ALGEBRA AND GEOMETRY IN PIETRO MENGOLI (1625-1686)[1]

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ABSTRACT

One of the most important steps in the research carried out in the seventeenth century into new ways of calculating quadratures was the proposal of algebraic procedures. Pietro Mengoli (1625-1686), probably the most original student of Bonaventura Cavalieri (1598-1647), was one of the scholars who developed algebraic procedures in their mathematical studies.

Algebra and geometry are closely related in Mengoli's works, particularly in *Geometriae Speciosae Elementa* (Bologna, 1659). Mengoli used algebraic procedures to deal with problems of quadratures of figures determined by coordinates which are now represented by $y = K.\ x^m.\ (t-x)^n$. This paper analyses the interrelation between algebra and geometry in the above-mentioned work, showing the complementary nature of the two disciplines, and how their conjunction allowed Mengoli to calculate these quadratures in an innovative way.

L'un des plus grands pas en avant, au XVIIe siècle, dans la recherche de nouvelles méthodes de quadrature fut l'introduction des procédures algébriques.

1 A first version of this work was presented at the University Autònoma of Barcelona on June 26, 1998 for my Doctoral Thesis in the history of sciences.

Pietro Mengoli (1625-1686), probablement le plus intéressant des élèves de Bonaventura Cavalieri (1598-1647), fut l'un de ceux qui développa ce type de procédures dans ses travaux mathématiques.

Algèbre et géométrie sont étroitement liées dans les ouvrages de Mengoli, en particulier dans les *Geometriae Speciosae Elementa* (Bologna, 1659). Mengoli emploie des procédures algébriques pour résoudre des problèmes de quadrature de figures déterminées par des ordonnées que nous noterions par $y = K. x^m. (t-x)^n$. Le but de cet article est d'analyser les rapports entre algèbre et géométrie dans l'ouvrage ci-dessus, de montrer leur complémentarité et d'indiquer comment celle-ci a permis à Mengoli de mettre en oeuvre une nouvelle méthode dans le calcul des quadratures.

MSC 2000 subject classifications 01 A45, 14-03, 40 A25.

Key Words: Mengoli, 17th century, algebra, geometry, quadrature.

INTRODUCTION

One of the greatest innovations in seventeenth century mathematics was the introduction of algebraic procedures to solve geometric problems. Two of the most important advances in mathematics during that century were the foundation of what is now called analytic geometry, and the development of infinitesimal calculus. Both attained their exceptional power by establishing connections between algebraic expressions and figures, and between algebraic operations and geometrical constructions[2].

The publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540-1603) drew attention to these connections. Viète used symbols not only to represent unknown quantities but also to represent known ones. In this way he was able to deal with equations in a completely general form. Viète solved equations creating a link with geometry through the Euclidean theory of proportions; he equated algebraic equations with proportions by means of the product of medians and extremes of one proportion, thus introducing a new way of working out equations[3]. So, as Viète's work became known during the early years of the seventeenth century, some mathematicians began to consider the utility of algebraic procedures in solving geometric problems. Among these scholars was Pierre de Fermat (1601-1665)[4], although the most influential figure in the research on the relationship between algebra and geometry was René Descartes (1596-1650) who wrote *La Géométrie* in 1637[5].

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[2] In the early seventeenth century a tradition had already developed in Italy of using algebra as an "art" to solve equations. The connection between algebra and geometry is present in most Italian algebrists: Leonardo da Pisa (1180-1250), Luca Pacioli (1445-1514), Niccolò Tartaglia (1500-1557), Girolamo Cardano (1501-1576) and Rafael Bombelli (1526-1573), but these algebrists of the "cinquecento" only made geometric demonstrations to justify the solutions of algebraic equations.

[3] On Viète see Viète, 1970, 12, Freguglia, 1999 and Giusti, 1992.

[4] Fermat did not publish during his lifetime and his works circulated in the form of letters and manuscripts. On Fermat see Fermat, 1891-1922, 65-71 and 286-292; Mahoney, 1973, 229-232. However, parts of his work are explained in other publications. For instance Hérigone's course contains an exposition of Fermat's work on tangents, see Hérigone, 1644, 59-69 and Cifoletti, 1980, 129.

[5] The interpretation of Descartes' program causes conflicting opinions even today. On the one hand Bos, Boyer and Lenoir state that, for Descartes, algebra is merely a labor-saving instrument. "For Descartes the equation of a curve was primarily a tool and not a means of definition or representation." (Bos, 1981, 323). Besides, the equation is a tool that permits classification of the curves. For these historians, Descartes' purpose in writing *La Géométrie* was to find a method for solving geometric problems, as was usual at that time, and the equation is not the last step on the way towards the solution. Giusti, on the other hand, says that for Descartes the curve is the equation. Giusti emphasizes the algebraic component of *La Géométrie* as key to Descartes' program. Among the many studies on this program the following are particularly useful: Mancosu, 1996, 62-84; Bos, 2001, 225-412; Giusti, 1987, 409-432.

From that moment on, over a period of roughly a hundred years, mathematics became algebraized[6]. This process marked a change from a mainly geometrical way of thinking to a more algebraic or analytical approach and was implemented in a slow and irregular manner[7]. Not all mathematicians in this period practised algebraic procedures. Some of them accepted these new techniques as an "art" and tried to justify them according to a more "classical" form of mathematics; others disregarded algebra because their research evolved along other paths. Finally, a few accepted these new techniques as a complement to their mathematical procedures[8].

To some extent, Pietro Mengoli (1625-1686)[9], a mathematician from Bologna, student of Bonaventura Cavalieri (1598-1647), can be included in the last of these groups. As we will see, in Mengoli's work *Geometriae Speciosae Elementa* (1659), algebra and geometry are complementary to each other. Mengoli dealt with problems of quadrature of figures using algebraic procedures to obtain his results.

At the beginning of *Geometriae Speciosae Elementa* Mengoli claimed that his geometry was a conjunction of those of Cavalieri and Archimedes, which he already knew, together with the tools that Viete's "specious algebra" offered him,

> Both geometries, the old form of Archimedes and the new form of indivisibles of my tutor, Bonaventura Cavalieri, as well as Viète's algebra, are regarded as pleasurable by the learned. Not through their confusion nor through their mixture, but through their perfect

[6] On this process of algebraization see Bos, 1998, 291-317; Mancosu, 1996, 84-86; Pycior, 1997, 135-166; Panza, 2004,1-30.
[7] An exhaustive analysis of this change in thought can be found in Mahoney, 1980, 141-155.
[8] On this subject see Hoyrup, 1996, 3-4, Massa, 2001, 708-710.
[9] The name of Pietro Mengoli appears in the register of the University of Bologna in the period 1648-1686. He studied with Bonaventura Cavalieri and ultimately succeeded him in the chair of mechanics. He graduated in philosophy in 1650 and three years later in canon and civil law. He took holy orders in 1660 and was prior of the church of Santa Maria Madalena in Bologna until his death. For more biographical information on Mengoli, see Natucci, 1970, 303; Massa, 1998, 9-26; Baroncini-Cavazza, 1986,1.

> conjunction, a somewhat new form [of geometry will arise] – our own – which cannot displease anyone[11].

He established a new form of geometry in order to calculate quadratures. Mengoli knew the value of these quadratures by the method of the indivisibles and he tried to prove these results using a new algebraic approach. This algebraic method was based on the underlying ideas of the method of indivisibles and Archimedes' method of exhaustion, that is to say through a perfect conjunction, not through the confusion or the mixture, as will be explained below.

In Section 1 we will analyse the "specious" language in Mengoli's works and describe his notation and algebraic tools. In Section 2 we will show the relationship between algebra and geometry through his system of coordinates, the figures or "forms", the triangular tables of figures and the calculation of their quadratures.

MENGOLI'S "SPECIOUS" LANGUAGE

In 1655, Mengoli wrote a work in verse dedicated to Queen Christina of Sweden[12], *Via Regia ad Mathematicas per Arithmeticam, Algebram Speciosam, & Planimetriam, ornata Maiestatae Serenissimae D. Christinae Reginae Suecorum* in which he showed her a "royal path" for understanding mathematics. It is divided into three parts: arithmetic, in which he explains operations with numbers; "specious" algebra, where he shows how to use letters to solve equations; and planimetry, in which he deals with plane figures and their properties. It is clear that he assumed algebra as a part of mathematics alongside arithmetic and geometry.

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[11] " Ipsae satis amabiles litterarum cultoribus visae sunt utraque Geometria, Archimedis antiqua, & Indivisibilium nova Bonaventura Cavallerij Praeceptoris mei, necnon & Viettae Algebra: quarum non ex confusione, aut mixione, sed coniuntis perfectionibus, nova quaedam, & propria laboris nostri species, nemini poterit displicere."(Mengoli, 1659, 2-3)

In this work he did not define the terms arithmetic and planimetry, but he did explain "specious algebra" and stressed its usefulness[13]. Indeed his position in regard to algebra differed sharply from that of his master Cavalieri, Torricelli and others in whose works algebraic calculus was deliberately excluded[14]. So, at the beginning of the second part of *Via Regia*, that is, the part devoted to "specious" algebra, Mengoli describes it as an art in the following way:

About the utility of Specious [*Speciosa*] Algebra

> One alone among Mathematics will be called Specious Algebra, an art in which nothing is hidden to the investigator. If one asks if it is or else it is not, [the question involves] telling the truth; if one asks how much it is, this art is enough for that. Now that [this art] gives suitable methods to the general numbers to make and to prove made and proved notions. That is, it will operate with two types of general numbers, those one looks for (unknown quantity) and those one can give arbitrarily (data)[15].

Mengoli considered algebra as an art that provided suitable methods of demonstration for results that are already known. On this definition, here algebra does not seem to be of any use in obtaining new results.

In this work Mengoli adopted Viète's algebraic symbols. He explained that numbers would be represented by letters and presented algebra as a language; metaphorically he compared linguistic and algebraic figures: consonants represented dates; vowels, unknowns; syllables, algebraic expressions of one

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[12] This work for the Queen was commissioned on occasion of her visit to Bologna.

[13] Mengoli thought that the Queen already knew the significance of arithmetic and planimetry, but felt that "specious algebra" was a new part of mathematics that required a supplementary explanation.

