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Cospectral digraphs from locally line digraphs

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Abstract

A digraph $\Gamma = (V, E)$ is a line digraph when every pair of vertices $u, v \in V$ have either equal or disjoint in-neighborhoods. When this condition only applies for vertices in a given subset (with at least two elements), we say that Γ is a locally line digraph. In this paper we give a new method to obtain a digraph Γ' cospectral with a given locally line digraph Γ with diameter D , where the diameter D' of Γ' is in the interval $[D - 1, D + 1]$. In particular, when the method is applied to De Bruijn or Kautz digraphs, we obtain cospectral digraphs with the same algebraic properties that characterize the formers.

Mathematics Subject Classifications: 05C20, 05C50.

Keywords: Digraph, adjacency matrix, spectrum, cospectral digraph, diameter, De Bruijn digraph, Kautz digraph.

1 Preliminaries

In this section we recall some basic terminology and simple results concerning digraphs and their spectra. For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks and papers on the subject; for instance, Chartrand and Lesniak [1] and Diestel [3].

Through this paper, $\Gamma = (V, E)$ denotes a digraph, with set of vertices $V = V(\Gamma)$ and set of arcs (or directed edges) $E = E(\Gamma)$, that is strongly connected (namely, every vertex is connected to any other vertex by traversing the arcs in their corresponding direction). An arc from vertex u to vertex v is denoted by either (u, v) or $u \rightarrow v$. As usual, we call *loop* an arc from a vertex to itself, $u \rightarrow u$, and *digon* to two opposite arcs joining a pair of vertices, $u \rightleftarrows v$. The set of vertices adjacent to and from $v \in V$ is denoted by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively. Such vertices are referred to as *in-neighbors* and *out-neighbors*

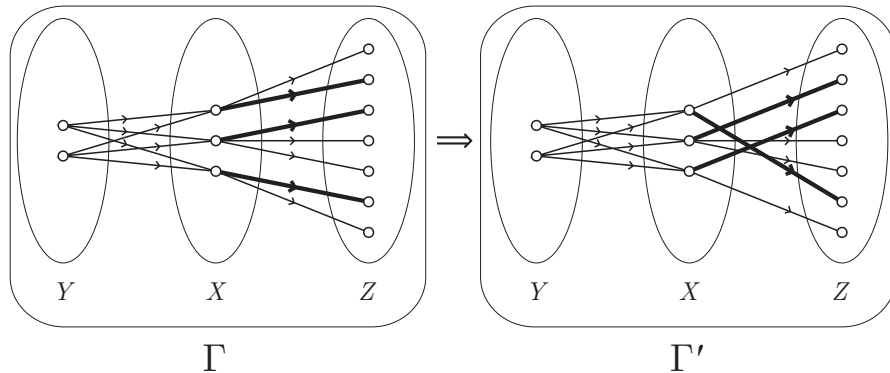


Figure 1: Scheme of the sets of Theorem 2.1. The arcs that change from Γ to Γ' are represented with a thick line.

of v , respectively. Moreover, $\delta^-(v) = |\Gamma^-(v)|$ and $\delta^+(v) = |\Gamma^+(v)|$ are the *in-degree* and *out-degree* of vertex v , and Γ is *d-regular* when $\delta^+(v) = \delta^-(v) = d$ for any $v \in V$. Similarly, given $U \subset V$, $\Gamma^-(U)$ and $\Gamma^+(U)$ represent the sets of vertices adjacent to and from (the vertices of) U . Given two vertex subsets $X, Y \subset V$, the subset of arcs from X to Y is denoted by $e(X, Y)$.

