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MATHEMATICAL MODELING OF POLYDISPERSE SUSPENSIONS SEDIMENTATION

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For a batch sedimentation test of a suspension formed by k particle categories, the settling velocity of a particle belonging to a category i, i=1,2,...,k, at a given time t and a given height x, from the bottom of the column, is a function of the terminal settling velocity of each category and of the concentration distribution by categories at (x,t). The mathematical model for this system is represented by k hyperbolic partial differential equations with the given initial conditions and the appropriate boundary conditions. The model allows a realistic description of the process and numerical simulations, using the donor-cell finite differences method, which are in good agreement with some previously published experimental data about bidisperse suspensions, when LOCKETT and AL-HABBOOBY or SELIM et al. relative fluid-particle velocity expressions were used.

Introduction

Many phenomena in nature and many separation techniques in industry are governed by the sedimentation process, and suspensions found usually show a distribution of particles by sizes and even by densities. Models capable of describing mixtures settling are needed to assess industrial operations and to understand natural processes involving sedimentation.

Sedimentation of polydisperse suspensions results in a particle stratification due to different relative settling velocities. Considering a suspension formed by k categories of particles, the sedimentation process will classify suspension in k vertical zones, where the top zone will contain smallest particles only, and the bottom zone will contain particles of all categories. Usual models predict zone interfaces settling velocities by considering that the bottom zone contains all initial biggest particles and the initial concentration for the other categories, and by proceeding sequentially from the bottom to the top.

During sedimentation, the surrounding fluid is displaced upwards by the settling particles and in polydisperse suspensions the displaced fluid could transport smallest particles upwards. In this situation the smallest particles concentration in the bottom zone will be less than the initial concentration, and will increase in the upper zones where its settling velocity will decrease under the calculated velocity using the consideration of constancy in smallest particles concentration in the bottom zone.

As a dynamical system, the proposed method in this work is the calculation of the interfaces velocities by solutions approximation of the k partial differential equations system obtained by applying the continuity equation to the k particle categories.

Monodisperse Suspensions

The terminal settling velocity, u_t, of a single particle in an infinite fluid medium, for low Reynolds numbers, is given by Stokes law. A more general expression for terminal settling velocities calculation, at any fluid flow regime, is given by

$$u_t^{2-e} = \frac{4g}{3f}\frac{(\rho-\rho_l)d^{1+e}}{\nu_l^e\rho_l}\,, \tag{1}$$

$$\mathrm{Re} \le 2 \Rightarrow f = 24,\ e = 1\,,$$

$$2 < \mathrm{Re} \le 500 \Rightarrow f = 18.5,\ e = 0.6\,,$$

$$\mathrm{Re} > 500 \Rightarrow f = 4/9,\ e = 0\,.$$

Table 1 Some functional forms for the relation between sedimentation velocity and the terminal settling velocity in a monodisperse suspension

Reference	Correlation $F(C,\mathrm{Re})$	Eqs.	Application
RICHARDSON and ZAKI (1954) [1]	$(1-C)^n$	(6)	$0<\mathrm{Re}\le 0.2$ $\quad n = 4.6+20\dfrac{d}{d_r}$ $0.2<\mathrm{Re}\le 1$ $\quad n = (4.4+18\dfrac{d}{d_r})\mathrm{Re}^{-0.03}$ $1<\mathrm{Re}\le 200$ $\quad n = (4.4+18\dfrac{d}{d_r})\mathrm{Re}^{-0.1}$ $200<\mathrm{Re}\le 500$ $\quad n = 4.4\,\mathrm{Re}^{-0.1}$
BATCHELOR (1972) [6]	$1-6.55C$	(7)	$C<0.03$ $\mathrm{Re}<<0.01$
BARNEA and MIZRAHI (1973) [3]	$1/1+kC^{1/3}$	(8)	$\mathrm{Re}<0.01$ $C<0.01$
GARSIDE and AL-DIBOUNI (1977) [2]	$(1-C)^n$	(9)	$\dfrac{5.1-n}{n-2.7}=0.1\,\mathrm{Re}^{0.9}$
BUSCALL et al. (1982) [5]	$(1-C/p)^{kp}$	(10)	$\mathrm{Re}<<0.01$ $p=0.58$ $k=5.4$
AL-NAAFA and SELIM (1992) [4]	$(1-C)^{6.55}$	(11)	$C<0.37$ $\mathrm{Re}<<0.01$

