A fully syntactic AC-RPO*

Albert Rubio

Universitat Politècnica de Catalunya, Dept. L.S.I.
Modul C6, C/ Jordi Girona, 1-3, 08034 Barcelona, Spain.
E-mail: rubio@lsi.upc.es
phone/fax: +34 93 4017988 / 4017014

Abstract. We present the first fully syntactic (i.e., non-interpretation-based) AC-compatible *recursive path ordering* (RPO). It is very simple, and hence easy to implement, and its behaviour is intuitive as in the standard RPO. The ordering is AC-total, and defined uniformly for both ground and non-ground terms, as well as for partial precedences. More importantly, it is the first one that can deal *incrementally* with partial precedences, an aspect that is essential, together with its intuitive behaviour, for interactive applications like Knuth-Bendix completion.

1 Introduction

Rewrite-based methods with built-in associativity and commutativity (AC) properties for some of the operators are well-known to be crucial in theorem proving and programming. Therefore a lot of work has been done on the development of suitable *AC-compatible* reduction or simplification orderings, like [DHJP83, BP85, GL86, BCL87, KSZ90, NR91, Bac92, DP93, RN95, KS97]. An essential additional property of the ordering that is needed in order to preserve the completeness of most rewrite-based theorem proving techniques (modulo AC) is *AC-totality*, i.e. the totality on (AC-different) ground terms.

Since the initial attempts, it has always been an aim to obtain AC-compatible versions of Dershowitz' *recursive path ordering* [Der82], as it is simple, easy to automatize and use, and normally orients the rules in an adequate direction. In [RN95] we gave the first RPO-based AC-total and AC-compatible reduction ordering without any restriction on the number of AC-symbols or on the precedence over the signature. Unfortunately, although being defined in terms of RPO, it does not behave like RPO; e.g. it does not orient the distributivity rule in the "right" (i.e. distributing) way, since a transformation on the terms is applied before using RPO (this approach, with different transformations, is also used in [BP85] among others). Therefore, a better approach seems to be to directly apply an RPO-like scheme, treating as the only special case the *AC-equal-top case*, that is, when both terms to be compared are headed by the same AC-symbol. In this direction the first AC-compatible simplification ordering with an RPO scheme was defined in [KSZ90] and the first one AC-total on ground terms in

* Partially supported by the ESPRIT working group CCL-II, ref. WG # 22457.

[KS97]. Other simpler proposals for AC-orderings with RPO scheme were given in [Rub97] and in [KS98].

However, all these AC-orderings need to interpret terms (apart from *flattening*) in some way, which makes their behaviour less intuitive, unlike it happens with the standard RPO, whose simple fully syntactic definition has been an important reason for its success.

In this paper we propose the first fully syntactic AC-RPO, i.e., no interpretation is needed apart from flattening. It is very simple, and hence easy to implement, and its behaviour is intuitive as for the standard RPO. The ordering is AC-total, and defined uniformly for both ground and non-ground terms, as well as for partial precedences.

Moreover, precisely due to the fact that it is not interpretation-based, it is the first AC-RPO that can deal *incrementally* with partial precedences, i.e. if $s \succ t$, then $s \succ t$ under any extension of the precedence. This aspect is essential, together with its intuitive behaviour, for interactive applications like Knuth-Bendix completion. Of course, previously existing orderings could work with partial precedences, but in a useless way, simply by considering an arbitrarily chosen total extension of the partial precedence, and hence loosing incrementality.

In order to introduce the concepts smoothly we give the ordering in three steps, first for ground terms and total precedences, then for terms with variables and total precedences and finally for terms with variables and partial precedences, each definition strictly extending the previous one. For this reason we prove all properties only for the last one, showing that it is indeed an AC-compatible simplification ordering.

The paper is organized as follows. In the following section we give some basic notions and definitions. In section 3 we introduce the ordering for ground terms and total precedences. Section 4 is devoted to terms with variables and total precedences. In section 5 we generalize the previous ordering for dealing with partial precedences and in section 6 we prove that it is an AC-compatible simplification ordering. Conclusions are given in section 7.

2 Preliminaries

In the following we consider that $\mathcal{F}$ is a finite set of function symbols that is totally ordered by a precedence $\succ_{\mathcal{F}}$, where $\mathcal{F}_{AC}$ is the subset containing all AC-symbols of $\mathcal{F}$.

The arity of a function symbol f is a natural number that indicates the number of arguments that f may take. If $f \in \mathcal{F}_{AC}$ then its arity is greater than or equal to 2. $\mathcal{T}(\mathcal{F})$ and $\mathcal{T}(\mathcal{F}, \mathcal{X})$ are defined as usual according to these arities, if $\mathcal{X}$ is a set of variables, whose elements will be denoted by $x, y, z, \ldots$, possibly with subscripts.

We denote by $=_{AC}$ the congruence generated on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ by the associativity and commutativity axioms for the symbols in $\mathcal{F}_{AC}$. In what follows we will ambiguously use $=_{AC}$ to also denote the standard extension of AC-equality to multisets (and in fact to any other structure).

A term rewriting system (TRS) is a (possibly infinite) set of rules $l \to r$ where l and r are terms. Given a TRS R, s rewrites to t with R, denoted by $s \to_R t$, if there is some rule $l \to r$ in R, $s|_p = l\sigma$ for some position p and substitution σ and $t = s[r\sigma]_p$.

In the following terms are flattened wrt. the AC-symbols. The flattening of t, denoted by $\overline{t}$, is the normal for of t wrt. the infinite TRS containing the rules

$$f(x_1, \ldots, x_n, f(y_1, \ldots, y_m), z_1, \ldots, z_r) \to f(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r)$$

for every $f \in \mathcal{F}_{AC}$ and $n, m, r \geq 0$. Due to flattening, the AC-symbols have a variable arity. We assume that all other symbols have a fixed arity.

Let s and t be two terms such that $\overline{s} = f(s_1, \ldots, s_m)$ and $\overline{t} = g(t_1, \ldots, t_n)$. If $s =_{AC} t$ then $f = g$, $m = n$ and $\overline{s}$ is equal to $\overline{t}$ up to permutation of arguments for the AC-symbols. We will denote this equality up to permutation of arguments also by $=_{AC}$. The top-flattening of a term s wrt. an AC-symbol f, denoted by $tf_f(s)$, is a string of terms defined as $tf_f(f(s_1, \ldots, s_n)) = s_1, \ldots, s_n$ and $tf_f(s) = s$ if $top(s) \neq f$.

Let s, t, s' and t' be arbitrary terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, let u be a non-empty context in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and let σ be a substitution. Then an ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ (a transitive irreflexive relation) $\succ$ is *monotonic* if $s \succ t$ implies $u[s] \succ u[t]$, and *stable under substitution* if $s \succ t$ implies $s\sigma \succ t\sigma$. Monotonic orderings that are stable under substitution are called *rewrite orderings*. An ordering $\succ$ fulfills the *subterm property* if $u[t] \succ t$ and the *deletion property* if $f(\ldots s \ldots) \succ f(\ldots\ \ldots)$ for every variadic symbol f. A rewrite ordering that fulfills the subterm property and the deletion property is called a *simplification ordering* and is *well-founded*: there are no infinite sequences $t_1 \succ t_2 \succ \ldots$ An ordering is AC-total on ground terms if when s and t are ground terms, either $s \succ t$ or $t \succ s$ or $s =_{AC} t$. Finally an ordering $\succ$ is *AC-compatible* if $s' =_{AC} s \succ t =_{AC} t'$ implies $s' \succ t'$.

Given a TRS R, a terms s rewrites to t with R modulo AC, denoted by $s \to_{R/AC} t$, if $s =_{AC} s'$, $s'|_p = l\sigma$ for some term s', position p and substitution σ, and $t =_{AC} s'[r\sigma]_p$.

Theorem 1. *Let $\succ$ be an AC-compatible simplification ordering and let R be a term rewriting system. If $l \succ r$ for all rules $l \to r$ in R then R is terminating for rewriting modulo AC.*

Given a relation $\succ$, the (AC-)lexicographic extension of $\succ$ on sequences, denoted by $\succ_{lex}$, is defined by: $\langle s_1, s_2, \ldots \rangle \succ_{lex} \langle t_1, t_2, \ldots \rangle$ if there is some s_j s.t. $s_j \succ t_j$ and we have $s_i =_{AC} t_i$ for all $i < j$.

Given a relation $\succ$, the (AC-)multiset extension of $\succ$ on finite multisets, denoted by $\succ\!\!\succ$, is defined as $M = \{s_1, \ldots, s_m\} \succ\!\!\succ \{t_1, \ldots, t_n\} = N$ if (i) $M \neq \emptyset$ and $N = \emptyset$; or (ii) $s_i =_{AC} t_j$ and $M \setminus \{s_i\} \succ\!\!\succ N \setminus \{t_j\}$, for some i in $1 \ldots m$ and j in $1 \ldots n$; or (iii) $s_i \succ t_{j_1} \wedge \ldots \wedge s_i \succ t_{j_k}$ and $(M \setminus \{s_i\} \succeq\!\!\succeq N \setminus \{t_{j_1}, \ldots, t_{j_k}\}$ for some i in $1 \ldots m$ and $1 \leq j_1 < \ldots < j_k \leq n$ $(k \geq 0)$, where $\succeq\!\!\succeq$ is the union of $\succ\!\!\succ$ and $=_{AC}$. Alternatively (and equivalently if $\succ$ is AC-compatible), it can be defined as the smallest transitive relation containing

$$X \cup \{s\} \succ\!\!\succ Y \cup \{t_1, \ldots, t_n\} \qquad \text{if } X =_{AC} Y \text{ and } s \succ t_i \text{ for all } i \in \{1 \ldots n\}$$

In general we will consider that $\succeq$ is the union of a given ordering $\succ$ and $=_{AC}$.

If $\succ$ is an AC-compatible ordering on a set S then $\succ\!\!\succ$ and $\succ_{lex}$ are respectively an AC-compatible ordering on multisets of elements in S and an AC-compatible ordering on sequences of elements in S. Being more precise, in order to fulfil transitivity we need $\succ$ to be both transitive and AC-compatible.

3 The ordering for ground terms

In this section we consider only ground terms, and assume that the precedence is total on the set of function symbols. First we introduce two different sets of terms obtained from a term headed by an AC-symbol.

Definition 2. Let s be a term of the form $f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$.

- The set of terms embedded in s through an argument headed by a small symbols, denoted by $EmbSmall(s)$, is defined as

  $$\{f(s_1, \ldots, tf_f(v_j), \ldots, s_n) \mid s_i = h(v_1, \ldots, v_r) \wedge f \succ_{\mathcal{F}} h \wedge j \in \{1 \ldots r\}\}$$

- The set of arguments of s headed by a big symbol, denoted by $BigHead(s)$, is defined as $\{s_i \mid 1 \leq i \leq n \wedge top(s_i) \succ_{\mathcal{F}} f\}$

We will now give the definition of the ordering $\succ$ on ground terms.

