Integrable nonholonomic geodesic flows on compact Lie groups *

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Abstract

This paper is a review of recent results on integrable nonholonomic geodesic flows of left–invariant metrics and left- and right–invariant constraint distributions on compact Lie groups.

Contents

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1 Introduction

This paper is a review of recent results on integrable flows on compact Lie groups under nonholonomic constraints. We mostly follow papers [24, 30, 31, 26], trying to present their results within a unified framework. Furthermore, some new examples of integrable nonholonomic systems are given.

1.1 Nonholonomic Geodesic Flows

We start with basic definitions and settings. Let (Q, ds^2) be n–dimensional Riemannian manifold Q with a nondegenerate matric $ds^2$and a Levi–Civita connection ∇, D be a non-integrable k–dimensional distribution on the tangent bundle TQ. A smooth path $\gamma(t) \in Q,\ t \in \Delta$ is called *admissible* (or allowed by constraints) if the velocity $\dot\gamma(t)$ belongs to $D_{\gamma(t)}$ for all $t \in \Delta$. There are two approaches to define geodesic lines among admissible paths: by induced connection as "straightest" lines and by the variation principle as "shortest" lines. We shall deal with the first approach which arises from mechanics.

The admissible path $\gamma(t)$ is called *a nonholonomic geodesic* if it satisfies the equations

$$\pi(\nabla_{\dot\gamma(t)}\dot\gamma(t)) = 0, \tag{1.1}$$

where $\pi : T_qQ \to D_q$, $q \in Q$ is the orthogonal projection.

Equivalently, we can introduce the Lagrangian function $l = \frac{1}{2}(K\dot q, \dot q)$, where K is the metric on Q also regarded as a mapping $K : TQ \to T^*Q$. Let $q = (q_1, \dots, q_n)$ be some local coordinates on Q. The trajectory of the system $q(t)$ that satisfies the constraints is a solution to the Lagrange–d'Alambert equations

$$\left(\frac{\partial l}{\partial q} - \frac{d}{dt}\frac{\partial l}{\partial \dot q}, \eta\right) = \sum_i \left(\frac{\partial l}{\partial q_i} - \frac{d}{dt}\frac{\partial l}{\partial \dot q_i}\right)\eta_i = 0, \quad \text{for all} \quad \eta \in D_q. \tag{1.2}$$

One can also write the Lagrange-d'Alambert equations as a first-order system on the $(n+k)$-dimensional *constraint submanifold* $\mathcal{M} = K(D)$ of the cotangent bundle T^*Q. Let D be locally defined by $\rho = n - k$ independent 1-forms α^i

$$D_q = \{\xi \in T_qQ,\ (\alpha^j_q, \xi) = \sum_i \alpha^j_i \xi_i = 0,\ j = 1, \dots, \rho\}.$$

Then $\mathcal{M}$ is locally given by the equations $(\alpha^i_q, K_q^{-1}p) = 0,\ i = 1, \dots, \rho$. Let $p_i = \partial l/\dot q_i$, $i = 1, \dots, n$ be momenta which together with q provide canonical coordinates on T^*Q. Let $h(q,p) = \frac{1}{2}(p, K_q^{-1}p)$ be the Hamiltonian function (the usual Legendre transformation of L). The equations (1.2) are equivalent to

$$\dot p_i = -\frac{\partial h(q,p)}{\partial q_i} + \sum_{i=1}^{\rho} \lambda_j \alpha^j(q)_i, \quad \dot q_i = \frac{\partial h(q,p)}{\partial p_i}, \qquad i = 1, \dots, n, \tag{1.3}$$

where Lagrange multipliers are chosen such that the solutions $(q(t), p(t))$ belong to $\mathcal{M}$.

As for the Hamiltonian systems, the Hamiltonian function is always the first integral of the system. There is also a nonholonomic version of the Noether theorem (see [38, 2, 24, 6]).

The Noether theorem. Suppose that a Lie group $\mathfrak{G}$ acts on the configuration space Q and that the action is naturally extended to TQ and T^*Q. The momentum mappings $\Psi_l : TQ \to \mathfrak{g}^*$ and $\Psi^* : T^*Q \to \mathfrak{g}^*$ are defined by

$$\Psi_l(q, \dot q \,|\xi) = \left(\frac{\partial l}{\partial \dot q}, \xi_Q\right) = (K_q \dot q, \xi_Q), \quad \Psi^*(q, p\,|\xi) = (p, \xi_Q), \tag{1.4}$$

where ξ_Q is the vector field on Q associated to the action of one-parameter subgroup $\exp(t\xi)$, $\xi \in \mathfrak{g} = T_{Id}\mathfrak{G}$.

Theorem 1.1. *Assume that ξ_Q is a section of the distribution D and the one-parameter subgroup* $\exp(t\xi)$ *preserves l (or h). Then $\Psi_l(\xi)$ is the first integral of the system (1.2), or equivalently, $\Psi^*(\xi)$ is the first integral of (1.3).*

Invariant measure and integrability. The equations (1.3) are not Hamiltonian. This is why it is still not clear how to define the notion of complete integrability for nonholonomic systems (see [4]). However, in some cases they have an invariant measure, a rather strong property, which puts the system close to Hamiltonian systems. In particular, if apart from the Hamiltonian there exist $\dim \mathcal{M} - 3$ additional independent integrals, then, by the Euler–Jacobi theorem, the solutions of (1.3) can be found by quadratures.

The importance of an invariant measure for integrability of nonholonomic systems was indicated by Kozlov in [39], where various examples were discussed (see also [2]). Namely, consider a non-Hamiltonian system

$$\dot x = f(x), \qquad x \in \mathbb{R}^m, \tag{1.5}$$

having an invariant measure $\mu(x)\,dx$ and $m-2$ first integrals $F_1(x), \dots, F_m(x)$. If the latter are independent on the invariant set $M_c = \{x \in \mathbb{R}^m,\ F_i(x) = c_i,\ i = 1, \dots, m-2\}$, then M_c is a two-dimensional submanifold and the flow on M_c has also an invariant measure. Then, accoring to the Euler–Jacobi theorem, solutions of (1.5) lying on M_c can be found by quadratures. Moreover, if L_c is a compact connected component of M_c and $f(x) \neq 0$ on L_c, then, if orientable, L_c is diffeomorphic to two-dimensional torus.

According to Kolmogorov's theorem on reduction of differential equations with a smooth invariant measure on a torus ([37]), one can find angular coordinates φ_1, φ_2 on L_c, in which the reduction of equations (1.5) takes the form similar as in the Liouville theorem:

$$\dot\varphi_1 = \frac{\Omega_1}{\Phi(\varphi_1, \varphi_2)}, \quad \dot\varphi_2 = \frac{\Omega_2}{\Phi(\varphi_1, \varphi_2)},$$

where Ω_1, Ω_2 depend on the constants of motion $c_1, \dots, c_{m-2}$ only and Φ is a smooth positive 2π–periodic function in φ_1, φ_2, the density of the induced invariant measure on L_c.

Therefore, it is natural to call the system (1.5) *completely integrable* if it can be integrated by the Jacobi theorem; or, more generally (see [54, 55]), if the phase space is almost everywhere foliated by invariant tori $\mathbb{T}^k\{\varphi_1, \dots, \varphi_k\}$ with the dynamics of the form

$$\dot\varphi_1 = \frac{\Omega_1}{\Phi(\varphi_1, \dots, \varphi_k)}, \quad \dots, \quad \dot\varphi_k = \frac{\Omega_k}{\Phi(\varphi_1, \dots, \varphi_k)}. \tag{1.6}$$

The above definition of complete integrability is slightly different from the definition of complete integrability of non-Hamiltonian systems given in [8, 60]. Namely, here we have quasi-periodic motions after the time substitution $d\tau = \Phi^{-1}(\varphi)dt$.

The existence of an invariant measure for smooth dynamical systems and for a class of nonholonomic systems with symmetries is studied in [40] and [59], respectively. Various mechanical examples with an invariant measure can be found in [12]. The authors of the paper [54, 55] constructed nonholonomic systems on unimodular Lie groups with right-invariant nonintegrable constraints and a left-invariant metric (so called *LR systems*), and showed that they always possess an invariant measure, whose density can be effectively calculated. In particular, the motion of a rigid body around a fixed point under a nonholonomic constraint (projection of the angular velocity to the fixed vector in space is constant) is described by an integrable LR system ([54]). Similar integrable problems on Lie groups with left–invariant constraints are studied in [26, 30, 31]. Also, an important example of an integrable nonholonomic mechanical system, the problem of rolling of a homogeneous ball on a surface of revolution (the Routh problem), was treated in detail in [27, 57].

1.2 Chaplygin Systems

Another approach to the integrability of nonholonomic systems is based on their reduction to a Hamiltonian form after an appropriate time rescaling. First, following [36] and [6], let us recall some basic facts about the Chaplygin systems.

Let (Q, l, D) be a nonholonomic system with a Lagrangian l of the natural mechanical type, with kinetic energy that correspods to the metric ds^2 and the potential function v. Assume that there is a bundle structure $\pi : Q \to N$ with the base manifold N and let the map π be a submersion, such that $T_qQ = D_q \oplus V_q$ for all q. Here V_q is the kernel of $T_q\pi$ called the *vertical space* at q. Then the distribution D can be seen as a collection of *horizontal spaces* of the *Ehresmann connection* associated to $\pi : Q \to N$. Given a vector $X_q \in T_qQ$, there is a decomposition $X_q = X_q^h + X_q^v$, where $X_q^h \in D_q$, $X_q^v \in V_q$. The *curvature* of the connection is the vertical valued 2-form B on Q defined by

$$B(X_q, Y_q) = -[\bar{X}_q^h, \bar{Y}_q^h]_q^v,$$

where $\bar{X}$ and $\bar{Y}$ are smooth vector fields on Q obtained by extending of X_q and Y_q.

By applying the Ehresmann connection the Lagrange–d'Alambert equations (1.2) can be represented in the form (see [6])

$$\left(\frac{\partial l_c}{\partial q} - \frac{d}{dt}\frac{\partial l_c}{\partial \dot{q}}, \eta\right) = \left(\frac{\partial l}{\partial \dot{q}}, B(\dot{q}, \eta)\right), \quad \text{for all} \quad \eta \in D_q, \tag{1.7}$$

where $l_c(q, \dot{q}) = l(q, \dot{q}^h)$ is the *constrained Lagrangian.*

Now, suppose that $\pi : Q \to N = Q/\mathfrak{G}$ is a principal bundle with respect to the *left* action of a Lie group $\mathfrak{G}$, and D is a principal connection, i.e., D is a $\mathfrak{G}$-invariant distribution. Let the Lagrangian l be also $\mathfrak{G}$-invariant, i.e., $\mathfrak{G}$ acts by isometries on Riemannian manifold (Q, ds^2) and v is a $\mathfrak{G}$–invariant function. Then the constrained Lagrangian l_c induces a well defined reduced Lagrangian $L : TQ \to \mathbb{R}$ via identification $TN \approx D/\mathfrak{G}$. The reduced Lagrangian L is of the natural mechanical type as well. Its kinetic energy is given by metric ds_D^2 and its potential energy will be denoted by V.

Under the above assumtions, equations (1.7) are $\mathfrak{G}$-invariant and induce *reduced Lagrange–d'Alambert* equations on the tangent bundle TN,

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \eta\right) = \Sigma(\dot{q}, \eta), \quad \text{for all} \quad \eta \in T_qN. \tag{1.8}$$

Here Σ is semi-basic two-form given by the right hand side of (1.7) and $q = (q_1, \dots, q_k)$ are some local coordinates on the base space N. From (1.4) we see that Σ depends on the curvature of the connection D and on the momentum mapping Φ_l.

The system $(Q, l, D, \mathfrak{G})$ is referred to as *a (generalized) Chaplygin system* (see [36, 6]), as a generalization of classical Chaplygin systems with Abelian symmetries [17].

Remark 1.1. Note that horizontal and vertical spaces do not need to be orthogonal with respect to the metric ds^2. In fact, if D is ds^2-orthogonal to the leaf of $\mathfrak{G}$-action, then D will be an invariant submanifold of the nonconstrained geodesic flow of the metric ds^2, and the right hand sides of equations (1.7) will be zero. In this case, ds^2_D coincides with the *submersion metric* induced from ds^2.

Chaplygin's reducing multiplier. Let $p_i = \partial L/\partial \dot q_i$, $i = 1, \dots, k$ be momenta, g_{ij} the metric tensor of ds^2_D and g^{ij} the dual metric on T^*N. Then the reduced Lagrangian has the form $L(q, \dot q) = \frac{1}{2}\sum g_{ij}\dot q_i \dot q_j - V(q)$. We also introduce the Hamiltonian function $H(q, p) = \frac{1}{2}\sum g^{ij} p_i p_j + V(q)$. The reduced system (1.8) can be rewritten as a first-order dynamical system on T^*N:

$$\dot q_i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q_i} + \Pi_i(q, p), \qquad i = 1, \dots, k. \tag{1.9}$$

The functions Π_i are quadratic in momenta and can be regarded as non-Hamiltonian perturbations of the equations of motion of a particle on N.

Let $\Omega = \sum dp_i \wedge dq_i$ be the standard symplectic form on T^*N. The equations (1.9) have an invariant measure $f\Omega^k$ if $\sum_i \left(\frac{\partial(f\dot q_i)}{\partial q_i} + \frac{\partial(f\dot p_i + f\Pi_i)}{\partial p_i}\right) = 0$. Since the finction f depends only on the coordinates q, this is equivalent to condition

$$d(\ln f) + \alpha = 0, \tag{1.10}$$

where the one-form α is given by $\sum_i \frac{\partial \Pi_i}{\partial p_i}|_{\dot q = gp} = (\alpha, \dot q)$.

Remark 1.2. The paper [50] (see also [13]) contains a nontrivial observation about the density of the invariant measure, which in our terms reads as follows. Suppose that system (1.9) has an invariant measure with density $f(q, p)$ in the case of absence of potential ($V(q) = 0$). Then one can check that the function $f_0(q) = f(q, 0)$ is also a solution of (1.10). In other words, if the reduced system (1.9) has an invariant measure for $V = 0$, one can take this measure to be of the form $f(q)\Omega^k$. Then, since (1.10) does not depend on the potential, the reduced system (1.9) has the same invariant measure in the presence of a potential field $V(q)$ as well.