In the line digraph $L\Gamma$ of a digraph Γ , each vertex represents an arc of Γ , $V(L\Gamma) = \{uv : (u, v) \in E(G)\}$, and a vertex uv is adjacent to a vertex wz when $v = w$, that is, when in Γ the arc (u, v) is adjacent to the arc (w, z) : $u \rightarrow v(=w) \rightarrow z$. By the Heuchenne's condition [9], a digraph Γ is a line digraph if and only if, for every pair of vertices u, v , either $\Gamma^+(u) = \Gamma^+(v)$ or $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$. Since the line digraph of the converse digraph $\bar{\Gamma}$ (obtained from Γ by reversing the directions of all the arcs) equals the converse of the line digraph, $L\bar{\Gamma} = \overline{L\Gamma}$, the above condition can be restated in terms of the in-neighborhoods $\Gamma^-(u)$ and $\Gamma^-(v)$. In particular, we say that a digraph is a (*U*-)locally line digraph if there is a vertex subset U with at least two elements such that $\Gamma^-(u) = \Gamma^-(v)$ for every $u, v \in U$.

In the case of graphs instead of digraphs, the Godsil-McKay switching given in [8] is a technique to obtain cospectral graphs.

2 Main result

The following result describes the basic transformation of a digraph Γ into another digraph Γ' modifying slightly the walk properties of the former (see Figure 1).

Theorem 2.1. *Let $\Gamma = (V, E)$ be a digraph with diameter $D \geq 2$. Consider a subset of vertices $X = \{x_1, \dots, x_r\} \subset V$, $r \geq 2$, such that the sets of the in-neighbors of every x_i are the same for every x_i , say, $Y = \Gamma^-(x_i)$ for $i = 1, \dots, r$. Let $Z = \Gamma^+(X)$. Let Γ' be the modified digraph obtained from Γ by changing the set of arcs $e(X, Z)$ by another set of arcs $e'(X, Z)$ in such a way that the two following conditions are satisfied:*

- (i) The loops remain unchanged, that is, with $e'(Y, X)$ being the set of arcs from Y to X in Γ' , we must have $e'(Y, X) \cap e'(X, Z) = e(Y, X) \cap e(X, Z)$.
- (ii) For the arcs that are not loops, every vertex of X has some out-going arcs to a vertex of Z , and every vertex of Z gets some in-going arcs from a vertex of X .

Assume that there is a walk of length $\ell \geq 2$ from u to v ($u, v \in V$) in Γ .

- (a) If $u \notin X$, then there is also a walk of length ℓ from u to v in Γ' .
- (b) If $u \in X$, then there is a walk of length at most $\ell + 1$ from u to v in Γ' .

Proof. (a) Let $u_0(= u), u_1, \dots, u_{\ell-1}(= v)$ be an ℓ -walk from u to v in Γ . We distinguish two cases:

1. If $u_i \notin X$ for every $i = 1, \dots, \ell - 2$, the result is trivial as the walk in Γ' is the same as that in Γ .
2. If $u_i \in X$ for some $i = 1, \dots, \ell - 2$, then from the hypothesis on X we must have $u_{i-1} \in Y$ and $X \cap \Gamma^+(u_{i-1}) = X$. Moreover, by (ii), in Γ' there is a vertex $u'_i \in X$ adjacent to u_{i+1} . Thus, the required ℓ -walk in Γ' is just $u_0, \dots, u_{i-1}, u'_i, u_{i+1}, \dots, u_{\ell-1}$.

(b) If $u \in X$, the result is a simple consequence of (a). Indeed, by (ii) there is a vertex $u' \in Z \setminus X$ adjacent from u (otherwise, Γ would not be strongly connected). Then, it suffices to consider the walk u, u', \dots, v . This completes the proof. \square

If we consider shortest walks, the following consequence is straightforward.

Corollary 2.2. *If Γ is a digraph with diameter D , the modified digraph Γ' (in the sense of Theorem 2.1) has diameter D' satisfying $D - 1 \leq D' \leq D + 1$.*

Note that the case $D' = D - 1$ could happen when, in Γ , all vertices not in X have eccentricity $D - 1$ and, in Γ' all vertices in X result with the same eccentricity $D - 1$.

Examples of the case when the diameter remains unchanged, $D' = D$, are provided by the modified De Bruijn digraphs (see Section 4).