Because of the possible significance of the bed wall effect in laboratory experiments, the terminal velocity should be corrected. RICHARDSON and ZAKI [1] proposed the correction

$$\log_{10}(u_t / u_0) = d / d_r, \tag{2}$$

where u_0 is the corrected velocity, d is the particle diameter and d_r is the vessel diameter. GARSIDE and AL-DIBOUNI [2] recommend the equation of FRANCIS,

$$\frac{u_t}{u_0} = \left[\frac{1-0.475(d/d_r)}{1-(d/d_r)}\right]^4, \tag{3}$$

for Re<0.2, and the linear relation, given by *Eq.(4)*, for $0.2<\mathrm{Re}<10^3$.

$$u_t / u_0 = 1 + 2.35(d / d_r). \tag{4}$$

Eq.(4) gives results that are very similar to those predicted by *Eq.(2)* [2].

When a suspension of particles of equal sizes and densities sediments in a finite fluid medium, there is a decrease in its velocity due to the displaced fluid and due to the interaction among them. The relation between the settling velocity u of a particle in the suspension and its velocity u_0 can be expressed as a function of C, the volumetric concentration of particles, and its Reynolds number,

$$\frac{u}{u_0} = F(C, \mathrm{Re}) . \tag{5}$$

Table 1 shows some functional forms for this relation.

BARNEA and MIZRAHI [3] noted that fitted curves of measured data of u/u_0 vs. Re, for some C values, tend to flatten for low values of Reynolds, and do so for low C values at any Re. The model of BATCHELOR [6], based on a theoretical study of pairwise interparticle interactions, integrates previous theoretical and empirical information in this range of application. AL-NAAFA and SELIM model [4] extends that model to higher values of C at low Re.

For non colloidal suspensions, the correlation of RICHARDSON and ZAKI [1], *Eq.(6)*, is the most extensively used, both in fluidization and sedimentation. GARSIDE and AL-DIBOUNI [2] realised that Richardson-Zaki correlation was not in good agreement with experimental data for low values of C (less than 0.1), that it underestimates the mutual influence of particles at high voidage ε and so gives values of u that are too high. They proposed the *Eq.(9)*, that gives higher values of n for low Re ($\mathrm{Re} \to 0$, $n \to 5.1$). It is important to note here that progressive refinements in correlations tend to give values of n tending to increase and approach Batchelor values for low values of C and Re, but there is not a general and unified correlation for any situation. The Richardson-Zaki general expression, given by $F(C, \mathrm{Re}) = \varepsilon^n$, and used by GARSIDE and AL-DIBOUNI, will be assumed as the more realistic for non colloidal suspensions.

The upward fluid velocity u_F at a given height of a column of constant section A, where a suspension is settling with a concentration C and velocity u, is given by

$$u_F = -\frac{uC}{\varepsilon}, \tag{12}$$

and the relative particle fluid velocity $u_{/F}$ is given by

$$u_{/F} = u - u_F = u_0 \varepsilon^{n-1} \tag{13}$$

The evolution of a suspension in a column of height L and constant section, with an initial uniform concentration C_0, will be represented by the continuity equation, taking the following assumptions:

1. The particle distribution in a transversal section of the column at height x is uniform.
2. Physical properties of each particle remain constant during settling (no phenomena of aggregation or fragmentation are present).
3. Settling velocity for each particle depends on its physical properties and the local suspension concentration.
4. Particles are large enough not to consider transport by diffusion.