Now consider time substitution $d\tau = \mathcal{N}(q)dt$, where $\mathcal{N}(q)$ is a differentiable nonvanishing function on Q, and denote $q' = dq/d\tau$. Then we have the following commutative diagram

$$\begin{array}{ccc} TN\{q, \dot q\} & \xrightarrow{q' = \dot q/\mathcal{N}(q)} & TN\{q, q'\} \\ {\scriptstyle p = g\dot q}\downarrow & & \downarrow{\scriptstyle \tilde p = \mathcal{N}^2 g q'} \\ T^*N\{q, p\} & \xrightarrow{\tilde p = \mathcal{N}p} & T^*N\{q, \tilde p\}. \end{array}$$

The Lagrangian and Hamiltonian functions in the coordinates $\{q, q'\}$ and $\{q, \tilde p\}$ take the form

$$L^*(q, q') = \frac{1}{2}\sum \mathcal{N}^2 g_{ij} q'_i q'_j - V(q), \quad H^*(q, \tilde p) = \frac{1}{2}\sum \frac{1}{\mathcal{N}^2} g^{ij}\tilde p_i \tilde p_j + V(q).$$

There is a remarkable relation between the existence of an invariant measure of the reduced system (1.9) and its reducibility to a Hamiltonian form (see [26]).

Theorem 1.2. *1). Suppose that after the time substitution $d\tau = \mathcal{N}(q)dt$ the equations (1.9) become Hamiltonian,*

$$q'_i = \frac{\partial H^*}{\partial \tilde p_i}, \quad \tilde p'_i = -\frac{\partial H^*}{\partial q_i}. \tag{1.11}$$

Then the function $f(q) = \mathcal{N}(q)^{k-1}$ satisfies the equation (1.10), i.e., the original system (1.9) has the invariant measure with density $f(q)$.

2). For $k = 2$, the above statement can also be inverted: the existence of the invariant measure with the density $\mathcal{N}(q)$ implies that in the new time $d\tau = \mathcal{N}(q)dt$, the system (1.9) gets the Hamiltonian form (1.11).

In nonholonomic mechanics the factor $\mathcal{N}$ is known as the *reducing multiplier*, item 2) of this theorem is referred to as *Chaplygin's reducibility theorem* (see [16, 17] or section III.12 in [45]). Notice that for $k > 2$, the multiplier $\mathcal{N}(q)$ and the density of the invariant measure of system (1.9) do not coincide. Also, the existence of the multiplier do not depends on the potential V.

There are many examples of the Chaplygin reducing multiplier for $k = 2$. Since many conditions on the metric and constraints are imposed, until recently there were no nontrivial examples of multidimensional systems, appart of several examples for $k = 3, 4$ with the property that factor $\mathcal{N}(q)$ depends only on one coordinate, that are reducible to a Hamiltonian form by the Chaplygin procedure ([45, 20, 28, 44]).

As an alternative, in the reduction of Chaplygin systems one can use the symplectic (or Poisson) framework (see [50, 3, 13, 14]). Such systems can be represented in a Hamilton-like form with respect to an nondegenerate (almost-symplectic) 2-form, which however may be not closed Namely, let Ξ be the Legandre transformation of the semi-basic form Σ. Then one can write (1.9) as

$$\Omega_{nh}(X_H, \cdot) = dH(\cdot), \quad \text{where} \quad \Omega_{\text{nh}} = \Omega + \Xi.$$

In this framework, the Chaplygin multiplier is a function $\mathcal{N}$ such that the form $\tilde{\Omega} = \mathcal{N}\Omega_{nh}$ is closed. Then, after rescaling $Y = X/\mathcal{N}$, we obtain the Hamiltonian system $\tilde{\Omega}(Y, \cdot) = dH(\cdot)$ (see [50, 27, 13, 21]). Contrary to the procedure described in Theorem 1.2, here the vector field Y has no direct mechanical description.

Recently, necessary and sufficient conditions for the existence of an invariant measure of the reduced system in case when the Lagrangian of the system is of a pure kinetic energy type are given in [13, 14].

1.3 Contents of the Paper

In section 2 we consider the systems with left–invariant metrics and left–invariant constraint distributions, so called *LL systems*. The equations of the motion reduce to the Euler–Poincaré–Suslov equations on the corresponding Lie algebra. Although such equations generally are not Hamiltonian, their nice algebraic structure allows us to construct various integrable examples with an invariant measure.

In section 3 we consider a class of LR systems (left–invariant metrics and right–invariant constraint distributions), which can be regarded as Chaplygin systems on the principle bundle $G \to Q = G/H$, H being a Lie subgroup. We show that, in contrast to generic Chaplygin systems, the reductions of our LR systems onto the homogeneous space Q always possess an invariant measure. Then we study the case $G = SO(n)$, when LR systems are multidimensional generalizations of the Veselova problem of a nonholonomic rigid body motion, which admit a reduction to the system with an invariant measure on the (co)tangent bundle on the unit sphere S^{n-1}. For a special choice of the left-invariant metric on $SO(n)$, we prove that under a time reparameterization, the reduced system becomes an integrable Hamiltonian system describing a geodesic flow on the unit sphere S^{n-1}. This provides a first multidimensional example of a nonholonomic system for which the celebrated Chaplygin reducibility theorem is applicable. Lastly, we present an explicit reconstruction of the motion on the group $SO(n)$.

Finally, in section 4 we present another class of systems on an unimodular Lie group G, which always possess a non-trivial invariant measure and which are obtained as modifications of a geodesic flow on G with respect to a sum of a left- and a right-invariant metrics, so called L+R systems. It appears that a nonholonomic LR system on a group G can be obtained as a limit case of an appropriate L+R system on this group. As an example, we consider a nonholonomic mechanical system called the spherical support.

2 LL Systems

2.1 Euler–Poincaré–Suslov Equations

In this section we consider nonholonomic systems (G, l, D) with a left-invariant distributions D and a left-invariant Lagrangians l that describes left-invariant metrics on a compact connected Lie group G. Let $\mathfrak{g} = T_{Id}G$ be the Lie algebra of G. In what follows we shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by Ad_G invariant scalar product $\langle\cdot,\cdot\rangle$, and TG and T^*G by bi-invariant metric on G. For clearness, we shall use the symbol ω for the elements in $\mathfrak{g}$ and the symbol x for the elements in $\mathfrak{g}^* \cong \mathfrak{g}$.

Let

$$\mathfrak{d} = \{\omega \in G,\ \langle\omega, a^i\rangle = 0,\ i = 1, \dots, \rho\} \subset \mathfrak{g}$$

be the restriction of the left-invariant distribution D to the algebra, for some constant and linearly independent vectors a^i in $\mathfrak{g}$. From the left invariance condition we have $D_g = g \cdot \mathfrak{d}$. The distribution is nonintegrable if and only if $\mathfrak{d}$ is not a subalgebra. Also, it is sufficient to give a Lagrangian at one point of the group, for instance the identity $l(g, \dot{g}) = \frac{1}{2}\langle\mathcal{I}\omega, \omega\rangle$, $\omega = g^{-1} \cdot \dot{g}$. Here $\mathcal{I} : \mathfrak{g} \to \mathfrak{g}$ is a symmetric positive definite (with respect to $\langle\cdot,\cdot\rangle$) operator. The Hamiltonian in the left-trivialization is given by $H(x) = \frac{1}{2}\langle\mathcal{A}(x), x\rangle$, $\mathcal{A} = \mathcal{I}^{-1}$. The corresponding left-invariant metric will be denoted by $ds^2_{\mathcal{I}}$.

Let $\mathfrak{m}$ be the restriction of the constraint submanifold $\mathcal{M}$ to $\mathfrak{g}$, that is $\mathfrak{m} = \mathcal{I}(\mathfrak{d})$. Equations (1.3) are G–invariant and reduce to $\mathfrak{m}$,

$$\dot{x} = [x, \nabla H(x)] + \sum_{i=1}^{\rho} \lambda_i a^i = [x, \mathcal{A}(x)] + \sum_{i=1}^{\rho} \lambda_i a^i, \tag{2.1}$$

where λ_i are Lagrange multipliers chosen such that x belongs to $\mathfrak{m} = \mathcal{I}(\mathfrak{d})$, i.e., such that $\omega = \mathcal{A}(x)$ belongs to $\mathfrak{d}$: $\langle\mathcal{A}(x), a^i\rangle = 0$, $i = 1, \dots, \rho$. In other words, the following commutative diagram holds

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathcal{P}^t} & \mathcal{M} \\ \Lambda\big\downarrow & & \big\downarrow\Lambda \\ \mathfrak{m} & \xrightarrow{P^t} & \mathfrak{m}, \end{array} \tag{2.2}$$

where $\mathcal{P}^t$ and P^t are phase flows of the nonholonomic geodesic flow and the system (2.1) respectively, and Λ maps $g \cdot x \in T_gG$ to $x \in \mathfrak{g}$.

Following [24], we shall call (2.1) the *Euler–Poincaré–Suslov (EPS) equations*, as a generalization of the Suslov problem of the nonholonomic rigid body motion (see the example below).

These equations have a quite different nature in comparison with the Euler–Poincaré equations $\dot{x} = [x, \mathcal{A}(x)]$. In particular, as indicated in [41], in the case of only one constraint $\langle a, \mathcal{A}(x)\rangle = 0$, they have a smooth invariant measure if and only if $[a, \mathcal{A}(a)] = \mu a$.

Reconstruction of the motion on the group. In the Hamiltonian case, the integrability of the reduced system implies generally a non-commutative integrability of the original system, namely the phase space is foliated by invariant isotropic tori with quasi-periodic dynamic (see [60]). However there is no such analog in the nonholonomic setting. To reconstruct the motion $(g(t), \dot{g}(t))$ on the whole phase space, we have to solve the kinematic equation

$$g^{-1}(t) \cdot \dot{g}(t) = \omega(t) = \mathcal{A}(x(t)),$$

where $x(t)$ are solutions of (2.1), i.e., to find all trajectories in $\mathcal{M}$ that projects to the given trajectory $x(t)$ in $\mathfrak{m}$. In particular, if $x(t)$ is a relative equilibrium ($x(t) = x(t_0)$ for all t) or if $x(t)$ is a relative periodic orbit ($x(t+T) = x(t)$ for all t), then the invariant set $\Lambda^{-1}(\{x(t), t \in \mathbb{R}\}) \subset \mathcal{M}$ is foliated by invariant tori of maximal dimension $\operatorname{rank} G$ or $\operatorname{rank} G + 1$, respectively (e.g., see [27]).

Multidimensional Suslov problem. The most natural example of LL systems is the nonholonomic Suslov problem, which describes the motion of an n-dimensional rigid body with a fixed point, that is, the motion on the Lie group $SO(n)$, with certain left-invariant nonholonomic constraints.

For a path $g(t) \in SO(n)$, the angular velocity of the body is defined as the left-trivialization $\omega(t) = g^{-1} \cdot g(t) \in so(n)$. The matrix $g \in SO(n)$ maps a coordinate system fixed in the body to a coordinate system fixed in the space. Therefore, if $e_1 = (e_{11}, \dots, e_{1n})^T \dots, e_n = (e_{n1}, \dots, e_{nn})^T$ is the orthogonal frame of unit vectors fixed in the space *and regarded in the moving frame*, we have

$$E_1 = g \cdot e_1, \ \dots, \ E_n = g \cdot e_n,$$

where $E_1 = (1, 0, \dots, 0)^T, \ \dots, \ E_n = (0, \dots, 0, 1)^T$. From the conditions $0 = \dot{E}_i = \dot{g} \cdot e_i + g \cdot \dot{e}_i$, we find that the vectors $e_1, \dots, e_n$ satisfy the Poisson equations

$$\dot{e}_i = -\omega e_i, \qquad i = 1, \dots, n. \tag{2.3}$$

The left-invariant metric on $SO(n)$ is given by non-degenerate inertia operator $\mathcal{I} : so(n) \to so(n)$. Then the Lagrangian of the free motion of the body reads $l = \frac{1}{2}\langle \mathcal{I}\omega, \omega \rangle$, where now $\langle \cdot, \cdot \rangle$ denotes the Killing metric on $so(n)$, $\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr}(XY)$, $X, Y \in so(n)$. For a "physical" rigid body, $\mathcal{I}\omega$ has the form $I\omega + \omega I$, where I is a symmetric $n \times n$ matrix called *mass tensor* (see [24]). However, since we are interested mainly in nonholonomic geodesic flows, we shall consider other inertia operators as well.

Recall that in the three-dimensional case, the Suslov problem describes the motion of a rigid body with the constraint: the projection of the angular velocity to a vector fixed in the moving frame (for example E_3) is equal to zero [52, 2]. In other words, only infinitesimal rotations in the planes $\operatorname{span}(E_1, E_2)$ and $\operatorname{span}(E_1, E_3)$ are allowed. Hence, it is natural to define its n-dimensional analog as follows: only infinitesimal rotations in the fixed 2-planes spanned by $(E_1, E_2), \dots, (E_1, E_n)$ (i.e., in the planes containing the vector E_1) are allowed. Following [24], one can relax these constraints by assuming that the angular velocity matrix has the following structure

$$\omega = \begin{pmatrix} 0 & \cdots & \omega_{1r} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ -\omega_{1r} & \cdots & 0 & \cdots & \omega_{rn} \\ \vdots & & \vdots & \mathbf{O} & \\ -\omega_{1n} & \cdots & -\omega_{rn} & & \end{pmatrix},$$

where $\mathbf{O}$ is zero $(n-r)\times(n-r)$ matrix. This implies the left–invariant constraints

$$\langle \omega, E_i \wedge E_j \rangle = 0, \quad r+1 \le i < j \le n. \tag{2.4}$$

As a result, the Suslov problem is described by the EPS equations

$$\frac{d}{dt}(\mathcal{I}\omega) = [\mathcal{I}\omega, \omega] + \sum_{r<p<q\le n} \lambda_{pq}\, E_p \wedge E_q, \tag{2.5}$$

together with Poisson equations (2.3). Here the components of the vectors $e_1, \dots, e_n$ play the role of redundant coordinates on $SO(n)$.

Various integrable cases of the Suslov problem with additional potential fields and their multidimensional generalization are given in [34, 39, 2, 47] and [32], respectively.

2.2 Some Integrable Cases of EPS Equations

EPS equations on symmetric pairs. Let $\mathfrak{h}$ be the subspace of the algebra $\mathfrak{g}$ spanned by a^i, $i = 1, \dots, \rho$.