Definition 3. Let s and t be terms in $\mathcal{T}(\mathcal{F})$. Then $s = f(s_1, \ldots, s_n) \succ g(t_1, \ldots, t_m) = t$ if and only if

1. $s_i \succeq t$ for some $i \in \{1 \ldots n\}$.
2. $f \succ_{\mathcal{F}} g$ and $s \succ t_i$ for all $i \in \{1 \ldots n\}$.
3. $f = g \notin \mathcal{F}_{AC}$ and $\langle s_1, \ldots, s_n \rangle \succ_{lex} \langle t_1, \ldots, t_n \rangle$ and $s \succ t_i$ for all $i \in \{1 \ldots n\}$.
4. $f = g \in \mathcal{F}_{AC}$ and $s' \succeq t$ for some $s' \in EmbSmall(s)$.
5. $f = g \in \mathcal{F}_{AC}$ and $s \succ t'$ for all $t' \in EmbSmall(t)$ and $BigHead(s) \succeq\!\!\succeq BigHead(t)$ and either
   (a) $BigHead(s) \succ\!\!\succ BigHead(t)$ or
   (b) $n > m$ or
   (c) $n = m$ and $\{s_1, \ldots, s_n\} \succ\!\!\succ \{t_1, \ldots, t_m\}$.

The first three cases of this definition of course correspond to the standard RPO. Cases 4 and 5 apply when both terms are headed by the same AC-symbol f. The intuition behind them is very simple. On the one hand, in order to obtain AC-compatibility, terms are considered in flattened form. On the other, the symbols that disappear under flattening must still be taken into account in order to obtain monotonicity. Let us consider an example.

Assume $f \succ_{\mathcal{F}} g$. Then, as in the standard RPO, we have of course $f(a,a) \succ g(a)$. By monotonicity, if we add the context $f(a,[])$ and flatten, we must have $f(a,a,a) \succ f(a,g(a))$, that is, the symbol f that has been removed under flattening is important in order to "take care" of the g. The number of such *implicit* f's depends of course on the number of arguments.

But, similarly, if $g \succ_{\mathcal{F}} f$, then $g(a) \succ f(a,a)$ and by monotonicity we should have $f(a,g(a)) \succ f(a,a,a)$. Clearly, in this kind of situations where the comparison of arguments headed by big symbols is conclusive, the number of such implicit f's is not important.

This motivates the three stage hierarchy in case 5: (a) first consider the multisets of arguments headed by symbols bigger than f; (b) if these sets coincide, then compare the number of arguments (i.e., the number of implicit f's); (c) finally, if both terms are equal under the previous two measures, then we can safely compare the multisets of all (or only the small-headed ones) arguments in the usual (multiset) way.

Of course, since any simplification ordering must contain the *embedding* relation, we must have $s[g(\ldots t \ldots)]_p \succ s[t]_p$ for all s, t, g and p. This indicates that the use of $\mathit{EmbSmall}(s)$ and $\mathit{EmbSmall}(t)$ in cases 4 and 5 are no real restriction.

But the ideas of the three stage approach of case 5 can be safely applied precisely due to the precondition stating that $s \succ t'$ for all $t' \in \mathit{EmbSmall}(t)$, which prevents situations where t is a term like $f(a, h(\ldots u \ldots))$, and where by removing h (with $f \succ_{\mathcal{F}} h$) we get $f(a,u)$, where u can be headed by a big symbol, or, if u is headed by f, the number of arguments increases.

The following examples show the behaviour of the ordering when comparing terms headed by the same AC-symbol.

Example 1. Let $h \succ_{\mathcal{F}} f \succ_{\mathcal{F}} g \succ_{\mathcal{F}} a \succ_{\mathcal{F}} b$ be the precedence. Then we have

1. $f(g(f(h(a),a)),a) \succ f(h(a),a,a)$ by case 4.
2. $s = f(h(a),g(a)) \succ f(g(h(a)),a) = t$ by case 5a, since $s \succ f(h(a),a) \in \mathit{EmbSmall}(t)$ by case 4, and $\mathit{BigHead}(s) = \{h(a)\} \succ\!\succ \emptyset = \mathit{BigHead}(t)$.
3. $s = f(g(h(a)),b,b,b) \succ f(g(f(h(a),a)),a) = t$ by case 5b, since $n = 4 > 2 = m$ and $\mathit{BigHead}(s) = \emptyset = \mathit{BigHead}(t)$ and $s \succ f(h(a),a,a) = t' \in \mathit{EmbSmall}(t)$ by applying first case 4 and then $s' = f(h(a),b,b,b) \succ t'$ by case 5b, since $\mathit{BigHead}(s') = \{h(a)\} = \mathit{BigHead}(t')$, $\mathit{EmbSmall}(t') = \emptyset$ and $n = 4 > 3 = m$.
4. $s = f(h(a),a) \succ f(h(a),b) = t$, by case 5c, since we have $\mathit{EmbSmall}(t) = \emptyset$, $\mathit{BigHead}(s) = \{h(a)\} = \mathit{BigHead}(t)$, $n = m = 2$ and $\{h(a),a\} \succ\!\succ \{h(a),b\}$.

Lemma 4. *If $\succ_{\mathcal{F}}$ is total on the set of function symbols then $\succ$ is AC-total on ground terms.*

Proof. Let $s = f(s_1,\ldots,s_n)$ and $t = g(t_1,\ldots,t_m)$ be ground terms. Then either $s \succ t$ or $t \succ s$ or $s =_{AC} t$. We proceed by induction on $|s| + |t|$.

By induction hypothesis for every s_i we have either $s_i \succeq t$ or $t \succ s_i$, and for every t_j we have either $s \succ t_j$ or $t_j \succeq s$. If either there is some $s_i \succeq t$ or there is some $t_j \succeq s$, we have $s \succ t$ or $t \succ s$ by case 1. Otherwise $s \succ t_j$ for all t_j and $t \succ s_i$ for all s_i. Since the precedence is total on the function symbols and the terms are ground, then either $f \succ_{\mathcal{F}} g$ or $g \succ_{\mathcal{F}} f$ or $f = g$. If either $f \succ_{\mathcal{F}} g$ or $g \succ_{\mathcal{F}} f$ then, we have $s \succ t$ or $t \succ s$ by case 2. Otherwise $f = g$.

1. If $f \notin \mathcal{F}_{AC}$ then, by induction hypothesis, either $\langle s_1, \ldots, s_n \rangle \succ_{lex} \langle t_1, \ldots, t_n \rangle$ or $\langle t_1, \ldots, t_n \rangle \succ_{lex} \langle s_1, \ldots, s_n \rangle$ or $\langle s_1, \ldots, s_n \rangle =_{AC} \langle t_1, \ldots, t_n \rangle$, and hence either $s \succ t$ or $t \succ s$ by case 3 or $s =_{AC} t$.
2. If $f \in \mathcal{F}_{AC}$ then by induction hypothesis, either $s \succ t'$ or $t' \succeq s$ for all $t' \in \mathit{EmbSmall}(t)$; and either $t \succ s'$ or $s' \succeq t$ for all $s' \in \mathit{EmbSmall}(s)$. Therefore either $s \succ t$ or $t \succ s$ by case 4 or $s \succ t'$ for all $t' \in \mathit{EmbSmall}(t)$ and $t \succ s'$ for all $s' \in \mathit{EmbSmall}(s)$. Now by induction hypothesis, either $\mathit{BigHead}(s) \succ\!\!\succ \mathit{BigHead}(t)$ or $\mathit{BigHead}(t) \succ\!\!\succ \mathit{BigHead}(s)$ or $\mathit{BigHead}(s) =_{AC} \mathit{BigHead}(t)$. Therefore either $s \succ t$ or $t \succ s$ by case 5a or $\mathit{BigHead}(s) =_{AC} \mathit{BigHead}(t)$. In the latter case, if $m > n$ or $n > m$ then $s \succ t$ or $t \succ s$ by case 5b. Otherwise $m = n$, and by induction hypothesis, we have either $\{s_1, \ldots, s_n\} \succ\!\!\succ \{t_1, \ldots, t_n\}$ or $\{t_1, \ldots, t_n\} \succ\!\!\succ \{s_1, \ldots, s_n\}$, and hence $s \succ t$ or $t \succ s$ by case 5c, or $\{s_1, \ldots, s_n\} =_{AC} \{t_1, \ldots, t_n\}$, and hence $s =_{AC} t$. $\square$

The following theorem follows from corollary 15 and theorem 16.

Theorem 5. *$\succ$ is an AC-total AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F})$.*

4 Terms with variables

In this section we consider terms with variables, but we still assume that the precedence is total on the set of function symbols. First, due to the presence of variables, the counting of arguments has to be adapted, since one cannot know how many arguments a variable will include when instantiated and flattened.

Therefore in cases 5b and 5c, instead of n and m we will use the following notion of $\#(s)$ and $\#(t)$, and $n > m$ and $n = m$ become diophantine inequations over the positive integers.

Definition 6. Let s be a term. Then $\#(s)$ is an expression with variables on the positive integers, defined as $\#(f(s_1, \ldots, s_n)) = \#_v(s_1) + \ldots + \#_v(s_n)$, where $\#_v(x) = x$ and $\#_v(t) = 1$ if t is not a variable.

For example, we have $\#(f(x, y, g(x))) = x + y + 1 > x + y = \#(f(x, y))$, which is necessary to achieve stability under substitution.

In addition we have to replace the set $\mathit{BigHead}(s)$ by $\mathit{NoSmallHead}(s)$, which may include variables, in one of its uses.

Definition 7. Let s be a term of the form $f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$. The set of arguments of s headed by a symbol not smaller than f, denoted by $\mathit{NoSmallHead}(s)$, is defined as $\{s_i \mid 1 \leq i \leq n \land f \not\succ_{\mathcal{F}} top(s_i)\}$

Definition 8. Let s and t be terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$. Then $s = f(s_1,\ldots,s_n) \succ_v g(t_1,\ldots,t_m) = t$ if and only if

1. $s_i \succeq_v t$ for some $i \in \{1\ldots n\}$.
2. $f \succ_{\mathcal{F}} g$ and $s \succ_v t_i$ for all $i \in \{1\ldots n\}$.
3. $f = g \notin \mathcal{F}_{AC}$ and $\langle s_1,\ldots,s_n\rangle(\succ_v)_{lex}\langle t_1,\ldots,t_n\rangle$ and $s \succ_v t_i$ for all $i \in \{1\ldots n\}$.
4. $f = g \in \mathcal{F}_{AC}$ and there is some $s' \in \mathit{EmbSmall}(s)$ s.t. $s' \succeq_v t$.
5. $f = g \in \mathcal{F}_{AC}$ and $s \succ_v t'$ for all $t' \in \mathit{EmbSmall}(t)$ and $\mathit{NoSmallHead}(s) \underline{\succ\!\succ}_v \mathit{NoSmallHead}(t)$ and either
   (a) $\mathit{BigHead}(s) \succ\!\succ_v \mathit{BigHead}(t)$ or
   (b) $\#(s) > \#(t)$ or
   (c) $\#(s) \geq \#(t)$ and $\{s_1,\ldots,s_n\} \succ\!\succ_v \{t_1,\ldots,t_m\}$.

