent complete, we have an inclusion $\mathcal{S} \subseteq \mathcal{W}$ ([B] 10.9), so $(\mathbf{C}_+(\mathcal{E}), \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

Proposition 4.4.1. $(\mathbf{C}_+(\mathcal{E}), \mathcal{S}, \mathcal{W})$ *is a left Cartan-Eilenberg category.*

Proof. Let $\mathbf{K}_+(\mathcal{P})$ be the subcategory of complexes in $\mathbf{K}_+(\mathcal{E})$ with projective components. The result follows, as in the classical case of complexes in an abelian category, from the following two statements:

(1) any complex in $\mathcal{P}$ is cofibrant,
(2) there are enough $\mathcal{P}$-complexes, that is, for any complex A, there is a complex P with projective components and a quasi-isomorphism $\varepsilon : P \longrightarrow A$.

(1) follows easily from [Bu] Corollary 12.7, while (2) is the content of [Bu], Theorem 12.8, or [K], Example 12.2 for the dual situation for injective resolutions. For sake of completeness we

indicate the proof of (1), which follows the classical scheme for complexes on abelian categories, (cf. for example [W], 2.2.6 and 2.2.7).

Let P be a non-negative complex with projective components. We will deduce that P is cofibrant after the following three assertions:

Assertion 1: Let S be an acyclic non-negative complex of $\mathcal{E}$. Then,

$$Hom_{\mathbf{K}_+(\mathcal{E})}(P,S)=0.$$

In fact, non-negative projective complexes have a lifting property: if $P\longrightarrow A$ is a morphism such that the composition $P_1\longrightarrow P_0\longrightarrow A$ is zero and $B_*\longrightarrow B$ is an augmented acyclic complex, then any morphism $A\longrightarrow B$ may be lifted, uniquely up to homotopy, to a morphism $P\longrightarrow B_*$. The lifting is constructed inductively from P_0 and also the homotopy between two liftings.

Assertion 2: For any complex $A\in\mathbf{C}_+(\mathcal{E})$ we have

$$Hom_{\mathbf{K}_+(\mathcal{E})}(P,A)=Hom_{\mathbf{D}_+(\mathcal{E})}(P,A).$$

Let $\widetilde{f}:P\longrightarrow A$ be a morphism in the derived category. By the definition of the derived category, $\widetilde{f}$ is represented by morphisms of complexes

$$P\xleftarrow{\ s\ }B\xrightarrow{\ f\ }A$$

with s a quasi-isomorphism. Let $c(s)$ be the cone of s, which is acyclic. From the triangle, in $\mathbf{K}_+(\mathcal{E})$,

$$B\longrightarrow P\longrightarrow c(s)\longrightarrow B[1],$$

we deduce an exact sequence

$$Hom_{\mathbf{K}_+(\mathcal{E})}(P,c(s)[-1])\longrightarrow Hom_{\mathbf{K}_+(\mathcal{E})}(P,B)\longrightarrow Hom_{\mathbf{K}_+(\mathcal{E})}(P,P)\longrightarrow Hom_{\mathbf{K}_+(\mathcal{E})}(P,c(s)).$$

Hence, by Assertion 1, s induces an isomorphism

$$s_*:Hom_{\mathbf{K}_+(\mathcal{E})}(P,B)\cong Hom_{\mathbf{K}_+(\mathcal{E})}(P,P).$$

Now let g be the morphism corresponding to if_P under the composition

$$Hom_{\mathbf{K}_+(\mathcal{E})}(P,P)\xleftarrow{\ \sim\ }Hom_{\mathbf{K}_+(\mathcal{E})}(P,B)\xrightarrow{\ f_*\ }Hom_{\mathbf{K}_+(\mathcal{E})}(P,A).$$

The morphism g of $\mathbf{K}_+(\mathcal{E})$ represents also f in $\mathbf{D}_+(\mathcal{E})$ and, it is easy to prove that it is unique.

Assertion 3: P is cofibrant.