Consider the case when the tensor $\mathcal{A}$ preserves the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{d}$, i.e., $\mathcal{A} = \mathcal{A}_{\mathfrak{h}} + \mathcal{A}_{\mathfrak{d}}$, where $A_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$, $A_{\mathfrak{d}} : \mathfrak{d} \to \mathfrak{d}$ are positive definite operators. Then $\mathfrak{m} = I(\mathfrak{d}) = \mathfrak{d}$, and we can write (2.1) in the following way

$$\dot{x} = [x, \mathcal{A}_{\mathfrak{d}}(x)]_{\mathfrak{d}}, \qquad x \in \mathfrak{d}, \tag{2.6}$$

where $\xi_{\mathfrak{d}}$ denotes the orthogonal projection of $\xi \in \mathfrak{g}$ to the subspace $\mathfrak{d}$ (with respect to $\langle \cdot, \cdot \rangle$).

Equation (2.6) preserve the standard measure on $\mathfrak{d}$. Also the Hamiltonian function $H(x) = \frac{1}{2}\langle x, \mathcal{A}_D(x)\rangle$ and the invariant $F(x) = \langle x, x\rangle$ are always first integrals of the system. Therefore, by the Jacobi theorem the equation (2.6) is always integrable for $\dim \mathfrak{d} \le 4$.

Remark 2.1. Note that, in general, the invariant $F(x) = \langle x, x \rangle$ is not the integral of (2.1), although it is always an integral of non-constrained system. Namely, a first integral $f(x)$ of the Euler–Poincaré equations $\dot{x} = [x, \mathcal{A}(x)]$ is the integral of (2.1) if and only if the following condition holds

$$\sum_i \lambda_i \langle \nabla f(x), a^i \rangle|_{x\in\mathfrak{m}} = 0. \tag{2.7}$$

In our case $\nabla F(x) = 2x$, $x \in \mathfrak{m} = \mathfrak{d}$ is orthogonal to $\mathfrak{h}$ and therefore the invariant $F(x)$ remains to be an integral.

Example 2.1. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{w}$ the orthogonal complement of $\mathfrak{k}$. Suppose that $(\mathfrak{g}, \mathfrak{k})$ is a *symmetric pair*, i.e., the following conditions are satisfied:

$$[\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{w}] \subset \mathfrak{w}.$$

Then, in the special case $\mathfrak{d} = \mathfrak{w}$, we have $[\mathfrak{d}, \mathfrak{d}]_{\mathfrak{d}} = 0$. Therefore all the solutions of equations (2.6) are constants. As a result, the solution of the original system on G (nonholonomic geodesic lines of the metric $ds^2_{\mathcal{I}}$) is given by the motion along one-parameter subgroups,

$$g(t) = g_0 \exp(t\xi), \qquad \xi \in \mathfrak{d}.$$

This simplest situation occurs for the multidimensional Suslov equations (2.5) with $r = n-1$ and $\mathcal{I}\omega = I\omega + \omega I$, where $I = \operatorname{diag}(I_1, \dots, I_n)$. Then $\mathfrak{h} = so(n-1)$, $(so(n), so(n-1))$ is a symmetric pair, and $\mathcal{I}$ preserves the decomposition $so(n) = \mathfrak{d} + so(n-1)$. Hence the solutions $\omega(t)$ are just constants.

Motivated by the above observation and by another integrable case of the multidimensional Suslov problem (see below), let us assume that there is a chain of subalgebras

$$\mathfrak{l} \subset \mathfrak{k} \subset \mathfrak{g},$$

where $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, and consider the adjoint representation of $\mathfrak{l}$ onto the linear space $\mathfrak{w}$: $\eta \in \mathfrak{l} \mapsto [\eta, \cdot] \in End(\mathfrak{w})$. With respect to this representation, decompose $\mathfrak{w}$ into irreducible subspaces $\mathfrak{w} = \mathfrak{w}_0 + \mathfrak{w}_1 + \cdots + \mathfrak{w}_m$, $\mathfrak{w}_0$ being the subspace with the trivial representation. Next, assume that $\mathfrak{d}$ has the form

$$\mathfrak{d} = \mathfrak{u} + \mathfrak{d}_0 + \mathfrak{d}_1 + \cdots + \mathfrak{d}_g, \quad \mathfrak{u} = \mathfrak{d} \cap \mathfrak{l}, \quad \mathfrak{d}_k = \mathfrak{d} \cap \mathfrak{w}_k. \tag{2.8}$$

with $\dim \mathfrak{u} \geq 1$, $g \leq m$. Suppose also that $A_\mathfrak{u} = s \cdot Id_\mathfrak{u}$, $s \in \mathbb{R}$ and that the operator $\mathcal{A}_\mathfrak{d}$ preserves the decomposition (2.8), that is, $\mathcal{A}_\mathfrak{d} = A_\mathfrak{u} + A_0 + \cdots + A_g$. Let x_k denote the orthogonal projection of x to $\mathfrak{d}_k$, $k = 0, \dots, g$. Then the equation (2.6) reads

$$\frac{d}{dt}(x_\mathfrak{u} + x_0 + \cdots + x_g) = [x_\mathfrak{u} + x_0 + \cdots + x_g, s x_\mathfrak{u} + A_0(x_0) + \cdots + A_g(x_g)]_\mathfrak{d}.$$

In view of conditions $[\mathfrak{u}, \mathfrak{d}_k]_\mathfrak{d} \subset \mathfrak{d}_k$, $[\mathfrak{d}_i, \mathfrak{d}_j]_\mathfrak{d} \subset \mathfrak{u}$, the above system splits into $g+2$ equations

$$\begin{aligned} &\dot{x}_0 = 0, \\ &\dot{x}_\mathfrak{u} = [x_0 + \cdots + x_g, A_0(x_0) + \cdots + A_g(x_g)]_\mathfrak{u}, \\ &\dot{x}_k = [x_\mathfrak{u}, B_k(x_k)]_{\mathfrak{d}_k}, \qquad k = 1, \dots, g, \end{aligned}$$

where $B_k = A_k - s \cdot Id_k$.

Thus, apart from $H(x) = \frac{1}{2}\langle \mathcal{A}_\mathfrak{d}(x), x\rangle$ and $F(x) = \langle x, x\rangle$, the system (2.6) has a set of the first integrals given by the projection of x to $\mathfrak{d}_0$, $F_0(x) = x_0$ and the functions

$$F_k(x) = \langle B_k(x_k), x_k\rangle, \qquad k = 1, \dots, g.$$

In this case the following theorem holds (see [31]).

Theorem 2.1. *1). If the operators B_k are positive definite, then invariant varieties*

$$M_c = \{x \in D \mid x_0 = c_0, F_1(x) = c_1, \dots, F_n(x) = c_g, F(x) = c_{g+1}\},$$

$c_1, \dots, c_{g+1}$ being constants of motion, are diffeomorphic to the product of spheres $S^{\dim \mathfrak{d}_1 - 1} \times \cdots \times S^{\dim \mathfrak{d}_g - 1} \times S^{\dim \mathfrak{u} - 1}$, provided that c_{g+1} satisfies inequality

$$c_{g+1} > |c_0|^2 + \sum_{k=1}^{g} \frac{c_k}{b_k}, \quad b_k = \min_{|\xi_k|=1} \langle B_k(\xi_k), \xi_k\rangle. \tag{2.9}$$

2) If $\dim \mathfrak{d}_k \leq 2$, $k = 1, \dots, g$, $\dim \mathfrak{u} = 1$, and all the constants c_i are nonzero, then M_c is diffeomorphic to the disjoint union of two g–dimensional tori with a quasi-periodic dynamic of the form $\dot{\varphi}_i = \Omega_i/\Phi(\varphi)$, $i = 1, \dots, g$.

The Fedorov–Kozlov case. The above construction applied to the symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (so(n), so(2) \times so(n-2))$ gives the Fedorov–Kozlov integrable case of the multidimensional Suslov problem [24]. As above, let $\mathfrak{w}$ be the orthogonal complement of $\mathfrak{k}$:

$$so(n) = \begin{pmatrix} so(2) & \mathfrak{w} \\ -\mathfrak{w}^t & so(n-2) \end{pmatrix}.$$

We take $\mathfrak{u} = \mathfrak{l} = so(2) = \mathrm{span}\{E_1 \wedge E_2\}$, $\mathfrak{d} = \mathfrak{u} + \mathfrak{w}$, i.e., the constraint are given by relations (2.4) with $r = 2$. Then

$$\mathfrak{d}_1 = \mathfrak{w}_1 = \mathrm{span}\{E_1 \wedge E_3, E_2 \wedge E_3\}, \ \dots, \ \mathfrak{d}_{n-2} = \mathfrak{w}_{n-2} = \mathrm{span}\{E_1 \wedge E_n, E_2 \wedge E_n\}.$$

In the Suslov problem there is a natural choice of the inverse inertia operator $\mathcal{A} = \mathcal{I}^{-1}$ which preserves the decomposition $\mathfrak{d} = so(2) + \mathfrak{d}_1 + \cdots + \mathfrak{d}_{n-2}$. Namely, we take the left-invariant metric on $so(n)$ determined by the kinetic energy of the multidimensional rigid body,

$$\mathcal{A} : E_i \wedge E_j \mapsto \frac{1}{I_i + I_j} E_i \wedge E_j . \tag{2.10}$$

If $I_1 > I_2 > I_3 > \cdots > I_n$, then, under condition (2.9), the integrals $F_1, \dots, F_{n-2}$ are positive definite and the invariant submanifolds

$$\{F_1 = c_1, \ \dots, \ F_{n-2} = c_{n-2}, \ F = c_{n-1}\}$$

are union on two disjoint $(n-2)$–dimensional tori. Moreover, as shown in [24], the motion on the tori is straight-line but not uniform and in appropriate angle coordinates φ_i it is described by equations

$$\dot\varphi_i = \frac{\Omega_i}{\omega_{12}(\varphi)}, \qquad \Omega_i = \sqrt{\frac{(I_1 - I_{i+2})(I_2 - I_{i+2})}{(I_1 + I_{i+2})(I_2 + I_{i+2})}}, \quad i = 1, \dots, n-2 . \tag{2.11}$$

For this integrable case the reconstruction problem was studied in [58]. As follows from (2.11), if the trajectories are periodic on one torus, they are periodic on the rest of the tori. Then the trajectories $(g(t), \dot g(t))$ which correspond to the given periodic trajectory $\omega(t)$ are quasi-periodic (e.g., see [27]).

According to [58], in the opposite case, if for some constants $c > 0$ and $\gamma > n - 3$, the frequencies satisfy Diophantine conditions

$$|l + \sqrt{-1}(k, \Omega)| \ge c/|k|^\gamma, \quad l = 0, 1, 2, \quad \text{for all} \quad k \in \mathbb{Z}^{n-2}$$

and the value of the integrals F (or F and F_1) are dominant with respect to those of other integrals ($c_i/c_{n-1} \sim \epsilon$, $i \ne n-1$ or $c_i/c_1, c_i/c_{n-1} \sim \epsilon$, $i \ne 1, n-1$), then the dynamics on the whole phase space can be approximated by quasi–periodic dynamics on the time interval of length $\sim \exp(1/\epsilon)$.

The Suslov problem on so(4). Now we concentrate on the integrable case when $\mathfrak{d}$ is not an eigenspace of $\mathcal{A}$. Let $\mathfrak{g} = so(4)$. Then $\mathfrak{k} = \mathrm{span}\{E_1 \wedge E_2, E_3 \wedge E_4\}$ is a Cartan subalgebra. As above, take the inertia operator in the form (2.10), which implies $\mathcal{A}(\mathfrak{k}) = \mathfrak{k}$. Further, take $a = a_1 E_1 \wedge E_2 + a_2 E_3 \wedge E_4$, $a_1 a_2 \ne 0$ and the constraint $\langle a, \mathcal{A}(x)\rangle = 0$. Then $[a, \mathcal{A}(a)] = 0$ and the Euler–Poincaré–Suslov equations

$$\dot x = [x, \mathcal{A}(x)] + \lambda a$$

preserve the standard measure on $\mathfrak{m} = \{x \mid \langle a, \mathcal{A}(x)\rangle\} = 0$ (see [41]). Next, one can always chose a linear combination of quadratic invariants on $so(4)$, $c_1 I_1 + c_2 I_2$, such that the condition (2.7) holds. Thus our system on five-dimensional space $\mathfrak{m}$ has the integrals $H = \frac{1}{2}\langle \mathcal{A}(x), x\rangle$ and $F_1 = c_1 I_1 + c_2 I_2$. For the integrability one needs one more independent integral. It can be taken in the form of a quadratic function on the orthogonal complement of $\mathfrak{k}$.

This approach is a special case of the method of construction of integrable EPS equations on six-dimensional unimodular Lie algebras given in [30].

Chains of subalgebras. Suppose there is a chain of subalgebras

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}.$$

Let $\mathfrak{g}_i = \mathfrak{g}_{i-1} + \mathfrak{w}_i$ be the corresponding orthogonal decompositions. Then $\mathfrak{g}_i = \mathfrak{g}_0 + \mathfrak{w}_1 + \cdots + \mathfrak{w}_i$. Following [7], consider $\mathcal{A}$ of the form:

$$\mathcal{A} = \mathcal{A}_0 + s_1 \cdot Id_{\mathfrak{w}_1} + \cdots + s_n \cdot Id_{\mathfrak{w}_n}, \qquad s_i > 0, \quad i = 1, \ldots, n, \tag{2.12}$$

where $\mathcal{A}_0$ is a symmetric positive operator defined in the subalgebra $\mathfrak{g}_0$. Suppose that $\mathfrak{d}$ has orthogonal decomposition

$$\mathfrak{d} = \mathfrak{d}_0 + \mathfrak{d}_1 + \cdots + \mathfrak{d}_n, \quad \mathfrak{d}_k = \{\omega_k \in \mathfrak{w}_k,\ \langle a_k^i, \omega_k\rangle = 0,\ i = 1, \ldots, \rho_k\}. \tag{2.13}$$

Then $\mathfrak{d}_k$, $k > 0$ are invariant subspaces of $\mathcal{A}$. By x_k denote the orthogonal projection of x to $\mathfrak{d}_k$, $k > 0$; and by x_0 denote the orthogonal projection to $\mathfrak{g}_0$.

Now we can formulate the following theorem (see [31]).

