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Combinatorial Recurrences
and Linear Difference Equations

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Abstract

In this work we introduce the triangular arrays of depth greater than 1 given by linear recurrences, that generalize some well known recurrences that appear in enumerative combinatorics. In particular, we focussed on triangular arrays of depth 2, since they are closely related with the solution of linear three–terms recurrences. We show through some simple examples how this triangular arrays appear as essential components in the expression of some classical orthogonal polynomials and combinatorial numbers.

Keywords: Combinatorial identities, triangular matrices, finite difference equations, orthogonal polynomials.

1   Introduction

In this work we are interested in obtaining the unique solution of the initial value problem

(1) $\qquad a_k z_{k+1} - b_k z_k + c_{k-1} z_{k-1} = 0, \; k \in \mathbb{N}^*, \; z_0 = 1, \; z_1 = q,$

where $q \in \mathbb{R}^*$, $(a_k), (b_k)$ and (c_k) are sequences of real numbers such that $a_k, c_k \neq 0$ for all k. Since (z_k) doesn't depend on a_0 and b_0 we always assume that $b_0 = q$. In addition, we also assume the usual convention that empty sums and empty products are defined as 0 and 1, respectively.

As we'll show, the solution of (1) can be expressed through double sequences determined by recurrences that are related with combinatorial recurrences. Specifically, given $n \in \mathbb{N}^*$ we call *triangular array of depth n* to any double sequence, indexed from 0, such that $t_{k,m} = 0$ when $0 \leq k < nm$. When necessary, we also assume that $t_{k,m} = 0$ when k or m are negative integers. If we identify double sequences with infinite matrices, then the subspace of triangular arrays of depth n is identity with a subspace of lower triangular matrices, that in column m the first $nm - 1$ entries are null. The triangular arrays we consider, are given by recurrence relations. Specifically, given $g = (g_{k,m}), h = (h_{k,m})$, for any $n \in \mathbb{N}^*$ we define the triangular array of depth n, $(t_{k,m})$, as $t_{0,0} = 1$ and

(2) $\quad t_{k,m} = g_{k-1,m} t_{k-1,m} + h_{k-n,m-1} t_{k-n,m-1}, \; k \geq nm; \; m \geq 0, \; k+m \geq 1.$

The linear character of the above recurrence allows us to solve it by determining each *column* in terms of the preceding ones.

Theorem 1.1 *Given the double sequences $(g_{k,m}), (h_{k,m})$, for any $n \in \mathbb{N}$ the unique triangular array of depth n satisfying the recurrence (2) is*

$$t_{k,0} = \prod_{j=0}^{k-1} g_{j,0}, \quad t_{k,m} = \sum_{s=nm}^{k} h_{s-n,m-1} t_{s-n,m-1} \prod_{j=s}^{k-1} g_{j,m}, \; k \geq mn, \; m \geq 1.$$

In particular, $t_{nm,m} = \prod\limits_{j=0}^{m-1} h_{nj,j}$ for any $m \in \mathbb{N}$.

We denote by $\mathscr{G}_n(g,h)$ the unique triangular array of depth n determined by the recurrence (2) and moreover, the double sequences g and h are called

its *generators*.

The class triangular arrays of depth 1 were introduced by E. Neuwirth in [6] under the denomination of *pure Galton arrays* with the aim of solving a question raised by R. Graham, D, Knuth and O. Patashnik in [3, Problem 6.94], at least in some particular cases. Whereas triangular arrays of depth 1 have been well studied and completely determined in some cases, see [1,8], the triangular arrays of depth greater than 1 are still unknown. This work is devoted to study some triangular arrays of depth 2, whose generators are column independent; that is, are given by sequences $g = (g_k)$ and $h = (h_k)$.

2 Some classical second order difference equations

The relation between the second order initial value problems and triangular arrays of depth 2 is given for the following result.

Theorem 2.1 *The unique solution of the initial value problem (1) is*

$$z_k = \Big(\prod_{j=1}^{k-1} a_j\Big)^{-1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \Phi_{k,m}, \ \ k \in \mathbb{N},$$

where $\Phi = \mathscr{G}_2(b, ac)$.

We remark that the above theorem implies that obtaining $\mathscr{G}_2(b, ac)$ leads to solve the initial value problem (1); that is, an irreducible second order linear difference equation can be solved by means of a sequence of first order linear difference equations. This represent a huge distinction between the treatment of differential and difference linear equations.

Now we study some particular cases of the above Theorem that recover many classical solutions of both difference and differential linear equations of second order. First we treat a general framework and then we specialize the results. Therefore, we fix $x \in \mathbb{R}$ and suppose that $b_k = 2x\hat{b}_k$. Under these hypotheses, we have that

$$\mathscr{G}_2(b, ac)_{k,m} = (2x)^{k-2m} \mathscr{G}_2(\hat{b}, ac)_{k,m}, \ \ k, m \in \mathbb{N},$$

and hence, if $\widehat{\Phi} = \mathscr{G}_2(\hat{b}, ac)$ and we denote by $P_k(x)$ the k–th term of the solution of (1), then

$$(3) \qquad P_k(x) = \Big(\prod_{j=1}^{k-1} a_j\Big)^{-1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m (2x)^{k-2m} \widehat{\Phi}_{k,m}, \ \ k \in \mathbb{N},$$

which implies that $P_k \in \mathbb{R}[x]$ has degree less than or equal to k and moreover $P_k(-x) = (-1)^k P_k(x)$ for any $k \in \mathbb{N}$. In addition, P_k has degree equal to k iff $\hat{b}_j \neq 0$ for $j = 0, \ldots, k-1$.