If we have morphisms in $\mathbf{K}_+(\mathcal{E})$,

$$\begin{tikzcd} & A \arrow[d, "w"] \\ P \arrow[ur, dotted, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

with w a quasi-isomorphism, then w induces an isomorphism in $\mathbf{D}_+(\mathcal{E})$, so we can invert it to obtain a morphism $\widetilde{g}:P\longrightarrow A$ in $\mathbf{D}_+(\mathcal{E})$ and, after Assertion 2, we have a unique morphism $g:P\longrightarrow A$ in $\mathbf{K}_+(\mathcal{E})$ over $\widetilde{g}$, which satisfies $wg=f$. $\square$

4.5. **Filtered categories.** Let $\mathcal{A}$ be an abelian category with enough projectives. We denote by $\mathcal{FA}$ the category of filtered objects of $\mathcal{A}$: its objects are pairs (A, W) where A is an object of $\mathcal{A}$ and W is an increasing finite filtration of A, that is, $W_p \subseteq W_{p+1}$, $p \in \mathbb{Z}$, and $W_p = 0$ for $p \ll 0$ and $W_p = A$ for $p \gg 0$. The morphisms are the morphisms of $\mathcal{A}$ compatible with the filtrations.

We define an exact sequence of $\mathcal{FA}$ as a sequence

$$0 \longrightarrow (A', W) \longrightarrow (A, W) \longrightarrow (A'', W) \longrightarrow 0,$$

such that the induced sequences

$$0 \longrightarrow W_p A' \longrightarrow W_p A \longrightarrow W_p A'' \longrightarrow 0,$$

are exact in $\mathcal{A}$, for all $p \in \mathbb{Z}$.

Lemma 4.5.1. *$\mathcal{FA}$ is an exact category with enough projectives. A filtered object (P, W) is projective if and only if the graduated pieces $Gr_p^W P = W_p P / W_{p-1} P$ are projective objects of $\mathcal{A}$, $p \in \mathbb{Z}$.*

Proof. The proof of this lemma is an easy exercise. We only indicate the main features relating projective objects.

Let (P, W) be a filtered object. Observe that $Gr_p^W P = W_p P / W_{p-1} P$ are projective objects of $\mathcal{A}$, $p \in \mathbb{Z}$, if and only if the objects $W_p P$ are projective and the inclusions $W_{p-1} \longrightarrow W_p$ are split, for all $p \in \mathbb{Z}$. Now, given such an object, we can prove that it is projective in $\mathcal{FA}$: if $\pi : A \longrightarrow B$ is a surjection of filtered objects and $f : P \longrightarrow B$ is a morphism, we can lift f to a morphism $g : P \longrightarrow A$ with $f = \pi g$ inductively over each piece $W_p P \cong W_{p-1} P \oplus Gr_p P$.

The existence of enough projectives in $\mathcal{FA}$ is also easy: consider a filtered object (A, W), and suppose that $W_p A = 0$ for $p < 0$. As $\mathcal{A}$ has enough projective objects, there are projectives $W_0 P$ and $Gr_1 P$ in $\mathcal{A}$ and surjections $W_0 P \longrightarrow W_0 A$, $Gr_1 P \longrightarrow Gr_1 A$. Now take $W_1 P = W_0 P \oplus Gr_1 P$, the induced morphism $W_1 P \longrightarrow W_1 A$ is a surjection, and we can proceed inductively. $\square$

We can easily identify the classes of homotopy equivalences and quasi-isomorphism in $\mathbf{C}_+(\mathcal{FA})$:

- the class $\mathcal{S}$ is the class of filtered homotopy equivalences,
- the class $\mathcal{W}$ is the class of filtered quasi-isomorphisms, that is morphisms f such that the induced graduated morphisms $Gr_p f$ are quasi-isomorphisms (equivalently, after the finiteness assumptions made on the filtrations, the morphisms $W_p f$ are quasi-isomorphisms for all p).

Corollary 4.5.2. *Let $\mathcal{S}$ be the class of filtered homotopy equivalences and $\mathcal{W}$ the class of filtered quasi-isomorphisms, then $(\mathbf{C}_+(\mathcal{FA}), \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.*

5. Two examples of Sulivan categories

Let $\mathcal{C}$ be a category with two distinguished classes of morphisms $\mathcal{S} \subseteq \mathcal{W}$. It is clear that if $\mathcal{S} = \mathcal{W}$, then the triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is trivially a left and right Cartan-Eilenberg category. However this extreme case is not without interest if we consider Sullivan's structures given by minimal models.