The model is given by

$$\frac{\partial C}{\partial t} = \frac{\partial \varphi}{\partial x}, \tag{14}$$

with the boundary conditions

$$\begin{aligned} C &= C_0(x,t), \quad t = 0, \\ \varphi(x,t) &= 0, \quad x = 0, \\ \varphi(x,t) &= 0, \quad x = L, \end{aligned} \tag{15}$$

where $\varphi = uC$ is the flux density. Using the dimensionless variables $\hat{x} = x / L$ and $\hat{t} = u_0 t / L$, the model can be expressed in the dimensionless form

$$\frac{\partial C}{\partial \hat{t}} = \frac{\partial [C(1-C)^n]}{\partial \hat{x}}, \tag{16}$$

$$\begin{aligned} C &= C_0(\hat{x}, \hat{t}), \quad \hat{t} = 0, \\ \varphi(\hat{x}, \hat{t}) &= 0, \quad \hat{x} = 0, \\ \varphi(\hat{x}, \hat{t}) &= 0, \quad \hat{x} = L. \end{aligned}$$

Polydisperse Suspensions

Let us consider now a suspension formed by particles that can be classified in k categories. Each category can be characterized by its representative particle density ρ_i and its representative diameter d_i, $1 \le i \le k$. For each category, its terminal settling velocity u_{0i} and its dimensionless parameter n_i of the general Richardson-Zaki expression can be calculated.

Applying the continuity equation for each category, at a given height of the column, k hyperbolic partial differential equations are obtained,

$$\begin{aligned} &\frac{\partial C_i}{\partial t} = \frac{\partial \varphi_i}{\partial x}, \ i = 1, 2, ..., k, \\ &\varphi_i = u_i C_i, \ i = 1, 2, ..., k, \\ &\varepsilon = 1 - \sum_{j=1}^{k} C_j, \end{aligned} \tag{17}$$

where u_i is the settling velocity of particles of i category. The boundary conditions are

$$\begin{aligned} C_i &= C_{0i}(x, t), \quad t = 0, \quad i = 1, 2, ..., k, \\ \varphi_i(x, t) &= 0, \quad x = 0, \quad i = 1, 2, ..., k, \\ \varphi_i(x, t) &= 0, \quad x = L, \quad i = 1, 2, ..., k. \end{aligned} \tag{18}$$

The upward fluid velocity, at a height x of the column where the particle concentration of category i is C_i, $1 \le i \le k$, is

$$u_F = -\frac{1}{\varepsilon} \sum_{j=1}^{k} u_j C_j = -\sum_{j=1}^{k} u_{0j} \varepsilon^{n_j - 1} C_j. \tag{19}$$

The settling velocity u_i for the i category at a given height of the column will be the addition of the fluid velocity u_F (*Eq.(19)*) and the relative fluid particle velocity $u_{i/F}$. Many researchers have worked on expressions for $u_{i/F}$ calculation. *Table 2* summarizes some of these expressions.

There are two basic ways to express the sedimentation velocity for each category: 1) extrapolating the methodological way used for monodisperse suspensions, and 2) considering a change in fluid density due to the suspension. LOCKETT and AL-HABOOBY's model [7] and the improvement due to MIRZA and RICHARDSON [8] , where the change in the exponent in *Eq.(21)* is a correction for better fit experimental data, are in the first category of models.