3 Cospectral digraphs

First notice that, because of the condition $Y = \Gamma^-(x_i)$, $i = 1, \dots, r$, the spectrum of Γ contains the eigenvalue 0 with multiplicity $m(0) \geq r - 1$. Indeed, suppose that its adjacency matrix \mathbf{A} is indexed in such a way that the first r rows correspond to the vertices of X . Then, the $r - 1$ (column) vectors $(1, -1, 0, 0, 0, \dots, 0)$, $(0, 1, -1, 0, 0, \dots, 0)$, \dots , $(0, \dots, 0, 1, -1, 0, \dots, 0)$ are clearly linearly independent, and they are also eigenvectors with eigenvalue 0. For more details, see Fiol and Mitjana [5].

Another interesting consequence of Theorem 2.1 is the following relationship between the adjacency matrices of Γ and Γ' , in the particular case when the in-degrees of the vertices of Z are preserved.

Proposition 3.1. *Assume that in the modified digraph Γ' from Γ , every vertex of Z gets the same number of in-going arcs as in Γ . That is, $\Gamma'^-(v) = \Gamma^-(v)$ for every $v \in Z$. Let $\mathbf{A} = (a_{uv})$ and $\mathbf{A}' = (a'_{uv})$ be the adjacency matrices of Γ and Γ' , respectively. Then, for any polynomial $p \in \mathbb{R}[x]$ without constant term, say, $p(x) = xq(x)$, with $\deg q = \deg p - 1$, we have*

$$p(\mathbf{A}') = \mathbf{A}'q(\mathbf{A}). \quad (1)$$

Proof. We only need to prove that $\mathbf{A}'\mathbf{A} = \mathbf{A}'\mathbf{A}'$. Since the only modified arcs are those adjacent from the vertices of X , we have

$$\begin{aligned} (\mathbf{A}'\mathbf{A})_{uv} &= \sum_{x \in X} a'_{ux}a_{xv} + \sum_{x \notin X} a'_{ux}a_{xv} = |X \cap \Gamma^-(v)| + \sum_{x \notin X} a'_{ux}a_{xv} \\ &= |X \cap \Gamma'^-(v)| + \sum_{x \notin X} a'_{ux}a_{xv} = \sum_{x \in X} a'_{ux}a'_{xv} + \sum_{x \notin X} a'_{ux}a'_{xv} = (\mathbf{A}'\mathbf{A}')_{uv}, \end{aligned}$$

where we used that every vertex of Z in Γ' gets the same number of in-going arcs as in Γ . \square

Proposition 3.2. *Within the conditions of Proposition 3.1, the digraphs Γ and Γ' are cospectral.*

Proof. First, note that Eq. (1) is equivalent to state that, for any polynomial $q \in \mathbb{R}[x]$,

$$\mathbf{A}'q(\mathbf{A}') = \mathbf{A}'q(\mathbf{A}).$$

In particular, if $q(x) = \phi_\Gamma(x)$ is the characteristic polynomial of Γ , the above equation gives

$$\mathbf{A}'\phi_\Gamma(\mathbf{A}') = \mathbf{A}'\phi_\Gamma(\mathbf{A}) = 0,$$

so that the polynomial $x\phi_\Gamma(x)$ is a multiple of the characteristic polynomial $\phi_{\Gamma'}(x)$ of Γ' , say, $x\phi_\Gamma(x) = r(x)\phi_{\Gamma'}(x)$ with $\deg r = 1$. Analogously, since Γ can be seen as a modified digraph of Γ' (G satisfies Proposition 3.1), we get $x\phi_{\Gamma'}(x) = s(x)\phi_\Gamma(x)$ with $\deg s = 1$. Then, we deduce that $\phi_\Gamma(x)$ and $\phi_{\Gamma'}(x)$ can only differ by a constant, but, as they are both monic polynomials, $\phi_\Gamma(x) = \phi_{\Gamma'}(x)$ and $\text{sp } \Gamma = \text{sp } \Gamma'$, as claimed. \square

