

REGULARIZATION OF SLIDING GLOBAL BIFURCATIONS DERIVED FROM THE LOCAL FOLD SINGULARITY OF FILIPPOV SYSTEMS

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ABSTRACT. In this paper we study the Sotomayor-Teixeira regularization of a general visible fold singularity of a planar Filippov system. Extending Geometric Fenichel Theory beyond the fold with asymptotic methods, we determine the deviation of the orbits of the regularized system from the generalized solutions of the Filippov one. This result is applied to the regularization of global sliding bifurcations as the Grazing-Sliding of periodic orbits and the Sliding Homoclinic to a Saddle, as well as to some classical problems in dry friction.

Roughly speaking, we see that locally, and also globally, the regularization of the bifurcations preserve the topological features of the sliding ones.

1. Introduction. The main goal of this paper is to study how codimension-one global bifurcations of discontinuous Filippov systems evolve when the Sotomayor-Teixeira regularization [23] of the system is considered. We will mainly focus in the so called grazing-sliding bifurcation of periodic orbits and the sliding homoclinic bifurcation to a saddle. Both bifurcations involve a tangency between the periodic (or homoclinic) orbit of one of the adjacent vector fields with the discontinuity manifold. Therefore, although we are studying a global phenomenon, its behavior relies on the local behaviour of the regularized Filippov System near a so-called visible tangency point.

Summarizing, the problem addressed in this paper is the following: has the regularized vector field a similar behaviour as the Filippov one?

In recent years there has been an increasing research in piecewise differentiable vector fields. This kind of systems model many phenomena in control theory, in mechanical friction and impacts, in hysteresis in electrical circuits and plasticity, etc... See [4] for a general scope of the matter. In a piecewise differentiable vector field the phase space is divided into several regions where the system takes different smooth forms. The degree of discontinuity in the edge between two adjacent regions,

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usually called switching manifold, is used to classify them. Vector fields with jump discontinuities are usually named Filippov Systems.

In Filippov Systems the derivatives of the state variables are no longer uniquely determined since at the switching manifold they can take values in a whole interval. For the study of these systems, it has been generalized the concept of differential equation to a more general differential inclusion. The theory developed for these systems has succeeded to prove, under general conditions, theorems related to the existence and uniqueness of solutions ([16]). Moreover, over the switching manifold, using the Filippov convention ([7]), one can define a vector field made up from a certain linear convex combination of two adjacent equations.

The non-smooth mathematical models are often a discontinuous idealization of regular phenomena where the phase space is divided into regions in which the variables have different orders of behavior (slow-fast regions, for example). It is natural to ask if the generalized solutions of these discontinuous models are close to the solutions of the corresponding real regular ones. A natural question is whether a discontinuous system can be embedded in a set of parametric regular systems in such a manner that the discontinuous one will be, in some sense, their limit. But as noted in [26], not only there is not an unambiguous regularization technique but different regularization techniques can lead to different ways of defining the edge solutions. The way chosen will depend on their suitability to model the problem. For example in the case of dry friction systems that we deal with in Section 2.4, the regularization should be different if we use the stiction friction model or the Coulomb model, in spite of both models are identical outside the switching manifold.

Nevertheless, in the cases where the Filippov convention is used, it seems natural to consider the Sotomayor-Teixeira regularization, based in replacing the two adjacent fields by an ε -parametric field built as a linear convex combination of them in a ε -neighbourhood of the switching manifold. The regularized system so obtained is a slow-fast system on the plane.

We want to study how global bifurcations involving sliding are affected by the regularization. Therefore, we need to understand how the regularization affects tangency points with the switching manifold. It is known [2, 25] that, under general conditions, in some compact regions near the switching manifold (the so-called sliding and escaping zones which do not contain these tangency points) the regularized system has, for small values of the parameter ε , a normally hyperbolic invariant manifold (attracting near the sliding region or repelling near the escaping one) which is ε -close to the switching manifold. Furthermore, the flow of the regularized vector field reduced to this invariant manifold tends to the Filippov flow.

Therefore, the results in [2, 25] give a partial positive answer to the main question of this paper: the solutions of the regularized vector field are well approximated by the Filippov ones in these regions. This result can be proved in several ways but we stress the methods issued from the geometrical theory of singular perturbation of N.Fenichel and others [6, 11, 12].

But as one approaches to a boundary of the sliding (or escaping) region, that is, a point of tangency of one of the vector fields with the switching manifold (called in [9] fold-regular point) this theory fails because the tangency point of the Filippov vector field creates a fold point in the critical manifold of the regularized vector field and, therefore, the invariant manifold loses its hyperbolicity. At this stage, the theory needs to be combined with other tools, like asymptotic or blow-up methods to understand the behavior of the manifold near the fold point.

In [17, 9], a systematic topological classification and normal forms for different types of tangency points of Filippov vector fields and their bifurcations is made. It is therefore natural to study the regularization of these normal forms to determine in which cases the dynamics of the regularized normal forms moves towards the corresponding one in the Filippov system. Although in this paper we only examine in detail the regularization of the normal form of a visible tangency, we think that the same approach can be used to study other tangencies.