Note that the difference between $\mathit{NoSmallHead}(s)$ and $\mathit{BigHead}(s)$ is that the latter does not include the variables. Then, on the one hand, the condition $\mathit{NoSmallHead}(s) \underline{\succ\!\succ}_v \mathit{NoSmallHead}(t)$ ensures that every variable in t is taken care of by a variable in s or by a argument of s headed by a big symbol. Then if, by instantiation, some variable becomes a term headed by a big symbol, we know that some argument of the (instantiation) of s headed by a big symbol takes care of it. On the other, the condition $\mathit{BigHead}(s) \succ\!\succ_v \mathit{BigHead}(t)$, prevents us from using variables that can become small terms by instantiation. The combination of both conditions is crucial to prove stability under substitutions.

Example 2. Let $h \succ_{\mathcal{F}} f \succ_{\mathcal{F}} g$ be the precedence. Then we have

1. $f(g(f(h(x),x)),x) \succ_v f(h(x),x,x)$ by case 4.
2. $s = f(h(x),g(x)) \succ_v f(g(h(x)),x) = t$ by case 5a, since we have $s \succ_v f(h(x),x) \in \mathit{EmbSmall}(t)$ by case 4, and $\mathit{NoSmallHead}(s) = \{h(x)\} \succ\!\succ_v \{x\} = \mathit{NoSmallHead}(t)$ and $\mathit{BigHead}(s) = \{h(x)\} \succ\!\succ_v \emptyset = \mathit{BigHead}(t)$.
3. $s = f(g(h(x)),x,x,y) \succ_v f(g(f(h(x),y)),x) = t$ by case 5b, since we have $\#(s) = 2x+y+1 > x+1 = \#(t)$ and $\mathit{NoSmallHead}(s) = \{x,x,y\} \succ\!\succ_v \{x\} = \mathit{NoSmallHead}(t)$ and $s \succ f(h(x),y,x) = t' \in \mathit{EmbSmall}(t)$ by applying first case 4 and then $s' = f(h(x),x,x,y) \succ t'$ by case 5b, since $\mathit{NoSmallHead}(s') = \{h(x),x,x,y\} \succ\!\succ_v$
   $\{h(x),y,x\} = \mathit{NoSmallHead}(t')$, $\mathit{EmbSmall}(t') = \emptyset$ and $\#(s') = 2x+y+1 > x+y+1 = \#(t')$ (since x is a positive integer).
4. $s = f(g(g(x)),x) \succ f(g(x),g(x)) = t$, by case 5c, since $s \succ_v f(g(x),x) \in \mathit{EmbSmall}(t)$ by case 4, $\mathit{NoSmallHead}(s) = \{x\} \succ\!\succ_v \emptyset = \mathit{NoSmallHead}(t)$ and $\#(s) = x+1 \geq 2 = \#(t)$ and $\{g(g(x)),x\} \succ\!\succ_v \{g(x),g(x)\}$.

Lemma 9. *Let s and t be ground terms. Then $s \succ t$ if and only if $s \succ_v t$.*

Proof. The result is trivial since both definitions coincide when applied to ground terms. Note that if $s = f(s_1,\ldots,s_n)$ is ground then we have $\#(s) = n$ and $\mathit{NoSmallHead}(s) = \mathit{BigHead}(s)$. $\square$

The following theorem follows from lemma 14 and theorem 16.

Theorem 10. *$\succ_v$ is an AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F},\mathcal{X})$, that is AC-total on ground terms.*

Example 3. Rings. With $+, * \in \mathcal{F}_{AC}$ and $* \succ_{\mathcal{F}} I \succ_{\mathcal{F}} + \succ_{\mathcal{F}} 0$, the ordering orients (and hence proves termination of) the following TRS:

$$
\begin{array}{lll}
x+0 & \to & x \\
x+I(x) & \to & 0 \\
I(0) & \to & 0 \\
I(I(x)) & \to & x \\
I(x+y) & \to & I(x)+I(y) \\
x*(y+z) & \to & (x*y)+(x*z) \\
x*0 & \to & 0 \\
x*I(y) & \to & I(x*y)
\end{array}
$$

Example 4. With $+, * \in \mathcal{F}_{AC}$ and $* \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$, the ordering orients (and hence proves termination of) the following TRS:

$$
\begin{array}{lll}
x+0 & \to & x \\
x+s(y) & \to & s(x+y) \\
x*0 & \to & 0 \\
x*s(y) & \to & x*y+x \\
x*(y+z) & \to & (x*y)+(x*z)
\end{array}
$$

5 Partial precedences

First, in order to deal with partial precedences, we now weaken the multiset extension of the ordering when applied to the sets *NoSmallHead*(s) and *NoSmallHead*(t), since otherwise we cannot ensure incrementality.

Definition 11. Let $\succ$ be an ordering on terms and let $\succ_{\mathcal{F}}$ be a (partial) precedence. The multiset extension of $\succ$ wrt. an AC-symbol f in the precedence $\succ_{\mathcal{F}}$, denoted by $\ggcurly_f$, is defined as the smallest transitive relation containing

$$X \cup \{s\} \ggcurly_f Y \cup \{t_1,\ldots,t_n\} \text{ if} \begin{cases} X =_{AC} Y \text{ and } top(s) \succ_{\mathcal{F}} f \text{ and} \\ s \succ_p t_i \text{ for all } i \in \{1\ldots n\} \end{cases}$$

$$X \cup \{s\} \ggcurly_f Y \cup \{t_1,\ldots,t_n\} \text{ if} \begin{cases} X =_{AC} Y \text{ and } top(s) \not\succ_{\mathcal{F}} f \text{ and } s \succ_p t_i \\ \text{and } top(s_i) \succeq_{\mathcal{F}} top(t_i) \text{ for all } i \in \{1\ldots n\} \end{cases}$$

Now we adapt the set *EmbSmall*(s) to allow embeddings through symbols not bigger than the head.

Definition 12. The set of terms embedded in s through an argument headed by a non-big symbol, denoted by *EmbNoBig*(s), is defined as

$$\{f(s_1,\ldots,tf_f(v_j),\ldots,s_n) \mid s_i = h(v_1,\ldots,v_r) \wedge h \not\succ_{\mathcal{F}} f \wedge j \in \{1\ldots r\}\}$$

Definition 13. Let s and t be terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$. Then $s = f(s_1,\ldots,s_n) \succ_p g(t_1,\ldots,t_m) = t$ if and only if

1. $s_i \succeq_p t$ for some $i \in \{1\ldots n\}$.
2. $f \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1\ldots n\}$.
3. $f = g \notin \mathcal{F}_{AC}$ and $\langle s_1,\ldots,s_n\rangle(\succ_p)_{lex}\langle t_1,\ldots,t_n\rangle$ and $s \succ_p t_i$ for all $i \in \{1\ldots n\}$.
4. $f = g \in \mathcal{F}_{AC}$ and there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p t$.
5. $f = g \in \mathcal{F}_{AC}$ and $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(t)$ and $\mathit{NoSmallHead}(s) \underline{\ggcurly}_{pf}\, \mathit{NoSmallHead}(t)$ and either
   (a) $\mathit{BigHead}(s) \ggcurly_p \mathit{BigHead}(t)$ or
   (b) $\#(s) > \#(t)$ or
   (c) $\#(s) \geq \#(t)$ and $\{s_1,\ldots,s_n\} \ggcurly_p \{t_1,\ldots,t_m\}$.

The reason to ask for $\mathit{NoSmallHead}(s) \underline{\ggcurly}_{pf}\, \mathit{NoSmallHead}(t)$, instead of using simply $\underline{\ggcurly}_p$ is that if, by extending the precedence, an argument t' of t headed by a symbol incomparable with f becomes a term headed by a big symbol then the argument in s that takes care of t' becomes a term headed by a big symbol as well; and if, by extending the precedence, an argument s' of s headed by a symbol incomparable with f becomes a term headed by a small symbol, then all arguments in t taken care of by s' become terms headed by a small symbol as well.

Lemma 14. *Let s and t be terms. If the precedence is total then $s \succ_v t$ if and only if $s \succ_p t$.*

Proof. The result is trivial since both definitions coincide when applied with a total precedence. Note that if the precedence is total $\mathit{EmbNoBig}(s) = \mathit{EmbSmall}(s)$ and that for all $s' \in \mathit{NoSmallHead}(f(s_1,\ldots,s_n))$ either $top(s') \succ_{\mathcal{F}} f$ or s' is a variable, which implies that $\ggcurly_{pf}$ and $\ggcurly_p$ coincide. ☐

Corollary 15. *Let s and t be ground terms. If the precedence is total then $s \succ t$ if and only if $s \succ_p t$.*

The proof of the following theorem is given in the next section.

Theorem 16. *$\succ_p$ is an AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F},\mathcal{X})$, AC-total on ground terms and incremental wrt. the precedence.*

Example 5. Let f be an AC-symbol.

1. With any precedence we have $s = f(g(g(x)),x) \succ_p f(g(x),g(x)) = t$ by case 5c, since $s \succ_p t' = f(g(x),x) \in \mathit{EmbNoBig}(t)$, by case 4, and $\mathit{NoSmallHead}(s) = \{g(g(x)),x\} \underline{\ggcurly}_{pf} \{g(x),g(x)\} = \mathit{NoSmallHead}(t)$, and $\#(s) = 1 + x \geq 2 = \#(t)$ and $\{g(g(x)),x\} \underline{\ggcurly} \{g(x),g(x)\}$.
2. With precedence $g \succ_{\mathcal{F}} h$ we have $s = f(x,x,g(x)) \succ_p f(x,h(x)) = t$ by case 5b, since $s \succ_p t' = f(x,x) \in \mathit{EmbNoBig}(t)$, by case 5b, and $\mathit{NoSmallHead}(s) = \{x,x,g(x)\} \underline{\ggcurly}_{pf} \{x,h(x)\} = \mathit{NoSmallHead}(t)$, and $\#(s) = 2x + 1 > x + 1 = \#(t)$ (note that x is a positive integer).