[14] Also, in England, Thomas Hobbes (1588-1679) in his *Examinatio et emendatio mathematicae hodiernae* (1660) emphatically condemned the new algebra. In his opinion geometry and its subordinate arithmetic were sciences, whereas algebra which he essentially regarded as symbolic reasoning, was an art able to record the inventions of geometry efficienfly and quickly, but not a science. Isaac Barrow (1630-1677) who also opposed algebra, considered arithmetic as one part of geometry, geometry the only science par excellence and algebra as a tool of logic. On this subject see Pycior, 1997, 135-166.

[15] "De Utilitate Algebra Speciosa. Una, Mathematicas inter, Speciosa vocatur Algebra: quaerenti qua nihil arte latet. Sive rogas, utrum sic, vel non, dicere verum est; sive rogas, quantum est: ars facit ista satis. Utpote quae numeris generalibus

letter; punctuation signs, rules of addition, subtraction...; words, algebraic expressions of several letters; text, equalities; verses, equations. He did not make examples with letters or with numbers of these metaphoric comparisons[16]. His originality lies in this explicit presentation. However, in this first study he did not make contribution to the formation of symbolic language.

We may wonder why Mengoli did these metaphorical comparisons. We think that his aim was didactic, that is to say, the "royal path" shown to the Queen for her to understand mathematics would be the easier way. However Mengoli's view on symbolic language was to be better understood in his later following work where, as it will be explained here-below, Mengoli developed Viète's algebra which allowed him to obtain new results.

Mengoli published *Geometriae Speciosae Elementa,* in 1659, a 472-page text on pure mathematics with six *Elementa,* in which algebra became an essential element. The title already suggests this development of "specious" algebra, which Mengoli named it "Specious Geometry"[17]. Taking Viète's symbolic language as a base, he created new algebraic tools to work with geometric figures that allowed

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instruit aptos, ad facere, ad facta,& dicta probare, modos. Scilicet interetit generalis uterque fuisse; Quem-quaeris numerus, quem-dare cunque potes."[Mengoli, 1655, 19]

[16] For instance Mengoli defined word as an algebraic expression this way: "One word is composed of a certain number of letters, the same number of exponents, only one sign and one multiple. So the character that is produced by the product of letters I have pleasure in calling word". (Mengoli, 1655, 22). Finally, Mengoli made a classification of equations up to the third degree in accordance with the degree and with the signs. Although Viète's classification was more complete there are some coincidences in the words used: *antithesi*, which meant transposition of terms of one equation; *subgraduales*, which referred to the terms with a lesser degree than the equation, etc.

[17] *Geometriae Speciosae Elementa* (1659) has an introduction entitled *Lectori elementario*, which provides an overview of the six *Elementa*, or individually titled chapters, that follow. In the first *Elementum*, *De potestatibus, à radice binomia, et residua* (pp. 1-19), Mengoli shows the first 10 powers of a binomial given with letters for both addition and subtraction, and says that it is possible to extend his result to higher powers. The second, *De innumerabilibus numerosis progressionibus* (pp. 20-94), contains calculations of numerous summations of powers and products of powers in Mengoli's own notation, as well as demonstrations of certain identities. In the third, *De quasi proportionibus* (pp. 95-147), he defines the ratios "quasi zero", "quasi infinity", "quasi equality" and "quasi a number". With these definitions, he constructs a theory of quasi proportions on the basis of the theory of proportions found in the fifth book of Euclid's Elements. The fourth *Elementum*, *De rationibus logarithmicis* (pp. 148-200), provides a complete theory of logarithmical proportions. He constructed a theory of proportions between the ratios in the same manner as Euclid did with the magnitudes in the fifth book of *Elements*. From this new theory in the fifth *Elementum*, *De propriis rationum logarithmis* (pp. 201-347) he found a method of calculation of the logarithm of a ratio and deduced many useful properties between the ratios and their powers. Finally, the sixth *Elementum, De*

him to develop a new method for determining quadratures, as we will show below. We might even think that Mengoli wished to create a new field: the "specious geometry" taking Viète's "specious algebra" as a model. But in our opinion Mengoli had unintentionally created a new part of the new mathematical field, which was beginning to emerge at that time, inspired by the works of Descartes and Fermat.

Mengoli's main algebraic sources were the texts by Viète, Pierre Hérigone (1580-1643)[18] and Jean Beaugrand (1595-1640),[19] as he tells us at the beginning of the book:

> On the other hand as François Viète and other Analysts...; To those symbols that Viète, Hérigone, Beaugrand...[20].

In the second book of his six-volume textbook *Cursus Mathematicus* Hérigone wrote a part of 296- page entitled *Algebra* composed of twenty chapters. He dealt with equations and their solutions, his algebra is clearly inspired en Viète but his notation and presentation was very different[21].

At this point we may wonder whether Mengoli knew Descartes' *Géométrie* or the works of Fermat. He did not cite Descartes as a source; nor does the treatment of

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innumerabilibus quadraturis (pp. 348-392) involves calculating the quadratures of figures determined by coordinates now represented by $y = K.\ x^m.\ (t-x)^n$. An exhaustive analysis of this work can be found in Massa [1998, 1-300].

[18] Hérigone, mathematician, wrote a six-volume textbook entitled *Cursus mathematicus* (Paris, between 1634 and 1644) which contained a book of algebra. On Hérigone's algebra see Hérigone (1644, second and sixth book) and Cifoletti [1990, p. 129].

[19] Beaugrand was also a mathematician; in 1635 he spent an entire year in Italy and visited Cavalieri in Bologna. He published a version of *In Artem analyticem Isagoge,* which was in fact the work of Viète extended with some "scolies" and a mathematical compendium. More references in Cifoletti [1990, pp. 114-128]

[20] "Porrò cum Francisco Viettae, alijsque placuerit Analystis,..."; "Quibus characteribus à Vietta, Herigonio, Beaugrand..."[Mengoli, 1659, 11-12].

[21] Notice that Hérigone distinguished between vulgar algebra, which dealt with numbers and specious algebra, which dealt with species. He defined Algebra in this way: "La doctrine analytique ou l'Algebra est l'art de trouver la grandeur incognue, en la prenant comme si elle estoit cognue,& trouvant l'egalité entre icelle & les grandeurs données". He also defined specious algebra: "Mais l'Algebre Specieuse n'est pas limitée par aucune genre de probleme, & n'est pas moins utile à inventer toutes sortes de theoremes, qu'à trouver les solutions & demonstrations des problemes." (Hérigone, 1644, 1) Also in the sixth book of his *Cursus* Hérigone wrote two parts about algebra, "supplement of algebra"(73 page) and "isagoge of algebra" (74-98). In this supplement Hérigone published Fermat's method of maximum and minimum.(Hérigone, 1644, 59-69).

algebraic symbols throughout this work suggest that he had read him[22]. As for Fermat, his manuscripts and letters circulated among Parisian mathematicians and reached Italy through Beaugrand and Mersenne[23]. So, it is possible that Mengoli knew Fermat's results: Ricci, Torricelli and Cavalieri certainly did. He may also have known Fermat's method of maximum and minimum, which was published by Hérigone in the *Sixth Book* of his *Cursus Mathematicus* (Hérigone, 1644, 59-69). Mengoli did not cite Fermat as a source in his *Geometriae* but conceivably this work could be inspired by a reading of Fermat's method in Hérigone or in Fermat's manuscripts.

a) Mengoli's notation

One of the main difficulties in understanding Mengoli's book concerns the notation; it is original and becomes more complicated as the text progresses[24].

On a separate page, under the title *Explicationes quarundam notarum,* before the first Theorem in the *Elementum primum* of *Geometriae Speciosae Elementa,* Mengoli outlined the basic notation that he would use throughout the book: addition, subtraction, the equals sign, and ratio. He also named all the letters and algebraic expressions that his analysis would involve.

There are certain differences between these signs and those defined by Viète, Descartes and Hérigone. For instance, equality was represented with two points, whereas Viète used the abbreviation of the word *aequalis*, Descartes wrote the

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[22] According to Luigi Pepe, Descartes' *Géométrie* did not reach a wide readership in Italy. Pepe claimed to have found two references, one in Giannantonio Rocca (1607-1659), a pupil of the Jesuit College of Parma, who had the translation of Descartes`*Géométrie* .(Pepe, 1982,263]. Mengoli wanted to square the figures as an answer to a question proposed by Rocca [Mengoli, 1659, 348]. This is the only association with Descartes' *Géométrie* that we have found.

[23] On the difussion of Fermat's works in Italy see Mahoney, 1973, 56.

[24] In a letter to Collins, Isaac Barrow said that Mengoli's style was harder than Arabic (Gregory, 1939,49).

symbol ∝ and Hérigone wrote 2/2. To multiply, Viète used the word *in*, whereas Mengoli, Descartes and Hérigone wrote one letter next to the other. Mengoli used a semicolon to express the ratio between two quantities; Viète, used the expression *ad ,* Descartes *à* and Hérigone wrote the symbol π.

To represent the quantities by symbols Mengoli did not distinguish between vowels and consonants, which could represent data, unknowns or variables. He used both capitals and lower case letters; in general, lower case represented data and capital letters variables. He invented names for the letters and expressions he used. In some cases these names were the same as Viète's, such as the word *radix* (the first power); others, such as *triprimam* (a^3r), *unisextam* ($a\ r^6$), and so forth, are original creations. To represent powers, Viète retained the words *A quadratus, A cubus,* and so on. Descartes wrote the exponents as they are written today, with one exception: on occasion, he wrote *xx* to represent the square. Mengoli neither used words like Viète, nor used the exponents like Descartes, but he wrote the exponents on the right side of the letter, x2, like Hérigone[25]. For instance, to represent one proportion Mengoli (1659, 8) wrote

"*a ; r : a2 ; ar*" for $a : r = a^2 : ar$.

Indeed in the seventeenth century there were no standard criteria either for the symbols or for the names[26].