Given a digraph Γ , its converse digraph $\bar{\Gamma}$ has the same vertex set as Γ , but all the directions of the arcs are reversed. Then, the walks of Γ and $\bar{\Gamma}$ are in correspondence, and, as the adjacency matrix of $\bar{\Gamma}$ is the transpose of that of Γ , both digraphs are cospectral. These facts leads us to the symmetric-like result of Theorem 2.1 and Proposition 3.2:

Corollary 3.3. *Let $\Gamma = (V, E)$ be a digraph with diameter $D \geq 2$. Consider a subset of vertices $X' = \{x_1, \dots, x_r\} \subset V$, $r \geq 2$, such that the sets of the out-neighbors of every x_i are the same for every x_i , say, $Y' = \Gamma^+(x_i)$ for $i = 1, \dots, r$. Let $Z' = \Gamma^-(X')$. Let Γ' be the modified digraph obtained from Γ by changing the set of arcs $e(Z', X')$ by another set of arcs $e'(Z', X')$ in such a way that the two following conditions are satisfied:*

- (i) *The loops remain unchanged, that is, with $e'(X', Y')$ being the set of arcs from X' to Y' in Γ' , we must have $e'(X', Y') \cap e'(Z', X') = e(X', Y') \cap e(Z', X')$.*
- (ii) *For the arcs that are not loops, every vertex of X' has some in-going arcs from a vertex of Z' , and every vertex of Z' gets some out-going arcs to a vertex of X' .*

Then, the following hold.

- (a) *The diameter D' of Γ' lies between $D - 1$ and $D + 1$.*
- (b) *If, in the modified digraph Γ' , every vertex of Z gets the same out-going arcs as in Γ , then Γ' and Γ are cospectral.*

Proof. Modify the converse digraph of Γ according to Theorem 2.1 and Proposition 3.2, and then take the converse digraph of the result. \square

4 The modified De Bruijn digraphs

The results of the preceding section can be used to obtain digraphs with specific distance-related or walk properties. Let us begin with the case of the so-called equi-reachable digraphs, of which the well-known De Bruijn digraphs are a particular example.

Let $\Gamma = (V, E)$ be a digraph with diameter D , and suppose that, for every pair of vertices $u, v \in V$, there is a walk of constant length $m (\geq D)$ from u to v . If ℓ is the smallest of such an m , we say that Γ is ℓ -reachable. Some times the term *equi-reachable* is used for digraphs that are ℓ -reachable (for some ℓ), that is, for digraphs with walks of equal length between vertices.

If Γ is ℓ -reachable and has maximum out-degree d , then its order is at most $N = d^\ell$, since this is the maximum number of different walks of length ℓ from a given vertex. To attain this bound there should be just one walk of length ℓ between any two vertices. Then, the adjacency matrix \mathbf{A} of Γ must verify the matrix equation

$$\mathbf{A}^\ell = \mathbf{J}, \tag{2}$$

and, therefore, Γ must be d -regular, see Hoffman and McAndrew [10]. Note also that these digraphs must be geodesic (that is, with just one shortest path between any two vertices).

The ℓ -reachable digraphs with d^ℓ vertices were studied by Mendelsohn in [12] as *UPP digraphs* (digraphs with the unique path property of order ℓ), and by Conway and Guy [2],