Example 6. Milners's nondeterministic machines. With $+ \in \mathcal{F}_{AC}$ and $T \succ_{\mathcal{F}} +$ and $L \succ_{\mathcal{F}} +$, the ordering orients (and hence proves termination of) the following TRS. Note that the decision about the precedence relation between T and $+$ is not needed until the last rule.

$$\begin{array}{lll}
0 + x & \to & x \\
x + x & \to & x \\
L(T(x)) & \to & L(x) \\
L(T(y) + x) & \to & L(x + y) + L(y) \\
T(T(x)) & \to & T(x) \\
T(x) + x & \to & T(x) \\
T(x + y) + x & \to & T(x + y) \\
T(T(y) + x) & \to & T(x + y) + T(y)
\end{array}$$

6 The Properties of the Ordering

Here we include the properties of the ordering $\succ_p$. All proofs are quite simple, except the one for stability under substitutions, which involves some technical problems caused by the arguments of t that are variables instantiated by terms headed by a non-big symbol.

Property 17 *If* $s =_{AC} t$ *then* NoSmallHead$(s) =_{AC}$ NoSmallHead(t), BigHead$(s) =_{AC}$ BigHead(t) *and* $\#(s) = \#(t)$.

Lemma 18. $\succ_p$ *is AC-compatible.*

Proof. Since after flattening AC-equal terms are equal up to permutation of arguments of AC-symbols, we have to prove the compatibility of $\succ_p$ wrt. this permutative equality, which we also call $=_{AC}$. Then $s' =_{AC} s \succ_p t =_{AC} t'$ implies $s' \succ_p t'$. We proceed by induction on $|s| + |t|$ and case analysis on the proof of $s \succ_p t$.

1. If $s = f(s_1, \ldots, s_n) \succ_p t$ by case 1 then there is some $s_i \succeq_p t$, and since $s' =_{AC} s$, there is some $s'_j =_{AC} s_i$, and by induction hypothesis $s'_j \succeq_p t'$, and hence $s' \succ_p t'$ by case 1.
2. If $s \succ_p g(t_1, \ldots, t_m) = t$ by case 2 then $top(s) \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all t_i, and by induction hypothesis $s' \succ_p t'_j$ for all t'_j and $top(s') \succ_{\mathcal{F}} g$, which implies $s' \succ_p g(t'_1, \ldots, t'_m) = t'$ by case 2.
3. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_n) = t$ by case 3 then $f \notin \mathcal{F}_{AC}$ and $\langle s_1, \ldots, s_n \rangle (\succ_p)_{lex} \langle t_1, \ldots, t_n \rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots n\}$. Then by induction hypothesis, $\langle s'_1, \ldots, s'_n \rangle (\succ_p)_{lex} \langle t'_1, \ldots, t'_n \rangle$ and $s' \succ_p t'_i$ for all $i \in \{1 \ldots n\}$. Note that in this case there is no permutation.
4. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by case 4 then $f \in \mathcal{F}_{AC}$ and there is some $u \in EmbNoBig(s)$ s.t. $u \succeq_p t$ then there is some $u' \in EmbNoBig(s')$ s.t. $u =_{AC} u'$ and hence, by induction hypothesis $u' \succeq_p t'$, which implies $s' \succ_p t'$ by case 4.

5. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by case 5 then $f \in \mathcal{F}_{AC}$ and $s \succ_p u$ for all $u \in \mathit{EmbNoBig}(t)$ and $\mathit{NoSmallHead}(s) \underline{\succ\!\!\succ}_{pf} \mathit{NoSmallHead}(t)$. Since $\mathit{EmbNoBig}(t) =_{AC} \mathit{EmbNoBig}(t')$ and $\mathit{NoSmallHead}(s) = \mathit{NoSmallHead}(s')$ and $\mathit{NoSmallHead}(t) =_{AC} \mathit{NoSmallHead}(t')$, by induction hypothesis $s' \succ_p u'$ for all $u' \in \mathit{EmbNoBig}(t')$ and $\mathit{NoSmallHead}(s') \underline{\succ\!\!\succ}_{pf} \mathit{NoSmallHead}(t')$. Finally, if $s \succ_p t$ by case 5a then we have $\mathit{BigHead}(s') =_{AC} \mathit{BigHead}(s) \succ\!\!\succ_p \mathit{BigHead}(t) =_{AC} \mathit{BigHead}(t')$, which implies, by induction hypothesis, that $\mathit{BigHead}(s') \succ\!\!\succ_p \mathit{BigHead}(t')$, and hence $s' \succ_p t'$ by case 5a. If $s \succ_p t$ by case 5b, then, since $\#(s') = \#(s) > \#(t) = \#(t')$, we have $s' \succ_p t'$ by case 5b. Otherwise, $s \succ_p t$ by case 5c, and $\#(s') = \#(s) = \#(t) = \#(t')$ and $\{s'_1, \ldots, s'_n\} =_{AC} \{s_1, \ldots, s_n\} \succ\!\!\succ_p \{t_1, \ldots, t_n\} =_{AC} \{t'_1, \ldots, t'_n\}$, which implies by induction hypothesis that $\{s'_1, \ldots, s'_n\} \succ\!\!\succ_p \{t'_1, \ldots, t'_n\}$, and hence $s' \succ_p t'$ by case 5c. □

Lemma 19. *Let f be an AC-symbol. If $n > m \geq 2$ and $1 \leq i_1 < \ldots < i_m \leq m$ then $f(s_1, \ldots, s_n) \succ_p f(s_{i_1}, \ldots, s_{i_m})$.*

Proof. Let s be $f(s_1, \ldots, s_n)$ and let t be $f(s_{i_1}, \ldots, s_{i_m})$. We proceed by induction on $|s| + |t|$.

First we have that for all $t' \in \mathit{EmbNoBig}(f(s_{i_1}, \ldots, s_{i_m}))$ of the form $f(s_{i_1}, \ldots, \mathit{tf}_f(v_j), \ldots, s_{i_m})$ there is some $s' \in \mathit{EmbNoBig}(f(s_1, \ldots, s_n))$ of the form $f(s_1, \ldots, \mathit{tf}_f(v_j), \ldots, s_n)$, s.t. by induction hypothesis, $s' \succ_p t'$ and, hence, by case 4, we have $s \succ_p t'$.

Then since trivially $\mathit{NoSmallHead}(s) \underline{\succ\!\!\succ}_{pf} \mathit{NoSmallHead}(t)$ and $\#(s) > \#(t)$, by case 5b it holds. □

Lemma 20. *Let f be an AC-symbol. If $n > m \geq 2$ and $1 \leq i_1 < \ldots < i_m \leq m$ then $s \succeq_p f(t_1, \ldots, t_n)$ implies $s \succ_p f(t_{i_1}, \ldots, t_{i_m})$.*

Proof. We proceed by induction on $|s|$. If $s =_{AC} f(t_1, \ldots, t_n)$ then it holds by lemma 19 and AC-compatibility. Otherwise there are several cases acording to the cas applied in the proof of $s \succ_p f(t_1, \ldots, t_n)$.

1. If case 1 applies then $s_i \succeq_p f(t_1, \ldots, t_n)$ and by induction hypothesis $s_i \succ_p f(t_{i_1}, \ldots, t_{i_n})$, which implies $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by case 1.
2. If case 2 applies then trivially $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by case 2.
3. If case 4 applies then $s = f(s_1, \ldots, s_k)$ and there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p f(t_{i_1}, \ldots, t_{i_m})$ and by induction hypothesis $s' \succ_p f(t_{i_1}, \ldots, t_{i_m})$, which implies $s \succ_p f(t_{i_1}, \ldots, t_{i_m})$ by case 4.
4. If case 5 applies, we have that for all $t'' \in \mathit{EmbNoBig}(f(t_{i_1}, \ldots, t_{i_m}))$ of the form $f(t_{i_1}, \ldots, \mathit{tf}_f(v_j), \ldots, t_{i_m})$ there is some $t' \in \mathit{EmbNoBig}(f(t_1, \ldots, t_n))$ of the form $f(t_1, \ldots, \mathit{tf}_f(v_j), \ldots, t_n)$, and therefore, since $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(f(t_1, \ldots, t_n))$, by induction hypothesis, we have $s \succ_p t''$ for every $t'' \in \mathit{EmbNoBig}(f(t_{i_1}, \ldots, t_{i_m}))$.
   On the other hand, $\mathit{NoSmallHead}(s) \underline{\succ\!\!\succ}_{pf} \mathit{NoSmallHead}(f(t_1, \ldots, t_n))$ implies $\mathit{NoSmallHead}(s) \underline{\succ\!\!\succ}_{pf} \mathit{NoSmallHead}(f(t_{i_1}, \ldots, t_{i_m}))$.

Now if case 5a applies then we have $BigHead(s) \gg_p BigHead(f(t_{i_1}, \ldots, t_{i_m}))$, and hence $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by case 5a. Otherwise, if cases 5b or 5c apply then, since $n > m$ we have $\#(s) \geq \#(t) > \#(f(t_{i_1}, \ldots, t_{i_n}))$ and we conclude by case 5b. □

Lemma 21. *If $s \succeq_p t$ then $s \succ_p t_i$, for all t_i argument of t.*

Proof. By induction on the $|s| + |t|$. Let s be $f(s_1, \ldots, s_n)$ and t be $g(t_1, \ldots, t_m)$. If $s =_{AC} t$ then for every t_i there is some s_j s.t. $s_j =_{AC} t_i$ and therefore by case 1 we have $s \succ_p t_i$. Otherwise $s \succ_p t$ and we distinguish several cases according to the definition.

1. If $s \succ_p t$ by case 1 then $s_j \succeq_p t$ for some $j \in \{1 \ldots n\}$, and by induction hypothesis $s_j \succ_p t_i$ which implies by case 1 $s \succ_p t_i$.
2. If $s \succ_p t$ by case 2 or case 3 then $s \succ_p t_i$ by definition.
3. If $s \succ_p t$ by case 4, then there is some $s' \in EmbNoBig(s)$ s.t. $s' \succeq_p t$. By induction hypothesis $s' \succ_p t_i$, and therefore by case 4 we have $s \succ_p t_i$.
4. If $s \succ_p t$ by case 5 then there are three cases
   (a) If $f \succ_{\mathcal{F}} top(t_i)$ and t_i is a constant, it holds trivially, by case 2.
   (b) If $f \succ_{\mathcal{F}} top(t_i)$ and $t_i = h(v_1, \ldots, v_r)$ with $r > 0$, then for all $j \in \{1 \ldots r\}$ there is some $t' \in EmbNoBig(t)$ s.t. $t' = f(t_1, \ldots, tf_f(v_j), \ldots, t_m)$ and $s \succ_p t'$. Now, for all v_j either $top(v_j) \neq f$ and $tf_f(v_j) = v_j$ and hence by induction hypothesis $s \succ_p v_j$, or $top(v_j) = f$, and then by lemma 20 we have $s \succ_p f(tf_f(v_j)) = v_j$. Therefore, since $f \succ_{\mathcal{F}} h$, by case 2, we have $s \succ_p t_i$.
   (c) If $f \not\succ_{\mathcal{F}} top(t_i)$ then, since $NoSmallHead(s) \underline{\gg}_{pf} NoSmallHead(t)$ there is some s_j (with $f \not\succ_{\mathcal{F}} top(s_j)$), s.t. $s_j \succeq t_i$ and hence $s \succ_p t_i$ by case 1. □

Corollary 22. *$\succ_p$ fulfils the subterm property.*

Property 23 *Let s and t be terms. Then* $\mathrm{NoSmallHead}(s) \underline{\gg}_{pf} \mathrm{NoSmallHead}(t)$ *implies* $\mathrm{BigHead}(s) \underline{\gg}_p \mathrm{BigHead}(t)$.