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[25] On the same page Mengoli also explained how he represented a proportion, a composition of ratios and a power of a ratio. He defined the composition of ratios as a ratio obtained by multiplying the antecedents and the consequents.
[26] On the origins of algebraic language see Malet, A. (1984), 169-179.

b) Algebraic tools

As far as the definitions of the *Elementum primum* are concerned, Mengoli defined the powers of a quantity in continuous proportion to the unity, *u,* like Descartes (Descartes, 1979, 138). When Mengoli used these definitions in the demonstrations, he wrote

$$u : a = a : a^2 = a^2 : a^3 = \ldots$$

In the fourth definition, he introduced the "rationalis", which we can call the unity $u = 1$,

> 4. Quantity, from which the progression of continuously proportional is ordered, in infinity, will be called "Rationalis" and it will be represented by the symbol *u*[28].

Then in the fifth definition, Mengoli introduced the radix *a,* and in the sixth the powers of *a*[29].

> 5. And the first following quantity from "Rationali" will be called Radix or first Power and it will be represented by a letter of the alphabet.
>
> 6. And the following remainders will be called the second, third and so on Powers, in accordance with their order. And any [power] will be represented by the letter of their radix with the number of the order on the right side. For example from radix "a", second power "a^2", third "a^3", and so on[31].

[28] "4. Quantitas, unde progressio continuè proportionalium, ordinatur in infinitum, dicetur, Rationalis.& significabitur charactere *u*"[Mengoli, 1659, 4].

[29] Curiously, though Mengoli never mentioned zero, either as a power or as a number, he defined the order of *u* as one unit less than the first power [Mengoli, 1659, 4].

[31] " 5 . Et prima consequens à rationali, dicetur, Radix, vel Potestas prima.& significabitur, charactere cuiusq; litterae alphabeti. 6. Et reliquae consequentes, dicentur Potestates radicis, Secunda, Tertia, & deinceps, iuxta suum cuiusque ordinem. Et significabitur unaquaeque, eidem litterae suae radicis, adscriptoque ordinis numero. Ut radicis a, secunda potestas a2, tertia a3, & sic deinceps" [Mengoli, 1659, 4].

Mengoli put these quantities in a triangular table, the table "of proportionals" [*proportionalium*], to make their identification easier[32]. The table presents numbers expressed by letters so that in every row the first two elements always have the same ratio *a:r*, *a* and *r* both being integers. They also have the same ratio in the diagonals *1:a* and *1:r*, respectively, because the letter *u* placed in the vertex represents the unity (see Figure 1).

$$
\begin{array}{ccccccc}
 & & & u & & & \\
 & & a & & r & & \\
 & a^2 & & ar & & r^2 & \\
a^3 & & a^2r & & a\,r^2 & & r^3
\end{array}
$$

Tabula Proportionalium[33]

FIGURE 1

Throughout the book the triangular tables were useful algebraic tools for Mengoli's calculations. In the *Elementum primum,* the terms of triangular tables are numbers and they are used to obtain the development of any binomial power. In the *Elementum secundum,* the terms of triangular tables are summations used to obtain the sum of the pth-powers of the first *t-1* integers. Finally, in the *Elementum sextum,* the terms of triangular tables are figures and they are used to obtain the quadrature of these figures. Therefore, Mengoli's originality stems not from the definition of these tables but from his treatment of them. On the one hand, he uses them and Viète's algebra to create other tables with algebraic expressions stating

[32] Mengoli noted its similarity to a table shown in Euclid VII.2. We have not found this table in Euclid's *Elements*, but there is a reference to a similar table in a 13th-century Latin edition of the *Elements* published by Johan Ludvig Heiberg and H. Menge in Bosmans [1924, 22].

[33] He composed this table "of proportionals" with the table of combinatory numbers to obtain a new triangular table. Its elements are the development of the powers of the binomial *a + r* or *a - r*, adding the corresponding signs depending on

clearly their formation laws; on the other hand, he employs the relations between these expressions and the combinatory numbers of the arithmetic triangle to prove the results. It is significant that he used the symmetry of triangular tables and the regularity of their rows in order to generalise the proofs. Mengoli took it for granted that if one result was true for one row of the table, this very result was also true for all rows and there was no need to prove it in the remaining rows. For instance, he proved the development of the powers of the binomial $a+r$, for the second row,

- $u : a = a : a^2 = r : ar = a + r : a^2 + ar.$
- $u : r = r : r^2 = a : ar = a + r : r^2 + ar.$
- $u : a+ r = a + r : a^2 +2 ar + r^2 .$
- $a^2 +2 ar + r^2$ *is a second power of* $a + r$[34].

The arithmetic manipulation of the algebraic expressions helped Mengoli to obtain new results and new procedures. In *Elementum secundum* he invented a manner of writing and calculating finite summations of powers and products of powers. He did not write them giving values or writing the numbers using the sign + and suspension points, but by representing the numbers by letters. In this way he created an innovative and useful construction that would allow him to calculate these summations, which he assumed as new algebraic expressions. He considered an arbitrary number or *tota*, represented by the letter *t*., and divided it into two parts, *a* (abscissa) and $r = t-a$ (residua)[35]. In his words,

wheter the binomial contains an addition or subtraction. He demonstrated these developments in Theorems 8 and 10 of the first *Elementum* (Mengoli, 1659, 15).

[34] It's noteworthy that here Mengoli is using propositions of the theory of proportions of the fifth book of Euclid's *Elements*. (Mengoli, 1659, 16).

[35] Mengoli stated an "arbitrary number" [quantitas utcunque] although here he only puts examples of integers. As we will show later in the quadratures he divided the unit into *t* parts of side $1/t$; ;that is to say, $a = 1/t$,and $r = 1-1/t$.

> The parts of *tota* will be called the separated part [*abscissa*] and the remaining part [*residua*] and the separated part will be represented by the letter *a* and the remainder by *r* [36].

He then took *tota* equal to 2, 3,... and gave examples up to 10. That is to say, if *t* is 2, *a* is 1, and *r* is 1. If *t* is 3, *a* may be 1 or 2 and *r* is then 2 or 1, respectively. He also calculated the squares and cubes of *a*, the products of *a* and *r*, of the squares of *a* and *r*, etc. He then proceeded to add all the numbers *a* that he separated from the same number *t* , for instance if *t* is 3, the summation will be 3, because it is the sum of 1 and 2; if *t* is 4, the summation will be 6, because it is the sum of 1, 2, and 3, and so on. He wrote *O.a*[37] to express this sum from *a*=1 to *a*=*t*-*1*

$$O\,.\,a = \sum_{a=1}^{a=t-1} a$$

Mengoli put all these summations of powers and products of powers in a triangular table which he called the table "of symbols" [*Speciosa*] (See figure 2).

$$\begin{array}{c} O.u \\ O.a \quad O.r \\ O.a^2 \quad O.ar \quad O.r^2 \\ O.a^3 \quad O.a^2r \quad O.ar^2 \quad O.r^3 \end{array}$$

Tabula Speciosa

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[36] "Et partes totae, dicentur, Abscissa, &Residua: &significabitur abscissa, charactere a; & residua, r."[Mengoli, 1659, 21].

FIGURE 2

The terms of this table that he called "species" are summations of the type

O.u = (t -1)

O.a = 1 + 2 + 3+...+(t-1)

O.r = (t-1) + (t-2) + (t-3)+...+1

$O.a^2 = 1^2 + 2^2 + 3^2 + ... + (t-1)^2$

O.ar =1.(t-1) + 2.(t-2) + 3.(t-3) +...+(t-1).1

Mengoli composed his table "of symbols" with the table of combinatory numbers to obtain a new table. Then he used new relations between the terms of these tables to calculate the summations of positive integers and summations of products of powers indefinitely[38]. Specifically in Theorem 22 of *Elementum Secundum* he

proved [39]

$$(m+n+1).\binom{m+n}{n}.\sum_{a=1}^{a=t-1} a^m .(t-a)^n = t^{m+n+1} - (\sum_{s<m+n} t^s)$$

He took advantage of the properties of the combinatory numbers to find and demonstrate the value of the sum of the *p*th powers of the first *t-1* integers using the number *t* as the starting-point for their construction. Mengoli reached this result by using Viète's algebra to express the summations. Algebra allows him to obtain a certain level of generalisation.

[37] Obviously "*O.* " meant *Omnes* and was due to his master Cavalieri and his *Omnes lineae*.
[38] The formula was, in fact, not new. The first recognition as a general rule was apparently made in 1636 by Fermat, who announced that he had solved "what is perhaps the most beautiful problem of all arithmetic"(Fermat; 1891-1922, 69), namely, given an arithmetic progression, to find the sum of any power. Fermat stated the rules but wrote neither the formula nor the demonstration. Mengoli stated the rule, demonstrated it and performed 36 calculations. He ended with the statement: "And in infinity, it can be demonstrated, with the method shown above, that every summation is equal to some *totae*" (Mengoli; 1659, 44).
[39] On this demonstration see Massa, 1997, 266-268.

Another of Mengoli's original contributions was the justification and use of the notion of variable in the *Elementum tertium*. His idea was that letters could represent not only given numbers or unknown quantities, but variables as well: that is, determinable indeterminate quantities. For example, the summations were indeterminate quantities but they were determinate when the value of *t* was known. To clarify this idea Mengoli stated:

> When I write *O.a, ...*you have the summation [*massa]* of all the abscissae: but what value this summation is you do not yet know if I do not write of which number the summation is. But if I assign *O.a* to the summation of the number *t*, you do not know either how much it is if at the same time I do not assign the value of the letter *t*. But when I allow you to fix a value for the letter *t*, and you, using this licence, say that *t* is equal to 5, immediately you will accurately assign *O.a* equal to 10, t^2 equal to 25, t^3 equal to 125, and *O.r* equal to 10, and if the letters *t* are determinate, the quantities *O.a*, *O.r*, t^2 ,t^3 , [which are] determinable [but] indeterminate quantities[40].

Mengoli applied his idea of variable to calculate the "quasi ratios" of these summations. The value of the ratio between summations is also indeterminate but is determinable by increasing the value of *t*. The ratio does not really reach this value, which we can be interpreted as its actual value; instead, it tends towards it as *t* increases. It is in this sense that Mengoli understood the expression "determinable indeterminate ratio".