Therefore, given sequences of real numbers $(a_k), (\hat{b}_k)$ and (c_k) such that $a_k, \hat{b}_k, c_k \neq 0$ for all $k \in \mathbb{N}$, the solution of the initial value problem

$$(4) \qquad a_k z_{k+1} - 2x\hat{b}_k z_k + c_{k-1} z_{k-1} = 0, \ k \in \mathbb{N}^*, \ z_0 = 1, \ z_1 = 2xq,$$

where $q = \hat{b}_0$ is a polynomial $P_k(x)$ for any $k \in \mathbb{N}$. Next we list some examples in which these polynomials can be explicitly obtained. In all cases, we obtain $\widehat{\Phi} = \mathscr{G}_2(\hat{b}, ac)$ from the recurrence relation (2); that is, by applying the Theorem 1.1.

Chebyshev and related Polynomials

This, corresponds to the case in which the coefficients of (4) are constant; that is, $a_k = a, \hat{b}_k = q, c_k = c$, where $a, q, c \in \mathbb{R}^*$. We have that

$$\widehat{\Phi}_{k,m} = (ac)^m b^{k-2m} \binom{k-m}{m}, \quad \text{for any } k \geq 2m$$

and hence,

$$P_k(x) = a^{1-k} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m (ac)^m (2xb)^{k-2m} \binom{k-m}{m}, \quad k \in \mathbb{N}$$

that when $a = b = c = 1$ is the classical k–th *Chebyshev polynomial of second kind*, see [5]. On the other hand, when $a = 1, c = -1$ and $b = \frac{1}{2}$, then

$$P_k(x) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} x^{k-2m} \binom{k-m}{m}, \quad k \in \mathbb{N}$$

is known as k–th *Fibonacci polynomial*.

Hermite Polynomials

This, corresponds to the case in which the coefficients of (4) are given by $a_k = 1, \hat{b}_k = q, c_k = 2q(k+1)$. We have that

$$\widehat{\Phi}_{k,m} = q^{k-m} \frac{(2m)!}{m!} \binom{k}{2m}, \quad \text{for any } k \geq 2m$$

and hence,

$$P_k(x) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m q^{k-m} \frac{k!(2x)^{k-2m}}{m!(k-2m)!}, \quad k \in \mathbb{N}$$

that when $q = 1$ is the classical k–th *Hermite polynomial*, see [7].

Gegenbauer Polynomials

This, corresponds to the case in which the coefficients of (4) are given by $a_k = k+1, \hat{b}_k = k+q, c_k = r(k+2q)$, where $r \in \mathbb{R}^*$ and $-2q \notin \mathbb{N}$. We have

$$\widehat{\Phi}_{k,m} = r^m (q)_{k-m} \frac{(2m)!}{m!} \binom{k}{2m}, \quad \text{for any } k \geq 2m.$$

where for any $x \in \mathbb{R}$, $(x)_n = \prod_{k=0}^{n-1} (k+x)$ is the *factorial function*. Therefore,

$$P_k(x) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-r)^m (q)_{k-m} \frac{(2x)^{k-2m}}{m!(k-2m)!}, \quad k \in \mathbb{N},$$

that when $r = 1$ coincides with the k–th *Gegenbauer polynomial*, see [7]. In particular, the choice $q = \frac{1}{2}$ leads to the k–th *Legrendre Polynomial*.

Linear coefficients

It would be desirable to obtain $\mathscr{G}_2(b, ac)$ explicitly when all the sequences $(a_k), (b_k)$ and (c_k) are linear functions with respect to k; that is, when

$$a(k) = rk + s, \quad b(k) = \tau k + \rho, \quad c(k) = \mu k + \nu, \quad k \in \mathbb{N},$$

where $s, \rho, \nu \in \mathbb{R}^*$ and moreover $-r^{-1}s, -\tau^{-1}\rho, -\mu^{-1}\nu \notin \mathbb{N}$ when r or τ or μ are non null. Under these hypotheses, the initial value problem (1) encompasses the recurrences generating many famous combinatorial numbers, such as *Fine, central Delannoy, Schröder, Motzkin, and derangements numbers*, to mention only a few, that play an important role in enumerative combinatorics and count several combinatorial objects, see for instance [4] and references therein for the origin and the meaning of these sequences. Of course, the case of linear coefficients doesn't exhaust this class of problems, since other combinatorial sequences as *Franel numbers of order* 3 *or* 4 or *Apéry numbers* satisfy a three-term recurrence, but the corresponding coefficients are polynomials in k of order greater that 1, see [2].

As an example, we consider the case when $a_k = k + 3$, $c_k = -3(k + 1)$, $k \in \mathbb{N}$, $b_k = 2k + 3$, $k \in \mathbb{N}^*$ and $b_0 = 1$. The solution of the initial value problem (1) is called *Motzkin sequence*. In this case, we have that

$$\Phi_{k,m} = (-1)^m \frac{2^{k-m}}{3} \left(\frac{3}{2}\right)_{k-m} \left(\frac{3}{2}\right)^m \frac{(2m)!}{m!} \binom{k}{2m}, \quad \text{for any } k \geq 2m$$

and hence, the Moztkin sequence is given by

$$M_k = \frac{2^{k+1}}{(k+1)(k+2)} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left(\frac{3}{4}\right)^m \frac{\left(\frac{3}{2}\right)_{k-m}}{(k-2m)!m!}, \quad k \in \mathbb{N}.$$

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