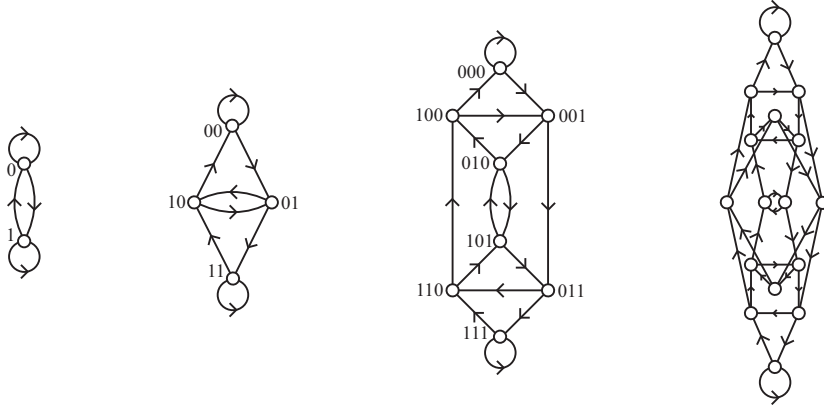


Figure 2: The De Bruijn digraphs $B(2, 1)$, $B(2, 2)$, $B(2, 3)$, and $B(2, 4)$.

unaware of the work of Mendelsohn, as *tight precisely ℓ -steps digraphs*, using them to construct large transitive digraphs of given diameter. Equi-reachable digraphs were also studied by Fiol, Alegre, Yebra, and Fàbrega [4].

Among the UPP digraphs, there are the well-known De Bruijn (or Good-De Bruijn) digraphs $B(d, \ell)$, whose set of vertices consists of all words of length ℓ from an alphabet of d symbols, say $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, and a vertex x is adjacent to a vertex y if the last $\ell-1$ symbols of x coincide with the first $\ell-1$ symbols of y . The De Bruijn digraphs $B(2, \ell)$ for $\ell = 1, 2, 3, 4$ are shown in Figure 2. In general, it is well-known that the digraph $B(d, \ell)$ is d -regular with diameter $D = \ell$, and it is the line digraph of $B(d, \ell-1)$. Moreover, its adjacency matrix satisfies Eq. (2), which, as said before, it is the algebraic condition for being ℓ -reachable. For more details, see Fiol, Yebra and Alegre [6, 7].

The De Bruijn digraphs are not the only UPP digraphs. For instance, for $d = 3$ and $\ell = 2$ Mendelsohn presented in [12] five other nonisomorphic such digraphs that can be seen as models of groupoids. More generally, UPP digraphs can be seen as models of a universal algebra, for more information see Mendelsohn [11].

To obtain a UPP digraph by modifying $B(d, \ell)$ according to Proposition 3.1, we need the modified digraph $B'(d, \ell)$ to have the same diameter ℓ , as shown in the following result.

Proposition 4.1. *Let $\Gamma = B(d, \ell)$. For some fixed values $x_i \in \mathbb{Z}_d$, $i = 1, 2, \dots, \ell-1$, not all of them being equal (to avoid loops), consider the vertex set $X = \{x_1 x_2 \dots x_{\ell-1} k : k \in \mathbb{Z}_d\}$. Let α_j , for $j \in \mathbb{Z}_d$, be d -permutations of $0, 1, \dots, d-1$. Let $\Gamma' = B'(d, \ell)$ be the modified digraph obtained by changing the out-going arcs of X in such a way that every vertex $x_1 x_2 \dots x_{\ell-1} k \in X$ is adjacent to the d vertices*

$$x_2 x_3 \dots x_{\ell-1} \alpha_j(k) j, \quad k = 0, 1, \dots, d-1. \quad (3)$$

Then Γ' is a d -regular digraph with the same diameter $D' = \ell$ as $\Gamma = B(d, \ell)$, and it is ℓ -reachable.

Proof. First, we only need to prove in-regularity (that is, constant in-degree) for every

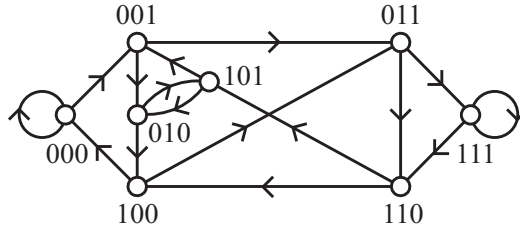


Figure 3: The modified De Bruijn digraph $B'(2, 3)$.