Theorem 2.2. *The Euler–Poincaré–Suslov equations (2.1), with D and operator $\mathcal{A}(x)$ of the form (2.13) and (2.12), are equivalent to the Euler–Poincaré–Suslov equations on the Lie subalgebra $\mathfrak{g}_0$:*

$$\dot{x}_0 = [x_0, \mathcal{A}_0(x_0)] + \sum_{i=1}^{\rho_0} \mu_i a_0^i, \tag{2.14}$$

$$\langle \mathcal{A}_0(x_0), a_0^i\rangle = 0, \quad i = 1, \ldots, \rho_0,$$

together with a chain of linear differential equations on the subspaces $\mathfrak{d}_k$:

$$\dot{x}_k = [x_k, \mathcal{A}_0(x_0) - s_k x_0 + (s_1 - s_k)x_1 + \cdots + (s_{k-1} - s_k)x_{k-1}]_k. \tag{2.15}$$

If the Euler–Poincaré–Suslov equations (2.14) on $\mathfrak{g}_0$ are solvable, then the integration of original equations (2.1) reduces to consequitive integration of the chain of linear dynamical systems (2.15) for $k > 0$. In the most simplest case the solutions of (2.14) are constants. Then the components of the vector x_1 satisfy a system of linear equations with constant coefficients, hence they are elementary functions of the time t. This happens if $\mathcal{A}_0 = Id_{\mathfrak{g}_0}$ or if $\mathfrak{g}_0$ is a commutative subalgebra. In particular, if $\dim D_0 = 0$, then we have $x_0 = 0$. In this case $\dot{x}_1 = 0$ and x_2 is given by elementary functions of t.

2.3 Hamiltonian Flows

In some cases, the nonholonomic geodesic flow (1.3) on $\mathcal{M}$ can be obtained as a restriction of a Hamiltonian flow on T^*Q to the invariant submanifold $\mathcal{M}$. In most examples this happen when the Lagrange multipliers in (1.3) vanish, i.e., when $\mathcal{M}$ is an invariant submanifold of the unconstrained geodesic flow.

There are also cases of nonzero Lagrange multipliers. This means that $\mathcal{M}$ is the invariant submanifold of some other Hamiltonian system. In particular, in Example 2.1 one can take a geodesic flow of a bi-invariant metric. Note that the Lagrange multipliers, in general, are different from zero ($\sum_{i=1}^{\rho} \lambda_i a^i = -[x, \mathcal{A}_{\mathfrak{d}}(x)]$) and $\mathcal{M}$ is not an invariant submanifold of the unconstrained geodesic flow of the left–invariant metric $ds^2_{\mathcal{I}}$.

Below we concentrate on the first case (zero Lagrange multipliers). Suppose that the orthogonal complement $\mathfrak{h}$ of $\mathfrak{d}$ is the Lie algebra $\mathfrak{k}$ of a Lie subgroup $\mathfrak{H}$ and that the operator $\mathcal{A}$ also preserves orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{d}$, i.e., $\mathcal{A} = \mathcal{A}_{\mathfrak{h}} + \mathcal{A}_{\mathfrak{d}}$. Then $\mathfrak{m} = \mathcal{I}(\mathfrak{d}) = \mathfrak{d}$ and EPS equations take the form (2.6).

Further, suppose that $\langle x, \mathcal{A}_{\mathfrak{d}}(x)\rangle$ is an invariant of the adjoint action of $\mathfrak{H}$ on $\mathfrak{d}$,

$$\langle [\mathfrak{h}, x], \mathcal{A}_{\mathfrak{d}}(x)\rangle = 0, \quad \text{or, equivalently,} \quad [x, \mathcal{A}_{\mathfrak{d}}(x)]_{\mathfrak{h}} = 0, \quad x \in \mathfrak{d}.$$

Then one can easily check that $\mathfrak{d}$ is the invariant subspace of the Euler equations

$$\dot{x} = [x, \nabla H(x)] = [x, \mathcal{A}(x)], \quad x \in \mathfrak{g}. \tag{2.16}$$

Therefore the Lagrange multipliers vanish. Moreover, one can consider the new Hamiltonian function $H^*(x) = \frac{1}{2}\langle x, \mathcal{A}_{\mathfrak{d}}(x)\rangle$ on $\mathfrak{g}$ and the Euler equations

$$\dot{x} = [x, \nabla H^*(x)] = [x, \mathcal{A}_{\mathfrak{d}}(x)], \quad x \in \mathfrak{g}. \tag{2.17}$$

In both cases, the restriction of the systems to $\mathfrak{d}$ coincides with the Euler–Poincaré-Suslov equations (2.6). For example, after projection to $\mathfrak{h}$ and $\mathfrak{d}$ the system (2.17) becomes

$$\dot{x}_{\mathfrak{d}} = [x_{\mathfrak{d}}, \mathcal{A}_{\mathfrak{d}}(x_{\mathfrak{d}})] + [x_{\mathfrak{k}}, \mathcal{A}_{\mathfrak{d}}(x_{\mathfrak{d}})], \quad \dot{x}_{\mathfrak{h}} = 0.$$

Let $h,\ h^* : T^*G \to \mathbb{R}$ be the functions obtained by left translations from H and H^*. While h is the Hamiltonian of the geodesic flow of the left-invariant metric $ds^2_{\mathcal{I}}$, the function h^* is degenerate in momenta and has another geometric meaning.

Suppose that $\mathfrak{d}$ generates the Lie algebra $\mathfrak{g}$ by commutations. Then, by the Chow–Rashevski theorem, any two points on G can be joined by a piecewise smooth admissible curve $g(t)$. Locally shortest admissible curves are called *sub-Riemannian geodesic lines* of the sub-Riemannian metric obtained by restriction of the given left-invariant metric $ds^2_{\mathcal{I}}$ to D. The Hamiltonian flow of h^* on T^*G is a sub-Riemannian geodesic flow. In other words, the projection of the flow to G give us sub-Riemannian geodesic lines (for more details see [51, 53]). Such systems are also known as vaconomic systems [2].

We summarize previous considerations in the following proposition (see [31]).

Proposition 2.3. *Suppose that $H|_{\mathfrak{d}}$ is an invariant of $\mathfrak{H}$-adjoint action and that $\mathfrak{d}$ generates the algebra $\mathfrak{g}$ by commutations. Then on the constrained submanifold $\mathcal{M}$ the following three different problems have the same flow: the nonholonomic geodesic flow, the geodesic flow with Hamiltonian h and the sub-Riemannian geodesic flow with Hamiltonian h^*.*

An example on the Lie group SU(n). Let us illustrate how the special case of the construction given in the Theorem 2.2 produces a nonholonomic geodesic flow with the above property. Namely, consider the chain of subalgebras

$$su(2) \subset su(3) \subset \cdots \subset su(n)$$

given by the natural matrix embedding. Let $su(2+i) = su(2) + \mathfrak{w}_i$ be the orthogonal decompositions and let $\mathcal{A}$ has the form

$$\mathcal{A} = s_0 Id_{su(2)} + s_1 \cdot Id_{\mathfrak{w}_1} + \cdots + s_{n-2} \cdot Id_{\mathfrak{w}_{n-2}}, \quad s_i > 0, \quad i = 0, \ldots, n-2\,. \tag{2.18}$$

We take $\mathfrak{d}$ to be the orthogonal complement to the Lie algebra of the maximal torus $\mathbb{T}^{n-1} \subset SU(n)$ consisting of diagonal matrices. Then the Hamiltonian $H = \frac{1}{2}\langle x, \mathcal{A}x\rangle$ will be an invariant of the adjoint action of $\mathbb{T}^{n-1}$ on $su(n)$ (see, e.g., [10]) and $\mathfrak{d}$ will generate $so(n)$ by commutations. Thus the system satisfies the conditions of Proposition 2.3.

Furthermore the system is integrable and can be considered as a Chaplygin system as well. Namely, let h be the corresponding left invariant Hamiltonian function on $T^*SU(n)$. Since H is adjoint $\mathbb{T}^{n-1}$–invariant, we have that h is also *right* $\mathbb{T}^{n-1}$–invariant function.

Thus, the group $\mathbb{T}^{n-1}$ acts on Riemannian manifold $(SU(n), ds^2_{\mathcal{I}})$ by isometries. By submersion, the metric $ds^2_{\mathcal{I}}$, induces the $SU(n)$-invariant metric $ds^2_{\mathcal{I},sub}$ on the flag manifold $F_n = SU(n)/\mathbb{T}^{n-1}$

$$\begin{array}{ccc} \mathbb{T}^{n-1} & \longrightarrow & SU(n) \\ & & \downarrow \quad \pi \ . \\ & & F_n = SU(n)/\mathbb{T}^{n-1} \end{array} \tag{2.19}$$

Note that the horizontal spaces of the submersion coinside with those of the distribution D. In other words, we can also consider $(SU(n), ds^2, D, \mathbb{T}^{n-1})$ as an example of a Chaplygin system such that the right hand side of (1.7) is equal to zero (see Remark 1.1).

The geodesic flow of the metric $ds^2_{\mathcal{I},sub}$ on F_n is completely integrable (see [10]). To describe the motion on the whole phase space D one must solve the reconstruction problem. Since the group $\mathbb{T}^{n-1}$ is Abelian, this can easily be done by quadratures (e.g., see [42]).

3 LR Systems

3.1 LR Systems as Generalized Chaplygin Systems

Following [54, 55], one defines an *LR system* on a compact Lie group G as a nonholonomic Lagrangian system (G, l, D) where l is a left-invariant Lagrangian and D is a *right-invariant* distribution on TG. As in LL systems, the Lagrangian is defined by a left-invariant metric $ds^2_{\mathcal{I}}$, $l(g, \dot g) = \frac{1}{2}\langle \mathcal{I}\omega, \omega\rangle$, $\omega = g^{-1}\cdot \dot g$.

The distribution D is determined by its restriction $\mathfrak{d}$ to the Lie algebra,

$$D_g = \mathfrak{d}\cdot g = g\cdot (g^{-1}\cdot \mathfrak{d}\cdot g) \subset T_g G.$$

Let $\mathfrak{h} = \operatorname{span}\{a^1, \dots, a^\rho\}$ be the orthogonal complement of $\mathfrak{d}$ with respect to $\langle\cdot,\cdot\rangle$. Then the right-invariant constraints can be written as

$$\begin{gathered} \omega \in g^{-1}\cdot \mathfrak{d}\cdot g, \quad \text{or} \quad \langle \omega, g^{-1}\cdot a^i\cdot g\rangle = 0, \qquad i = 1, \dots, \rho, \\ \text{or, equivalently,} \quad \langle \alpha^i, \mathcal{A}(x)\rangle = 0, \qquad \alpha^i = g^{-1}\cdot a^i \cdot g. \end{gathered} \tag{3.1}$$

Equations (1.3) in the left trivialization take the form

$$\dot x = [x, \mathcal{A}(x)] + \sum_{i=1}^{\rho} \lambda_i \alpha^i, \tag{3.2}$$

$$\dot g = g\cdot \omega = g\cdot \mathcal{A}(x). \tag{3.3}$$

Here the Lagrange multipliers λ_i are determinated by differentiating the constraints.

The system (3.2), (3.3) is actually defined on the whole phase space TG and has first integrals

$$f_i(g, x) = \langle \mathcal{A}(x), g^{-1}\cdot a^i\cdot g\rangle, \qquad i = 1, \dots, \rho.$$

Then the nonholonomic geodesic flow is just the restriction of (3.2), (3.3) onto the invariant submanifold $\mathcal{M} = \{(g, x) \mid f_i = 0,\ i = 1, \dots, \rho\}$.

Instead of (3.2), (3.3), one can consider the following closed system on the direct product $\mathfrak{g}^{1+\rho}$ in the variables $\{x, \alpha^1, \dots, \alpha^\rho\}$,

$$\dot x = [x, \mathcal{A}(x)] + \sum_{i=1}^{\rho} \lambda_i \alpha^i, \tag{3.4}$$

$$\dot\alpha^i = [\alpha^i, \mathcal{A}(x)], \qquad i = 1, \dots, \rho, \tag{3.5}$$

where the multipliers λ_i are determined from the conditions $\frac{d}{dt}\langle \alpha^i, \mathcal{A}(x)\rangle = 0$.

Equations (3.5) imply that $\alpha^i(t)$ belongs to the adjoint orbit $O_G(\alpha^i(t_0))$. Then, if $(x(t), \alpha^1(t), \dots, \alpha^\rho(t))$ is a solution of (3.4), (3.5) and $g(t)$ is a solution of the kinematical equation (3.3) (with appropriate initial conditions), we conclude that $(g(t), x(t))$ is a solution of the system (3.2), (3.3).

One of remarkable properties of LR systems is the existence of an invariant measure, which puts them rather close to Hamiltonian systems. Veselov and Veselova [55] proved that the system (3.4), (3.5) has an invariant measure with density

$$\sqrt{\det\left(\langle \mathcal{A}(\alpha^i), \alpha^j\rangle\right)}. \tag{3.6}$$

This implies that the original system (3.2), (3.3) on TG also has an invariant measure of the form $\mu(g)\cdot d\sigma$, where $d\sigma$ is the canonical volume form on TG and

$$\mu(g) = \sqrt{\det\left(\langle \mathcal{A}(g^{-1}\cdot a^i \cdot g), g^{-1}\cdot a^j\cdot g\rangle\right)}.$$

In particular, our nonholonomic geodesic flow on $\mathcal{M}$ also has an invariant measure described in the following way. Let $d\,\Sigma$ be a volume form on $\mathcal{M}$. Then

$$d\sigma = \theta\ df_1 \wedge \cdots \wedge df_\rho \wedge d\,\Sigma, \quad (g,x)\in\mathcal{M} \tag{3.7}$$

for some positive function θ. Next, let $\mathcal{L}$ be the Lie derivative with respect to the flow (3.2), (3.3). Since the functions f_s are first integrals, we have $\mathcal{L}df_s = 0$, $s = 1,\dots,\rho$. As a result, from the condition $\mathcal{L}(\mu\ d\sigma) = 0$ and (3.7) we obtain $df_1\wedge\cdots\wedge df_\rho \mathcal{L}(\mu\theta\ d\Sigma) = 0$. Hence, the restriction of the flow onto $\mathcal{M}$ has the invariant measure $\mu\theta\ d\Sigma$.