Table 2 Expressions for the calculation of the relative fluid particle velocity $u_{i/F}$ for the i category in a suspension formed by k categories

Reference	$u_{i/F}$	*Eqs.*	Application
LOCKETT and AL-HABBOOBY (1973) [7]	$u_{0i}\varepsilon^{n_i-1}$	(20)	
MIRZA and RICHARDSON (1979) [8]	$u_{0i}\varepsilon^{n_i-0.6}$	(21)	
MASLIYAH (1979) [9]	$u_{0i}\dfrac{\rho_i-\rho_{susp}}{\rho_i-\rho_l}\varepsilon^{n_i-2}$	(22)	$\rho_{susp}=\rho_l\varepsilon+\sum_{j=1}^{k}\rho_j C_j$
SELIM et al. (1983) [10]	$u_{0i}\dfrac{\rho_i-(\rho_{susp})_i}{\rho_i-\rho_l}\varepsilon^{n_i-1}$	(23)	$(\rho_{susp})_i=\dfrac{\rho_l\varepsilon+\sum_{j=1}^{i-1}C_j\rho_j}{1-\sum_{j=i}^{k}C_j}$
PATWARDHAN and TIEN (1985) [11]	$u_{0i}\dfrac{\rho_i-\rho_{susp}}{\rho_i-\rho_l}\varepsilon_i^{n_i-2}$	(24)	$\varepsilon_i=1-\left(\dfrac{d_i}{d_i+d_\varepsilon}\right)^3$ $d_\varepsilon=\dfrac{\sum_{j=1}^{k}C_j d_j}{\sum_{j=1}^{k}C_j}\left[(1-\varepsilon)^{-1/3}-1\right]$

MASLIYAH [9] introduced *Eq.(22)* and explained, with it, some experimental phenomena observed by previous authors. SELIM et al. [10] introduced the concept of change in suspension density for each particle category due to categories of smaller sizes. PATWARDHAN and TIEN [11] improved the Masliyah model introducing the concept of the apparent porosity of suspension defined as a function of the representative diameter d_i of each category.

It is common to express sedimentation data as interfaces velocities versus voidage in the suspension, and expressing these velocities from the model considering that the suspension has been segregated in k zones, containing different categories depending of each position in the column. The upper zone contains particles of the smallest size and the lower zone contains particles of all categories. Considering that all initial particles of the k category are in the zone k, considering that the concentrations in this zone are the initial concentration for the other categories, and proceeding in this manner from level k to level 1, authors, using different correlations of *Table 2*, calculate the interfaces velocities ([4], [8], [9], [10], [11], [12])

AL-NAAFÁ and SELIM [12] calculated the interfaces settling velocities for suspensions of two or three categories and they found that the SELIM et al. model [10] represents the experimental data accurately as compared to other models. They found that n_i values were better calculated by using Garside and Al-Dibouny correlation.

The consideration of maintenance of initial concentration in lower zones for smallest particles is very risky, taking into account that sedimentation velocity for some particles can be negative, moving upwards as has been noted by some researches [9]. For instance, using *Eq.(20)* for a two species suspension (k=2), and considering that $u_{01} > u_{02}$, u_2 becomes negative if the following condition is accomplished, at a given height x of the column,

$$\frac{u_{02}}{u_{01}} < (1 - C_1 - C_2)^{n_1-n_2}\frac{C_1}{1 - C_2} \tag{25}$$

where C_1 and C_2 are the particles concentration of each category at (x,t).

This consideration leads to the need to solve or approximate *Eq.(17)* in order to take into account the dynamic evolution of suspension.

Method of Approximation

The donor-cell discretisation method, and finite differences, will be used in the present work in order to approximate solutions of *Eqs.(17)* and *(18)*. This is characterised by [13],