[40] "Cum scripsero *O.a*... habes massam ex omnibus abscissi: sed quota sic haec massa, nondum habes, nisi scripsero cuius numeri sit massa. Quod si assignavero *O.a*, numeri *t* massam esse; neque sic habes, quota sit, nisi simul assignavero, quotus est numerus, valor litterae *t*...Cum verò licentiam dedero, ut quotum quemque litterae *t* valorem taxes; tuque huiusmodi usus licentia dixeris, *t* valere quinario: statim profecto assignabis & *O.a*, valere 10; & *t*2, valere 25;& *t*3 , valere 125; & *O.r*, valere 10; & determinatae litterae *t*, determinatas esse quantitates *O.a*, *O.r*, *t*2 , *t*3 . Quare data licentia antequam usus fueris, habebas profecto *O.a*, *O.r*, *t*2 ,*t*3 , quantitates indeterminatas determinabiles"[Mengoli, 1659, 61].

Mengoli proceeded to give examples and clarified his notion of "ratio quasi a number", as it will be explained below. From this idea he constructed the theory of "quasi proportions", which was useful to calculate the quadratures.

ALGEBRAIC TREATMENT OF GEOMETRIC FIGURES

Mengoli developed algebraic analysis in geometric figures in the *Elementum sextum* of *Geometriae*[41]. This chapter, entitled *De innumerabilibus quadraturis* involves calculating quadratures of plane figures in the interval (0,t) determined by coordinates now represented by $y = K\, x^m . (t-x)^n$.

In a preliminary calculation, using his master Cavalieri's method of indivisibles[42] Mengoli demonstrated the value of quadratures of these figures[43]. Immediately afterwards he went on to acknowledge that he did not publish the results on account of the attacks levelled against that method:

> Meanwhile I left aside this addition that I had made to the Geometry of Indivisibles, because I was afraid of the authority of those who think false the hypothesis that the infinity of all the lines of a plane figure is the same as the plane figure. I did not publish it not because I agreed with them, but because I was doubtful of it, and I tried...to establish new and secure

[41] This sixth *Elementum*, with the title *De innumerabilibus quadraturis* (pp. 348-392) contains (besides a letter to Cassini), three triangular tables, 36 definitions, 11 propositions (4 of them he named problems) and lastly, two pages on barycenters.

[42] Cavalieri's method of indivisibles is set forth basically in two works: *Geometria indivisibilibus continuorum nova quadam ratione promota* (Bologna, 1635) and *Exercitationes geometricae sex* (Bologna, 1647). The demonstration of quadratures of parabolas $y= x^{m}$ for m any positive integer was published by Cavalieri in this last book. On Cavalieri's indivisibles, see: Andersen, K. (1984/85); Giusti, E. (1980); Malet, A. (1996) and Massa, Mª R. (1994).

43 Interestingly, he did this using a lemma and three quasi-algebraic propositions of Jean Beaugrand and he stated that he would use one algebraic path because the procedure was shorter. These Beaugrand's propositions are found in Cavalieri's *Exercitatione quarta*. In the introduction to this part Cavalieri explained that when he was working on quadratures father Nicerone went to Paris, and Cavalieri told him of his discoveries; Nicerone then passed on the information to Beaugrand. Later Cavalieri learnt of Beaugrand's death, from Mersenne; Mersenne also told him of the solutions that Beaugrand had found to the proposed quadratures. Cavalieri incorporated these solutions so that they should not be lost (Cavalieri, 1647, 243-245).

foundations for the same method of indivisibles or for other methods, which were equivalent[44].

Mengoli recognised that the basis of Cavalieri's method of indivisibles was not sufficiently sound. As he wanted to provide a solid foundation for the method, he started out on a new path to square the figures[45]. He sought to make his procedure for introducing algebra into geometry clear from the beginning. First, using his own system of coordinates, he expressed the figures he wanted to square by algebraic expressions. Second, he put these algebraic expressions in a triangular table for their classification. Third, he associated algebraic expressions to figures by presenting their geometrical construction, and finally he used triangular tables and quasi proportions to find and prove the values of the quadratures.

a) Mengoli's system of coordinates

In the first definitions of *Elementum Sextum* Mengoli described his own system of coordinates. He proposed a line segment, which he named "Rationalis", whose measure is any quantity. He then put this segment in a straight line and named it "Tota".

[44] "Ipsam interim accessionem, quam Geometriae Indivisibilium feceram, praeterivi: veritus eorum authoritatem, qui falsum putant suppositum, omnes rectas figurae planae infinitas, ipsam esse figuram planam: non quasi hanc sequens partem; sed illam quasi non prorsus indubiam debitans: tentandi animo, si possem demum eandem indivisibilium methodum, aut aliam equivalentem novis, & indubijs prorsus constituere fundamentis."[Mengoli, 1659, 364]

[45] As we have noted above after 1650 through the influence of Viète and, above all, Descartes, algebraic methods became increasingly accepted in the field of geometry. Other mathematicians of the period - such as Fermat, Gilles Personne de Roberval (1602-1675), John Wallis (1616-1703), and Blaise Pascal (1623-1662) - also used these methods in a different approach. They aimed, among other things, to calculate the result which today would look like $lim\frac{1^p+...+t^p}{t^{p+1}}=\frac{1}{p+1}$ for *t* tending to infinity. This would have allowed them to square the parabolas $y = x^p$, for *p* any positive integer. Mengoli also calculated this as we will explain, using the theory of quasi proportions and the triangular tables. Informations on these subject may be found in the following sources: on Fermat, 1973,230; on Roberval see Auger, 1962, 18-21 and Walker, 1986, 41-44; on Wallis, 1972, 365-392 and on Pascal see Boyer, 1943, 240 and Pascal, 1954, 171.

> 1. One of the line segments will be taken, of any quantity, which will be called Rationalis. 2. And [one] will be put in a straight line equal to Rationali, which will be called Tota[46].

Next Mengoli defined a base as a straight-line segment the length of which is *t* or one. He used the word abscissa[47] as our x, but in a segment measuring the unit *u* or *t.* Mengoli always worked within a finite base in which the abscissa was represented by the letter "*a*" and the remainder was represented by the letter "*r*= *t*-*a*" or "*1-a*", depending on whether the base was a given value *t* or the unit *u*.

> 3. And a position is given, which will be called Base. 4. And one of the ends [of the base] will be called the end of the abscissae. 5. And the other one the end of the remainders. 6. And the quantity [that goes] from any point of the base to the end of the abscissae, as far as the same base is extended, will be called abscissa[48].

He considered a base AR,

A---------B------------------------R

A is the end of the abscissae, R is the end of the remainders, AB is the abscissa and BR is the remainder.

As for the word ordinate[49], Mengoli first defined the ordinates of known figures, such as the square (or rectangle) and the triangle, from his construction on every point of the base. For instance in the square (or rectangle) he stressed how to draw these lines

---

[46] "1.Assumatur inter lineas, una quaelibet quantitas; quae, Rationalis, dicetur. 2. Et exponatur quaedam recta linea, rationali aequalis; quae dicetur, Tota."[Mengoli, 1659, 367]

[47] The word abscissa appears in Fermat, 1891-1922, 195; in Torricelli, 1919, III, 366, in Cavalieri, 1966, 858-859 and in Stefano Degli Angeli, 1659, 175-179. Another word used with the same meaning was "diameter".

[48] "3. Sitque data positione; quae dicetur, Basis. 4. Eiusque alterum extremorum punctorum, dicetur, Finis abscissarum. 5. Alterum, Finis residuarum. 6.Et ab unoquoque puncto in basi sumpto, usque ad finem abscissarum, quatenus ipsa basis extenditur, quantitas dicetur Abscissa."[Mengoli, 1659, 367]

[49] Mengoli used the word "ordinata" instead of the word "applicata" which was used at that time. Descartes defined the ordinates as "celles qui s'appliquent par ordre", (Descartes,1954, 67). Here there is a note that affirmes: "The equivalent of "ordination application" was used in the fifteenth century on translating Apollonius". The note also states that Hutton's Mathematical Dictionary of 1796 gave "applicata" as the word corresponding to the ordinate and explained that the expression "ordinata applicata" was also used. In fact Fermat and Cavalieri used "applicata". Mengoli in *Circolo* (1672) named them "ordinatamente applicate" (Mengoli, 1672,5).

> 10. Over a base is described a square, and I suppose that from any of the points of the base a straight line will be traced to the opposite side, maintaining itself parallel at all times to the sides of the square; this will be called ordinate in [the] square[50].

He defined the ordinates traced in a triangle made as half of a square.

> 15. The diagonal of the square, traced from the end of the abscissae, makes a half-square triangle. [In] which I suppose that from any of the points of the base a straight line will be traced to the aforementioned diagonal, once again parallel to the sides [of the square]; this [line] will be called ordinate in triangle.[51]

Mengoli did not define the ordinates in the case of mixed-line figures through his constructions, but he explained that they are equal to abscissae or power of abscissae and named them "ordinate in form". The equality between ordinates and the power of abscissae was used by means of proportions such as

$$1 : y = (1 : x)^n$$

b) Geometric figures as algebraic expressions

Mengoli defined the figures that he wanted to square as "extended by their ordinates". He called them "forms" and expressed them by an algebraic expression beginning with FO. He never mentioned the word "curve" - only the word figure or *forma* which dates from the previous century and was identified by measuring the quantity of one quality. The word appears in the work of Oresme (1323-1382) *Tractatus de latitudinibus formarum (1346)* among others[52]. A form was any quantity or quality that was variable in nature. The intensity or latitude was

[50] "10 . Super basi describatur quadratum: & ab uno quolibet puncto in basi sumpto, recta ducatur, usque ad oppositum latus, reliquis lateribus quadrati parallela: quae dicetur, Ordinata in quadrato."[Mengoli, 1659,368]
[51] "15. A fine abscissarum ducta diameter quadrati, facit semiquadratum triangulum: cuius ab unoquolibet puncto in basi sumpto recta ducatur, usque ad praedictam diametrum, alteri lateri parallela, quae dicetur, Ordinata in triangulo. "[Ibid,368]
[52] On Oresme see Clagett, 1968, 91-92; Lindberg, D.C. (ed.), 1978, 231-241; Crombie, A.C., 1980, 82-95.

measured vertically over a base that measured the longitude, and the area of the described figure measured the quantity.