vertex of $Z = \Gamma^+(X)$ given by (3). But such a vertex is adjacent from the vertices

$$hx_2 \dots x_{\ell-1} \alpha_j(k), \quad h \neq x_1, \quad \text{and} \quad x_1 x_2 \dots x_{\ell-1} k.$$

Moreover, according to Theorem 2.1, it suffices to show that, from each vertex $u = x_1 x_2 \dots x_{\ell-1} k \in X$, there is an ℓ -walk from u to every other vertex $v = z_1 z_2 \dots z_{\ell-1} z_\ell$ in Γ' . To this end, we consider the following walk $u_0 (= u), u_1, \dots, u_{\ell-1}, u_\ell$ with

$$\begin{aligned} u_0 &= x_1 x_2 \dots x_{\ell-1} k, \\ u_1 &= x_2 x_3 x_4 \dots x_{\ell-1} \alpha_{y_1}(k) y_1, \\ u_2 &= x_3 x_4 \dots x_{\ell-1} \alpha_{y_1}(k) \alpha_{y_2}(y_1) y_2, \\ u_3 &= x_4 \dots x_{\ell-1} \alpha_{y_1}(k) \alpha_{y_2}(y_1) \alpha_{y_3}(y_2) y_3, \\ &\vdots \\ u_\ell &= \alpha_{y_2}(y_1) \alpha_{y_3}(y_2) \dots \alpha_{y_\ell}(y_{\ell-1}) y_\ell, \end{aligned}$$

where, if $u_i \notin X$ for some i , it is assumed that in u_{i+1} all the α_j 's are the identity (since there are no changes in the out-going arcs of the former), and

$$y_\ell = z_\ell, \quad y_{\ell-1} = \alpha_{y_\ell}^{-1}(z_{\ell-1}), \quad \dots, \quad y_2 = \alpha_{y_3}^{-1}(z_2), \quad y_1 = \alpha_{y_2}^{-1}(z_1),$$

so giving $u_\ell = z_1 z_2 \dots z_{\ell-1} z_\ell = v$, as desired. \square

By way of example, consider the modified De Bruijn digraph of Figure 3, obtained from $B(2, 3)$ by considering the set $X = \{100, 101\}$ (so that $Y = \{010, 110\}$), and removing the arcs $100 \rightarrow 001$ and $101 \rightarrow 011$ to set $100 \rightarrow 011$ and $101 \rightarrow 001$. (This corresponds to take the permutations $\alpha_0 = \iota$ (the identity) and $\alpha_1 = (01)$). Such a digraph was first shown by Fiol, Alegre, Yebra, and Fàbrega in [4].

The adjacency matrices of $B(2, 3)$ and $B'(2, 3)$, with the modified 1's in bold, are, respectively,

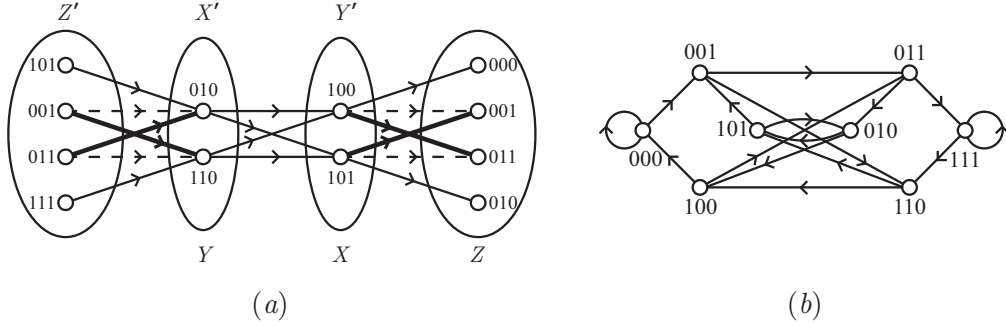


Figure 4: (a) The “left and right modifications” of $B^*(2, 3)$; (b) The doubly modified De Bruijn digraph $B^*(2, 3)$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

According to Proposition 3.2, both digraphs are cospectral with

$$\text{sp } B(2, 3) = \text{sp } B'(2, 3) = \{0^7, 2^1\},$$

where the superscripts denote the (algebraic) multiplicities of the eigenvalues.