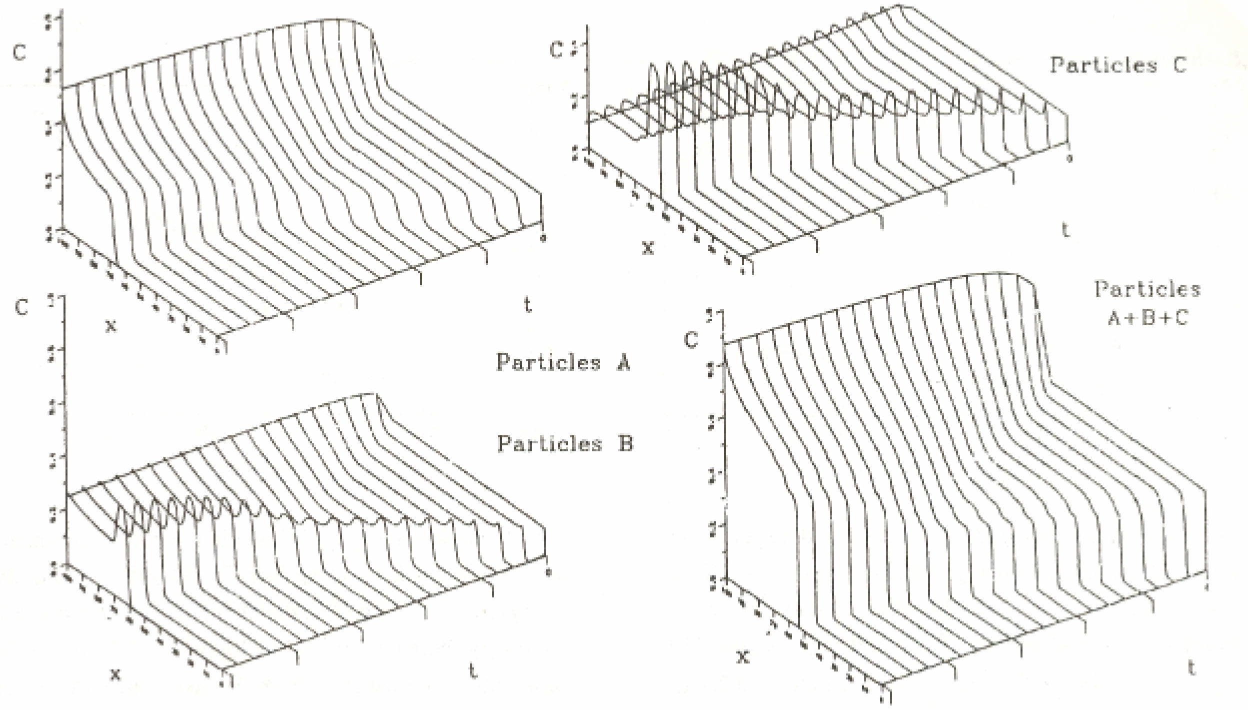

Fig.1 Evolution of particle concentration in a tridisperse suspension in water. Numerical approximation of *Eqs.(17)* and (18) with: $C_{0A}=C_{0B}=C_{0C}=0.1$, $u_{0A}=1.109\cdot 10^{-3}$ m/s, $u_{0B}=0.75u_{0A}$, $u_{0C}=0.5$, u_{0A}, $d_r=0.04$, $\rho_A=\rho_B=\rho_c=1070$ kg/m^3, $\nu_l=7.73\cdot 10^{-7}$ m^2/s, n values calculated by *Eq.(6)*

$$\begin{aligned}&\frac{1}{\Delta t}\left[C_i(x,t+\Delta t)-C_i(x,t)\right]- \\ &-\frac{\varphi_i(x+\Delta x/2,t)-\varphi_i(x-\Delta x/2,t)}{\Delta x}=0\,,\end{aligned} \qquad (26)$$

$$\varphi_i(x+\Delta x/2,t)=u_i(x+\Delta x/2,t+\Delta t/2)C_i(x+\Delta x/2,t)\,,$$

$$C_i(x+\Delta x/2,t)=\begin{cases}C_i(x+\Delta x,t), & u_i(x+\Delta x/2,t)>0,\\ C_i(x,t), & u_i(x+\Delta x/2,t)<0.\end{cases}$$

This is a first order method, it is conservative and it is stable if $\frac{|u_i|\Delta t}{\Delta x}\le 1$.