Mengoli began with known figures such as the square and the triangle and then progressed to mixed-line figures. He expressed the square and the triangle algebraically:

> 12. And the square, extended by its ordinates, is called "Form all rationals", and "Form all "totals"" and it will be represented by the characters *FO. u* and *FO. t* [53].
>
> 17. And in the same manner the triangle [made] by its ordinates extended will be called "Forma *omnes abscissae*" [Form all abscissae] and it will be represented by the character *FO.a*[54].

The first mixed-line figure that he defined was determined by one branch of parabola, $y=x^2$, the base and heights (ordinates) being within the square.

> 20. If over the base a figure is constructed, not extended more than by ordinates within the square but in which any ordinate is the "second" abscissa [a^2], it will be called "Form all second abscissae", and it will be represented by the character *FO.* a^2 [55].

When he used this definition in the demonstrations he explained:

The ratio of the base AR [u] to the ordinate by B [y] is "the double" of the ratio AR [u] to AB [x]. [56]

(In modern notation $1 : y = (1 : x)^2$). In the same way, he also defined the "Form all products of the abscissa and the remainder" and the "Form all second remainders", representing them by the characters *FO. ar, FO.* r^2. The ordinates of these figures

---

[53] " 12. Et quadratum, per suas ordinatas extensum, dicetur, Forma omnes rationales, & Forma omnes totae. & significabitur characteribus *FO.u,* & *FO.t."*[Mengoli,1659, 368]

[54] "17. Ipsumque triangulum per suas ordinatas extensum, dicetur, Forma omnes abscissae. & significabitur charactere, FO.a."[Ibid, 368]

[55] "20. Si super basi concipiatur figura extensa non nisi per ordinatas in quadrato: sed in qua, unaquaelibet ordinata est abscissa secunda, dicetur, Forma omnes abscissae secundae. & significabitur charactere FO a2." [Mengoli, 1659, 369]

[56] "Basis AR, ad ordinatam per B, duplicata habet rationem eius, quàm habet ad AB."[Mengoli, 1659, 372].

verify the proportions *1 : y = (1 : x). (1 :(1-x))* and *1: y =* $(1:(1-x))^2$ respectively. And generally he defined the figure extended by any ordinate[57]

> 23. And generalising, if over the base a figure is constructed, not extended more than by ordinates within the square, in which any ordinate is considered as some element of the proportional table [see figure 1]. [This figure] is called "Form all possible proportionals" and an appropriate character will represent it. For instance "Form all third abscissae", $FO.a^3$, "Form all products of the second abscissae and the remainders" *biprimae*, $Fo.\ a^2r$, "Form all products of the abscissa and second remainders", *unisecundae,* $FO.\ ar^2$, "Form all third remainders", $FO.\ r^3$ and so on[58].

c) Triangular tables of figures

After defining the figures and assigning new algebraic expressions he decided to work with these new algebraic objects. Mengoli's approach here was deeply original. He used these new symbols, such as *FO. a.,* which he had associated with geometric figures, to perform algebraic calculations. So, he put these algebraic symbols in triangular tables as he had done in other *Elementa* in order to work with them as if they were arithmetic expressions. Mengoli explained that when these figures [forms] constructed over a base are put in a triangular table as

[57] In a later work, *Circolo* (Bologna, 1672), Mengoli defined the same ordinates as powers of abscissae through other proportions and named them "ordinatamente applicate". "Et altresì sopra la Rationale s'intendano descritte tre figure, una nella quale le ordinatamente applicate alla base sono le terze proportionali della tota, e dell'abscissa, ch'io chiamo Abscisse seconde: l'altra nella quale le ordinatamente applicate alla base sono le quarte proportionali della tota dell'abscissa, e della residua, ch'io chiamo Uniprime: la terza nella quale le ordinatamente applicate alla base sono le terze proportionali della tota e della residua, ch'io chiamo Residue Seconde, ..." [Mengoli, 1672,, 5] Mengoli also defined the ordinates named third abscissae as the fourth proportional of the "tota", the abscissa and the second abscissa. Afterwards he defined the ordinates called the products of the second abscissae and the remainders again as the third proportional, and in this case he stressed that all the ordinates, "in infinity", could be defined in this way.

[58] "23. Et generaliter, si super basi concipiatur figura , extensa non nisi per ordinatas in quadrato; & in qua, unaquaelibet ordinata, est assumpta quaedam in tabula proportionalium: dicetur, Forma omnes tales proportionales aptoque significabitur charactere. Vt Forma omnes abscissae tertiae, FO.a3: Forma omnes biprimae, FO.a2r: Forma omnes unisecundae, FO.ar2: Forma omnes residuae tertiae, FO.r3. & sic deinceps."[Mengoli,1659, 369] Let me stress that Mengoli wrote the exponents in the right side of the letter.

he had done before, they become a new one which he called *Tabula Formosa,* table of "forms" (See figure 3).

FO.u.

FO.a. *FO.r.*

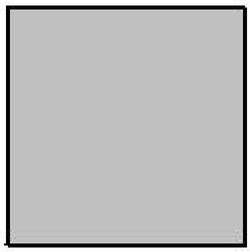
FO.a^2. *FO.ar.* *FO.r*^{2}.

FO.a^3. *FO.a*^{2}*r.* *FO.ar*^{2}. *FO.r*^{3}.

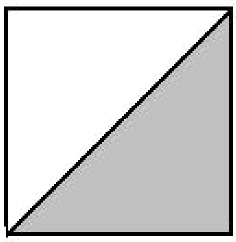
Tabula Formosa

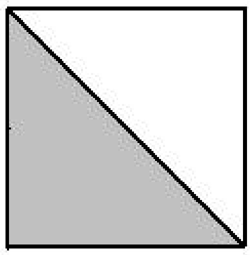
FIGURE 3

The figure at the vertex represented a square of side 1. The two figures of the first row represented two triangles. The first "FO.a" is determined by the bisectrix of the first quadrant $y=x$, the axis of abscissae and the straight line $x=1$, and the second triangle "FO. r" is determined by the straight $y=1-x$ traced from the point (1,0) to the point (0,1) and the axis of abscissae. The three figures of the second row are determined by the ordinates of a parabola, the axis of abscissae and the straight line $x=1$. The first figure, "FO.a$^{2"}$, is determined by the ordinates $y=x^2$, the second, "FO.ar", by the ordinates $y= x.(1-x)$ and the third, "FO.r^2", by the ordinates $y=(1-x)^2$ and so on in the other rows. Below are my designs of these figures arranged as a triangular table (see figure 4).



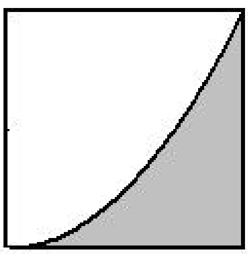
FO. u

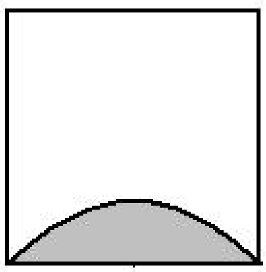


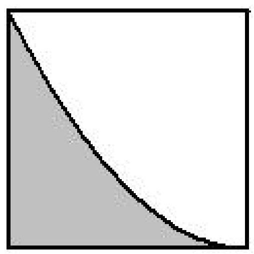


FO. a

FO. r







FO. a^2

FO. ar

FO. r^2

FIGURE 4

From this table "Formosa" [of forms or figures], Mengoli derived a second table multiplying its own elements, term by term, by those in the table of combinatorial numbers. He called this new one *Subquadraturarum,* "of subquadratures". (See figure 5).

1	*FO.u.*
1 1	*FO.a. FO.r.*
1 2 1	$FO.a^2$. $FO.2ar.$ $FO.r^2$.
1 3 3 1	$FO.a^3$. $FO.3a^2r.$ $FO.3ar^2$. $FO.r^3$.

Tabula Subquadraturarum

FIGURA 5

Mengoli called the first row of the triangular tables "of order one", the second "of order two", and so on. He then formed a third table multiplying each of the rows of the previous table by the order of the row plus one: so he multiplied the first row by two, the second one by three and so on. He called this new table *quadraturarum*, "of quadratures" (See figure 6).

	FO.u
First base	*FO.2a .* *FO.2r*
Second base	$FO.3a^2$. $FO.6ar.$ $FO.3r^2$
Third base	$FO.4a^3$. $FO.12a^2r.$ $FO.12ar^2$. $FO.4r^3$

Tabula quadraturarum

FIGURE 6

Mengoli put algebraic expressions of figures in triangular tables in order to classify them and to be able to work with these groups at the same time, as will be explained in the following paragraphs. The figures placed in these triangular tables could thus be infinite in number; it is only necessary to increase the degree of algebraic expression and to calculate the coefficients through the laws of formation of the table. The symmetry of the table and the regularity of its rows allowed Mengoli to generalise the proofs.

d) Representation and geometrical construction of figures

As far as the graphic representation of these figures is concerned, Mengoli only designed a horizontal axis as a base, which he called rational. He did not design a vertical axis, and he always drew the ordinates as lines perpendicular to the base. However, he made few drawings in *Geometriae Speciosae Elementa*[59].

In Mengoli's work the graphic representation of figures was not so much a design but an accurate description of a figure that was informative enough to allow a design to be made of it. Mengoli did not design figures but he made clear that the drawings of these figures could be deduced from their own definitions and their positions in the triangular table. He made three groups of figures and for each group he demonstrated their characteristics in one specific power of algebraic expression. So Mengoli considered the description proved for all the figures of the table due to the table’s symmetry and the regularity of its rows.

In the First Theorem of *Elementum Sextum*, he demonstrated that in all the figures of the lateral of the table *Formosa*, $FO.a^m$ (determined by $y=x^m$), the ordinates increase and the maximum ordinate is found at the end of the base and is equal to it.

[59] In a later work, Circolo (1672), in which he wished to calculate the quadrature of circle, he did not make any drawings.

The demonstration is based on its own definition of the ordinates: that is to say, for n=2, he established the proportion $1 : y = (1 : x)^2$. In order to make this demonstration, he started from the inequality of the abscissae and from there he obtained the inequality of the ordinates, through this same proportion.

Mengoli also demonstrated that all the figures in the opposite lateral of the table, *FO.* r^n , were determined by ordinates that were always decreasing.