In this section we reinterpret known results, in two different contexts, as examples of Sullivan categories for which $\mathcal{S} = \mathcal{W}$.

5.1. **Finite topological spaces.** Denote by $\mathbf{Top}_f$ the category of finite topological spaces. For such a space X, the intersection of all open sets containing a given point $x \in X$ is an open set, which we denote by U_x.

With the aid of the open sets U_x we define a pre-order in X by

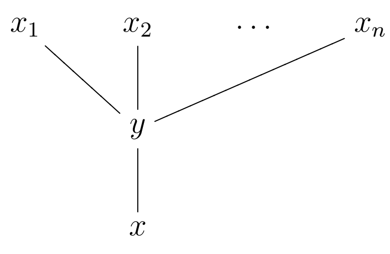
$$x \leq y \Longleftrightarrow U_x \subseteq U_y.$$

Many properties of the topology of X and of the continuous maps in $\mathbf{Top}_f$ may be expressed in terms of this pre-order $\leq$, cf. [M]. For example, the topology of X is T_0 if and only if $\leq$ is a partial order.

Definition 5.1.1. Let X be a finite space and $x \in X$.

(a) x is *upbeat* if there is a $y > x$ such that $z > x \Rightarrow z \geq y$.
(b) x is *downbeat* if there is a $y < x$ such that $z < x \Rightarrow z \leq y$.

The upbeat and downbeat points may be easily visualized if we associate a graph to a finite space as follows: draw a line upwards from x to y if $x < y$, where $<$ is the partial order above. Then, the graph at an upbeat point x looks as:



Proposition 5.1.2. *Let $\mathcal{S}$ be the class of homotopy equivalences in $\mathbf{Top}_f$ and take $\mathcal{W} = \mathcal{S}$. Then $(\mathbf{Top}_f, \mathcal{S})$ is a left Sullivan category, its minimal objects are the T_0 finite spaces without upbeat or downbeat points.*

Proof. The minimality of T_0 spaces without upbeat and downbeat points corresponds to Corollary 6.9 of [M]. As for the existence of sufficiently many minimal models, by Theorem 6.7 of loc. cit. any finite space has a deformation retract which is minimal. We can give an sketch of the construction of the minimal model: given a finite space X, define an equivalence relation by $x \sim y \Leftrightarrow U_x = U_y$ and let X_0 be the quotient (finite) space. Choosing a representative in

each class, one easily realizes X_0 as a subspace of X which is a deformation retract of X. So we can assume that X is a T_0-space. In this case, if x is an upbeat or downbeat point, it is easy to see that $X - \{x\}$ is a deformation retract of X, so the result follows inductively. $\square$

5.2. **Deformation theory: Schlessinger's theorem.** This example is based on Schlessinger's paper [S]. Let $\mathcal{C}$ be the category of local artinian $\mathbb{C}$-algebras, with residue field $\mathbb{C}$, and denote by $\widehat{\mathcal{C}}$ the category of complete local noetherian $\mathbb{C}$-algebras, with residue field $\mathbb{C}$.

We denote by $\mathbf{Cat}_*(\mathcal{C}, \mathbf{Sets})$ the category of covariant functors $F : \mathcal{C} \longrightarrow \mathbf{Sets}$ with $F(\mathbb{C}) = \{*\}$. There is a natural functor

$$\begin{array}{rcl} h : \widehat{\mathcal{C}} & \longrightarrow & \mathbf{Cat}_*(\mathcal{C}, \mathbf{Sets}) \\ R & \mapsto & h_R = Hom_{\widehat{\mathcal{C}}}(R, -). \end{array}$$

Its image defines the subcategory of *prorepresentable* functors.

Given a functor $F : \mathcal{C} \longrightarrow \mathbf{Sets}$, its *tangent space* is defined by

$$t_F = F(\mathbb{C}[\varepsilon]),$$

were $\varepsilon^2 = 0$. In general t_F is only a set, but for the deformation functors introduced below t_F will be a complex vector space. Any morphism $u : F \longrightarrow G$ in $\mathbf{Cat}_*(\mathcal{C}, \mathbf{Sets})$ induces a map between tangent spaces $t_u : t_F \longrightarrow t_G$, that will be linear for deformation functors.