Lemma 24. *$\succ_p$ is transitive.*

Proof. We prove $s \succ_p t$ and $t \succ_p u$ implies $s \succ_p u$ by induction on $|s| + |t| + |u|$ and case analysis on the definition. Let s be $f_1(s_1, \ldots, s_n)$, t be $f_2(t_1, \ldots, t_m)$ and u be $f_3(u_1, \ldots, u_p)$.

1. If $s \succ_p t$ by case 1 then there is some $s_i \succeq_p t$ and by induction hypothesis and AC-compatibility $s_i \succ_p u$, which implies by case 1 $s \succ_p u$.
2. If $t \succ_p u$ by case 1 then there is some $t_i \succeq_p u$. By lemma 21 we have $s \succ_p t_i$ and by induction hypothesis and AC-compatibility $s \succ_p u$.
3. If $s \succ_p t$ by case 2 and $t \succ_p u$ by cases 2–5 then $f_1 \succ_{\mathcal{F}} f_3$. By lemma 21, $t \succ_p u_i$ for all i, which by induction hypothesis implies $s \succ_p u_i$ for all i, and therefore by case 2 $s \succ_p u$.
4. If $s \succ_p t$ by case 3–5 and $t \succ u$ by case 2, then $f_1 \succ_{\mathcal{F}} f_3$, and we conclude as in the previous case.

5. If both $s \succ_p t$ and $t \succ_p u$ by case 3 then $\langle s_1, \ldots, s_n \rangle \succ_{lex} \langle t_1, \ldots, t_n \rangle$ and $\langle t_1, \ldots, t_n \rangle \succ_{lex} \langle u_1, \ldots, u_n \rangle$ and $t \succ_p u_i$ for all i, and by induction hypothesis $\langle s_1, \ldots, s_n \rangle \succ_{lex} \langle u_1, \ldots, u_n \rangle$ and $s \succ_p u_i$ for all i, which implies by case 3 $s \succ_p t$.
6. If $s \succ_p t$ by case 4 and $t \succ_p u$ by case 4 or 5, then there is some $s' \in EmbNoBig(s)$ s.t. $s' \succeq_p t$, and by induction hypothesis and AC-compatibility $s' \succ_p u$, which implies $s \succ_p u$ by case 4.
7. If $s \succ_p t$ by case 5 and $t \succ_p u$ by case 4 then there is some $t' \in EmbNoBig(t)$ s.t. $t' \succeq_p u$, and since $s \succ_p t'$ for all $t' \in EmbNoBig(t)$, by induction hypothesis and AC-compatibilty we have $s \succ_p u$.
8. If $s \succ_p t$ by case 5 and $t \succ_p u$ by case 5 then $f_1 = f_2 = f_3 = f$ and by induction hypothesis and AC-compatibility we have $s \succ_p u'$ for all $u' \in EmbNoBig(u)$ and $NoSmallHead(s) \underline{\gg}_{pf} NoSmallHead(u)$.
   Now if either $s \succ_p t$ or $t \succ_p u$ by case 5a then, by property 23, induction hypothesis and AC-compatibility, we have $s \succ_p u$ by case 5a. Otherwise, if either $s \succ_p t$ or $t \succ_p u$ by case 5b then we have $s \succ_p u$ by case 5b. Otherwise, $s \succ_p t$ and $t \succ_p u$ by case 5c, and then $\{s_1, \ldots, s_n\} \gg_p \{t_1, \ldots, t_m\}$ and $\{t_1, \ldots, t_m\} \gg_p \{u_1, \ldots, u_p\}$, implies, by induction hypothesis and AC-compatibility $\{s_1, \ldots, s_n\} \gg_p \{u_1, \ldots, u_p\}$, and hence $s \succ_p u$ by case 5c. □

Lemma 25. *$\succ_p$ is irreflexive.*

Proof. We proceed by induction on $|s|$. We show that we cannot apply any case to prove $s \succ_p s$.

1. If we apply case 1 then we have $s_i \succeq_p s$ then since by case 1 we have $s \succ_p s_i$, which by transitivity (and AC-compatibility) implies $s_i \succ_p s_i$, which cannot be by induction hypothesis.
2. Cases 2 and 5b cannot trivially apply.
3. If we apply case 4 then there is some $s' \in EmbNoBig(s)$ s.t. $s' \succeq_p s$, but by case 4 we have $s \succ_p s'$, which by transitivity (and AC-compatibility) implies $s' \succ_p s'$, which cannot be by induction hypothesis.
4. Finally by using the induction hypothesis on the multiset and lexicographic extension of $\succ_p$ applied on subterms, we have that neither case 3 nor 5a nor 5c can be applied. □

Lemma 26. *$\succ_p$ is monotonic.*

Proof. If $s \succ_p t$ then $\overline{f(\ldots s \ldots)} \succ_p \overline{f(\ldots t \ldots)}$ for every flattened context $f(\ldots [] \ldots)$, by induction on $|\overline{f(\ldots s \ldots)}| + |\overline{f(\ldots t \ldots)}|$. Note that if the context is not flattened, we can flatten it and then apply the result.

If f is not in $\mathcal{F}_{AC}$ then $\overline{f(\ldots s \ldots)} = f(\ldots s \ldots)$ and $\overline{f(\ldots t \ldots)} = f(\ldots t \ldots)$ and by case 3 it holds.

If f is in $\mathcal{F}_{AC}$ then $\overline{f(\ldots s \ldots)} = f(\ldots tf_f(s) \ldots) = cs$ and $\overline{f(\ldots t \ldots)} = f(\ldots tf_f(t) \ldots) = ct$. Let $u_1, \ldots, u_p$ be the arguments of the context. Then, by induction hypothesis, for every $u_i = h(v_1, \ldots, v_r)$ with $h \not\succ_{\mathcal{F}} f$ and for every $j \in \{1, \ldots, r\}$, we have that $f(\ldots tf_f(v_j) \ldots tf_f(s) \ldots) \succ f(\ldots tf_f(v_j) \ldots tf_f(t) \ldots)$, and hence by case 4 we have $f(\ldots tf_f(s)\ldots) \succ_p f(\ldots tf_f(v_j)\ldots tf_f(t)\ldots)$. Therefore $cs \succ_p t'$ for all terms $t' \in \mathit{EmbNoBig}(ct)$ which are embedded through a terms in the context (we will consider the ones that are embedded through t afterwards). Now, let $s = g_1(s_1,\ldots,s_n)$ and $t = g_2(t_1,\ldots,t_m)$, there are several cases to be considered according to the proof of $s \succ_p t$ and the head symbols of s and t.

1. If $g_1 \not\succeq_{\mathcal{F}} f$ then $tf_f(s) = s$. We consider two cases.
   (a) $s \succ_p t$ by case 1. Then $s_i \succeq t$ for some s_i, and by induction hypothesis $f(\ldots tf_f(s_i)\ldots) \succeq f(\ldots tf_f(t)\ldots)$, which by case 4 implies $cs \succ_p ct$.
   (b) $g_1 \succeq_{\mathcal{F}} g_2$. Then $g_2 \not\succeq_{\mathcal{F}} f$ and $tf(t) = t$. Since, by lemma 21, $s \succ_p t_i$ for all t_i, by induction hypothesis $cs \succ_p f(\ldots tf_f(t_i)\ldots)$ for all t_i, and hence (together with what proved above) we have $cs \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(ct)$. Then, if $f \succ_{\mathcal{F}} g_1$ then $f \succ_{\mathcal{F}} g_2$ and $\mathit{NoSmallHead}(cs) = \mathit{NoSmallHead}(ct)$. Otherwise if $f \not\succ_{\mathcal{F}} g_1$ then we have $\mathit{NoSmallHead}(cs) = X \cup \{s\} \succeq\!\!\succeq_{pf} X \cup \{t\} \subseteq \mathit{NoSmallHead}(ct)$, for some multiset X. Finally, since neither s nor t are variables, we have $\#(cs) = \#(ct)$, and then $\{\ldots,s,\ldots\} \succ\!\!\succ_p \{\ldots t \ldots\}$ implies $cs \succ_p ct$ by case 5c.
2. If $f = g_1$ then $tf_f(s) = s_1 \ldots s_n$. We consider three cases.
   (a) $s \succ_p t$ by case 1. Then $s_i \succeq t$ for some s_i, and by induction hypothesis $f(\ldots s_i \ldots) \succeq ct$ (note that $tf_f(s_i) = s_i$). By lemma 19 $cs = f(\ldots s_1 \ldots s_n \ldots) \succeq f(\ldots s_i \ldots)$ (note that the context is not empty), and therefore, by transitivity, $cs \succ_p ct$.
   (b) $g_1 \succ_{\mathcal{F}} g_2$. Then $f \succ_p g_2$ and $tf_f(t) = t$. As in case 1b of this proof, we have $cs \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(ct)$. Then, since $\mathit{NoSmallHead}(cs) \supseteq \mathit{NoSmallHead}(ct)$ and $\#(f(\ldots s_1 \ldots s_n \ldots)) > \#(f(\ldots t \ldots))$ (note that $n \geq 2$), we have $cs \succ_p ct$ by case 5b.
   (c) $g_1 = g_2$ and $s \succ_p t$ by case 4 or by case 5. Then $tf_f(t) = t_1 \ldots t_m$ and it is easy to prove that $f(\ldots s_1 \ldots s_n \ldots) \succ_p f(\ldots t_1 \ldots t_m \ldots)$ holds by the same case as $s \succ_p t$.
3. If $g_1 \succ_{\mathcal{F}} f$ then $tf_f(s) = s$. We consider three cases.
   (a) $g_2 \not\succeq_{\mathcal{F}} f$ then $tf_f(t) = t$. Again, as in case 1b, we have $cs \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(ct)$. Since $\mathit{NoSmallHead}(cs) = X \cup \{s\} \succeq\!\!\succeq_{pf} X \cup \{t\} \subseteq \mathit{NoSmallHead}(ct)$, for some multiset X, and $\mathit{BigHead}(cs) \supset \mathit{BigHead}(ct)$, we have $cs \succ_p ct$ by case 5a.
   (b) $g_2 \succ_{\mathcal{F}} f$ then $tf_f(t) = t$. Then, since $\mathit{BigHead}(cs) = \{s\} \cup X \succ\!\!\succ \{t\} \cup X = \mathit{BigHead}(ct)$ for some multiset X, we have $cs \succ_p ct$ by case 5a.
   (c) $g_2 = f$ then $g_1 \succ_{\mathcal{F}} g_2$ and $tf_f(t) = t_1 \ldots t_m$. By lemma 21 we have $s \succ_p t_i$ for all t_i, and by transitivity if $t_i = h(v_1,\ldots,v_r)$ with $f \succ_p h$ and then $s \succ_p v_j$ for all $j \in \{1 \ldots r\}$ and $s \succ_p w$ for every $w \in tf(v_j)$. Therefore since $g_1 \succ_{\mathcal{F}} f = g_2$, by case 2 we have $s \succ_p f(t_1,\ldots,tf(v_j),\ldots,t_m)$ and, by induction hypothesis, $cs \succ_p f(\ldots t_1,\ldots,tf_f(v_j),\ldots,t_m \ldots)$, and hence (together with what proved at the begining) we have $cs \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(ct)$. Finally, since we have $\mathit{NoSmallHead}(cs) = \{s\} \cup X \succeq\!\!\succeq_{pf} \mathit{NoSmallHead}(f(t_1,\ldots,t_m)) \cup X = \mathit{NoSmallHead}(ct)$, for some