As for the figures in the middle of the table, in the Second Theorem he demonstrated that in the figures *FO.* a^m r^n, determined by $y = x^m .(1-x)^n$, the ordinates first increase and then decrease, reaching their maximum value in an abscissa that divides the base AR in the ratio $m : n$. The demonstration is made with the figure *FO.* a^2r^3, where the abscissa B that verifies AB : BR = 2 : 3 has the maximum ordinate, A is the end of the abscissae, R is the end of the remainders and D is any division of the base AR.

A---------------D-----B-----------------R

He proved that the ordinates of the figure increased to this maximum value and then decreased to the ordinate of the end of the base. Using modern notation, the demonstration can be summed up as follows. We know that $u=1$, $a = x$ = abscissa, $r = 1-x$ = residua, and we will denote by Ord B= the ordinate of the abscissa B= y, AR = 1 = base. The following proportions are thus established:

AR : AB = $1 : x$; AR : BR = $1 : 1-x$;

AR : Ord B = 1 : Ord B = $(1 : y) = (1 : x)^2 . (1 : (1-x))^3$

Moreover, taking the abscissa D, x_1= AD any division of the base smaller than x, and using the letter y_1 as the ordinate of this abscissa we realise that

Ord D : AR = Ord D : 1 = $(y_1 : 1) = (x_1 : 1)^2 . ((1-x_1) : 1)^3$

And operating and composing the two proportions, it follows that

$$\text{(Ord D)} : \text{(Ord B)} = (y_1) : (y) = ((x_1)^2 . (1-x_1)^3) : ((x)^2 . (1-x)^3)$$

Now Mengoli proved that the antecedent - Ord D- is smaller than the consequent - Ord B - for any abscissa D, and he was thus able to affirm that the ordinate of the abscissa B is maximum[60].

Let us stress that these descriptions of the design of the figures do not depend on the unit of measure that one takes, but on the type of algebraic expression in accordance with the position in the triangular table[61].

According to Bos (2001,3), in the seventeenth century, a figure was "known" or "given" when one could construct it starting from given elements.[62] So Mengoli had to ensure that all the algebraic expressions in the triangular table, which were new algebraic objects, could be identified with one geometric figure. He enunciated this Proposition Three as a Problem, and demonstrated how to construct the ordinate in a proposed algebraic expression of a figure and a given point. Let us show the demonstration[63]:

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[60] For this demonstration he needed to use some results obtained in the *Elementum quintum* in which he constructed the logarithm of a ratio. Through the property of the logarithm that the product of the power of a ratio and its logarithm is equal to the logarithm of the ratio raised to this exponent, he obtained a relation between the ratios and their powers in certain specific conditions. The proposition that Mengoli used is "Given four quantities, disposed arithmetically, if it is verified that the first to the last is as one number to one number, then the first to the second raised to the number homologous to the first will be bigger than the third to the fourth raised to the number homologous to the fourth. If it is verified that the second to the third is as one number to one number, then the first to the second raised to the number homologous to the second will be smaller than the third to the fourth raised to the number homologous to the third." [Mengoli, 1659, 338] Mengoli applied this theorem to four quantities [segments], which have the same differences and of which two are in a specific ratio to each other. Mengoli named them arithmetical ordinates. So he considered AD, AB, BR, RD and proved that they were arithmetically ordinate quantities since
$AB - AD = x - x_1 = RD - BR = (1 - x_1) - (1 - x) = BD$ and besides $AB : BR = 2 : 3$. He could then apply the theorem and set up the following inequality $(AD : AB)^2 = (x_1 : x)^2 < [(1-x) : (1-x_1)]^3 = (BR : RD)^3$. Multiplying the antecedent of the first ratio by the consequent of the second ratio and vice versa, Mengoli demonstrated that the ordinate by D is smaller than the ordinate by B. For this demonstration Mengoli only used algebraic procedures and the Euclidean theory of proportions.

[61] Mengoli defined the figures like Roberval, Fermat and others throughout the proportion between ordinates and abscissa but he could use the same demonstration for any figure of the same type. Information on Roberval may be found in the following sources: Auger, 1962,18-21; Walker, 1986, 41-44.

[62] Today the geometrical construction of algebraic expressions of figures presents no difficulty, but in Mengoli's time the geometrical construction was a very important issue. On this subject see Bos, 1981 and 2001.

Probl. I. Prop. 3.

Find the ordinate of a proposed figure, of a given point and a given base[64].

Hypothesis

That is, given FO. 10 a^2r^3, over a given base AR, in which a given point B. It is necessary to find the ordinate of B[65].

Construction.[66]

Given AR, and given AB, BR, the recta BC will be found, to which AR is a ratio composed of given ratios AR to AB squared, AR to BR cubed, and of the ratio one tenth: and BC will be put perpendicular to AR. I say BC is the ordinate of B, in *FO.10* a^2r^3 [67].

Demonstration

The ratio AR to BC will be composed of ratios AR to AB squared, AR to BR cubed, and of one tenth; but AR is u; AB, is *a*; BR, is *r*. So the ratio AR to BC will be composed of ratios "*u* to *a"*, squared, "*u* to *r"*, cubed, and of one tenth. But *u* to $10\ a^2r^3$ will be composed of these: then AR to BC is like *u* to $10\ a^2r^3$. But AR is *u*, so BC is $10\ a^2r^3$: then BC is the ordinate of B, in *FO.* $10\ a^2r^3$ [68].

Note here that Mengoli not only worked with proportions of segments but also equated segments with the letters of the triangular table and consequently with the algebraic expression of the figures. He equated the product of segments with the composition of ratios because he knew exactly how to work with the Euclidean theory of proportions. But, unlike Descartes, he did not define an algebra of

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[63] Mengoli here drew one horizontal axis AR and a perpendicular line (not in the middle) with the letter B over the base and the letter C at the top of the perpendicular line.
[64] "Formae propositae, in data basi, per datum punctum, ordinatam invenire."[Mengoli, 1659, 377].
[65] "Esto proposita *FO.10ª2r3*, super data basi AR, in qua datum punctum B. Oportet per B ordinatam invenire."[Mengoli, 1659, 377]
[66] Throughout the book Mengoli presented Theorems and Problems. In this case he wrote the word Construction, as Euclid did, before the demonstration and explained the construction used in it.
[67] "Data AR, datisque AB, BR, inveniatur recta BC, ad quàm AR, rationem habet compositam ex datis rationibus, AR ad AB duplicata, AR ad BR triplicata, & ex ratione subdecupla:& collocetur BC perpendiculariter ad AR. Dico BC, esse ordinatam per B, in *FO.10 a2r3."*[Mengoli, 1659,377]
[68] " Ratio AR ad BC, componitur ex rationibus AR ad AB duplicata, AR ad BR triplicata, & ex subdecupla: sed AR, est *u*; AB est *a*; BR est *r*: Ergo AR ad BC ratio, componitur ex rationibus *u* ad *a* duplicata, *u* ad *r* triplicata, & ex subdecupla: sed ex ijsdem componitur *u* ad 10 *a2r3:* ergo AR ad BC est ut *u* ad *10 a2r3*: sed AR est *u*: ergo BC est *10 a2r3*: ergo BC est ordinata per B, in *FO. 10 a2r3.* Quod&c."[Ibid,378].

segments. Rather, he demonstrated, for a given measure, only how to construct the ordinate in a given algebraic expression of a figure by using the composition of ratios. In this way, he established an isomorphic relation between the new algebraic objects and the geometric figure that allowed him to study figures by their algebraic expressions.

e) Calculation and demonstration of the value of quadratures

Mengoli knew the value of these quadratures through Cavalieri's indivisibles, but he was keen to find another way to demonstrate them. From Viète's symbolic language he created new algebraic expressions and furthermore he made an innovative development of Viète's algebra through the triangular tables and the theory of "quasi proportions". Notice that the Euclidean theory of proportions is very important in this Mengoli's work. He considered Euclid's *Elements* as the book of mathematics by excellence and developed new theories, the theory of "quasi proportions" and the theory of logarithmic ratios taking as a model the Euclidean theory of proportions[69].

Therefore in order to understand this demonstration of the value of quadratures more clearly, let us now consider the basic ideas of this theory of "quasi proportions" used in it. He set up this theory on the notion of "ratio quasi a number", which he clarified thoroughly. He considered values up to 10 in the ratio $O.a$ to t^2; for instance, if $t = 3$, then the ratio $O.a$ to t^2 is 3 to 9; if $t = 4$, then the ratio is 6 to 16; if $t = 5$, then the ratio is 10 to 25; ...if $t = 10$, then the ratio is 45 to

[69] The knowledge of algebraic language granted Mengoli to extend the Euclidean theory of proportions in order to create his new theories. On the importance in Mengoli's work of the Euclidean theory of proportions see Massa, 2003, 472-474.

100. He argued that the ratio takes different values as the value of *t* increases. Moreover, these values are nearer to 1/2 than any other given ratio. So Mengoli called it ratio quasi 1/2. The difference between 1/2 and the ratio, which is determined when the value of *t* increases, is thus smaller than the difference between 1/2 and any other given ratio. The "limit" of this succession of ratios or of this ratio, as far as it is thus determinable, is 1/2, and Mengoli terms this "limit" ratio quasi 1/2. The idea of "ratio quasi a number" suggests, though in an imprecise way, the modern concept of limit[70].

This notion, together with the idea of determinable indeterminate ratio explained above (see page 16), allowed Mengoli to make the definitions of ratio "quasi infinite", "quasi null", "quasi equality" and "quasi a number" in the *Elementum tertium:*

> 1.A determinable indeterminate ratio, which, when determined, can be greater than any given ratio, as far as is thus determinable, will be called quasi infinite[71].
>
> 2.And one that can be smaller than any given ratio, as far as it is thus determinable, will be called quasi null.
>
> 3.And one that can be smaller than any given ratio greater than equality, and greater that any given ratio smaller than equality, as far as it is thus determinable, will be called quasi equality. Or otherwise, that which can be nearer to equality than any given ratio not equal to equality, as far as, it is thus determinable, will be called quasi equality.
>
> 4.And one that can be smaller than any ratio larger than a given ratio, and larger than any ratio smaller than the same given ratio, as far as is thus determinable, will be called quasi equal to this given ratio. Otherwise one that can be nearer to any given ratio than any other

[70] In his *Circolo* of 1672, Mengoli again uses quasi ratios and explains: "Dissi quasi, e volsi dire, che vadino accostandosi ad essere precisamente tali."(Mengoli, 1672, 49).