With the tools provided by singular perturbation theory and asymptotic expansions, following [21], we analyse how the normally hyperbolic invariant manifold deviates in passing around the fold and we determine regions close to the fold exponentially attracted to this variety. Then we conclude that the orbits issuing from these regions, after passing near the tangency, are concentrated in an exponentially small neighborhood of the extended invariant manifold provided by Fenichel theory. Moreover, the deviation of the invariant manifold is leaded by a distinguished solution of a Riccati equation, a typical result in singular perturbed systems around the singular points of the critical manifold ([21, 1, 15]). One can then conclude that, also close to a visible fold-regular point, the regularized system behaves closely to the Filippov one.

From the work of Dumortier, Krupa, Roussarie, Szmolian, Wechselberger ([5, 24, 14]) and others, the blow-up technique is used as a geometrical alternative to asymptotic methods. Nevertheless, we have decided to use these last methods because we only need to arrive until the lower half region of the fold and the calculations involved are not too difficult. Furthermore, the careful analysis needed to control the regions exponentially attracted by the invariant manifold is made comfortably with these methods. The recent paper [13] studies the problem of a two-fold singular point in \mathbb{R}^3 using blow up methods.

The qualitative results obtained in this work do not depend of the degree of smoothness of the regularized system but the quantitative ones do. In the case that the regularized system is \mathcal{C}^1 , that is, the contact of the regularized field and the two adjacent fields in the boundary of the regularization zone is strictly of order one, we prove the well known result [21, 14] that the deviation of the invariant manifold is $O(\varepsilon^{\frac{2}{3}})$. But we think is worth deriving it in the setting of piecewise differentiable systems and also as a basis to extend it to the \mathcal{C}^{p-1} contact, $p \geq 2$, where we find that the deviation is $O(\varepsilon^{\frac{p}{2p-1}})$. A crucial result in our work is to see that the invariant manifold attracts a region near the sliding region which contains points up to a distance of order ε^λ , $\lambda < \frac{p}{2p-1}$, to the tangency point.

The fact that the regularization only takes place in an ε -neighborhood of the switching manifold, remaining unaltered the adjacent fields outside, makes easier to analyze global properties of the system. If the field tangent to the switching manifold has any stable recurrence, such as a (sliding or grazing) periodic orbit or a sliding homoclinic orbit to a hyperbolic saddle, the exponential flattening to the slow manifold of sliding areas ε^λ -near the fold, will ensure recurrence also in the regularized system, and a return Poincaré map can be determined and computed.

All this will allow us to study the existence of global periodic orbits in the regularized system in different settings, like in one parameter Filippov families of vector fields having a grazing-sliding bifurcation of periodic orbits or a sliding homoclinic bifurcation. We will also apply our results to some classical examples as the dry friction models.

The paper is organized as follows.

In Section 2 we introduce the notation, the basic concepts of a Filippov vector field in the plane and we present the Sotomayor-Teixeira regularization. To study the dynamics near a fold-regular point we introduce Poincaré sections and a Poincaré map near the fold. The main technical result is Theorem 2.1, where we give the main asymptotic properties of this Poincaré map.

Using this local result, in Theorem 2.2 we analyze the existence of periodic orbits in the regularized system assuming that the Filippov vector field has some global recurrence which typically occurs near a grazing sliding bifurcation. Finally, Theorem 2.4 studies the possible global bifurcations of periodic orbits in the regularization of a one parameter family of Filippov vector fields undergoing a grazing-sliding bifurcation. As expected, we see that the grazing-sliding bifurcation of a hyperbolic attracting periodic orbit leads to a structurally stable periodic orbit in the regularized system and the grazing-sliding bifurcation of a hyperbolic repelling periodic orbit creates a bifurcation of periodic orbits in the regularized system.

In Section 2.4 we consider three basic models of dry friction in single degree of freedom systems, following the formulation described in [18, 19]. We see that the Stribeck model fulfills the hypotheses of the Theorem 2.2 to directly conclude the existence of attracting periodic orbits of the regularized system. In Theorem 2.3, we will see that our methods will be able to ensure the existence of periodic orbits also in the Coulomb model, in spite of the neutral character of the tangent orbit (it belongs to a center). The exponential concentration of the regularized field to a neighborhood of the Fenichel variety combined with the return that provides the center will guaranty that the unique orbit of the non-smooth system tangent to the border of the regularization zone is semi-stable, that is, attracts all the regularization strip.

The Sotomayor-Teixeira regularization does not apply for the Stiction model as the mechanical analysis in the switching manifold gives an equation different from the Filippov one. It is clear that a different regularization will be needed as the phase portrait of the slip Stiction model equations is identical to Coulomb and therefore the regularized system would tend to the Filippov dynamics. This case is beyond the scope of this article and will be studied later.

The last results of the paper deal with the existence of periodic orbits (and homoclinic ones) in the regularization of a Filippov system having a sliding homoclinic orbit to a saddle, creating a pseudo-separatrix connection between a saddle and a fold ([17]). This is a codimension one phenomena and therefore appears generically in some one-parameter families. Theorem 2.7 studies the general case, showing the existence, in the regularized system, of a so-called homoclinic bifurcation where the periodic orbit dies in a homoclinic one and then disappears. Theorem 2.8 studies the corresponding bifurcation in the Hamiltonian case where the existence of a homoclinic orbit is generic.