multiset X, and $BigHead(cs) = \{s\} \cup Y \gg_p BigHead(f(t_1, \ldots, t_m)) \cup Y = BigHead(ct)$ for some multiset Y, we conclude $cs \succ_p ct$ by case 5a. $\square$

Property 27 *Let x be a variable and let s and t be terms. If* $\text{NoSmallHead}(s) \underline{\gg}_{pf} \text{NoSmallHead}(t)$ *and* $\text{BigHead}(s) \gg_p \text{BigHead}(t)$ *then* $\text{BigHead}(s) \cup X_s \gg_p \text{BigHead}(t) \cup X_t$*, where X_s and X_t are respectively the multisets containing all x in* $\text{NoSmallHead}(s)$ *and* $\text{NoSmallHead}(t)$*.*

Proof. Let M be $NoSmallHead(s) \setminus BigHead(s)$ and let N be $NoSmallHead(t) \setminus BigHead(t)$. Since $BigHead(s) \gg_p BigHead(t)$ and $NoSmallHead(s) \underline{\gg}_{pf} NoSmallHead(t)$, by definition of $\underline{\gg}_{pf}$, we can split N in two sets N_1 and N_2 s.t. $BigHead(s) \gg_p BigHead(t) \cup N_1$ and $M \underline{\gg} N_2$. Then let X_1 be the variables x of N_1 and X_2 the variables x of N_2. Since $X_s \subseteq M$ and $M \underline{\gg} N_2$, we have $X_s \supseteq X_2$, and since $X_1 \subseteq N_1$ and $BigHead(s) \gg_p BigHead(t) \cup N_1$, we have $BigHead(s) \gg_p BigHead(t) \cup X_1$. Therefore, $BigHead(s) \cup X_s \gg_p BigHead(t) \cup X_1 \cup X_2 = BigHead(t) \cup X_t$. $\square$

Lemma 28. *Let f be in $\mathcal{F}_{AC}$. If $f(s_1, \ldots, x, \ldots, s_n) \succ_p f(t_1, \ldots, x, \ldots, t_m)$ then $f(s_1, \ldots\ldots, s_n) \succ_p f(t_1, \ldots\ldots, t_m)$.*

In fact if either n or m is equal to 1 then (due to the arity of f) we do not include the f on top for $f(s_1, \ldots\ldots, s_n)$ or $f(t_1, \ldots\ldots, t_m)$ and we just have s_1 or t_1.

Proof. Let $s = f(s_1, \ldots, x, \ldots, s_n)$, $t = f(t_1, \ldots, x, \ldots, t_m)$, $s' = f(s_1, \ldots\ldots, s_n)$ and $t' = f(t_1, \ldots\ldots, t_m)$. Note that, as said, if $n = 1$ then $s' = s_1$ and if $m = 1$ then $t' = t_1$. We proceed by induction on $|s| + |t|$ and case analysis.

1. $s \succ_p t$ by case 1. Then $s_i \succeq_p t$ and by lemma 20 if $m \geq 2$, or by lemma 21 otherwise, we have $s_i \succ_p t'$, and hence $s' \succ_p t'$ by case 1.
2. $s \succ_p t$ by case 4. Then there is some $u = f(u_1, \ldots, x, \ldots, u_k) \in EmbNoBig(s)$ s.t. $u \succeq_p t$, then if $k \geq 2$, by induction hypothesis $u' = f(u_1, \ldots\ldots, u_k) \succeq_p t'$. If $n \geq 2$ then, since $u' \in EmbNoBig(s')$, we have $s' \succ_p t'$ by case 4. Otherwise if $n = 1$, then $s_1 = h(\ldots, f(u_1, \ldots, u_k), \ldots)$, and therefore, by case 1 we have $s_1 \succ_p t'$. Finally if $k = 1$ then $n = 1$ and, by induction hypothesis, $u_1 \succeq_p t'$. Then since $s_1 = h(\ldots, u_1, \ldots)$ by case 1 we have $s_1 \succ_p t'$.
3. $s \succ_p t$ by case 5. There are three cases.
   If $m = 1$ then if $f \not\succ_{\mathcal{F}} top(t_1)$, since $NoSmallHead(s) \underline{\gg}_{pf} NoSmallHead(t)$, we have that $s_i \succeq_p t_1$ for some s_i (with $f \not\succ_{\mathcal{F}} top(s_i)$). Otherwise, if $f \succ_{\mathcal{F}} top(t_1)$ then $t_1 = h(v_1, \ldots v_r)$ and $s \succ_p f(tf_f(v_j), x) \in EmbNoBig(t)$, which, by induction hypothesis, implies $s' \succ_p v_j$ for all $j \in \{1 \ldots n\}$ (note that if $top(v_j) = f$ then $f(tf_f(v_j)) = v_j$). Therefore, if $n \geq 2$ or $n = 1$ and $top(s_1) \succ_{\mathcal{F}} f$, we have $s' \succ_p h(v_1, \ldots v_r)$, by case 2, otherwise, if $n = 1$ and $top(s_1) \not\succ_{\mathcal{F}} f$ then we should have $s \succ_p t$ by case 5c, since no other can apply, and hence $\{s_1, x\} \gg_p \{t_1, x\}$ implies $s_1 \succ_p t_1$.
   If $m > 1$ $n = 1$ then, since $n < m$, we cannot have $\#(s) \geq \#(t)$, and hence $s \succ_p t$ should hold by case 5a, which implies $top(s_1) \succ_{\mathcal{F}} f$. Now if there is some $t_i = h(v_1, \ldots, v_r)$ with $h \not\succ_{\mathcal{F}} f$ and then, since $s \succ_p$

$f(t_1,\ldots,tf_f(v_j),\ldots,x,\ldots,t_n)$ for all $j \in \{1\ldots n\}$, we have, by induction hypothesis, $s_1 \succ_p f(t_1,\ldots,tf_f(v_j),\ldots,t_n)$, and therefore by lemma 21 we have $s_1 \succ_p t_p$ for all $p \neq i$ and $s_1 \succ_p v'$ for all $v' \in tf_f(v_j)$ and all $j \in \{1\ldots n\}$, and since $top(s_1) \succ_{\mathcal{F}} f$, by case 2, we have $s_1 \succ_p t_i$, which implies $s_1 \succ_p t'$ by case 2. Finally, if $top(t_i) \succ_{\mathcal{F}} f$ for all $i \in \{1\ldots m\}$ then, since $BigHead(s) = \{s_1\} \ggg_p \{t_1,\ldots,t_n\} = BigHead(t)$, we have $s_1 \succ_p t_i$ for all $i \in \{1\ldots m\}$, and hence $s_1 \succ_p t'$ by case 2.
If $m \geq 2$ and $n \geq 2$ then for every $u' = f(u_1,\ldots,u_k) \in EmbNoBig(t')$ there is a $u = f(u_1,\ldots,x,\ldots,u_k) \in EmbNoBig(t)$ s.t. $s \succ_p f(u_1,\ldots,x,\ldots,u_k)$, and hence by induction hypothesis $s' \succ_p u'$. Then since $NoSmallHead(s') \cup \{x\} = NoSmallHead(s) \ggg\!\succeq_{pf} NoSmallHead(t) = NoSmallHead(t') \cup \{x\}$, we have $NoSmallHead(s') \ggg\!\succeq_{pf} NoSmallHead(t')$. Finally, if $s \succ_p t$ by case 5a then $BigHead(s') = BigHead(s) \ggg_p BigHead(t) = BigHead(t')$ implies $s' \succ_p t'$ by case 5a. If $s \succ_p t$ by case 5b then trivially $s' \succ_p t'$ by case 5b. Otherwise $s \succ_p t$ by case 5c, and hence, since $\{s_1,\ldots,x,\ldots,s_n\} \ggg_p \{t_1,\ldots,x,\ldots,t_n\}$ implies $\{s_1,\ldots\ldots,s_n\} \ggg_p \{t_1,\ldots\ldots,t_n\}$, we have $s \succ_p t$ by case 5c. □

The following lemma is used to prove stability under substitution, i.e. $s \succ_p t$ implies $s\sigma \succ_p t\sigma$. The proof of this lemma becomes rather technical due to the new embedded terms in $EmbNoBig(t\sigma)$ which come from an embedding trough an instantiated variable. Due to this problem we have to generalize the stability property in order to prove that if $s \succ_p t$ then $s\sigma \succ_p t'$ for every t' (equal to or) embedded in $t\sigma$ through the instantiated variables. On the other hand, we have first considered the simplest substitution σ.

Lemma 29. *Let σ be $\{x \mapsto q(y_1,\ldots,y_k)\}$ for some symbol q. If $s \succ_p t$ then $\overline{s\sigma} \succ_p \overline{t'}$ for $t' \in M(t)$, where $M(t)$ is defined as*

- *if $top(t) = f \notin \mathcal{F}_{AC}$ or $q \succeq_{\mathcal{F}} f$ then $M(t) = \{t\sigma\}$, and*
- *if $t = f(t_1,\ldots,t_m)$ and $f \in \mathcal{F}_{AC}$ and $q \not\succeq_{\mathcal{F}} f$ then*
  $$M(t) = \{f(w_1,\ldots w_m) \mid w_i \in \{q(y_1,\ldots,y_k), y_1,\ldots,y_k\} \text{ if } t_i = x, \text{ and } w_i = t_i\sigma \text{ otherwise, and } 1 \leq i \leq m\}$$

Proof. Note that $t\sigma \in M(t)$, and therefore, this property implies $\overline{s\sigma} \succ_p \overline{t\sigma}$.