[71] To clarify the notion of "ratio quasi infinite" Mengoli considered values up to 10 in the ratio *O.a to t* ; for instance, if *t*=4, then the ratio is 6 to 4; if *t*=7 then the ratio is 21 to 7; if *t*=10 then the ratio is 45 to 10. He argued that the ratio takes values

ratio not equal to it, as far as it is thus determinable, will be called quasi equal to the same (given) ratio.

5.And the terms of ratios quasi equal between them will be called quasi proportional.

6.And (the terms) of quasi equality ratios will be called quasi equal[72].

The sixth and final definition, in light especially of the third definition can be read as: "And the terms of ratios that are nearer to equality than any other given ratio other than equality, as far as these ratios are determinable, will be called quasi equal". In the demonstration of the values of quadratures Mengoli used this interpretation of the definition of quasi equality ratio. In fact, he considered a "maior inaequalitas" ratio[73] and proved that he could find a number that allowed him to set up a ratio smaller than the "maior inaequalitas" ratio given.

In *Elementum tertium* Mengoli following these six definitions established ratios between all sorts of summations and the number t. (Recall that these are all constructed using t and that these summations have $t-1$ addends with different exponents). He calculated what these ratios tend toward when the number is very large, obtaining in this way all possible quasi ratios. Specifically, in Theorem 42, Mengoli demonstrated that the summation expressed above (Theorem 22, see

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always greater as the value of t increases, so the ratio is quasi infinite. For the ratio quasi null he considered values up to 10 in the ratio $O.a$ to t^3. (Mengoli, 1659, 64-65).

[72] "1.Ratio indeterminata determinabilis, quae in determinari, potest esse maior, quam data, quaelibet, quatenus ita determinabilis, dicetur, Quasi infinita.2. Et quae potest esse minor, quàm data quaelibet, quatenus ita determinabilis, dicetur, Quasi nulla.3. Et quae potest esse minor, quàm data quaelibet minor inaequalitas; & maior, quàm data quaelibet minor inaequalitas, quatenus ita determinabilis, dicetur, Quasi aequalitas. Vel aliter, quae potest esse propior aequalitati, quàm data quaelibet non aequalitas, quatenus talis, dicetur, Quasi aequalitas. 4. Et quae potest esse minor, quàm data quaelibet non maior, proposita quadam ratione; & maior, quàm data quaelibet minor, propositâ eâdem ratione, quatenus ita determinabilis, dicetur, Quasi eadem ratio. Vel aliter, quae potest esse propior cuidam propositae rationi, quàm data quaelibet alia non eadem, quatenus talis, dicetur, Quasi eadem.5. Et rationum quasi earundem inter se, termini dicentur, Quasi proportionales.6. Et quasi aequalitatum, dicentur, Quasi aequales." (Mengoli, 1659, 97).

[73] The *inaequalitas* of a ratio denotes a number other than unity, and so ratios *minor inaequalitas* and *maior inaequalitas* correspond to numbers smaller and larger than unity, respectively.

page 15) "tends" to t^{m+n+1} when the number of addends increases[74]. This quasi ratio is used in the demonstration of value of the quadratures, as we will explain.

Let us return now to the procedure of Mengoli's demonstration of the value of quadratures of these geometric figures described above and to the use of triangular tables. (See Figure 6).

FO.u

FO.2a . *FO.2r*

FO.3a^2. *FO.6ar.* *FO.3r*^{2}

FO.4a^3. *FO.12a*^{2}*r.* *FO.12ar*^{2}. *FO.4r*^{3}

Tabula quadraturarum

FIGURE 6

The table of the quadratures of figures determined by coordinates now represented by "$y = x^m. (1-x)^n$ " has $(m + n+ 1) . \binom{m+n}{n}$ as coefficients. Notice that ($m + n$) is the order of the row and that this value coincides with the degree of the algebraic expression. Mengoli used the technique of multiplying all the elements of the table by $(m+ n +1)$ and by the corresponding combinatorial number to perform all quadratures of the table at the same time. He knew that the value of these quadratures is the inverse of these products.

$$\int_0^1 x^m .(1-x)^n = \frac{1}{(m+n+1).\binom{m+n}{n}}$$

[74] On this subject see Massa, 1997, 271-275.

So in the table of quadratures (see figure 6) he put these products as the coefficients of the figures which he wanted to square[75]. Thus, all that remained was to prove that the quadratures of these new figures (with these coefficients) were the area of the square of side 1 (if *t=1*). In modern notation, that is to say,

$$\int_0^1 (m+n+1).\binom{m+n}{n}.x^m.(1-x)^n = 1$$

Consider now Mengoli's techniques to prove this value of quadratures. For this demonstration Mengoli used the theory of "quasi proportions" explained above. He established two quasi equality ratios: the first one, between a new figure (the "ascribed" figure) and the figure or form which he wanted to square, and the other one, between this "ascribed" figure and the square of side 1[76]. After establishing these two quasi equality ratios, Mengoli used a theorem that he had previously demonstrated, which showed that given two quasi equality ratios with the same antecedents, therefore the consequent of each ratio is also equal.

For the first quasi equality ratio he used Archimedes' definitions of inscribed and circumscribed figures. The inscribed figure is determined by all the maximum rectangles included in the figure and the circumscribed figure, is determined by all the minimum rectangles including the figure[77]. The ascribed figure is determined by all the rectangles built over the ordinates of the divisions of the base. So, the

[75] Mengoli knew that one factor, the combinatorial number, corresponded to the coefficient of the bynomial developement of $[x + (1\text{-}x)]^{m+n} = [1]^{m+n}$ and the other factor could be found through the relation between the summation of powers and the degree. For instance if we wish to calculate the quadrature of figure *FO.* $X^{25}.(1\text{-}x)^{30}$ it will be necessary to multiply by 56 and by the combinatorial number $\binom{55}{25}$.

[76] For these demonstrations Mengoli used the definitions of *Elementum tertium* of quasi equality.

ascribed figure is determined by *t-1* rectangles when one divides the base in *t* parts.

> 33. The figure composed of just as many rectangles, as there are ordinates through the points of division and of the adjacent to these ordinates, will be called "ascribed" of the form[78].

Mengoli demonstrated that the circumscribed figure is larger than the ascribed one in a rectangular quantity (the area of the rectangle determined by the maximum ordinate and one of the equal parts of the base). He also proved that the ascribed figure is larger than the inscribed figure, but the difference in size is not greater than this rectangular quantity (Proposition 5). Immediately, using the theory of quasi proportions (Proposition 6), Mengoli proved that the circumscribed and inscribed figures of any of these figures in the table of quadratures are "quasi equal". That is to say, he demonstrated that it is possible to find a number of divisions of the base in a way that the ratio between the circumscribed and the inscribed figures can be nearer to equality than any other given ratio not equal to equality[79]. With this result he was already able to affirm that the ascribed figure, determined by rectangles, and the figure or form, determined by ordinates, were quasi equal (Proposition 7)[80].

This demonstration follows Archimedes but uses the quasi ratio's method instead of the reduction to the absurd. Another difference is that in Archimedes the

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[77] The circumscribed and inscribed figures were already known and used for instance by Luca Valerio (1604, 13-14), James Gregory in Malet (1996, 83) , Fermat, Newton and others.

[78] "33. Figura vero ex tot parallelogrammis, quot sunt ordinatae per puncta divisionum, & ad ipsas ordinatas iacentibus composita, dicetur, Adscripta formae." [Mengoli, 1659,.371]

[79] Notice that Mengoli's ascribed, inscribed and circumscribed figures are explicitly determined by a finite number of rectangles.

[80] He used the Proposition 67 of *Elementum quintum* , which established ratios of quasi equality between two magnitudes that are situated between two quasi equals.

figure between the inscribed and circumscribed figures is used directly, but Mengoli used a new figure, determined by a finite number of rectangles, which he named the ascribed figure. By this he meant the figure or form extended by its ordinates like the figure to which the ascribed figure, determined by rectangles, is "approximated" when the number of these rectangles increases. However we should not suppose that for Mengoli the rectangles of the ascribed figure became the ordinates of the figure, because the figure exists independently of the existence of the ascribed figure. Its previous existence is due to the fact that the ascribed figure is determined by the rectangles constructed over the ordinates of the divisions of the base of the figure.

In fact, like Newton in Lemma II of the *Principia* (Newton, 1972, 73-74), Mengoli might have stated that the ratios between the curvilinear, the inscribed and the circumscribed figures are ratios of equality. But it is evident that he needed the ascribed figure to be able to establish ratios with finite terms. For Mengoli the ascribed figure is a tool to clarify the nature of the figure, and furthermore to demonstrate the quasi ratio that gives the value of the quadratures.

As regards the second quasi equality ratio between the ascribed figure and the square of side 1, Mengoli first established a proportion. The proportion is established between the ratio of the square of side 1 to the ascribed figure and the ratio of one power of *t* to a summation of powers.

$$\frac{Square(side1)}{Figure\,ascribed} = \frac{t^{m+n+1}}{\binom{m+n}{n}\sum_{a=1}^{a=t-1} a^m.(t-a)^n}$$

He then applied the theory of quasi proportions to this proportion. That is to say, in one side, he made the number of rectangles infinite and on the other side he made the number of addends infinite. Since he knew the second ratio is a quasi equality ratio by the theory of quasi proportions, then the first ratio between the square and the ascribed figure is also a quasi equality ratio.

Now we will look at the foundation of this demonstration, which establishes the proportion between the ratio of the square of side 1 to the ascribed figure and the ratio of a power of "*t*" to the summation of powers. (See figure 7).