The proof of Theorem 2.1, rather cumbersome, is deferred to Section 3. The main idea is to use the fact that the regularized vector field and the Filippov one are identical everywhere except in a region near the switching manifold which is of order ε . So the main part of the proof is to study the behavior of the regularized system, which turns to be a slow-fast system, in this region. This study is done using geometric singular perturbation theory, which provides the existence of a normally attracting invariant manifold Λ_ε of the system. Once we have this invariant manifold we need to extend it to see two things: on the one hand we have that this manifold exponentially attracts a region which contains points which are at a distance of

order ε^λ , $\lambda < \frac{p}{2p-1}$, to the origin (see propositions 4, 8, 13). On the other hand, we need to give an asymptotic expression of this invariant manifold when it arrives to the border of the regularized region (see propositions 3, 5, 6, 9, 11). This last part is done using asymptotic expansions and matching methods to obtain a suitable inner equation.

Although in Section 3.4 we study in detail the \mathcal{C}^1 regularization of the normal form of the visible fold, in sections 3.5 and 3.6 we show that the techniques used and the results generalize straightforwardly to \mathcal{C}^{p-1} regularizations and generic folds.

Besides a greater complication of the computations, the only delicate issue to study the \mathcal{C}^{p-1} case, is the determination of a distinguished solution of the equation

$$y' = x + y^p$$

that appears as a dominant term in the asymptotic development near the fold. This equation is well studied in the case $p = 2$ (see [21]) but, as far as the authors know, the general case has not been done before. For this reason, in Proposition 10, we prove the existence and properties of this distinguished solution. Later, in Proposition 11 we prove that the Fenichel manifold is well approximated by this solution up to an error of order $O(\varepsilon^{\frac{p}{2p-1}})$.

We want to conclude by emphasizing that, although this work studies a generic visible fold-regular point in a Filippov vector field in the plane, we think that the methods used here can be useful to study local bifurcations as fold-fold points and also higher dimensional Filippov systems. We also expect to extend these results to the case where the regularized vector field is analytic. The main novelty of this case will be that the regularized vector field and the Filippov one are different in the whole phase space, but this is just a technical problem that will not change the final results.

2. Hypotheses and main results. The main goal of this section is to introduce the regularization of a Filippov vector field in the plane near a visible fold-regular point and give the main results of the paper. Therefore, we consider a non-smooth system in \mathbb{R}^2 :

$$Z(x, y) = \begin{cases} X^+(x, y), & (x, y) \in \mathcal{V}^+ \\ X^-(x, y), & (x, y) \in \mathcal{V}^-, \end{cases} \quad (1)$$

where: $\mathcal{V}^+ = \{(x, y) \in \mathcal{V}, y > 0\}$, $\mathcal{V}^- = \{(x, y) \in \mathcal{V}, y < 0\}$, and \mathcal{V} is a neighborhood of the origin, with a switching manifold given by:

$$\Sigma = \{(x, y) \in \mathcal{V}, y = 0\}.$$

We assume that the vector fields X^+ and X^- have an extension to Σ which is, at least \mathbb{C}^2 , and we denote their flows by ϕ_{X^+} and ϕ_{X^-} respectively.

We assume that the vector field X^- is transversal to Σ and that X^+ has a generic fold in Σ , that is:

$$\begin{aligned} X^+(0, 0) &= (X_1^+(0, 0), 0), & X_1^+(0, 0) &\neq 0, & \frac{\partial X_2^+}{\partial x}(0, 0) &\neq 0 \\ X^-(0, 0) &= (X_1^-(0, 0), X_2^-(0, 0)), & X_2^-(0, 0) &\neq 0. \end{aligned} \quad (2)$$

Without loss of generality we can assume that the fold point is at $(0, 0)$.

We will consider the case where:

$$X_2^-(0, 0) > 0, \text{ and } X_2^+(x, 0) < 0 \text{ for } x < 0, X_2^+(x, 0) > 0 \text{ for } x > 0. \quad (3)$$

These conditions ensure that $(0, 0)$ is a generic visible fold-regular point. As $X_1^+(0, 0) \neq 0$, we will deal with the case

$$X_1^+(0, 0) > 0, \quad (4)$$

which implies that X^+ goes “to the right”. Analogous results are true for $X_1^+(0, 0) < 0$.

The fold point divides, locally, the switching manifold Σ in two regions:

$$\begin{aligned} \Sigma^s &= \{(x, 0) \in \mathcal{V}, x < 0\} \text{ the sliding region,} \\ \Sigma^c &= \{(x, 0) \in \mathcal{V}, x > 0\} \text{ the crossing region.} \end{aligned} \quad (5)$$

Also, following [9], we define

$$W_+^s(0, 0) = \{\phi_{X^+}(t; 0, 0), t < 0\}, \quad W_+^u(0, 0) = \{\phi_{X^+}(t; 0, 0), t > 0\} \quad (6)$$

the stable and unstable pseudo-separatrices in \mathcal{V}^+ of the fold point $(0, 0)$. Under our hypotheses, the fold point also has a stable pseudoseparatrix in \mathcal{V}^- , but it does not play any role in our setting.

As usual in non-smooth vector fields, we consider the flow of a point $p \notin \Sigma$ as given by the flows of the vector fields X^+ or X^- , respectively, depending if $p \in \mathcal{V}^\pm$. If the point p belongs to the switching manifold Σ in the crossing region Σ^c we concatenate both flows in a consistent way. Moreover, with the Filippov convention [7], we can define a sliding vector field in the sliding region Σ^s , that, in our case, reads:

$$\dot{x} = \frac{X_1^+ X_2^- - X_1^- X_2^+}{X_2^- - X_2^+}(x, 0), \quad x < 0.$$

This allows us to define a flow in the whole neighborhood of $(0, 0)$ (see [9]).