We prove that $s \succ_p t$ implies $\overline{s\sigma} \succ_p \overline{t'}$ for every s, t and $t' \in M(t)$, by induction on the triple $(|s|, |t|, |t'|)$ ordered lexicographically and by case analysis on the proof of $s \succ_p t$.

1. $g(s_1,\ldots,s_n) = s \succ_p t$ by case 1. Then $s_i \succeq_p t$. If $s_i = x$ and $q = g \in \mathcal{F}_{AC}$ then $t = x$ and $t' = t\sigma$ and $\overline{s\sigma} = g(s_1,\ldots,y_1,\ldots,y_k,\ldots,s_n) \succ_p g(y_1,\ldots,y_k) = \overline{t\sigma}$, by lemma 19. Otherwise, by induction hypothesis, $\overline{s_i\sigma} \succeq_p \overline{t'}$, and hence $\overline{s\sigma} \succ_p \overline{t'}$ by case 1.
2. $s \succ_p t = f(t_1,\ldots,t_m)$ by case 2. Then $s \succ_p t_i$ for all $i \in \{1\ldots m\}$. By induction hypothesis $\overline{s\sigma} \succ_p \overline{t_i\sigma}$. Since all argument w_i of $\overline{t'}$ is a subterm of or equal to $\overline{t_i\sigma}$, by property 21, we have $\overline{s\sigma} \succ_p w_i$. Therefore $\overline{s\sigma} \succ_p \overline{t'}$ by case 2.

3. $s \succ_p t$ by case 3. Then $t' = t\sigma$. Let $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$ with $f \notin \mathcal{F}_{AC}$, and $\langle s_1, \ldots, s_n\rangle(\succ_p)_{lex}\langle t_1, \ldots, t_n\rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots n\}$. By induction hypothesis $\langle \overline{s_1\sigma}, \ldots, \overline{s_n\sigma}\rangle(\succ_p)_{lex}\langle \overline{t_1\sigma}, \ldots, \overline{t_n\sigma}\rangle$ and $\overline{s\sigma} \succ_p \overline{t_i\sigma}$ for all $i \in \{1 \ldots n\}$, which implies $\overline{s\sigma} \succ_p \overline{t\sigma} = \overline{t'}$ by case 3.
4. $s \succ_p t$ by case 4. Then $s = f(s_1, \ldots, s_n)$, $top(t) = f$ and $f \in \mathcal{F}_{AC}$ and there is some $s_i = h(v_1, \ldots, v_r)$ with $f \succ_{\mathcal{F}} h$ and some $j \in \{1 \ldots r\}$ s.t. $s' = f(s_1, \ldots, tf_f(v_j), \ldots, s_n) \in EmbNoBig(s)$ and $s' \succeq_p t$. Thus, by induction hypothesis $\overline{f(s_1\sigma, \ldots, tf_f(v_j\sigma), \ldots, s_n\sigma)} \succeq_p \overline{t'}$. Then if v_j is not a variable or $h \notin \mathcal{F}_{AC}$ or $q \neq h$, we have $\overline{s\sigma} \succ_p \overline{t'}$ by case 4, since $\overline{s'\sigma} \in EmbNoBig(\overline{s\sigma})$. Otherwise, $v_j = x$ and $q = h \in \mathcal{F}_{AC}$, by property 19 we have $\overline{h(v_1\sigma, \ldots, h(y_1, \ldots, y_k), \ldots, v_r\sigma)} \succ_p h(y_1, \ldots, y_k)$, and by monotonicity $\overline{s\sigma} \succ_p \overline{f(s_1\sigma, \ldots, v_j\sigma, \ldots, s_n\sigma)}$ and by transitivity $\overline{s\sigma} \succ_p \overline{t'}$.
5. $s \succ_p t$ by case 5. First we prove that for all $w' \in EmbNoBig(\overline{t'})$ we have $\overline{s} \succ_p w'$. Let w' be $f(\overline{w_1}, \ldots, tf_f(v'_j), \ldots, \overline{w_i})$ for some $\overline{w_i} = h(v'_1, \ldots, v'_{r'})$ with $f \succ_{\mathcal{F}} h$ and some $j \in \{1 \ldots r'\}$. There are two cases.
   (a) If $t_i \neq x$. Then if $t_i = h(v_1, \ldots, v_r)$ then either $v'_j = \overline{v_p\sigma}$ for some $p \in \{1 \ldots r\}$ or, $v'_j \in \{y_1, \ldots, y_k\}$ and some $v_p = x$ and $h = q \in \mathcal{F}_{AC}$. In the first case, we have that there is some $w \in EmbNoBig(t)$, s.t. $s \succ_p w = f(t_1, \ldots, tf_f(v_p), \ldots, t_m)$, which by induction hypothesis, implies $\overline{s\sigma} \succ_p \overline{f(w_1, \ldots, tf_f(v_p)\sigma, \ldots, w_m)} = w'$. In the second case $v_p = x$ and $q = h \in \mathcal{F}_{AC}$, then since for some $w \in EmbNoBig(t)$, we have that $s \succ_p w = f(t_1, \ldots, x, \ldots, t_m)$, and by induction hypothesis $\overline{s\sigma} \succ_p \overline{f(w_1, \ldots, q(y_1, \ldots, y_k), \ldots, w_m)}$ and then by subterm property and monotonicity, $\overline{f(w_1, \ldots, q(y_1, \ldots, y_k), \ldots, w_m)} \succ_p w'$ and, by transitivity,$\overline{s\sigma} \succ_p w'$.
   (b) If $t_i = x$. Then $w' = f(\overline{w_1}, \ldots, y_p, \ldots, \overline{w_m})$. for some $w_i = q(y_1, \ldots, y_k)$. Then since $f(w_1, \ldots, y_p, \ldots, w_m) \in M(t)$, by induction hypothesis $\overline{s\sigma} \succ_p \overline{f(w_1, \ldots, y_p, \ldots, w_m)} = w'$.

   Now we prove that $\overline{s\sigma} \succ_p \overline{t'}$. There are several cases:
   (a) If $t = f(t_1, \ldots, x, \ldots, t_m)$ and $s = f(s_1, \ldots, x, \ldots, s_n)$, then by lemma 28 we have $\underline{f(s_1, \ldots\ldots, s_n)} \succ_p f(t_1, \ldots\ldots, t_m)$. Then by induction hypothesis $\overline{f(s_1, \ldots\ldots, s_n)\sigma} \succ_p \overline{t''}$ for every $t'' \in M(f(t_1, \ldots\ldots, t_m))$. Then if some $w_i = y_p$ for some $p \in \{1 \ldots k\}$, then by AC-compatibility, monotonicity and either subterm or deletion property, we have that there is some t'' s.t. $\overline{f(t'', x\sigma)} \succ_p \overline{t'}$, and hence by monotonicity, AC-compatibility, we have $\overline{s\sigma} =_{AC} \overline{f(f(s_1, \ldots\ldots, s_n)\sigma, x\sigma)} \succ_p \overline{f(t'', x\sigma)} \succ_p \overline{t'}$, and by transitivity $\overline{s\sigma} \succ_p \overline{t'}$. If no $w_i = y_p$, then $t\sigma = t'$, and $\overline{s\sigma} =_{AC} \overline{f(f(s_1, \ldots\ldots, s_n)\sigma, x\sigma)} \succ_p \overline{f(f(t_1, \ldots\ldots, t_m)\sigma, x\sigma)} =_{AC} \overline{t'}$, as before, by monotonicity and AC-compatibility.
   (b) If $t_i \neq x$ for all $i \in \{1 \ldots m\}$ then we have $t' = t\sigma$ and therefore, since $NoSmallHead(s) \succeq\!\!\succ_{pf} NoSmallHead(t)$, we have $X = NoSmallHead(s) \setminus \{s_i \mid s_i = x \wedge 1 \leq i \leq n\} \succeq\!\!\succ_{pf} NoSmallHead(t) = Y$. Then by induction hypothesis $NoSmallHead(\overline{s\sigma}) \supseteq \{\overline{w\sigma} \mid w \in X\} \succeq\!\!\succ_{pf} \{\overline{w\sigma} \mid w \in Y\} = NoSmallHead(\overline{t\sigma}) = NoSmallHead(\overline{t'})$. Similarly, if $X = BigHead(s) \succ\!\!\succ_p$

$BigHead(t) = Y$ then, by induction hypothesis, $BigHead(\overline{s\sigma}) \supseteq \{\overline{w\sigma} \mid w \in X\} \ggcurly_p \{\overline{w\sigma} \mid w \in Y\} = BigHead(\overline{t\sigma}) = BigHead(\overline{t'})$. Therefore if $s \succ_p t$ by case 5a then $\overline{s\sigma} \succ_p \overline{t'}$ by case 5a. On the other hand, if $s \succ_p t$ by case 5b then $\#(s) > \#(t)$ implies $\#(\overline{s\sigma}) > \#(\overline{t\sigma}) = \#(\overline{t'}) = \#(t)$, since x is not an argument of t, and hence $\overline{s\sigma} \succ_p \overline{t'}$ by case 5b. Finally if $s \succ_p t$ by case 5c then similarly $\#(\overline{s\sigma}) \leq \#(\overline{t\sigma}) = \#(\overline{t'}) = \#(t)$, and by induction hypothesis $\{\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}\} \ggcurly_p \{\overline{t'_1\sigma}, \ldots, \overline{t'_m\sigma}\} = \{\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}\}$, since x is not an argument of t only for the $s_i = x$, $\overline{s_i\sigma}$ should be replaced by $y_1, \ldots, y_k$, but then we still have $\{\overline{s_1\sigma}, \ldots, y_1, \ldots, y_k, \ldots, \overline{s_n\sigma}\} \ggcurly_p \{\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}\}$, again since all $t_i \neq x$, and we conclude by case 5c.