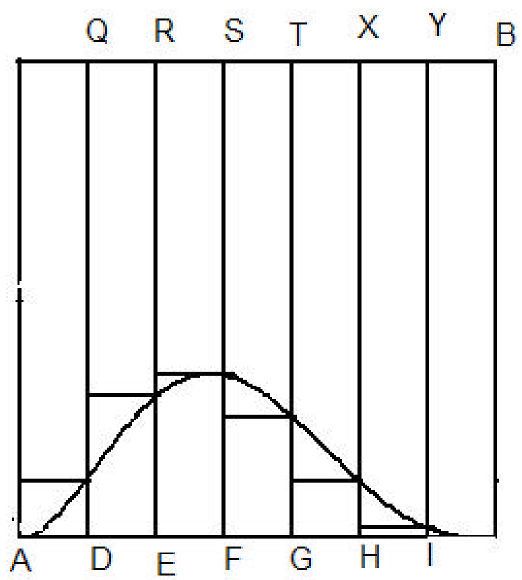

FIGURE 7

Mengoli made the demonstration using the specific figure *FO. 10* a^2r^3 from the table of "subquadratures" however, as we have explained above, it can be generalised for any figure of the table (Proposition 8). He divided the base of the square in *t* parts and on these constructed the ordinates of the figures and the square. He also constructed the rectangles of the ascribed figure and of the square of side 1. First, he established a proportion for each rectangle of these figures. For each division these rectangles were proportional to the ratio of the ordinates of each figure because each rectangle had the same base:

Rectangle of the square (AQ): rectangle of the ascribed figure (AK) = DQ : DK.

DQ = ordinate of the square; DK= ordinate of the ascribed figure

But the ordinate of the square is equal to the base of the square. He could then apply the proportion between the base of the square, that is one, and the ordinate of the figure.

In the case of the first division the value is

DQ : DK = $(1 : 10) .(1: (1/t))^2.(1: (1-1/t))^3 = 1 : [10.\ 1^2.\ (t-1)^3] / t^5 = t^5 : 10\ .\ 1^2.\ (t-1)^3$

Rectangle (square) = AQ; rectangle (ascribed) =AK, then

AQ : AK = DQ : DK = $t^5 : 10\ .\ 1^2.\ (t-1)^3$

AQ : AK = $t^5 : 10\ .\ 1^2.\ (t-1)^3$

In the case of the second division the value is:

rectangle (square) : rectangle (ascribed)=$1 : [10.\ 2^2.\ (t - 2)^3] / t^5 = t^5 :10.\ 2^2\ (t - 2)^3$

DR : DL = $t^5 :10.\ 2^2\ (t - 2)^3$

and so on.

Adding all the rectangles, in the antecedent, t rectangles because is the square and in the consequent, $t-1$ rectangles because is the ascribed figure. On the other side, in the antecedent, adding t^5, he obtained t^6 and in the consequent he obtained the algebraic expression of the finite sum[81]. Notice that the demonstration of this

[81] Mengoli then multiplied both consequents by ($m + n + 1$) and applied the theory of quasi proportions. As the second ratio is quasi equality (Theorem 42) then the first ratio, between the square of side 1 to the ascribed figure to one figure of the table of the quadratures, is also quasi equality.

proportion is based on the identification between the algebraic expression and the figure through the proportion between segments and quantities[82].

Following Malet's interpretation (Malet, 1996, 68-71) we would say that this proportion established by Mengoli aimed to give a solid foundation, to a certain degree, to Cavalieri's indivisibles method. This proportion can be interpreted as the ratio between the finite summation of ordinates and the ratio between figures. He then applied the quasi proportions, and thus did not have to establish proportions between infinity as Cavalieri did, because he established finite ratios which "tend" to other ratios, that is to say, quasi ratios.

One of the weak points of this demonstration is the step from a ratio of quasi equality between summation of powers and powers (numbers) to a ratio between figures. But Mengoli had based the theory of quasi proportions on the Euclidean theory of proportions, so for him it was valid for any magnitude, figure or number.

Let me emphasise that this demonstration does not depend on the degree and can be used in all cases where the quasi ratio of the summation of powers is known.

[82] It is obvious that Mengoli, like Roberval and Wallis, knew the result of the demonstration. But these authors made the summations of powers and proved their value in some cases. From these results they obtained the general rule and then applied it directly, making limits of ratios between sums of ordinates and areas of the figures. Mengoli, on the other hand, constructed the theory of quasi proportions to make the limits, and moreover to demonstrate the value of the area he did not apply them directly to the figures but made an intermediate step and used the ascribed figure. Mengoli could have used infinitesimals, as did Roberval, Wallis and others, but he seemed to find them difficult to use, and resorted to the ascribed figure and to Archimedes' method of exhaustion to base his demonstration of quadratures.

CONCLUDING REMARKS

Mengoli, like Viète, considered his algebra as a technique of symbols concerning abstract magnitudes. He dealt with species, forms, triangular tables, quasi ratios and logarithmic ratios. But the most innovative aspect of Mengoli's work is his use of letters to enable him to work directly with the algebraic expression of the figure. On the one hand, he expressed figures by an algebraic expression, in which the ordinate is related to the abscissa through proportional means, thus establishing the Euclidean theory of proportions as a link between algebra and geometry. On the other hand, he explained the geometrical construction of the algebraic expressions of these figures. This allowed him to study figures via their algebraic expressions and by constructing the tables of the expressions he was able to calculate countless areas of these figures at the same time.

The triangular table of quadratures that Mengoli constructed in *Elementum sextum* could be extended as far as he desired. He already knew the value of the quadratures and looked for a rule that allowed him to associate any figure to an algebraic expression. Putting them in the table, with the appropriate coefficients, the characteristics and the quadratures of these figures remained determined. He classified the figures in three types and studied the properties of each group, again using the theory of proportions. Besides, when he demonstrated the value of quadratures, the proof is independent of the graphic representation of the figure and can be used in all cases where the quasi ratio of the summation of powers is known.

At this point we may wonder whether Mengoli knew Descartes' *Géométrie*. We do not believe that he did: in terms of both the aims and the procedures the differences between the two are substantial. Mengoli introduced algebra into geometry to solve problems of quadratures; Descartes wanted to solve and classify many of geometrical problems and he used algebra as a tool. Mengoli did not make an algebra of segments, as Descartes did; that is to say, he did not make a geometrical interpretation of each of the algebraic operations that he defined. Furthermore, when he demonstrated the equality $(a + b)^2 = a^2 + 2ab + b^2$ he constructed the proof by using the properties of the proportions. His introduction of algebra into geometry bore more similarities with Viète's procedures. Viète also used the theory of proportions as a link, but he made diagrams without establishing coordinates system and he verified the constructions of the solutions of second degree's equations without establishing any connection between the ordinates and the abscissae. When the relation between the ordinates and abscissae in a geometric figure is mentioned, we immediately think of Fermat and his *Introduction to plane and solid loci* of early 1636. But although Mengoli may have drawn his inspiration from this work by Fermat, he only established this relation in specifically geometric figures such as $y = k.\ x^m .(t\text{-}x)^n$; he did not claim to have found a general principle, as Fermat did in his *Isagoge* (Fermat, 1891-1922, Book 1 91). Mengoli did not deal with solid problems, nor with problems of geometric locus, as Fermat did; what is more, his algebraic method cannot be applied to solve these other geometric problems.

Although Mengoli's contributions were a step forward in the process of algebraization of mathematics, his principal aim was not to demonstrate the equivalence of algebraic expressions and figures and their classification, but rather to provide a new method for solving problems of quadratures in an algebraic way. One should not forget that Mengoli wished to square the circle by interpolating these tables of quadratures. This procedure appeared in his later publication *Circolo* (1672) in which he calculated quadratures of figures determined by coordinates today represented by $y^p = k \cdot x^m \cdot (1-x)^n$. Any attempt to calculate these quadratures geometrically would have to be done case by case.

This study of Mengoli's work reveals that the basis of his new method of quadratures was not the method of indivisibles of his master Cavalieri, but the triangular tables and the theory of quasi proportions, as a development of Viète's algebra. In this way he created a numerical theory of summations of powers and products of powers and *limits* of these summations, which was unrelated to Cavalieri's *Omnes lineae*. It is not clear why Mengoli did not follow his master's path; perhaps it was because Cavalieri's method had received a great deal of criticism, something that Mengoli could not ignore. After showing that he knew the method of indivisibles and could apply it, Mengoli claimed that his purpose was to give solid foundations to a new method of calculating quadratures. To this end he constructed the triangular tables of figures and applied the theory of quasi proportions. Unlike Cavalieri, he never compared two figures through the comparison of lines, nor did he superimpose figures; rather, he established quasi ratios between figures. But what does it mean to say that a figure is quasi equal to another? Mengoli defined the ascribed, inscribed and circumscribed figures determined by rectangles built on the divisions of the base. He worked at all times with a finite number of divisions. He demonstrated that when the number of divisions increases it is possible to find a number of divisions that become the ratio between the circumscribed and inscribed figures nearer to equality than any given ratio not equal to equality. He also demonstrated that when the number of divisions increases the ascribed figure is quasi equal to the figure determined by the ordinates; that is to say, a figure determined by rectangles approximates to a mixed-line figure when the number of rectangles becomes infinite. To an extent, this first quasi equality recalls Archimedes ' method.

Mengoli also established a second quasi equality using algebraic procedures. He established a proportion in which the first ratio is between a summation of powers and a power and the other between the square and the ascribed figure. The step from the figure to an algebraic expression is essential in his demonstration. The Euclidean theory of proportions, once more, is the link to find this relationship and to allow him to operate with segments and to establish ratios and quasi ratios to demonstrate the value of quadratures of these figures.

The use of the two quasi equalities that Mengoli established (the ascribed figure and the square as well as the ascribed figure and the figure) allows us to understand better his words when he states that his geometry is a "perfect conjunction " of the geometry of indivisibles, the geometry of Archimedes and the algebra of Viète's. That is to say, the collaboration of algebra in its own geometry is an essential element in his method of quadratures.

For all these reasons we can conclude that in Mengoli's work the characteristics of algebraic and geometric thought by no means conflict: rather they complement each other, allowing him to obtain better results and to attain his goals. Furthermore Mengoli used an original development of Viète's symbolic language through triangular tables, quasi proportions in dealing with geometric figures and determining their quadratures.

ACKNOWLEDGMENTS

I am grateful to Javier Granados, Dolors Magret, Eberhard Knobloch, Marco Panza, Paolo Mancosu and Sabine Rommevaux, each of whom read earlier versions of this article and made many remarks concerning content and language.

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