Moreover, under conditions (2), (3) and (4), we also have, for $x < 0$, small enough:

$$X_1^+ X_2^- - X_1^- X_2^+ > 0 \quad (7)$$

which gives that the Filippov vector field also moves “to the right”.

To study the behavior near the fold, we consider any value $y_0 > 0$ and the Poincaré sections

$$\mathcal{S}_{y_0}^- = \{(x, y_0) \in \mathcal{V}, x < 0\}, \quad \mathcal{S}_{y_0}^+ = \{(x, y_0) \in \mathcal{V}, x > 0\}.$$

We denote by

$$(x_0^\pm, y_0) = W_+^{u,s}(0, 0) \cap \mathcal{S}_{y_0}^\pm$$

and we assume that y_0 is small enough in such a way that these intersections are transversal.

We consider the Poincaré map:

$$\begin{aligned} P_0 : D_0 \times \{y_0\} \subset \mathcal{S}_{y_0}^- &\rightarrow \mathcal{S}_{y_0}^+ \\ (x, y_0) &\mapsto (P_0(x), y_0). \end{aligned} \quad (8)$$

where $D_0 \subset \mathbb{R}$ is a suitable neighborhood of x_0^- . For the Filippov system (1), all the trajectories of the system beginning at $(x, y_0) \in D_0 \times \{y_0\}$ with $x \leq x_0^-$ arrive to the sliding region Σ^s (see (5)), then slide until they leave the switching manifold Σ at the fold $(0, 0)$ following its unstable pseudoseparatrix $W_+^u(0, 0)$ (see figure 1). Therefore the map P_0 satisfies:

$$\forall x \in D_0^- = \{x \in D_0, x \leq x_0^-\}, \quad P_0(x) = x_0^+.$$

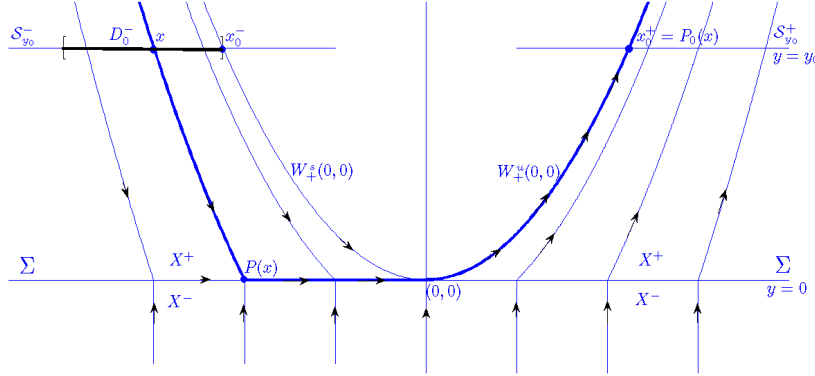


FIGURE 1. The Poincaré map P_0 for the Filippov system. The thick curve represents the trajectory passing through (x, y_0) .

2.1. The regularized system near the fold. As the non-smooth system Z in (1) can be written as:

$$Z(x, y) = \frac{X^+(x, y) + X^-(x, y)}{2} + \Xi(y) \frac{X^+(x, y) - X^-(x, y)}{2},$$

where the function Ξ is the discontinuous function: $\Xi : \mathbb{R} \rightarrow \mathbb{R}$, defined by:

$$\Xi(z) = \begin{cases} -1 & \text{if } z < 0 \\ 1 & \text{if } z > 0 \end{cases},$$

a classical way to regularize the vector field Z [23] is to consider vector fields Z_ε :

$$Z_\varepsilon(x, y) = \frac{X^+(x, y) + X^-(x, y)}{2} + \varphi\left(\frac{y}{\varepsilon}\right) \frac{X^+(x, y) - X^-(x, y)}{2}, \quad (9)$$

where we can take any increasing smooth function φ which approximates the discontinuous function Ξ and satisfies:

$$\varphi(v) = -1, \text{ for } v \leq -1, \quad \varphi(v) = 1, \text{ for } v \geq 1.$$

Let us point out that, with these smooth regularizations, outside the regularized zone $|y| \leq \varepsilon$, the regularized vector field Z_ε coincides with the non-smooth one Z . This would not be the case if we chose an analytic function φ in (9). In that case Z_ε and Z would be different everywhere and this will be the study of a future work.

To understand the orbits of the regularized vector field Z_ε we will study the Poincaré map

$$P_\varepsilon : D_\varepsilon \times \{y_0\} \subset \mathcal{S}_{y_0}^- \rightarrow \mathcal{S}_{y_0}^+,$$

that will be defined in a suitable domain $D_\varepsilon \subset \mathbb{R}$.