Reduction. Now, let the linear subspace $\mathfrak{h}$ be the Lie algebra of a subgroup $H\subset G$. Then the Lagrangian $l(g,\dot g)$ and the right-invariant distribution D are also invariant with respect to the left H-action. Consider homogeneous space $Q = H\backslash G$ of cossets $\{Hg\}$. The distribution D can be seen as a principal connection of the principal bundle

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \quad \pi\ . \\ & & Q = H\backslash G \end{array}$$

As a result, *the LR system* (G, l, D, H) *can naturally be regarded as a generalized Chaplygin system.* In order to write the reduced system on Q in a simple form, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Ad_G–invariant scalar product $\langle\cdot,\cdot\rangle$, and the spaces TQ, T^*Q by the *normal* metric, induced by the bi-invariant metric on G. Next, consider the moment mappings:

$$\phi : TG \cong T^*G \to \mathfrak{g}, \quad \Phi : TQ \cong T^*Q \to \mathfrak{g},$$

of the natural *right* actions of G on T^*G and T^*Q, respectively. We have $\phi(\dot g) = \omega = g^{-1}\cdot\dot g$ and the map Φ can be considered as a restriction of ϕ to D.

The reduced Lagrangian is, by definition, the constrained Lagrangian

$$l_c(g,\dot g) = \frac{1}{2}\langle \mathrm{pr}_{g^{-1}\mathfrak{d}g}\,\mathcal{I}(\phi(g,\dot g)), \phi(g,\dot g)\rangle\,,$$

considered on the orbit space $H\backslash D \cong T(H\backslash G)$. It follows that the reduced Lagrangian is simply given by

$$L(q,\dot q) = \frac{1}{2}\langle \mathcal{I}\Phi(q,\dot q), \Phi(q,\dot q)\rangle,$$

where $q = \pi(g)$ are local coordinates on Q (which may be redundant). This is a Lagrangian of the geodesic flow of metric which we shall denote by $ds^2_{\mathcal{I},D}$.

By using equations (1.7) one can prove the following proposition (see [24]), which is a special case of the general nonholonomic reduction procedure described in [36, 6].

Proposition 3.1. *The reduced Lagrange–d'Alambert equation describing the motion of the LR system (G, l, D) has the form*

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}, \xi \right) = \langle \mathcal{I}\Phi(q, \dot{q}), \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\Phi(q, \dot{q}), \Phi(q, \xi)] \rangle, \tag{3.8}$$

for all virtual displacements $\xi \in T_q Q$, where $\mathrm{pr}_{g^{-1}\mathfrak{h}g} : \mathfrak{g} \to g^{-1}\mathfrak{h}g$ is the orthogonal projection, and $q = \pi(g)$.

In addition, it appears that the reduced LR system (3.8) also possesses an invariant measure (note that a generic Chaplygin system does not have this property, see [13]). Namely, the following general statement holds (e.g., see [26]).

Lemma 3.2. *Suppose there is a compact group $\mathfrak{G}$ acting freely on a manifold N with local coordinates z and there is a $\mathfrak{G}$–invariant dynamical system $\dot{z} = Z(z)$ on N. If this system has an invariant measure (which is not necessary $\mathfrak{G}$-invariant), then the reduced system on the quotient manifold $N/\mathfrak{G}$ also has an invariant measure.*

3.2 Veselova Problem, an Integrable Geodesic Flow on the Sphere and the Neumann Problem

Veselova problem. The most descriptive illustration of an LR system is the *Veselova problem* on the motion of a rigid body about a fixed point under the action of nonholonomic constraint

$$(\Omega, \gamma) = 0, \tag{3.9}$$

where $\Omega \in \mathbb{R}^3$ is the angular velocity vector, $\gamma \in \mathbb{R}^3$ is a unit vector, which is fixed in a space frame, and $(\,,\,)$ denotes the scalar product in $\mathbb{R}^3$ [54]. Geometrically this condition means that the projection of the angular velocity of the body to a fixed vector must equal zero.

The equations of motion in the moving frame have the form

$$\mathcal{I}\dot{\Omega} = \mathcal{I}\Omega \times \Omega + \lambda\gamma, \quad \dot{\gamma} = \gamma \times \Omega, \tag{3.10}$$

where $\mathcal{I}$ is the inertia tensor of the rigid body, $\times$ denotes the vector product in $\mathbb{R}^3$, and λ is a Lagrange multiplier chosen such that $\Omega(t)$ satisfies the above constraint,

$$\lambda = -\frac{(\mathcal{I}\Omega \times \Omega, \mathcal{I}^{-1}\gamma)}{(\mathcal{I}^{-1}\gamma, \gamma)}\,. \tag{3.11}$$

The Veselova system (3.9), (3.10) is an LR system on the Lie group $SO(3)$, which is the configuration space of the rigid body motion. After identification of Lie algebras $(\mathbb{R}^3, \times)$ and $(so(3), [\cdot, \cdot])$, the operator $\mathcal{I}$ induces the left-invariant metric ds_I^2. The angular velocity correspond to $\Omega = g^{-1}\dot{g}$, the velocity in the left trivialization $TSO(3) \cong SO(3) \times so(3)$. The vector fixed in the space corresponds to the right-invariant vector field $\gamma_g = g \cdot (g^{-1} \cdot a \cdot g) \in T_g SO(3)$, $a \in so(3)$, and the nonholonomic constraint (3.9) has the form $\langle g^{-1} \cdot a \cdot g, \Omega \rangle = 0$. Once can check that the closed system (3.10), (3.11) has invariant measure with density $(\gamma, \mathcal{I}^{-1}\gamma)$, as predicted by formula (3.6). Note that integrable potential perturbations of the Veselova system can be found in [54, 26].

Multidimensional Veselova problem. Now we proceed to a n-dimensional generalization of the Veselova system, describing the motion on the Lie group $SO(n)$ with certain right-invariant nonholonomic constraints.

Let, as above, $e_1, \dots, e_n$ be unit vectors that form a fixed orthogonal frame in the space $\mathbb{R}^n$. Then, similarly to the generalized Suslov problem in Section 2, we define n-dimensional analog of (3.9) as follows: only infinitesimal rotations in the fixed 2-planes spanned by $(e_1, e_2), \dots, (e_1, e_n)$ are allowed. This implies the constraints

$$\langle \omega, e_i \wedge e_j \rangle = 0, \qquad 2 \le i < j \le n. \tag{3.12}$$

Equivalently, consider the right-invariant distribution D on $TSO(n)$ whose restriction to the algebra $so(n)$ is given by $\mathfrak{d} = \operatorname{span}\{E_j \wedge E_k,\ k = 1, \dots, r,\ j = 1, \dots, n\}$, where $E_i \wedge E_j$ form the basis in $so(n)$. Since $e_i \wedge e_j = g^{-1} \cdot E_i \wedge E_j \cdot g$, we have that constraints are $\omega \in \mathcal{D} = g^{-1} \cdot \mathfrak{d} \cdot g = \operatorname{span}\{e_1 \wedge e_i,\ 2 \le i \le n\}$.

Remark 3.1. As for the multidimensional Suslov problem, the constraints (3.12) can be relaxed. However, in this case, the existence of the integrable LR system is still not known. That is why we keep using the above constraints (see Theorem 3.4).

The LR system can be described by the Euler–Poincaré equations (3.2, 3.3, 3.4) on the space $so(n) \times SO(n)$ with indefinite multipliers λ_{pq},

$$\begin{aligned} &\frac{d}{dt}(\mathcal{I}\omega) = [\mathcal{I}\omega, \omega] + \sum_{2 \le p < q \le n} \lambda_{pq}\, e_p \wedge e_q, \\ &\dot{e}_i + \omega e_i = 0, \qquad i = 1, \dots, n. \end{aligned} \tag{3.13}$$

Here, as above, the components of $e_1, \dots, e_n$ play the role of redundant coordinates on $SO(n)$.

Reduction. The orthogonal complement $\mathfrak{h}$ of $\mathfrak{d}$ is a Lie algebra, namely

$$\mathfrak{h} = \operatorname{span}\{E_p \wedge E_q,\ 2 \le p < q \le n\} \cong so(n-1).$$

Therefore, the Veselova system can be treated as a generalized Chaplygin system on the principal bundle

$$\begin{array}{ccc} SO(n-1) & \longrightarrow & SO(n) \\ & & \downarrow \quad \pi\ , \\ & & S^{n-1} = SO(n-1)\backslash SO(n) \end{array} \tag{3.14}$$

where S^{n-1} is the n-dimensional sphere, realized as the unit sphere in $\mathbb{R}^n$,

$$S^{n-1} = \{q \in \mathbb{R}^{n-1},\ q_1^2 + \cdots + q_n^2 = 1\},$$

where we set $q = e_1$. The moment map is then $\omega = \Phi(q, \dot{q}) = q \wedge \dot{q}$. Thus, for solution $e_1(t)$, $\omega(t) = e_1(t) \wedge \dot{e}_1(t)$ of (3.13), $q(t) = e_1(t)$ is a motion of a reduced system on the sphere S^{n-1}.

The invariant measure. It appears that for some special inertia tensors, many of the calculations takes an especially simple form. Suppose that the operator $\mathcal{I}$ is defined by a diagonal matrix $A = \operatorname{diag}(A_1, \dots, A_n)$ in the following way

$$\mathcal{I}(E_i \wedge E_j) = \frac{A_i A_j}{\det A} E_i \wedge E_j. \tag{3.15}$$

Notice that for $n = 3$ this corresponds to the well known three-dimensional vector formula $I(x \times y) = (\det A)^{-1} Ax \times Ay$, $A = I^{-1}$.

Under the condition (3.15) the reduced Lagrangian $L(q,\dot q)$ and the right hand side of the Lagrange-d'Alambert equation (3.8) take the form

$$L = \frac{1}{2\det A}[(A\dot q,\dot q)(Aq,q) - (Aq,\dot q)^2]\,, \tag{3.16}$$

$$\begin{aligned}\langle \mathcal{I}\Phi(q,\dot q), \mathrm{pr}_{g^{-1}\mathfrak{h}g}[\Phi(q,\dot q),\Phi(q,\xi)]\rangle &= \frac{1}{\det A}\langle Aq\wedge A\dot q, \mathrm{pr}_{g^{-1}\mathfrak{h}g}\,\xi\wedge\dot q\rangle \\ &= \frac{1}{\det A}(\dot q, A\dot q)(Aq,\xi) - \frac{1}{\det A}(\dot q, Aq)(A\dot q,\xi) = \Psi(q,\dot q,\xi)\,. \end{aligned}\tag{3.17}$$

Here we used relation $\mathrm{pr}_{g^{-1}\mathfrak{h}g}\,\xi\wedge\dot q = \xi\wedge\dot q$ for any admissible vector $\xi = (\xi_1,\ldots,\xi_n)^T \in T_qS^{n-1}$. Below we shall keep using the redundant coordinates q_i and velocities $\dot q_i$, in which the Lagrange equations have the form

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \pi_i + \Lambda q_i\,, \qquad i = 1,\ldots,n, \tag{3.18}$$

$$\pi_i = \frac{\partial\Psi}{\partial\xi_i} = \frac{1}{\det A}(\dot q, A\dot q)A_i q_i - \frac{1}{\det A}(\dot q, Aq)A_i\dot q_i,$$

where Λ is a Lagrange multiplier.

Now we want to represent the reduced LR system on T^*S^{n-1} as a restriction of a system on the Euclidean space $\mathbb{R}^{2n} = \{q,p\}$. Note that $L(q,\dot q)$ is degenerate in the redundant velocities $\dot q$, hence they cannot be expressed uniquely in terms of the redundant moments

$$p_i = \frac{\partial L}{\partial \dot q_i} \equiv \frac{1}{\det A}(q, Aq)A_i\dot q_i - \frac{1}{\det A}(\dot q, Aq)A_i q_i. \tag{3.19}$$

In this case one can apply the Dirac formalism for Hamiltonian systems with constraints in the phase space (see, e.g., [18, 2, 43]). Namely, from (3.19) we find that $(q,p) = 0$, hence the cotangent bundle T^*S^{n-1} is realized as a subvariety of $\mathbb{R}^{2n} = (q,p)$ defined by constraints

$$\phi_1 \equiv (q,q) = 1, \quad \phi_2 \equiv (q,p) = 0.$$

Under these conditions, relations (3.19) can be uniquely inverted to yield

$$\dot q = \frac{\det A}{(q,Aq)}\left[A^{-1}p - (p, A^{-1}q)q\right]\,. \tag{3.20}$$

On the other hand, we note that $\partial L/\partial q_i = \pi_i$. Then, from (3.18) we obtain $\dot p = -\Lambda q$ and, from the condition $(\dot q, p) + (q,\dot p) = 0$,

$$\dot p = -\Lambda q, \qquad \Lambda = \det A\frac{(p,A^{-1}p) - (p,q)(q,A^{-1}p)}{(q,Aq)}\,. \tag{3.21}$$

The system (3.20), (3.21) on T^*S^{n-1} coincides with the restriction of the following system on $\mathbb{R}^{2n} = \{q,p\}$

$$\dot q_i = \{q_i,\hat H\}_*, \quad \dot p_i = \{p_i,\hat H\}_* - \hat\pi_i\,,$$

$$\hat\pi_i(q,p) = \pi_i(q,\dot q(q,p)), \quad \hat H = \frac{1}{2}\det A\frac{(p,A^{-1}p)}{(q,Aq)}\,,$$

which is quasi-Hamiltonian with respect to the following Dirac bracket on $\mathbb{R}^{2n}$

$$\{F,G\}_* = \{F,G\} + \frac{\{F,\phi_1\}\{G,\phi_2\} - \{F,\phi_2\}\{G,\phi_1\}}{\{\phi_1,\phi_2\}},$$

$\{\cdot,\cdot\}$ being the standard Poisson bracket on $\mathbb{R}^{2n}$. This system has explicit vector form

$$\begin{aligned} \dot q &= \frac{\det A}{(q,Aq)}\left[A^{-1}p - \frac{(p,A^{-1}q)}{(q,q)}q\right], \\ \dot p &= -\det A \frac{(p,A^{-1}p)(q,q)-(p,q)(q,A^{-1}p)}{(q,Aq)(q,q)^2}\, q\,. \end{aligned} \tag{3.22}$$

The bracket $\{\cdot,\cdot\}_*$ is degenerate and possesses Casimir functions ϕ_1,ϕ_2 specified above.