Observe that $B(2, 3)$ and $B'(2, 3)$, shown in Figs. 2 and 3 respectively, are not isomorphic since, for instance, the former has two cycles (closed walks without repeated vertices) of length 5:

$$000 \rightarrow 001 \rightarrow 011 \rightarrow 110 \rightarrow 100 \rightarrow 000, \quad \text{and} \quad 111 \rightarrow 110 \rightarrow 100 \rightarrow 001 \rightarrow 011 \rightarrow 111,$$

whereas the latter has three:

$$\begin{aligned} &000 \rightarrow 001 \rightarrow 011 \rightarrow 110 \rightarrow 100 \rightarrow 000, \quad 111 \rightarrow 110 \rightarrow 101 \rightarrow 001 \rightarrow 011 \rightarrow 111, \\ &\text{and} \quad 010 \rightarrow 100 \rightarrow 011 \rightarrow 110 \rightarrow 101 \rightarrow 010. \end{aligned}$$

Of course, the above situation is not the general case. Many digraphs obtained by using the modifications described in Proposition 4.1 are cospectral, but also isomorphic to the original digraph. So, an interesting open problem would be to determine the conditions on the d -permutations α_j , for $j = 0, \dots, d-1$, to obtain nonisomorphic cospectral digraphs.

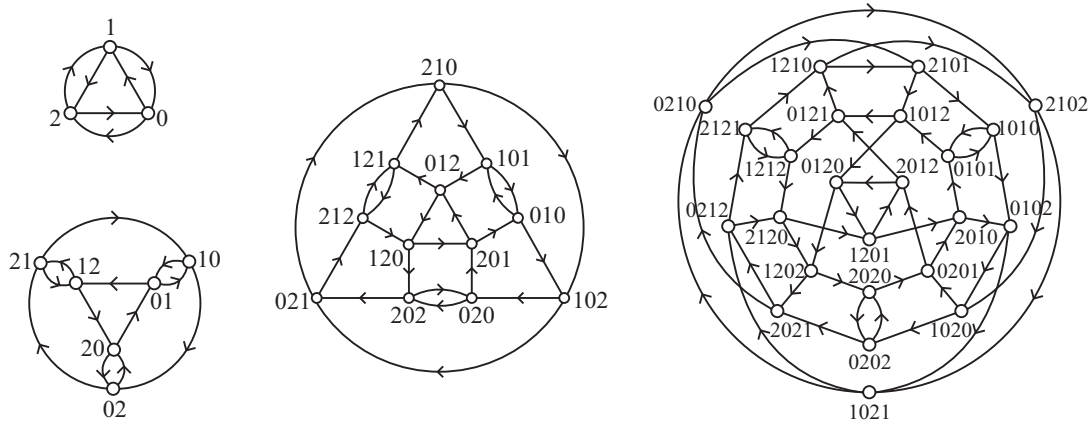


Figure 5: The Kautz digraphs $K(2, 1)$, $K(2, 2)$, $K(2, 3)$, and $K(2, 4)$.

In fact, a computer exploration shows that the only nonisomorphic 3-reachable 2-regular digraphs are $B(2, 3)$, $B'(2, 3)$, and $B''(2, 3) = \overline{B'(2, 3)}$, the converse digraph of $B'(2, 3)$, which can be also obtained by using our method. Indeed, it suffices to take $B(2, 3)$ and apply Corollary 3.3 with $X' = X = \{100, 101\}$ (so that $Y' = Y\{010, 110\}$), and change the same arcs as before, but now with opposite directions.