We denote by $(x_\varepsilon, \varepsilon)$ the point where the vector field X^+ has a tangency with the horizontal line $y = \varepsilon$, that is

$$X_2^+(x_\varepsilon, \varepsilon) = 0. \quad (10)$$

Clearly, by (2), $x_\varepsilon = O(\varepsilon)$, and we also consider $(\bar{x}_\varepsilon, y_0)$ the intersection of its orbit by X^+ with $\mathcal{S}_{y_0}^-$, that is

$$(\bar{x}_\varepsilon, y_0) = \phi_{X^+}(t; x_\varepsilon, \varepsilon) \in \mathcal{S}_{y_0}^- \quad (11)$$

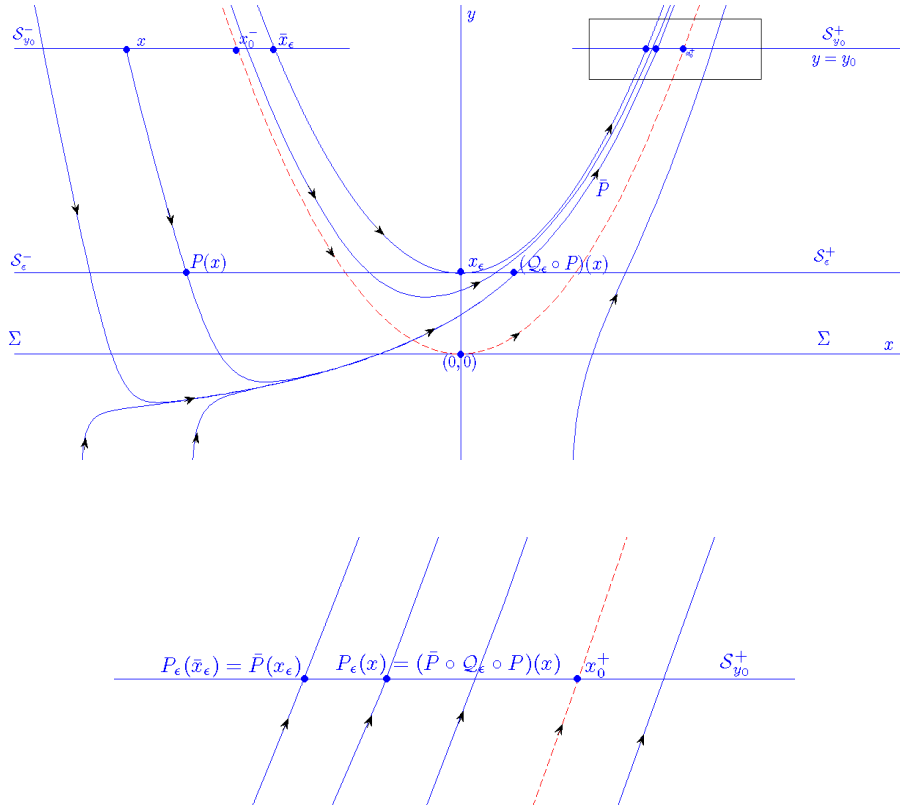


FIGURE 2. The Poincaré map $P_{\epsilon} = P \circ Q_{\epsilon} \circ \bar{P}$ for the regularized system Z_{ϵ} . The dotted red parabola represents the trajectory of X^+ passing through the fold. One can see the exponential attraction of the Fenichel manifold. The lower picture is a zoom of the neighbourhood of $P_{\epsilon}(x)$.

for some suitable $t < 0$ (see figure 2).

It is clear that, for $x \in D_{\epsilon}$ such that $x \geq \bar{x}_{\epsilon}$, one has $P_{\epsilon}(x) = P_0(x)$. Therefore, we will restrict our study of the Poincaré map P_{ϵ} to the points $x \in D_{\epsilon}$ such that $x \leq \bar{x}_{\epsilon}$.

In Theorem 2.1 we will give and asymptotic expansion, for ϵ small enough, of the Poincaré map P_{ϵ} in a suitable subset $\mathcal{I} \subset D_{\epsilon}$.

For $x \leq \bar{x}_{\epsilon}$, it will be convenient to write the map $P_{\epsilon} = \bar{P} \circ Q_{\epsilon} \circ P$ (see figure 2), where

$$\begin{aligned} P &: S_{y_0}^- \rightarrow S_{\epsilon}^- \\ Q_{\epsilon} &: S_{\epsilon}^- \rightarrow S_{\epsilon}^+ \\ \bar{P} &: S_{\epsilon}^+ \rightarrow S_{y_0}^+. \end{aligned}$$

The map Q_{ϵ} is defined in the region where the regularized system Z_{ϵ} and the original Filippov one Z are different. Its study will be one of the main goals of the paper and will be done using Geometric Singular Perturbation Theory in Section 3.

Clearly P and \bar{P} are the same for Z and the regularized system Z_ε . In fact, they are Poincaré maps associated to the vector field X^+ . Their asymptotic expressions for ε small enough are an easy consequence of next proposition.

Proposition 1. *Consider a vector field X^+ satisfying (2), (3) and (4). Consider the pseudoseparatrices of the fold $W_+^{u,s}(0,0)$, and the points $(x_0^\pm, y_0) = W_+^{u,s}(0,0) \cap \mathcal{S}_{y_0}^\pm$ and assume that these intersections are transversal, that is $X_2^+(x_0^\pm, y_0) \neq 0$. Denote by T^\pm the time such that $\phi_{X^+}(T^\pm; 0, 0) \in \mathcal{S}_{y_0}^\pm$, where $\phi_{X^+}(t; x, y)$ is the flow of the (regular) vector field X^+ . Consequently $\phi_{X^+}(T^\pm; 0, 0) = (x_0^\pm, y_0)$.*

Then, there exists a neighborhood U of the origin such that, for any $(x, y) \in U$, there exist regular functions

$$\begin{aligned} \tau^\pm : U &\rightarrow \mathbb{R} \\ (x, y) &\mapsto \tau^\pm(x, y) \end{aligned}$$

such that, $\phi_{X^+}(\tau^\pm(x, y); x, y) \in \mathcal{S}_{y_0}^\pm$. Moreover:

- $\tau^\pm(0, 0) = T^\pm$
- If $(x, y) \in U$, one has

$$\phi_{X^+}(\tau^\pm(x, y); x, y) = (x_0^\pm + \alpha^\pm y + \beta^\pm x^2(1 + \mathcal{O}(x)) + \mathcal{O}(xy, y^2), y_0)$$

with $\alpha^+ < 0$, $\beta^+ > 0$, $\alpha^- > 0$, $\beta^- < 0$.