Now, we can find the explicit form of the invariant measure of the reduced system. From (3.22) we find

$$\sum_{i=1}^n \left(\frac{\partial \dot q_i}{\partial q_i} + \frac{\partial \dot p_i}{\partial p_i}\right) = -(n-2)\frac{\det A\,(p,A^{-1}q)}{(q,q)\,(q,Aq)},$$

which, in view of (3.19), takes the form $(n-2)(q,A\dot q)/(q,Aq)$. Hence the extended system (3.22) possesses an invariant measure

$$\mathcal{J} = (Aq,q)^{-(n-2)/2}\, dp_1\wedge dq_1\wedge\cdots\wedge dp_n\wedge dq_n.$$

Next, at points of T^*S^{n-1}, the standard volume form in $\mathbb{R}^{2n}$ can be represented as

$$dp_1\wedge dq_1\wedge\cdots\wedge dp_n\wedge dq_n = \mathbf{w}^{n-1}\wedge d\Phi_1\wedge d\Phi_2,$$

where $\mathbf{w}$ is the restriction of the standard symplectic form $dp_1\wedge dq_1+\cdots+dp_n\wedge dq_n$ onto T^*S^{n-1} and Φ_1,Φ_2 are certain functions of the Casimir functions ϕ_1,ϕ_2. Since the latter are invariants of the vector field $V(p,q)$ given by (3.22), the Lie derivatives $\mathcal{L}_V d\Phi_1, \mathcal{L}_V d\Phi_2$ equal zero. Then, since $\mathcal{L}_V\mathcal{J}=0$, we conclude that on T^*S^{n-1},

$$\mathcal{L}_V\left[(Aq,q)^{-(n-2)/2}\,\mathbf{w}^{n-1}\right]=0.$$

As a result, we arrive at the following theorem.

Theorem 3.3. *The reduced LR system (3.20, 3.21) on T^*S^{n-1} possesses an invariant measure*

$$f(q) = (Aq,q)^{-(n-2)/2}\,\sigma, \qquad \sigma = \mathbf{w}^{n-1}$$

where σ is the canonical volume $2(n-1)$-form on T^*S^{n-1}.

Chaplygin reducing multiplier. As follows from Theorem 3.3, item 1) of Theorem 1.2, and the fact that the dimension of the reduced configuration manifold equals $n-1$, if our reduced LR system on T^*S^{n-1} were transformable to a Hamiltonian form by a time reparameterization, then the corresponding reducing multiplier $\mathcal{N}$ should be proportional to $1/\sqrt{(q,Aq)}$.

Although Chaplygin's reducibility theorem does not admit a straightforward multidimensional generalization, i.e., item 1) of Theorem 1.2 cannot be inverted, remarkably, for our reduced LR system on T^*S^{n-1} the inverse statement becomes applicable (see [26]).

Theorem 3.4. *1). Under the time substitution $d\tau=\sqrt{\det A/(Aq,q)}\,dt$ and an appropriate change of momenta, the reduced LR system (3.18) or (3.20), (3.21) becomes a Hamiltonian system describing a geodesic flow on S^{n-1} with the Lagrangian*

$$L^*(q,dq/d\tau) = \frac{1}{2}(q,Aq)^{-1}\left[\left(A\frac{dq}{d\tau},\frac{dq}{d\tau}\right)(Aq,q) - \left(Aq,\frac{dq}{d\tau}\right)^2\right]. \tag{3.23}$$

2). For $A_1 < A_2 < \cdots < A_n$ the latter system is algebraic completely integrable for any dimension n. In the spheroconic coordinates $\lambda_1, \ldots, \lambda_{n-1}$ on S^{n-1} such that

$$q_i^2 = \frac{(I_i - \lambda_1) \cdots (I_i - \lambda_{n-1})}{\prod_{j \neq i} (I_i - I_j)}, \qquad I_i = A_i^{-1} \tag{3.24}$$

the Lagrangian $L^*(q, dq/d\tau)$ takes the Stäckel form

$$L^* = \frac{1}{8} \sum_{k=1}^{n-1} \frac{\prod_{s \neq k} (\lambda_k - \lambda_s)}{(\lambda_k - I_1) \cdots (\lambda_k - I_n)\, \lambda_k} \left(\frac{d}{d\tau} \lambda_k \right)^2,$$

and the evolution of λ_k is described by the Abel–Jacobi quadratures

$$\frac{\lambda_1^{k-1} d\lambda_1}{2\sqrt{R(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^{k-1} d\lambda_{n-1}}{2\sqrt{R(\lambda_{n-1})}} = \delta_{k,n-1} \sqrt{2h}\, d\tau, \qquad k = 1, \cdots, n-1, \tag{3.25}$$

where

$$R(\lambda) = -(\lambda - I_1) \cdots (\lambda - I_n) \lambda (\lambda - c_2) \cdots (\lambda - c_{n-1}), \tag{3.26}$$

$h = L^*$ being the energy constant and $c_2, \cdots, c_{n-1}$ being other constants of motion (we set $c_1 = 0$). For generic values of these constants the corresponding invariant manifolds are $(n-1)$-dimensional tori.

The item 1) of Theorem 3.4 is based on the relation between the reduced LR system to the celebrated Neumann system (see Theorem 3.5 below).

Namely, consider the iso-energy submanifold $\mathcal{E}_h = \{L(q, \dot{q}) = h\} \subset TS^{n-1}$ of the reduced Veselova system (3.18) and introduce another new time τ_1 by formula

$$d\tau_1 = \sqrt{\det A \frac{2h}{(Aq, q)}}\, dt. \tag{3.27}$$

Theorem 3.5. *Under the time substitution (3.27), the solutions $q(t)$ of the reduced multidimensional Veselova system on S^{n-1} lying on the $\mathcal{E}_h$ transforms to the solution of the integrable Neumann problem with the potential $U(q) = \frac{1}{2}(A^{-1} q, q)$,*

$$\frac{d^2}{d\tau_1^2} q = -\frac{1}{A} q + \lambda q \tag{3.28}$$

corresponding to zero value of the integral

$$F_0 = \left(A \frac{dq}{d\tau_1}, \frac{dq}{d\tau_1} \right) (Aq, q) - \left(Aq, \frac{dq}{d\tau_1} \right)^2 - (Aq, q) \tag{3.29}$$

and vise versa.

For $n = 3$, Theorem 3.5 is proved by Veselov and Veselova [55]. The proof for arbitrary dimensions is given in [26].

3.3 Reconstructed Motion on D

Now we consider the integrability of the original (unreduced) LR system on the right-invariant distribution $D \subset TSO(n)$, which is specified by constraints (3.12) and the left-invariant metric given by (3.15). The relation between the reduced LR system and the

Neumann system described by Theorem 3.5 appears to be useful to reconstruct the motion on D exactly. For this purpose we also shall make use of the correspondence between the Neumann system and the geodesic flow on a quadric (see Knörrer [35]). Namely, consider a family of $(n-1)$-dimensional confocal quadrics in $\mathbb{R}^n$,

$$Q(\alpha) = \left\{ \frac{X_1^2}{\alpha - A_1} + \cdots + \frac{X_n^2}{\alpha - A_n} = -1 \right\}, \quad \alpha \in \mathbb{R}. \tag{3.30}$$

Theorem 3.6. ([35]). *Let $X(s)$ be a geodesic on the quadric $Q(0)$, s being a natural parameter. Then under the change of time*

$$ds = \sqrt{\frac{(dX/ds, A^{-1}dX/ds)}{(X, A^{-2}X)}}\, d\tau_1 \tag{3.31}$$

the unit normal vector $q(\tau_1) = A^{-1}X/|A^{-1}X|$ is a solution to the Neumann system (3.28) corresponding to zero value of the integral (3.29) and vise versa.

It is well known that the problem of geodesics on a quadric $Q(0)$ is completely integrable, and qualitative behavior of the geodesics is described by the remarkable *Chasles theorem* (see e.g., [35, 43]): the tangent line $\ell_s = \{X(s) + \sigma\, dX/ds \mid \sigma \in \mathbb{R}\}$ of a geodesic $X(s)$ on $Q(0)$ is also tangent to a fixed set of confocal quadrics $Q(\alpha_2), \dots, Q(\alpha_{n-1}) \subset \mathbb{R}^n$, where $\alpha_2, \dots, \alpha_{n-1}$ are parameters playing the role of constants of motion (we set $\alpha_1 = 0$). Now let $\mathfrak{n}_k$ be the normal vector of the quadric $Q(\alpha_k)$ at the touching point $\mathfrak{p}_k = \ell \cap Q(\alpha_k)$. Then another classical theorem of geometry says that the normal vectors $\mathfrak{n}_1, \dots, \mathfrak{n}_{n-1}$, together with the unit tangent vector $\gamma = dX/ds$, form an orthogonal basis in $\mathbb{R}^n$.

On the other hand, in [43], Moser made the following observation.

Proposition 3.7. *1). Let x be the position vector of a point on the line ℓ_s, which is tangent to geodesic $X(s)$. Then in the new parameterization s_1 such that $ds = -(X, A^{-2}X)\, ds_1$ the evolution of the line is described by the Lax equations in $n \times n$ matrix form*

$$\frac{d}{ds_1}\mathcal{L} = [\mathcal{L}, \mathcal{B}], \qquad \mathcal{L} = \Pi_\gamma (A - x \otimes x) \Pi_\gamma, \tag{3.32}$$

$$\mathcal{B} = A^{-1}x \otimes A^{-1}\gamma - A^{-1}\gamma \otimes A^{-1}x\,, \tag{3.33}$$

where $\Pi_\gamma = Id - (\gamma, \gamma)^{-1}\gamma \otimes \gamma$ is the projection onto the orthogonal complement of γ in $\mathbb{R}^n$.

2). The conserved eigenvalues of $\mathcal{L}$ are given by the parameters $\alpha_1 = 0, \alpha_2, \dots, \alpha_{n-1}$ of the confocal quadrics and by an extra zero. The corresponding eigenvectors are parallel to the normal vectors $\mathfrak{n}_1 = q, \dots, \mathfrak{n}_{n-1}$, and to γ.

Now we are ready to describe generic solutions of the original LR system on $D \subset TSO(n)$. Let $q(\tau_1)$ be the solution of the Neumann system (3.28) with $F_0(q, q') = 0$, which is associated to a solution $(q(t), p(t))$ of the reduced LR system as described by Theorem 3.5. Let

$$X = (q, Aq)^{-1/2} Aq(s),\ \mathfrak{n}_1 = q(s), \dots, \mathfrak{n}_{n-1}(s),\ \gamma(s) = \frac{dX}{ds} \tag{3.34}$$

be the corresponding geodesic on $Q(0)$ in the new parameterization s given by (3.31) and the unit eigenvectors of $\mathcal{L}$. Also, according to (3.27) and (3.31) we can treat s as a known functions of the original time t. Then we have the following reconstruction theorem (see [26]).

Theorem 3.8. *A solution $(g(t), \dot g(t))$ of the original LR system on the distribution D is given by the momentum map $\omega(t) = q \wedge \dot q$ and the orthogonal frame formed by the unit vectors*

$$e_1 = q(t),\ e_2 = \mathfrak{n}_2(t),\ \dots,\ e_{n-1} = \mathfrak{n}_{n-1}(t), \quad e_n = \gamma(t).$$

The other solutions $(g(t), \dot g(t))$ that are projected onto the same trajectory $(q(t), p(t))$ have the same ω, e_1, while the rest of the frame is obtained by the orthogonal transformations,

$$(e_2(t) \cdots e_n(t)) = (\mathfrak{n}_2(t) \cdots \mathfrak{n}_{n-1}(t)\, \gamma(t))\, \mathfrak{R}, \tag{3.35}$$

where the constant matrix $\mathfrak{R}$ ranges over the group $SO(n-1)$.

Thus, from Theorems 3.8, 3.5 and the integrability properties of the Neumann system on T^*S^{n-1} we conclude that the phase space $D \subset T\,SO(n)$ of the multidimensional Veselova LR system with the left-invariant metric defined by (3.15) is almost everywhere foliated by $(n-1)$-dimensional invariant tori, on which the motion is straight-line but not uniform.

3.4 Veselova Problem with Integrable Potentials and the Maupertuis Principle

The Maupertuis principle. Consider a natural mechanical system on a compact Riemannian manifold (Q, ds^2) with Hamiltonian $h(q,p) = \frac{1}{2}\sum g^{ij}(q)p_ip_j + v(q)$, where g^{ij} is the inverse of the metric tensor and $v(q)$ is a smooth potential on Q. Let By the classical *Maupertuis principle*, the integral trajectories of the Hamiltonian vector field X_h with $h(q,p) = c > \max v(q)$ coincide (up to a reparametrization) with the trajectories of another vector field X_{h^J} with Hamiltonian

$$h^J(q,p) = \frac{1}{2}\sum \frac{g^{ij}(q)}{c - V(q)} p_ip_j$$

on the fixed iso-energy level $\mathcal{E}_c = \{h(q,p) = c\} = \{h^J(q,p) = 1\}$. Namely, on $\mathcal{E}_c$ we have $dh = (c - v)dh^J$ (see [2]). The Hamiltonian flow of h^J is the geodesic flow of the Jacobi metric $ds_J^2 = (c - v(q))ds^2$,which is conformally equivalent to the original metric ds^2.

The Maupertuis principle can naturally be formulated for nonholonomic systems as well. Suppose the distribution D is locally defined by $\rho = n - k$ independent 1-forms α^i. Then the equations of the nonholonomic systems with Hamiltonians h and h^J subjected to the constraints $\dot q \in D_q$ are given by

$$\dot p_i = -\frac{\partial h}{\partial q_i} + \sum_{i=1}^{\rho} \lambda_j \alpha^j(q)_i\,, \quad \dot q_i = \frac{\partial h}{\partial p_i}\,, \qquad i = 1, \dots, n, \tag{3.36}$$

$$\dot p_i = -\frac{\partial h^J}{\partial q_i} + \sum_{i=1}^{\rho} \mu_j \alpha^j(q)_i\,, \quad \dot q_i = \frac{\partial h^J}{\partial p_i}\,, \qquad i = 1, \dots, n. \tag{3.37}$$

On the iso-energy level $\mathcal{E}_c$, the vector fields (3.36) and (3.37) are proportional and the Lagrange multipliers satisfy the relation $\lambda_i = \mu_i(c - v)$ (see [36]).

One can verify that the construction goes through the Chaplygin reduction. This property can be used in producing non-trivial nonholonomic geodesic flows on $SO(n)$ which, after the $SO(n-1)$-reduction, give rise to integrable systems on the sphere S^{n-1}.

In the case of Hamiltonian systems, under a similar reduction, the Kovalevskaya and Goryachev–Chaplygin integrable cases of rigid body dynamics result in integrable geodesic flows on S^2 that possess additional polynomial integrals of degree 4 and 3 in momenta respectively (see [9]).