Recall that a morphism $A \longrightarrow B$ is a *simple surjection* if its kernel is a simple A-module, that is, if there is an exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0,$$

with I isomorphic to $\mathbb{C}$.

For the definition of deformation functors we need to consider the following situation: given a functor F and morphisms of $\mathcal{C}$

$$A' \longrightarrow A \longleftarrow A'',$$

consider the induced map

$$\beta : F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'').$$

Definition 5.2.1. The *category of deformation functors* **Def** is the full subcategory category of $\mathbf{Cat}_*(\mathcal{C}, \mathbf{Sets})$ given by the functors satisfying the following properties:

(H1) β is surjective for any simple surjection $A'' \longrightarrow A$,
(H2) β is bijective for $A = \mathbb{C}$, $A'' = \mathbb{C}[\varepsilon]$,
(H3) $\dim_{\mathbb{C}} t_F < \infty$.

Observe that

$$Hom_{\mathcal{C}}(A, \mathbb{C}[\varepsilon]) = Der(A, \mathbb{C}),$$

where $Der(A, \mathbb{C})$ is the vector space of $\mathbb{C}$-valued derivation on A, so $\mathbb{C}[\varepsilon]$ is a vector space object of $\mathcal{C}$ and consequently $F(\mathbb{C}[\varepsilon])$ will be a $\mathbb{C}$-vector space for any F which commutes with the necessary products. Property (H2) guarantees this compatibility for deformation functors, so t_F is a complex vector space and (H3) makes sense.

In deformation theory one is interested in criteria for the prorepresentability of deformation functors. In order to state in which sense prorepresentable functors approximate deformation functors we recall the notion of étale morphism in $\mathcal{C}$, (see [S]).

Definition 5.2.2. A morphism $u : F \longrightarrow G$ in **Def** is said to be

- *unramified* if t_u is injective,
- *smooth* if for any surjection $A \longrightarrow B$ in $\mathcal{C}$, the induced map

$$\eta : F(A) \longrightarrow G(A) \times_{G(B)} F(B)$$

  is surjective,
- *étale* if it is unramified and smooth (in particular, t_u is bijective).

Definition 5.2.3. We say that F has a *prorepresentable hull* if there is an object $R \in Ob\widehat{\mathcal{C}}$ and an étale morphism $h_R \longrightarrow F$.

The main theorem of Schlessinger's paper [S], Theorem 2.11, is

Theorem 5.2.4. *Any deformation functor has a prorepresentable hull.* □

Schlessinger also observed that étale morphisms between prorepresentable functors are isomorphisms, in particular we have:

Lemma 5.2.5. *Let $R \in Ob\,\widehat{\mathcal{C}}$ and $u : h_R \longrightarrow h_R$ an étale morphism, then u is an isomorphism.*

Proof. If u is étale, the differential t_u is an isomorphism and consequently u induces an isomorphism of cotangent spaces, $u : m/m^2 \longrightarrow m/m^2$, were $m \subseteq R$ is the maximal ideal. It easily follows that $u : R \longrightarrow R$ is surjective, and as R is noetherian, it follows that it is an isomorphism. □

We can now resume Schlessinger theorem and the lemma above in our language of Sullivan categories as follows.

Corollary 5.2.6. *Let* **Def** *be the category of deformation functors and consider $\mathcal{S} = \mathcal{W}$ the class of étale morphisms of functors. Then,* $(\mathbf{Def}, \mathcal{S})$ *is a Sullivan category and its minimal models are the prorepresentable functors.*

6. An application to chain homology functors

In this section we prove the classical theorem of Burdick-Conner-Floyd which characterizes the generalized homology theories that come from a chain functor as an application of the models theorem 5.3.2 in [GNPR].

6.1. **Chain homology functors.** We denote by $\mathbf{CW}_f$ the category of finite CW-complexes and by $\mathbf{CW}^2_f$ that of finite CW-pairs. We denote by $\mathbf{C}_+(\mathbb{Z})$ the category of non-negative chain complexes of abelian groups.