Proof. Let's consider the flow of X^+ , $\phi_{X^+}(t; x, y)$ and denote by $\pi_x(\phi_{X^+}(t; x, y))$, $\pi_y(\phi_{X^+}(t; x, y))$ its x and y components respectively.

The existence of the functions $\tau^\pm(x, y)$ is a consequence of the Implicit Function Theorem applied to the equation $m(t, x, y) = \pi_y(\phi_{X^+}(t; x, y)) - y_0 = 0$ near $(T^+, 0, 0)$ and $(T^-, 0, 0)$ respectively: we know that $m(T^\pm, 0, 0) = 0$ and the transversality of the intersections of $W_+^u(0, 0) \cap \mathcal{S}_{y_0}^+$ and $W_+^s(0, 0) \cap \mathcal{S}_{y_0}^-$ gives $\frac{\partial m}{\partial t}(T^\pm, 0, 0) = X_2^+(x_0^\pm, y_0) \neq 0$.

We compute $\phi_{X^+}(t; x, y)$ developing by Taylor series at $(x, y) = (0, 0)$:

$$\phi_{X^+}(t; x, y) = \phi_{X^+}(t; 0, 0) + D\phi_{X^+}(t; 0, 0) \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}_2(x, y). \quad (12)$$

We observe that $D\phi_{X^+}(t; 0, 0)$ is the fundamental matrix of the variational equations:

$$z' = DX^+(\phi_{X^+}(t; 0, 0))z, \quad \text{satisfying} \quad D\phi_{X^+}(0; 0, 0) = \text{Id}.$$

We know that $\phi'_{X^+}(t; 0, 0)$ is a solution of the variational equations and that, by hypotheses (2), $\phi'_{X^+}(0; 0, 0) = (X_1^+(0, 0), 0)$, therefore, one can take $z_1(t) = \frac{1}{X_1^+(0, 0)}\phi'_{X^+}(t; 0, 0)$ and $D\phi_{X^+}(t; 0, 0) = \begin{pmatrix} z_1(t) & z_2(t) \end{pmatrix}$, where $z_2(t)$ is a suitable solution of the variational equation.

By the Implicit Function Theorem we know that:

$$D\tau^\pm(0, 0) = -\frac{1}{\partial_t m(T^\pm, 0, 0)} Dm(T^\pm, 0, 0) = -\frac{1}{y_0'(T^\pm)} \left(\frac{y_0'(T^\pm)}{X_1^+(0, 0)}, \pi_y(z_2(T^\pm)) \right)$$

where we have denoted by $(x_0(t), y_0(t)) = \phi_{X^+}(t; 0, 0)$.

Now, using (12), we compute:

$$\pi_x(\phi_{X^+}(\tau^\pm; x, y)) = x_0(\tau^\pm) + \frac{1}{X_1^+(0, 0)} x_0'(\tau^\pm)x + \pi_x(z_2(\tau^\pm))y + \mathcal{O}_2(x, y).$$

Using the Taylor expansion of τ^\pm and also expanding the above expression for $x_0(t)$ we obtain:

$$\begin{aligned} \pi_x(\phi_{X^+}(\tau^\pm; x, y)) &= x_0(T^\pm) - x'_0(T^\pm) \frac{1}{y'_0(T^\pm)} \left(\frac{y'_0(T^\pm)}{X_1^+(0, 0)} x + \pi_y(z_2(T^\pm)) y \right) \\ &+ \frac{1}{X_1^+(0, 0)} x'_0(T^\pm) x + \pi_x(z_2(T^\pm)) y + \mathcal{O}_2(x, y) \\ &= x_0^\pm + \alpha^\pm y + \mathcal{O}_2(x, y) = x_0^\pm + \alpha^\pm y + \beta^\pm x^2 (1 + \mathcal{O}(x)) \\ &+ \mathcal{O}(xy, y^2). \end{aligned}$$

One could write explicit expressions for the constants α^\pm and β^\pm but they are rather cumbersome and they are not necessary. We just stress here that the signs of α^\pm and β^\pm are a consequence of the fact that the orbits of a vector field on the plane can not intersect. \square

From this proposition, it is clear that, if $(x, \varepsilon) \in U$, and P and \bar{P} are defined:

$$\begin{aligned} P^{-1}(x) &= x_0^- + \alpha^- \varepsilon + \beta^- x^2 (1 + \mathcal{O}(x)) + \mathcal{O}(\varepsilon x, \varepsilon^2), \\ \bar{P}(x) &= x_0^+ + \alpha^+ \varepsilon + \beta^+ x^2 (1 + \mathcal{O}(x)) + \mathcal{O}(\varepsilon x, \varepsilon^2). \end{aligned} \quad (13)$$

Observe that, the domain of \bar{P} is $U^+ = [x_\varepsilon, k^+]$ where the point $(x_\varepsilon, \varepsilon)$ corresponds to the point (10) where the vector field X^+ has a tangency with the horizontal line $y = \varepsilon$, and k^+ is a suitable constant independent of ε . Analogously, the domain of P is $U^- = [K^-, \bar{x}_\varepsilon]$, where the point $\bar{x}_\varepsilon = P^{-1}(x_\varepsilon)$ was defined in (11).