Approximating solutions of *Eq.(16)*, the evolution surface of C vs. $\hat{x}$ and $\hat{t}$ is obtained. The intersection of this surface with the horizontal plan $C(\hat{x},\hat{t})=C_0$ is the sedimentation curve obtained by the suspension/clear liquid interface evolution.

Eqs.(17) with *(18)* can be expressed in a dimensionless form by using the dimensionless variables $\hat{x}=x/L$ and $\hat{t}=\bar{u}_0t/L$, where $\bar{u}_0=\max(u_{01}, u_{02}, ..., u_{0k})$, with any equation of *Table 2*. Solutions approximation gives k evolution surfaces and k intersecting curves corresponding to the k interfaces. *Fig.1* shows the surfaces obtained in a case study where suspension was formed by three categories.

In order to test the ability of the model, *Eqs. (17)* and *(18)*, and the donor-cell method, *Eq.(26)*, and to compare results obtained by using different expressions for slip velocity calculation, *Eqs.(20)*, *(22)* and *(23)*, or different methods for n_i calculation, *Eqs.(6)* and *(9)*, numerical simulations were carried out by using measured parameters of MIRZA and RICHARDSON [8] and AL-NAAFÁ and SELIM [12] about bidisperse suspensions, and results were compared with their published experimental data.

Results and Discussion

In all the suspensions studied, n calculated values using Garside and Al Dibouni correlation (9) provide better results than that obtained from Richardson Zaki correlation (6). Wall effect correction using *Eq.(2)* provides better results than *Eq.(4)*, and in the case of *Fig.2* data, the use of (9) and (4) provides practically the same results as (6) and (2). The best combination has been found with *Eqs.(2)* and *(9)* for calculations of corrected terminal settling velocities and corresponding n values.

In all simulations carried out, lower interface velocities have been lower than those obtained by MIRZA and RICHARDSON [8] or AL-NAAFÁ and SELIM [12] using *Eqs.(20)*, *(22)* or *(23)*.

Varying the concentration of big particles (*Fig.2*, MIRZA and RICHARDSON experiments) or varying the concentration of small particles (*Fig.3*, AL-NAAFÁ and SELIM experiments), velocities obtained by numerical simulation are in good agreement with experimental data in both cases. In *Fig.2*, velocities calculated from *Eqs.(20)*, *(22)* or *(23)* present a maximum deviation of 1.6%. In *Fig.3*, data calculated from *Eqs.(20)* and *(23)*

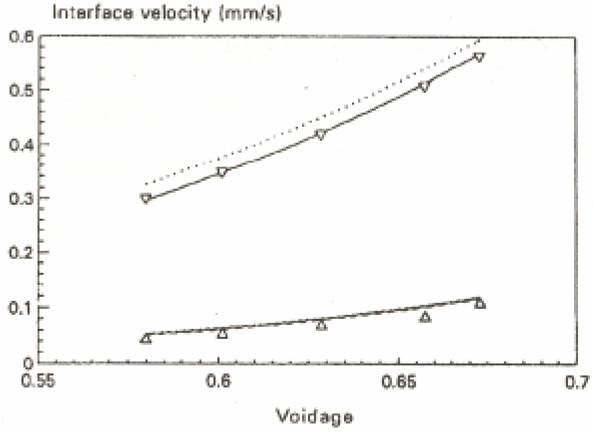

Fig.3 Comparison of *Eq.(17)* prediction with experimental data of [12], its *Fig.3*, by using *Eqs.(2)*, *(9)* and: —— *Eq.(20)*; - - - -*Eq.(22)* ; - · - · - *Eq.(23)*; d_1= 0.274·10^{-3} m, d_2= 0.081·10^{-3} m, ρ_1=2880 kg/m^3, ρ_2=2500 kg/m3, ρ_3=1113 kg/m^3, ν_l=1.563·10^{-5} m^2/s, d_r=0.03175 m, C_{01}=0.2201, C_{02}=0.1069-0.2. (∇: lower interface, Δ: upper interface)