(c) If $s_i \neq x$ for all $i \in \{1 \ldots n\}$ and $t_i = x$ for some $i \in \{1 \ldots m\}$ then since $X = NoSmallHead(s) \underline{\ggcurly}_{pf}$ $NoSmallHead(t) = Y$, we know that, by definition of $\underline{\ggcurly}_{pf}$ that all x are taken care by some $s_i = u[x]$, s.t. $top(s_i) \succ_{\mathcal{F}} f$. Which also implies that $BigHead(s) \ggcurly_p BigHead(t)$, since $BigHead(s) \underline{\ggcurly}_p BigHead(t)$ and they cannot be equal (otherwise s_i could not take care of the x). Then, we have, by induction hypothesis, $NoSmallHead(\overline{s\sigma}) = \{\overline{w\sigma} \mid w \in X\} \underline{\ggcurly}_{pf} \{\overline{w\sigma} \mid w \in Y\} = Z$, where the $x\sigma$ are taken care by $s_i\sigma$. Then we have that $NoSmallHead(\overline{t'})$ is equal to Z after removing some $x\sigma$ and replacing some other $x\sigma$ by some y_p and some others by $y_1, \ldots, y_k$ (if $q = f$), and hence $NoSmallHead(\overline{s\sigma}) \underline{\ggcurly}_{pf} NoSmallHead(\overline{t'})$. Finally, since $BigHead(s) \ggcurly_p BigHead(t)$, by property 27, we have $X = BigHead(s) \ggcurly_p BigHead(t) \cup X_t = Z$, where X_t is the multiset of arguments of t that are x. Now by induction hypothesis, $BigHead(\overline{s\sigma}) = \{\overline{w\sigma} \mid w \in X\} \ggcurly_p \{\overline{w\sigma} \mid w \in Z\} \supseteq BigHead(\overline{t'})$ (note that if $q \succ_{\mathcal{F}} f$ then some of the w_i, with $t_i = x$, should be in $BigHead(\overline{t'})$), which implies $BigHead(\overline{s\sigma}) \ggcurly_p BigHead(\overline{t'})$ and hence $\overline{s\sigma} \succ_p \overline{t'}$ by case 5a. □

Lemma 30. $\succ_p$ *is stable under substitution.*

Proof. We have to prove that if $s \succ_p t$ then $\overline{s\sigma} \succ_p \overline{t\sigma}$ for every substitution σ. We proceed by induction on the $|\sigma|$ defined as the multiset $\{|w| \mid (x \mapsto w) \in \sigma\}$ and compared by the multiset extension of $>$.

If σ is empty it holds trivially. Otherwise $\sigma = \{x \mapsto q(w_1, \ldots, w_k)\} \cup \sigma'$. Then, by lemma 29, we have $\overline{s\{x \mapsto q(y_1, \ldots, y_k)\}} \succ_p \overline{t\{x \mapsto q(y_1, \ldots, y_k)\}}$ and taking $\gamma = \{y_1 \mapsto w_1, \ldots, y_k \mapsto w_k\} \cup \sigma'$, $|\sigma| >> |\gamma|$, and by induction hypothesis $\overline{s\sigma} = \overline{(\overline{s\{x \mapsto q(y_1, \ldots, y_k)\}})\gamma} \succ_p \overline{(\overline{t\{x \mapsto q(y_1, \ldots, y_k)\}})\gamma} = \overline{t\sigma}$.

Lemma 31. $\succ_p$ *is incremental.*

Proof. We prove that if $s \succ_p t$ with the precedence $\succ_{\mathcal{F}}$ then $s \succ_p t$ with the precedence $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ where f' and g' are not related by $\succ_{\mathcal{F}}$. We proceed by induction on $|s| + |t|$ and case analysis on the proof of $s \succ_p t$ with the precedence $\succ_{\mathcal{F}}$.

1. $s = f(s_1, \ldots, s_n) \succ_p t$ by case 1. Then $s_i \succ_p t$ and by induction hypothesis $s_i \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, which implies $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by case 1.
2. $s = f(s_1, \ldots, s_n) \succ_p g(t_1, \ldots, t_m) = t$ by case 2. Then $f \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$. Then, by induction hypothesis, $s \succ_p t_i$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ for all $i \in \{1 \ldots m\}$, and hence $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by case 2.
3. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_n) = t$ by case 3. Then $\langle s_1, \ldots, s_n \rangle (\succ_p)_{lex} \langle t_1, \ldots, t_n \rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots n\}$, and by induction hypothesis, $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by case 3.
4. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by case 4. Then there is a term $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p t$. Then by induction hypothesis $s' \succeq_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. If $s' \in \mathit{EmbNoBig}(s)$ wrt. the new precedence $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, then $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by case 4. Otherwise, by monotonicity and subterm property $s \succ_p s'$ and by transitivity, $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$.
5. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by case 5. Then we have $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(t)$ and $\mathit{NoSmallHead}(s) \underline{\ggcurly}_{pf}$ $\mathit{NoSmallHead}(t)$. On the one hand, since $\mathit{EmbNoBig}(t)$ wrt. $\succ_{\mathcal{F}}$ is included in $\mathit{EmbNoBig}(t)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, then by induction hypothesis $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(t)$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. On the other hand, by definition of $\underline{\ggcurly}_{pf}$ every term s' in $\mathit{NoSmallHead}(s)$ s.t. $\mathit{top}(s') \not\succ_{\mathcal{F}} f$ only takes care of terms t' in $\mathit{NoSmallHead}(t)$ s.t. $\mathit{top}(s') \succeq_{\mathcal{F}} \mathit{top}(t')$, then if s' is not in $\mathit{NoSmallHead}(s)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ then none of the t' is in $\mathit{NoSmallHead}(t)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. Therefore, by induction hypothesis, $\mathit{NoSmallHead}(s) \underline{\ggcurly}_{pf} \mathit{NoSmallHead}(t)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$.
   Now if we have applied case 5b it holds trivially. If we have applied case 5c then by induction hypothesis $\{s_1, \ldots, s_n\} \ggcurly_p \{t_1, \ldots, t_n\}$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. Finally if we have applied case 5a then we have $\mathit{BigHead}(s) \ggcurly_p \mathit{BigHead}(t)$ wrt. $\succ_{\mathcal{F}}$. Now since $\mathit{NoSmallHead}(s) \underline{\ggcurly}_{pf} \mathit{NoSmallHead}(t)$, wrt. $\succ_{\mathcal{F}}$, by definition of $\underline{\ggcurly}_{pf}$, for every term t' in $\mathit{NoSmallHead}(t)$ wrt. $\succ_{\mathcal{F}}$ s.t. $t' \notin \mathit{BigHead}(t)$ wrt. $\succ_{\mathcal{F}}$ and $t' \in \mathit{BigHead}(t)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ there is some s' in $\mathit{NoSmallHead}(s)$ wrt. $\succ_{\mathcal{F}}$ s.t. $s' \in \mathit{BigHead}(s)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ that takes care of t' (note that for every t' in $\mathit{NoSmallHead}(t)$ wrt. $\succ_{\mathcal{F}}$ there is some s' which takes care of it and either $\mathit{top}(s') \succ_{\mathcal{F}} f$ or $\mathit{top}(s') \succeq_{\mathcal{F}} \mathit{top}(t')$). Therefore $\mathit{BigHead}(s) \ggcurly_p \mathit{BigHead}(t)$ wrt. $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, and hence $s \succ_p t$ with $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ by case 5a. □

7 Conclusions

We have presented the first fully syntactic AC-compatible *recursive path ordering* (RPO). The ordering is AC-total, and defined uniformly for both ground and non-ground terms, as well as for partial precedences, being the first incremental one.

As a simple improvement regarding efficiency of implementation, it is easy to show that some of the recursive comparisons in case 5 can be avoided. Being more precise, when comparing terms s and t some of the recursive comparisons

$s \succ_p t'$ for $t' \in EmbNoBig(t)$ are not necessary since they follow from other ones.

It is also straightforward to improve the ordering by adding the possibility to give status multiset to the non-AC-symbols. All proofs can be easily adapted since for the non-AC-symbols the ordering behaves exactly as RPO.

As a future development, due to its simplicity and, mainly, the fact that it is not interpretation-based, it opens the door to finding practically feasible ordering constraint solvers for the AC-case [CNR95].

References

[Bac92] Leo Bachmair. Associative-commutative reduction orderings. *Information Processing Letters*, 43:21–27, 1992.

[BCL87] Ahlem Ben-Cherifa and Pierre Lescanne. Termination of rewriting systems by polynomial interpretations and its implementation. *Science of Computer Programming*, 9:137–160, 1987.

[BP85] L. Bachmair and D. A. Plaisted. Termination orderings for associative-commutative rewriting systems. *Journal of Symbolic Computation*, 1:329–349, 1985.

[CNR95] Hubert Comon, Robert Nieuwenhuis, and Albert Rubio. Orderings, AC-Theories and Symbolic Constraint Solving. In *Tenth Annual IEEE Symposium on Logic in Computer Science*, pages 375–385, San Diego, California, USA, June 26–29, 1995. IEEE Computer Society Press.

[Der82] Nachum Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.

[DHJP83] N. Dershowitz, J. Hsiang, A. Josephson, and D. Plaisted. Associative-commutative rewriting. In *Int. Joint Conference on Artificial Inteligence*, pages 940–944, 1983.

[DP93] Catherine Delor and Laurence Puel. Extension of the associative path ordering to a chain of associative commutative symbols. In C. Kirchner, editor, *5th International Conference on Rewriting Techniques and Applications*, LNCS 690, pages 389–404, Montreal, Canada, 1993. Springer-Verlag.

[GL86] Isabelle Gnaedig and Pierre Lescanne. Proving termination of associative-commutative rewrite systems by rewriting. In Jörg H. Siekmann, editor, *Proc. 8th International Conference on Automated Deduction*, LNCS 230, pages 52–61, Oxford, England, 1986. Springer-Verlag.

[KS97] D. Kapur and G. Sivakumar. A total, ground path ordering for proving termination of ac-rewrite systems. In H. Comon, editor, *8th International Conference on Rewriting Techniques and Applications*, LNCS , 1997.

[KS98] D. Kapur and G. Sivakumar. A recursive path ordering for proving associative-commutative termination, 1998.

[KSZ90] D. Kapur, G. Sivakumar, and H. Zhang. A new method for proving termination of ac-rewrite systems. In *Conf. Found. of Software Technology and Theoretical Computer Science*, LNCS 472, pages 134–148, New Delhi, India, December 1990. Springer-Verlag.

[NR91] Paliath Narendran and Michael Rusinowitch. Any ground associative commutative theory has a finite canonical system. In *Fourth int. conf. on Rewriting Techniques and Applications*, LNCS 488, pages 423–434, Como, Italy, April 1991. Springer-Verlag.

[RN95] Albert Rubio and Robert Nieuwenhuis. A total AC-compatible ordering based on RPO. *Theoretical Computer Science*, 142(2):209–227, May 15, 1995.

[Rub97] Albert Rubio. A total AC-compatible ordering with RPO scheme. Technical Report UPC-LSI-97, Univ. Polit. Catalunya, September 1997.

This article was processed using the LaTeX macro package with LLNCS style