As $x_\varepsilon = O(\varepsilon)$, using the formulas given in (13):

$$\begin{aligned} \bar{P}(x_\varepsilon) &= x_0^+ + \alpha^+ \varepsilon + O(\varepsilon^2) \\ \bar{x}_\varepsilon = P^{-1}(x_\varepsilon) &= x_0^- + \alpha^- \varepsilon + O(\varepsilon^2). \end{aligned} \quad (14)$$

Summarizing, one has that

$$\begin{aligned} \bar{P} : [x_\varepsilon, k^+] &\rightarrow [\bar{P}(x_\varepsilon), K^+] \\ P : [K^-, \bar{x}_\varepsilon] &\rightarrow [k^-, x_\varepsilon]. \end{aligned}$$

Section 3 is devoted to study the Poincaré map \mathcal{Q}_ε after the regularization. Combining the behavior of \mathcal{Q}_ε with the maps P and \bar{P} we will obtain the asymptotics for P_ε .

We will consider different functions φ with different regularity and we will study how the properties of the regularized system depend on this regularity. Moreover, using geometric singular perturbation theory and matching asymptotic expansions, we will give asymptotic formulas for the Poincaré map \mathcal{Q}_ε .

There are two significantly different cases:

- φ is a continuous piecewise linear function:

$$\varphi(v) = \begin{cases} -1 & \text{if } v \leq -1 \\ v & \text{if } -1 < v < 1 \\ 1 & \text{if } v \geq 1. \end{cases} \quad (15)$$

- φ is a \mathbb{C}^{p-1} function, $p \geq 2$, such that:

$$\varphi(v) = \begin{cases} -1 & \text{if } v \leq -1 \\ 1 & \text{if } v \geq 1, \end{cases} \quad (16)$$

and is \mathbb{C}^∞ for $-1 < v < 1$. Therefore, locally, near $v = 1$, and for $v \leq 1$, it will behave as

$$\varphi(v) \simeq 1 + O(v - 1)^p. \quad (17)$$

Next theorem gives the asymptotic behavior of the Poincaré map P_ε in terms of the regularity of φ (see also figure 3):

Theorem 2.1. *Consider a Filippov vector field Z as in (1) satisfying (2), (3) and (4). Take $y_0 > 0$ small enough. Fix $p \geq 1$, $p \in \mathbb{N}$, and consider the regularized vector field Z_ε in (9) with φ a \mathbb{C}^{p-1} function as in (15) or (16). Fix any $0 < \lambda < \frac{p}{2p-1}$.*

There exist $\varepsilon_0 > 0$, $L^- < 0$, and $\alpha(\varepsilon) = x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + O(\varepsilon^{\lambda+1})$, where α^- , β^- are the constants given in Proposition 1, such that, for $0 < \varepsilon \leq \varepsilon_0$, the map P_ε restricted to the interval $\mathcal{I} := [L^-, \alpha(\varepsilon)]$ is a Lipschitz function with Lipschitz constant exponentially small in ε and satisfies:

- If φ is a piecewise linear function ($p = 1$):

$$P_\varepsilon(x) = x_0^+ + \alpha^+ \varepsilon + O(\varepsilon^2), \quad \forall x \in \mathcal{I}$$

- If φ is of class \mathbb{C}^{p-1} ($p \geq 2$):

$$P_\varepsilon(x) = x_0^+ + \alpha^+ \varepsilon + \beta^+ (\eta(0))^2 \varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^{\frac{2p+1}{2p-1}}), \quad \forall x \in \mathcal{I},$$

where $\eta(u)$ is the unique solution of equation:

$$\frac{d\eta}{du} = \frac{2}{4\eta - \frac{\varphi^{(p)}(1)}{p!} u^p} \quad (18)$$

satisfying $\eta(u) - \frac{\varphi^{(p)}(1)}{4p!} u^p \rightarrow 0$ as $u \rightarrow -\infty$. Here we denote as

$$\varphi^{(p)}(1) := \lim_{v \rightarrow 1^-} \varphi^{(p)}(v).$$

- $\eta(u)$ also satisfies:

$$\frac{\varphi^{(p)}(1)}{4p!} u^p < \eta_0(u) < \frac{\varphi^{(p)}(1)}{4p!} u^p + \frac{2K(p-1)!}{\varphi^{(p)}(1)} u^{1-p}, \quad K > \frac{1}{p}, \quad u \leq 0$$

The proof of Theorem 2.1 is done using geometric singular perturbation theory and asymptotic methods and is deferred to Section 3.