Veselova problem with potentials. Now let us go back to the n-dimensional Veselova problem and suppose that the n-dimensional rigid body is placed in an axisymmetric potential force field $v = v(e_1)$ (recall that $\{e_1, \dots, e_n\}$ are redundant coordinates on $SO(n)$). Then the equations of motion have the form

$$\frac{d}{dt}(\mathcal{I}\omega) = [\mathcal{I}\omega, \omega] + \frac{\partial v}{\partial e_1} \wedge e_1 + \sum_{2\le p<q\le n} \lambda_{pq}\, e_p \wedge e_q,$$
$$\dot{e}_i + \omega e_i = 0, \qquad i = 1, \dots, n, \tag{3.38}$$

togeteher with the constraints (3.12).

The potential is $SO(n-1)$–invariant and induces a well defined reduced potential $V(q)$ on the sphere S^{n-1}. Here $V(q) \equiv v(e_1)|_{e_1=q}$. The perturbed reduced system with the inertia tensor (3.15) has the same Chaplygin reducing multiplier as the nonperturbed one. Therefore, in the new time τ, the reduced system becomes a natural mechanical system on the sphere with the kinetic energy (3.23) and the potential $V(q)$.

Let $A_1 < \dots < A_n$. It is known, that the most general separable potentials compatible with the metric (3.23) in the variables $\{\lambda_1, \dots, \lambda_{n-1}\}$ have the form

$$V = \sum_{k=1}^{n-1} \frac{\Delta_k}{\prod_{s\neq k}(\lambda_k - \lambda_s)}, \tag{3.39}$$

where Δ_k are functions of the variable λ_k only (see [33]). Note that this potentials are of the same form as the potentials compatible with the standard metric in the same coordinates (e.g., see [56]). Then, if Δ_k is a Laurent polynomial in the variable λ_k, then the potential (3.39) is a Laurent polynomial in the coordinates variables $q_1, \dots, q_n$ (see, e.g., [33, 19, 56]). In particular, the reduced Veselova problem with potential

$$V(q) = \alpha_1(A^{-1}q, q) + \alpha_2((A^{-1}q, A^{-1}q) - (A^{-1}q, q)^2) + \sum_{i=1}^{n} \frac{\alpha_{i+2}}{q_i^2},$$

α_i being arbitrary constants, is completely integrable.

Now assume that $v(e_1) = \alpha_1(A^{-1}e_1, e_1) + \alpha_2((A^{-1}e_1, A^{-1}e_1) - (A^{-1}e_1, e_1)^2)$ and that the total energy is bigger than $\max_{SO(n)} v$. Let, as above, $ds^2_{\mathcal{I}}$ be the left–invariant metric given by the inertia operator (3.15) and introduce the Jacobi metric $ds^2_J = (c - v(e_1))ds^2_{\mathcal{I}}$. From the above considerations and the Maupertuis principle we get the following result.

Theorem 3.9. *The $SO(n-1)$-reduction of the the nonholonomic geodesic flow of the metric ds^2_J with the constraints (3.12) is completely integrable. The phase space T^*S^{n-1} is almost everywhere foliated by invariant $(n-1)$–dimensional Lagrangian tori with nonuniform quasi–periodic dynamics.*

The Lagrange case. In general, the operator (3.15) is not a physical inertia operator of a multidimensional rigid body. However, by taking $A_1 = \dots = A_{n-1}$, $A_n > A_1/2$ we get

$$\mathcal{I}\omega = I\omega + \omega I, \quad I = \mathrm{diag}(I_1, \dots, I_1, I_n), \quad I_1 = \frac{A_1^2}{2\det A}, \quad I_n = \frac{A_1 A_n}{\det A} - \frac{A_1^2}{2\det A}.$$

In this case the system (3.38) represents the motion of a symmetric rigid body under the nonholonomic constraints.

In the presence of the homogeneous gravitational force field in the direction e_1 we have $v = Mg(C, e_1)$, where g is the gravitational constant, M is the mass and $C = (C_1, \dots, C_n)$ is the position of the center of mass of the body. If the mass center is placed on the axis of the dynamical symmetry, then $v = MgC_n e_{1n}$ and the system (3.38) represents a multidimensional version of the Lagrange top (see [5]).

In the new time τ, the reduced system is completely integrable according to a non-commutative version of the Liouvilee theorem. Appart from the Hamiltonian function, there are integrals arrising from the $SO(n-1)$–symmetry of the system,

$$q_i\tilde{p}_j - q_j\tilde{p}_i, \qquad 1 \le i < j \le n-1.$$

As a result, the reduced phase space T^*S^{n-1} is foliated by two-dimensional invariant tori.

Note that there is an another generalization of a heavy rigid body ([48]), which is based on the generalization of the three–dimensional Euler–Poisson equations to the Euler–Poisson equations on the semi-direct product $so(n) \times so(n)$.

4 L+R Systems

4.1 Definition and Invariant Measure of L+R Systems

It appears that LR systems on a unimodular Lie group G can be viewed as a limit case of certain artificial systems on the same group, which also possess an invariant measure. The latter systems do not have a straightforward mechanical or geometric interpretation and arise as a "distortion" of a geodesic flow on G whose kinetic energy is given by a sum of a left- and right-invariant metrics.

Geodesic flow on G with L+R metric. In addition to the nondegenerate linear operator $\mathcal{I}$ defining the left-invariant metric $(\cdot,\cdot)_{\mathcal{I}}$, introduce a constant linear operator $\Gamma^0 : \mathfrak{g} \to \mathfrak{g}$ defining a right-invariant metric $(\cdot,\cdot)_\Gamma$ on the n-dimensional compact Lie group G: for any vectors $u, v \in T_gG$ we put $(u,v)_\Gamma = \langle ug^{-1}, \Gamma^0 vg^{-1}\rangle$. We take the sum of both metrics and consider the corresponding geodesic flow on G described by the Lagrangian

$$l(\omega, g) = \frac{1}{2}\langle \omega, \mathcal{I}\omega\rangle + \frac{1}{2}\langle g\omega g^{-1}, \Gamma^0\, g\omega g^{-1}\rangle \equiv \frac{1}{2}\langle \omega, \mathcal{I}\omega\rangle + \langle \omega, \Gamma(g)\omega\rangle,$$

where $\Gamma(g) = Ad_{g^{-1}}\Gamma^0 Ad_g$ and Ad_g is regarded as a matrix operator acting on $\mathfrak{g}$.

Suppose that the total inertia operator $\mathcal{B}(g) = \mathcal{I} + \Gamma(g)$ is nondegenerate and positive definite on the whole group G. The geodesic motion on the group is described by the Euler–Poincaré equations

$$\dot{x} = [x, \omega] + g^{-1}\frac{\partial l}{\partial g}, \qquad x = \frac{\partial l}{\partial \omega} = \mathcal{B}\omega, \tag{4.1}$$

together with the kinematic equation $\dot{g} = g \cdot \omega$.

In order to find explicit expression for $g^{-1}(\partial l/\partial g)$, we first note that for any $Y \in \mathfrak{g}$,

$$\langle Y, g^{-1}(\partial l/\partial g)\rangle = v_Y(l),$$

where v_Y is the left-invariant vector field on G generated by Y. Since the metric $(\cdot,\cdot)_{\mathcal{I}}$ is left-invariant, we have

$$v_Y(l) = \frac{1}{2}v_Y(\langle \omega, \Gamma\omega\rangle) = \frac{1}{2}\langle \omega, \Gamma \mathrm{ad}_Y\omega + \mathrm{ad}_Y^T\Gamma\omega\rangle = \langle \Gamma\omega, [Y,\omega]\rangle = \langle Y, \mathrm{ad}_\omega\,\Gamma\,\omega\rangle.$$

As a result, $g^{-1}(\partial l/\partial g) = \mathrm{ad}_\omega\,\Gamma\,\omega$.

Also, in view of the definition of Γ, its evolution is given by $n \times n$ matrix equation

$$\dot{\Gamma} = \Gamma \mathrm{ad}_\omega + \mathrm{ad}_\omega^T\Gamma. \tag{4.2}$$

Note that for compact group we have $\operatorname{ad}_\omega^T = -\operatorname{ad}_\omega$, and $\dot\Gamma = [\Gamma, \operatorname{ad}_\omega]$.

Equations (4.1), (4.2) form a closed system on the space $\mathfrak{g} \times \mathbf{Symm}(n)$ with the coordinates ω_i, Γ_{ij}, $i \le j = 1, \dots, n$. Indeed, since $\mathcal{B}$ is nondegenerate, the derivative $\dot\omega$ is uniquely defined from (4.1).

L+R systems. Now we modify equations (4.1) by rejecting the term $g^{-1}(\partial l/\partial g)$. As a result, we obtain another system on the space $\mathfrak{g} \times \mathbf{Symm}(n)$

$$\frac{d}{dt}(\mathcal{B}\omega) = \operatorname{ad}_\omega^T \mathcal{B}\omega, \quad \frac{d}{dt}\Gamma = \Gamma \operatorname{ad}_\omega + \operatorname{ad}_\omega^T \Gamma, \qquad \mathcal{B} = \mathcal{I} + \Gamma. \tag{4.3}$$

This is generally non a Lagrangian system, and, in contrast to equations (4.1), (4.2), it possesses the "momentum" integral $\langle \mathcal{B}\omega, \mathcal{B}\omega\rangle$. In view of the structure of the kinetic energy, we shall refer to the system (4.3) as *L+R system* on G.

Theorem 4.1. *The L+R system (4.3) possesses the kinetic energy integral* $\frac{1}{2}\langle \omega, \mathcal{B}\omega\rangle$ *and an invariant measure* $\mu\, d\omega_1 \wedge \cdots \wedge \omega_n \wedge d\Gamma_{11} \wedge \cdots \wedge \Gamma_{nn}$ *with density*

$$\mu = \sqrt{\det(\mathcal{I} + \Gamma)}\,. \tag{4.4}$$

Remark 4.1. The L+R systems can be also naturally considered on non-compact groups. Then Theorem 4.1 holds for unimodular groups as well. Recall that the group G is unimodular if $\operatorname{tr} \operatorname{ad}_\omega = 0$.

Proof of Theorem 4.1. First, replace $\frac{d}{dt}(\mathcal{B}\omega)$ with $\mathcal{B}\dot\omega + \dot\Gamma\omega$. Then, using (4.2) and the identity $\operatorname{ad}_\omega \omega = 0$, we can represent equations (4.3) in the form

$$\mathcal{B}\,\dot\omega = \operatorname{ad}_\omega^T \mathcal{I}\omega, \quad \dot\Gamma = \Gamma \operatorname{ad}_\omega + \operatorname{ad}_\omega^T \Gamma. \tag{4.5}$$

Using this form, we compute

$$\begin{aligned}\frac{d}{dt}\langle \omega, \mathcal{B}\omega\rangle &= 2\langle \omega, \mathcal{B}\dot\omega\rangle + \langle \omega, \dot{\mathcal{B}}\omega\rangle \\ &= 2\langle \omega, \operatorname{ad}_\omega^T \mathcal{I}\omega\rangle + \langle \omega, \Gamma \operatorname{ad}_\omega \omega + \operatorname{ad}_\omega^T \Gamma\omega\rangle = 0.\end{aligned}$$

i.e., $\langle \omega, \mathcal{B}\omega\rangle$ is a first integral.

Next, divergence Δ of the phase flow of the system is calculated by the formula

$$\Delta = \sum_{i \le j}^{n} \frac{\partial \dot\Gamma_{ij}}{\partial \Gamma_{ij}} + \sum_{i=1}^{n} \frac{\partial \dot\omega_i}{\partial \omega_i}. \tag{4.6}$$

In view of (4.2), the first sum equals $\sum_{i\le j}^n [(\operatorname{ad}_\omega)_{jj} + (\operatorname{ad}_\omega)_{ii}] = 0$. Then we can write

$$\Delta = \operatorname{tr}(\mathcal{B}^{-1}U), \qquad U_{ij} = \frac{\partial(\operatorname{ad}_\omega^T \mathcal{I}\omega)_i}{\partial \omega_j}, \quad i, j = 1, \dots, n.$$

As follows from the first equation in (4.3), here we can put $U = \operatorname{ad}_{\mathcal{I}\omega} + \operatorname{ad}_\omega^T \mathcal{I}$.

In view of symmetry of $\mathcal{B}^{-1}$, the skew symmetric part of U does not contribute to the expression for Δ. The symmetric part of U has the form

$$U^+ \equiv \frac{1}{2}(U + U^T) = \frac{1}{2}\left(\operatorname{ad}_\omega^T(\mathcal{B} - \Gamma) + (\mathcal{B} - \Gamma)\operatorname{ad}_\omega\right).$$

As a result, taking into account (4.2), we obtain

$$\begin{aligned}\Delta = \operatorname{tr}(\mathcal{B}^{-1}U^{+}) &= \frac{1}{2}\operatorname{tr}\left(\mathcal{B}^{-1}\operatorname{ad}_{\omega}^{T}\mathcal{B} + \operatorname{ad}_{\omega} - \mathcal{B}^{-1}\dot{\Gamma}\right) \\ &= -\frac{1}{2}\operatorname{tr}(\mathcal{B}^{-1}\dot{\Gamma}) = -\frac{1}{2}\operatorname{tr}(\mathcal{B}^{-1}\dot{\mathcal{B}}).\end{aligned}$$

Now, using the unimodularity condition $\operatorname{tr}\operatorname{ad}_{\omega} = 0$ and the well-known identity

$$\frac{d}{dt}\det\mathcal{B} = \det\mathcal{B}\,\operatorname{tr}(\mathcal{B}^{-1}\dot{\mathcal{B}}), \tag{4.7}$$

we conclude that $\mu = \sqrt{\det\mathcal{B}}$ satisfies the Liouville equation $\frac{d}{dt}(\ln\mu) + \Delta = 0$, which establishes the theorem.

Chaplygin's sphere. One of the best known examples of nonholonomic systems with an invariant measure is the celebrated Chaplygin sphere. It described a dynamically non-symmetric ball rolling without sliding on a horizontal plane. The center of the mass is assumed to be at the geometric center. Under these condition the motion is integrable ([15, 17]).