Another possible interesting perturbation is to apply a double modification: The one proposed in Theorem 2.1 (or Proposition 3.2) with the sets Y , X , and $Z(= \Gamma^+(X))$; and that of Corollary 3.3 with $Z' = \Gamma^-(X')$, $X'(= Y)$, and $Y'(= X)$. For example, in the case of $B(2, 3)$, these modifications are depicted in Fig. 4(a), where the dashed arcs are changed to the bold ones, and the obtained digraph is shown in Fig. 4(b). Notice that, in this case, we are not longer under the conditions of Proposition 4.1 and, hence, the resulting digraph $B^*(2, 3)$ with adjacency matrix \mathbf{A} , although still cospectral with $B(2, 3)$, is not a UPP digraph, that is, $\mathbf{A}^3 \neq \mathbf{J}$. (But, in fact, we have $\mathbf{A}^4 = 2\mathbf{J}$, which indicates the existence of exactly 2 walks of length 4 between any two vertices.)

5 The modified Kautz digraphs

The Kautz digraph $K(d, \ell)$ is defined as the De Bruin digraph $B(d, \ell)$ but now the consecutive symbols x_i and x_{i+1} , taken from the alphabet $\{0, 1, \dots, d\}$, must be different. The first four Kautz digraphs $K(2, \ell)$, for $\ell = 1, 2, 3, 4$, are represented in Figure 5. Again, it is well-known that any of these digraphs is the line digraph of the previous one (see Fiol, Yebra, and Alegre [7]). The adjacency matrix \mathbf{A} of the Kautz digraph $K(d, \ell)$ satisfies the matrix equation

$$\mathbf{A}^\ell + \mathbf{A}^{\ell-1} = \mathbf{J}, \quad (4)$$

so that between every pair of vertices u, v there is exactly one walk of length ℓ or $\ell - 1$.

Contrarily to the De Bruijn digraphs, some experimental results seems to show that all the modified Kautz digraphs $K'(d, \ell)$ have diameter $D' = \ell + 1$. For example, Figure 6 shows

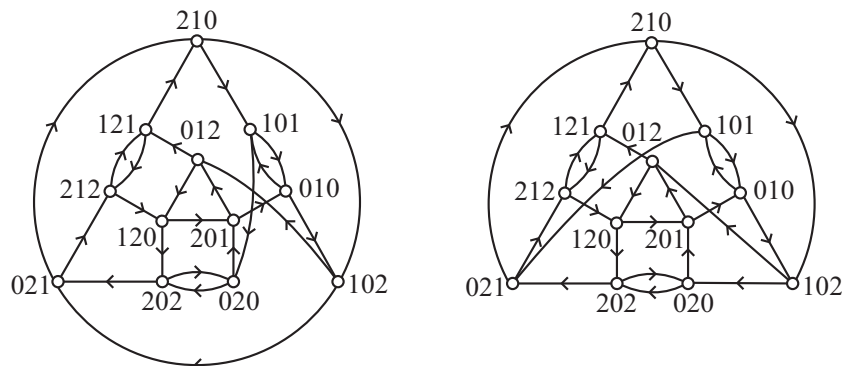


Figure 6: The modified Kautz digraphs $K'(2, 3)$ and $K''(2, 3)$.

two modified Kautz digraphs, $K'(2, 3)$ and $K''(2, 3)$, where, in both cases, $X = \{101, 102\}$ (so that $Y = \{010, 210\}$). Then, $K'(2, 3)$ is obtained by removing the arcs $101 \rightarrow 012$ and $102 \rightarrow 020$ to set the new arcs $101 \rightarrow 020$ and $102 \rightarrow 012$; whereas $K''(2, 3)$ is obtained by changing $101 \rightarrow 012$ and $102 \rightarrow 021$ to get $101 \rightarrow 021$ and $102 \rightarrow 012$.

In concordance with Proposition 3.2, all these digraph are cospectral with

$$\text{sp } K(2, 3) = \text{sp } K'(2, 3) = \text{sp } K''(2, 3) = \{-1^2, 0^9, 2^1\}.$$

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