Definition 6.1.1. A *chain homology functor* on $\mathbf{CW}_f^2$ is a (covariant) functor

$$L_* : \mathbf{CW}_f^2 \longrightarrow \mathbf{C}_+(\mathbb{Z}),$$

that satisfies the following two properties:

(1) for each CW-pair (X, A) the sequence of complexes
$$0 \longrightarrow L_*(A) \longrightarrow L_*(X) \longrightarrow L_*(X, A) \longrightarrow 0$$
is exact in each degree, (where, as usually, we use the notation $L_*(X) = L_*(X, \emptyset)$).

(2) the homology functors
$$h_n(X, A) := H_n(L_*(X, A)),\ n \in \mathbb{Z},$$
define a generalized homology theory (i.e. they satisfy all Eilenberg-Steenrod axioms except possibly the dimension axiom).

The main example of chain homology functor is given by the complex of singular chains of the CW-pair, $S_*(X, A)$, giving rise to ordinary singular homology.

6.2. **The Burdick-Conner-Floyd theorem.** The following theorem of Burdick-Conner-Floyd shows that the singular chains functor is the unique example of chain homology functor up to coefficients. More precisely:

Theorem 6.2.1. *Let L_* be a chain homology theory and $h_* = h_*(pt)$. Then there is an isomorphism of functors*
$$\tau : S_*(-) \otimes L_*(pt) \longrightarrow L_*(-),$$
in $\mathbf{Cat}(\mathbf{CW}_f^2, \mathbf{C}_+(\mathbb{Z}))[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the class of quasi-isomorphisms. This isomorphism induces, for any finite CW-pair (X, A), natural group isomorphisms

$$h_n(X, A) \cong \bigoplus_{p+q=n} H_p(X, A; h_q), \quad n \in \mathbb{Z}.$$

Proof. We begin by remarking that $K_* = S_* \otimes L_*(pt)$ is a chain homology functor: the exact sequence of a pair of property (1) follows from the exact sequence for singular chains, which being of free abelian groups remains exact after tensoring by $L_*(pt)$; while property (2) is a consequence of the isomorphisms
$$H_n(S_*(X, A) \otimes L_*(pt)) \cong \bigoplus_{p+q=n} H_p(X, A; h_q), \quad n \in \mathbb{Z},$$
which follow from the next lemma.

Lemma 6.2.2. *Over a PID any chain complex is formal.*

Proof of the lemma. Let R be a PID an K a chain complex of R-modules. As K is quasi-isomorphic to a free chain complex, we may assume that K is free. But then, the boundary submodules $B_* \subseteq K_*$ are also free, as R is a PID, so the exact sequences

$$0 \longrightarrow Z_* \longrightarrow K_* \longrightarrow B_{*-1} \longrightarrow 0,$$

split in each degree. We can interpret this as saying that the complex K_* is isomorphic to the cone of the inclusion $i : B_* \longrightarrow Z_*$, but $C(i)$ is formal since the projection $C(i) \longrightarrow Z$ induces a quasi-isomorphism of complexes $C(i) \longrightarrow H(K)$.

Let's continue the proof of the theorem. Since K_* is a homology theory, it is enough to define τ on the category of finite CW-complexes, because property (1) will determine τ over the category of CW-pairs.

We introduce a Cartan-Eilenberg structure on $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$ using a cotriple associated to a set of models on $\mathbf{CW}_f$. For the set of models we take the standard simplexs

$$\mathcal{M} = \{\Delta^m \ : \ m \geq 0\}.$$

The associated cotriple on $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$ is given by

$$GK(X) = \bigoplus_m \bigoplus_{(\Delta^m, X)} K(\Delta^m),$$

with augmentation morphism $\varepsilon(x, \sigma) = K(\sigma)(x)$.

This cotriple is additive and compatible with the summable class of quasi-isomorphisms $\mathcal{W}$, so by [GNPR] Theorem 5.2.2, there is a Cartan-Eilenberg structure on $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$ whose cofibrant objects are the functors $F : \mathbf{CW}_f \longrightarrow \mathbf{C}_+(\mathbb{Z})$ for which the natural augmentation $BF \Rightarrow F$ is a quasi-isomorphism, where BF is the functor associated to F via the standard construction applied to the cotriple G, (see [GNPR]).