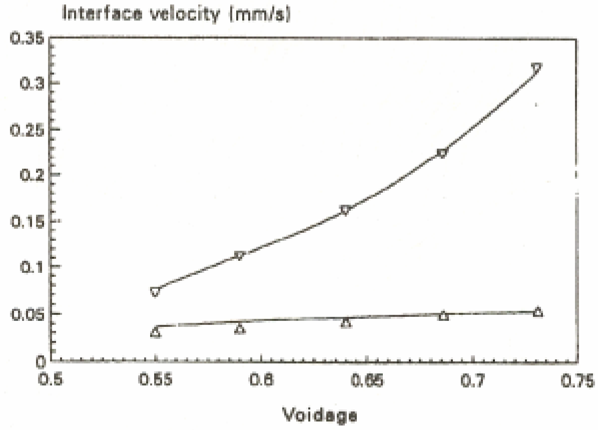

Fig.2 Comparison of *Eq.(17)* prediction with experimental data of [8], its *Fig.2*, by using *Eqs.(2)*, *(9)* and: —— *Eq.(20)*; - - - - *Eq.(22)*;- · - · - *Eq.(23)*; d_1= 0.462·10^{-3} m, d_2= 0.115·10^{-3} m, ρ_1=ρ_2=2958.6 kg/m^3, 909.2 kg/m^3, ν_l=1.78·10^{-4} m^2/s, d_r=0.028 m, C_{01}=0.201 - 0.3802, C_{02}=0.0679. (∇: lower interface, Δ: upper interface)

are practically the same, and *Eq.(22)* fails in experimental data prediction.

Numerical simulation predicts higher interface velocities for MIRZA and RICHARDSON experiments, when small particles concentration increases over 8%, increasing deviation as concentration is increased. This deviation has not been observed for simulations carried out with suspensions corresponding to AL-NAAFÁ and SELIM data, where small particles are faster and smaller than those used in [8], due to differences in fluid viscosity and density. It may be due to the evolution of small particles in compression zone where small particles can be entrapped, depending on the relative particles size concentrations.

Conclusions

The donor cell finite differences method can be used to approximate the continuity equation applied to the polydisperse suspension sedimentation problem, and predicted interfaces velocities are in good agreement with previously published experimental data in a wide range of relative concentrations. Predictions are more accurate when dimensionless parameter n is calculated by Garside and Al-Dibouni correlation and when relative particle fluid velocity is calculated by using expressions due to SELIM et al. [10] or LOCKETT and AL-HABBOOBY [7], in the range of tested parameters for bidisperse suspensions.

SYMBOLS

C	Particles concentration by volume
C_o	Initial particles concentration
d	Particles diameter
d_r	Diameter of the sedimentation vessel
e	Parameter in *Eq.(1)*
f	Parameter in *Eq.(1)*
k	Number of particles categories in a polydisperse suspension
L	Height of the sedimentation column
n	Dimensionless parameter in the general Richardson-Zaki expression
p	Parameter in *Eq.(10)*. Equals the close packed bed concentration
Re	Reynolds number
t	Time
$\hat{t}$	Dimensionless time
u	Settling velocity of particles
u_F	Fluid velocity
u_{fF}	Slip velocity. Relative velocity between particles and fluid
u_o	Terminal settling velocity of particles in a finite fluid medium
$\bar{u}_0$	Maximum terminal settling velocity among particles in a polydisperse suspension
u_t	Terminal settling velocity of particles in an infinite fluid medium
x	Distance from the top of the column
$\hat{x}$	Dimensionless length

Greek letters

ε	Suspension voidage
φ	Flux density of particles
ν_l	Kinematic viscosity of fluid
ρ	Density of particles
ρ_l	Density of fluid
ρ_{susp}	Density of a suspension

Subscripts

i	Corresponding to the ith particles category, $i \leq i \leq k$

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