It appears that a reduction of Chaplygin's sphere can be regarded as a $L + R$ system. Namely, the original configuration space is $\mathbb{R}^2 \times SO(3)$ and the nonholonomic constraints define a $SE(2)$-invariant three-dimensional distribution. Then one can regard the system as an a Chaplygin system on the trivial bundle $\mathbb{R}^2 \times SO(3) \to SO(3)$. After the $\mathbb{R}^2$-reduction we obtain a system on $TSO(3)$, which, written in the body frame, takes the following vector form

$$\begin{gathered}\dot{K} = K \times \Omega, \qquad K = J\Omega + ma^2\Omega - ma^2(\Omega,\gamma)\gamma \\ \dot{\alpha} = \alpha \times \Omega, \quad \dot{\beta} = \beta \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega,\end{gathered} \tag{4.8}$$

where J, a, m, are the inertia operator, radius, and mass of the ball respectively. Next, Ω is vector of the angular velocity and K is vector of the angular momentum at the contact point; α, β, γ are unit vectors forming a fixed orthonormal frame in space, γ is assumed to be vertical vector. The components of these vectors can be regarded as redundant coordinates on $SO(3)$.

Equations (4.8) can be resolved with respect to $\dot{\Omega}$ to give

$$\begin{gathered}\mathcal{I}\dot{\Omega} = \mathcal{I}\Omega \times \Omega + \frac{ma^2}{1 - ma^2(\gamma, \mathcal{I}^{-1}\gamma)}(\mathcal{I}\Omega \times \Omega, \mathcal{I}^{-1}\gamma)\,\gamma, \\ \dot{\alpha} = \alpha \times \Omega, \quad \dot{\beta} = \beta \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega.\end{gathered} \tag{4.9}$$

After usual identification of the Lie algebras $(\mathbb{R}^3, \times)$ and $(so(3), [\cdot,\cdot])$, the system (4.8) can be seen as a L+R system (4.3) on $SO(3)$ with left invariant metric given by the $\mathcal{I} : so(3) \to so(3)$ and right invariant degenerate operator $\Gamma = -ma^2\gamma \otimes \gamma$. According to Theorem 4.1, in the space (Ω, γ) the above equations have an invariant measure with density $\sqrt{\det(\mathcal{I} - ma^2\gamma \otimes \gamma)}$. Up to a constant factor, it equals $\sqrt{1 - ma^2(\gamma, \mathcal{I}^{-1}\gamma)}$, the expression given by Chaplygin in [15].

Note that, in contrast to what was belived earlier, Chaplygin's sphere cannot be represented as an LR system on the group $SE(3)$ (see [49]).

It is interesting that equations (4.8) are Hamiltonian with respect to a certain nonlinear brackets (see Borisov and Mamaev [11, 12]).

4.2 The spherical Support

The Chaplygin sphere admits an integrable generalization on the configuration space $SO(3)$. Namely, consider the motion of a dynamically nonsymmetric ball $\mathcal{S}$ with the unit radius around its fixed center. Suppose that the ball touches N arbitrary dynamically symmetric balls whose centers are also fixed, and there is no sliding at the contacts points. We call this mechanical construction *the spherical support* ([22, 23], see Figure 4.1).

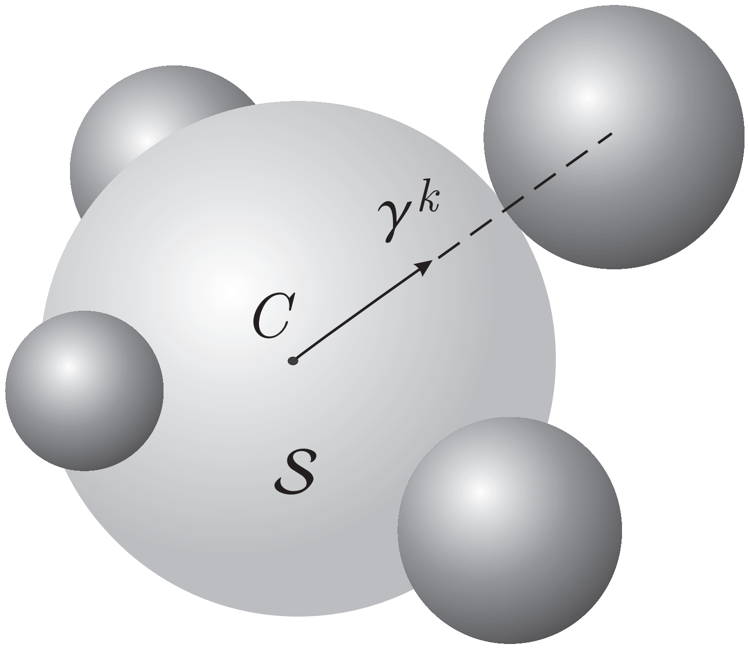

Figure 4.1: The spherical support

Let $\Omega \in \mathbb{R}^3$ and $J : \mathbb{R}^3 \to \mathbb{R}^3$ be respectively the angular velocity vector and the inertia tensor of the ball $\mathcal{S}$ in a frame attached to the ball. Next, let $\mathbf{w}^k \in \mathbb{R}^3, D_k, \rho_k \in \mathbb{R}$ be the angular velocity, the central inertia moment and the radius of the kth peripheral ball, γ^k be the fixed *unit* vector directed from the center C of the ball $\mathcal{S}$ to the point of contact with the kth ball, R^k be the reaction force at this point acting on $\mathcal{S}$. Then the equations of motion of the total mechanical system can be written in the form

$$J\dot{\Omega} + \Omega \times J\Omega = \sum_{k=1}^{N} \gamma^k \times R^k, \quad D_k \dot{\mathbf{w}}^k = -\rho_k \gamma^k \times R^k, \qquad k = 1, \dots, N, \tag{4.10}$$

where, as above, $\times$ denotes the standard vector product in $\mathbb{R}^3$. Note that the first equation is taken in the moving frame, whereas the other equations are taken in *a fixed* frame.

The reaction forces are due to nonholonomic constraints expressing the absence of sliding at the contact points. This means that velocity of the point of contact of the ball $\mathcal{S}$ with the kth ball, $\Omega \times \gamma^k$, is the same as the velocity of the corresponding point on the kth ball, i.e., $\mathbf{w}^k \times (-\rho_k \gamma^k)$. Multiplying the velocities by the vectors γ^k we obtain the constraints in form

$$\rho_k(\mathbf{w}^k, \gamma^k)\gamma^k - \rho_k \mathbf{w}^k = \Omega - (\Omega, \gamma^k)\gamma^k, \quad k = 1, \dots, N.$$

By differentiating the constraints *in the fixed frame* and taking into account $\dot{\gamma}^k = 0$,

$(\mathbf{w}^k, \gamma^k) = \text{const}$, we get

$$\rho_k \dot{\mathbf{w}}^k = -\dot{\Omega} + (\dot{\Omega}, \gamma^k)\gamma^k$$

and, in view of (4.10),

$$\gamma^k \times R^k = -\frac{D_k}{\rho_k^2}[(\dot{\Omega}, \gamma^k)\gamma^k - \dot{\Omega}].$$

Substituting this into the first equation in (4.10) and using the fact that the time derivatives of Ω in the moving and the fixed frames are the same, we obtain

$$\mathcal{I}\dot{\Omega} + \Omega \times J\Omega = -\Gamma\dot{\Omega}, \tag{4.11}$$

$$\mathcal{I} = J - \sum_{k=1}^{N} \frac{D_k}{\rho_k^2}\mathbf{I}, \quad \Gamma = \sum_{k=1}^{N} \frac{D_k}{\rho_k^2}\gamma^k \otimes \gamma^k, \tag{4.12}$$

where $\mathbf{I}$ is the 3×3 identity matrix and Γ is the 3×3 symmetric matrix, which is fixed in the space. For $N \geq 3$ and a general location of the peripheral balls, it is nondegenerate, hence its components can be regarded as redundant coordinates on the group $SO(3)$. Since the evolution of γ^k in the moving frame is described by the Poisson equations $\dot{\gamma}^k = \gamma^k \times \Omega$, from (4.12) we have

$$\dot{\Gamma} = [\Gamma, \omega], \tag{4.13}$$

where $\omega \in so(3)$ is the 3×3 skew-symmetric matrix such that $\omega_{ij} = \varepsilon_{ijk}\Omega_k$.

Now we consider the motion of the central ball $\mathcal{S}$ only. As follows from (4.11)-(4.13), equations of motion can be represented in the form of an L+R system on the group $SO(3)$,

$$\dot{K} = K \times \Omega, \quad \dot{\Gamma} = [\Gamma, \omega], \tag{4.14}$$
$$K = (\mathcal{I} + \Gamma)\Omega \in \mathbb{R}^3.$$

Notice that from here $\dot{\Omega}$ can be uniquely expressed in terms of the components of Ω, Γ, hence (4.14) represents a closed system of differential equations. One can say that it describes the free rotation of a "generalized Euler top", whose tensor of inertia is a sum of two components: one is fixed in the body and the other one is fixed in the space.

Theorem 4.2. *The spherical support system* (4.14) *is integrable by the Euler–Jacobi theorem, and its generic invariant manifolds are two-dimensional tori.*

Indeed, we can put $\Gamma = a\alpha \otimes \alpha + b\beta \otimes \beta + c\gamma \otimes \gamma$, where α, β, γ are unit vectors forming a fixed orthonormal frame in the space and a, b, c are some constants, which can be uniquely determined from (4.12). Then the matrix equation in (4.14) can be replaced by the vector equations

$$\dot{\alpha} = \alpha \times \Omega, \quad \dot{\beta} = \beta \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega. \tag{4.15}$$

From the form of equations (4.14), (4.15) we immediately obtain four first integrals

$$(K, K), \quad (K, \alpha) = (\mathcal{I}_a\omega, \alpha), \quad (K, \beta) = (\mathcal{I}_b\omega, \beta), \quad (K, \gamma) = (\mathcal{I}_c\omega, \gamma),$$
$$\mathcal{I}_a = \mathcal{I} + a\mathbf{I}, \quad \mathcal{I}_b = \mathcal{I} + b\mathbf{I}, \quad \mathcal{I}_c = \mathcal{I} + c\mathbf{I},$$

of which any three integrals are independent. In addition, the system has trivial geometric integrals

$$(\alpha, \alpha) = 1, \quad (\beta, \beta) = 1, \quad (\gamma, \gamma) = 1,$$
$$(\alpha, \beta) = 0, \quad (\alpha, \gamma) = 0, \quad (\beta, \gamma) = 0$$

and the kinetic energy integral

$$\frac{1}{2}(\omega, (\mathcal{I}+\Gamma)\omega) = \frac{1}{2}(\omega, \mathcal{I}\omega) + \frac{a}{2}(\omega,\alpha)^2 + \frac{b}{2}(\omega,\beta)^2 + \frac{c}{2}(\omega,\gamma)^2.$$

Next, according to Theorem 4.1, the system also possesses an invariant measure with density

$$\mu = \sqrt{\det(\mathcal{I}+\Gamma)} = \sqrt{\det\mathcal{I}}\biggl(1 + a(\alpha,\mathcal{I}^{-1}\alpha) + b(\beta,\mathcal{I}^{-1}\beta) + c(\gamma,\mathcal{I}^{-1}\gamma) + \\ + \frac{bc}{\det\mathcal{I}}(\alpha,\mathcal{I}\alpha) + \frac{ac}{\det\mathcal{I}}(\beta,\mathcal{I}\beta) + \frac{ab}{\det\mathcal{I}}(\gamma,\mathcal{I}\gamma) + \frac{abc}{\det\mathcal{I}}\biggr)^{1/2}. \qquad (4.16)$$

This together implies the integrability by the Euler–Jacobi theorem. Notice that for the case of only one peripheral ball, the L+R system (4.14) has the same form as Chaplygin's ball system (4.8).

4.3 Limits of L+R Systems

As mentioned above, a nonholonomic LR system on a Lie group G can be obtained as a limit case of a certain L+R system on this group. Indeed, suppose that the operator $\Gamma : \mathfrak{g} \to \mathfrak{g}$ defining a right-invariant metric on G is *degenerate* and has the form

$$\Gamma = \epsilon(\alpha^1 \otimes \alpha^1 + \cdots + \alpha^\rho \otimes \alpha^\rho), \qquad \rho < n, \quad D = \text{const} > 0, \qquad (4.17)$$

where, as in (3.1), $\alpha^1, \ldots, \alpha^\rho$ are orthonormal right-invariant vector fields $\alpha^i = g^{-1} \cdot a^i \cdot g$, $a^i = \text{const} \in g$, generating a right-invariant distribution D on TG.

Now consider the L+R system (4.3) on the space $(\omega, \alpha^1, \ldots, \alpha^\rho)$. In view of (4.5), it can be represented in form

$$\mathcal{I}\dot\omega = \mathcal{I}(\mathcal{I}+\Gamma)^{-1}\mathrm{ad}^T_\omega \mathcal{I}\omega, \quad \dot\Gamma = \Gamma \mathrm{ad}_\omega + \mathrm{ad}^T_\omega \Gamma. \qquad (4.18)$$

Then the following theorem holds (see [23]).

Theorem 4.3. *1). As $\epsilon \to \infty$, equations (4.18) transform to the Euler–Lagrange equations with multipliers (3.4) and constraints (3.1), where $x = \mathcal{I}\omega$.*

2). The density $\sqrt{\det\mathcal{B}}/\sqrt{\epsilon}$ of the invariant measure of the L+R system tends to the density (3.6) of the LR system multiplied by a constant factor.

Note that as $\epsilon \to \infty$, the original equations (4.3) become singular. For this reason, before taking the limit they must be transformed to the form (4.18).

As an illustration, consider the following $L+R$ system on $SO(3)$ (we use the usual vector notation):

$$\mathcal{I}\dot\Omega = \mathcal{I}(\mathcal{I} + \epsilon\gamma\otimes\gamma)^{-1}(\mathcal{I}\Omega \times \Omega), \qquad \dot\gamma = \gamma \times \Omega, \qquad (4.19)$$

which formally coisides with the Chaplygin sphere system (4.9) if we set $\epsilon = ma^2$. It can be easily verified that

$$\lim_{\epsilon\to\infty}(\mathbf{I} + \epsilon\gamma\otimes\mathcal{I}^{-1}\gamma)^{-1} = \mathbf{I} - \frac{1}{(\mathcal{I}^{-1}\gamma,\gamma)}\gamma\otimes\mathcal{I}^{-1}\gamma.$$

Therefore, as ϵ tends to infinity, the system (4.19) transforms to the Veselova rigid body problem (3.10).

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