The cotriple G does not come from a cotriple on $\mathbf{CW}_f$, it is directly defined on the functor category $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$, so the acyclic models theorem 5.3.2 of [GNPR] does not apply directly. Nevertheless, G induces a functor

$$G : \mathbf{Cat}(\mathcal{M}, \mathbf{C}_+(\mathbb{Z})) \longrightarrow \mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$$

given by

$$GK'(X) = \bigoplus_m \bigoplus_{(\Delta^m, X)} K'(\Delta^m),$$

that we can extend in the usual way to a functor

$$B_* : \mathbf{Cat}(\mathcal{M}, \mathbf{C}_+(\mathbb{Z})) \longrightarrow \mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z})).$$

Consequently, any morphism $\phi : K' \longrightarrow L'$ in $\mathbf{Cat}(\mathcal{M}, \mathbf{C}_+(\mathbb{Z}))$ induces a morphism $B_*(\phi) : B_*(K') \longrightarrow B_*(L')$, so we have a map

$$\lambda : \mathbf{Cat}(K', L') \longrightarrow \mathbf{Cat}(B_*(K'), B_*(L'))$$

and as B_* is compatible with quasi-isomorphisms, the map λ passes to the quasi-isomorphism classes

$$\lambda : [K', L'] \longrightarrow [B_*(K'), B_*(L')].$$

Once we have such a map, then we can follow the proof of [GNPR] to obtain the acyclic models theorems in our situation, that is, if K' is a cofibrant object of $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))$, then λ induces a bijection

$$[K, L] \xrightarrow{\sim} [K_{|\mathcal{M}}, L_{|\mathcal{M}}].$$

Back to our proof, observe that the functor $K = S_* \otimes L_*(pt)$ is cofibrant: we can define a section

$$\theta_X : K(X) \longrightarrow GK(X)$$

by

$$\theta(\sigma \otimes \alpha) = \langle \sigma \rangle \otimes \alpha,$$

where for a singular simplex $\sigma : \Delta^m \longrightarrow X$, we have written $\langle \sigma \rangle = i_\sigma(id_{\Delta^m})$, where $i_\sigma : K(\Delta^m) \longrightarrow GK(\mathcal{X})$ is the inclusion in the corresponding σ-factor.

So, in order to define τ it is enough to define a natural morphism on the models. Denote by $p_m : \Delta^m \longrightarrow \{pt\}$ the projection and consider the following solid diagram

$$\begin{tikzcd}
S(\Delta^m) \otimes L_*(pt) \arrow[r, dotted] \arrow[d, "p_m \otimes id"] & L_*(\Delta^m) \arrow[d, "p_{m*}"] \\
S_*(pt) \otimes L_*(pt) \arrow[r, "w"] & L_*(pt)
\end{tikzcd}$$

where w is the augmentation morphism. By property (2) of the chain homology theories, the vertical arrows are quasi-isomorphisms, so we can complete the diagram with the dotted morphism in $\mathbf{Cat}(\mathbf{CW}_f, \mathbf{C}_+(\mathbb{Z}))[\mathcal{W}^{-1}]$. It is clear that the morphisms above are natural in Δ^m.

It remains to see that τ is an isomorphism. For each $n \geq 0$ and each CW-pair (X, A), τ induces a morphism of abelian groups

$$\tau_* : H_n(S_*(X, A) \otimes L_*(pt)) \longrightarrow h_n(X, A).$$

which is an isomorphism over the point. Hence, by the classical Eilenberg-Steenrod theorem the result follows. $\square$

Remark 6.2.3. It is well known that generalized homology theories can be corepresented by spectra, that is, if h_* is a generalized homology theory there is a CW-spectrum E such that $h_*(-) = \pi_*(E \wedge -)$, cf. [Sw]. Bauer has introduced the *chain functors* that are a weakened

version of the notion of chain homology functors in order to represent all generalized homology theories with complex valued functors, see [B].

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