2.2. Global results: Existence of periodic orbits. Now suppose that the upper vector field X^+ has a global recurrence in such a way that there exists a *exterior* Poincaré map:

$$\begin{aligned} P^e : \mathcal{S}_{y_0}^+ &\rightarrow \mathcal{S}_{y_0}^- \\ (x, y_0) &\mapsto (P^e(x), y_0) \end{aligned} \quad (19)$$

which is smooth, and denote by:

$$P^e(x_0^+) = x_0^- + \gamma, \quad \frac{dP^e}{dx}(x_0^+) = c \leq 0, \quad (20)$$

where we remind that $x_0^\pm = W_+^{u,s}(0, 0) \cap \mathcal{S}_{y_0}^\pm$.

The existence of global recurrence is a natural assumption in the case that the vector field X^+ has a hyperbolic periodic orbit $\Gamma \subset \mathcal{V}^+$ which intersects the section $\{(x, y), y = y_0\}$. See figure 4.

We compose this external map with the Poincaré map P_ε studied in Theorem 2.1. Next theorem gives conditions to ensure the existence of fixed points of the return Poincaré map $P^e \circ P_\varepsilon$, which give rise to periodic orbits for the regularized system Z_ε .

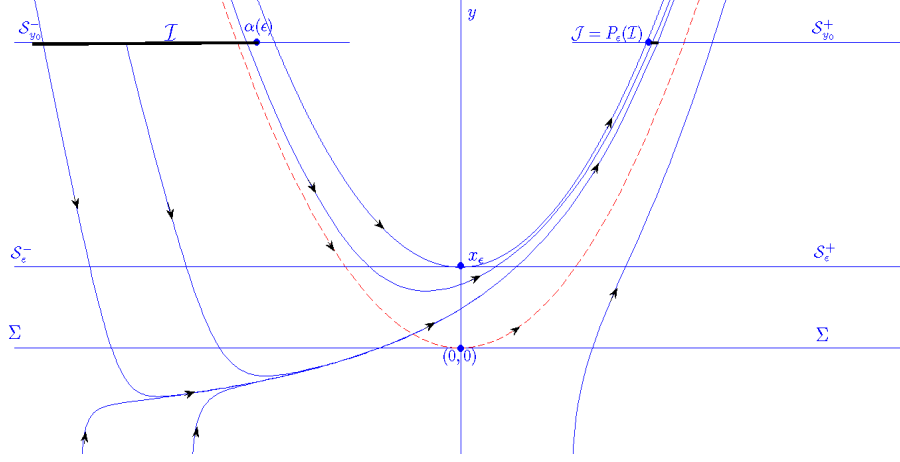


FIGURE 3. Dynamics of the Poincaré map P_ε for the regularized system Z_ε . The large domain \mathcal{I} is smashed to the small \mathcal{J} . The dotted red parabola is the trajectory of X^+ passing through the fold $(0,0)$.

Theorem 2.2. *Assume the hypotheses of Theorem 2.1 and the existence of a global Poincaré map as in (20). Consider the return map $P^e \circ P_\varepsilon$ restricted to the interval \mathcal{I} given in Theorem 2.1 with $\frac{1}{2} < \lambda < \frac{p}{2p-1}$. Let c and γ the constants given in (20), and let us call $\Delta = \alpha^- - c\alpha^+$, where α^\pm are the constants given in Proposition 1. Then, one has:*

- If $\gamma > 0$, or if $\gamma = 0$ and $\Delta < 0$, then, for $0 < \varepsilon < \varepsilon_0$,

$$P^e \circ P_\varepsilon(\mathcal{I}) \cap \mathcal{I} = \emptyset$$

and therefore $P^e \circ P_\varepsilon$ has no fixed points in the interval \mathcal{I} .

- If $\gamma < 0$, or if $\gamma = 0$ and $\Delta > 0$, the map $P^e \circ P_\varepsilon$ is a contraction in \mathcal{I} for $0 < \varepsilon < \varepsilon_0$ and therefore it has a unique fixed point in this interval.

Let us call Γ_ε the corresponding periodic orbit of the regularized system Z_ε .

- If $\gamma < 0$ the periodic orbit Γ_ε approaches, as $\varepsilon \rightarrow 0$, to the sliding periodic orbit Γ_0 of the Filippov system Z given by $\Gamma_0 = W_+^u(0,0) \cup \{(x,0), x^* \leq x \leq 0\}$, where $(x^*,0) = W_+^u(0,0) \cap \Sigma$.
- If $\gamma = 0$ and $\Delta > 0$, the periodic orbit Γ_ε approaches, as $\varepsilon \rightarrow 0$, to a grazing periodic orbit Γ_0 of the Filippov system Z given by $\Gamma_0 = W_+^u(0,0) = W_+^s(0,0)$, which is a hyperbolic attracting periodic orbit of the vector field X^+ .
- The limit $\Gamma_\varepsilon \rightarrow \Gamma_0$ is not uniform in the following sense:
 - In the region $(x,y) \in \mathcal{V}^+$, $y \geq y_0$, for any $y_0 > 0$ one has that Γ_ε is ε -close to Γ_0 .
 - If we call $(\gamma_0^\varepsilon, \varepsilon) = \Gamma_0 \cap \{(x,\varepsilon), x > 0\}$, and $(\gamma_\varepsilon^\varepsilon, \varepsilon) = \Gamma_\varepsilon \cap \{(x,\varepsilon), x > 0\}$, one has that

$$\gamma_\varepsilon^\varepsilon = \mathcal{O}(\varepsilon^{\frac{p}{2p-1}}), \quad \gamma_0^\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{2}}).$$

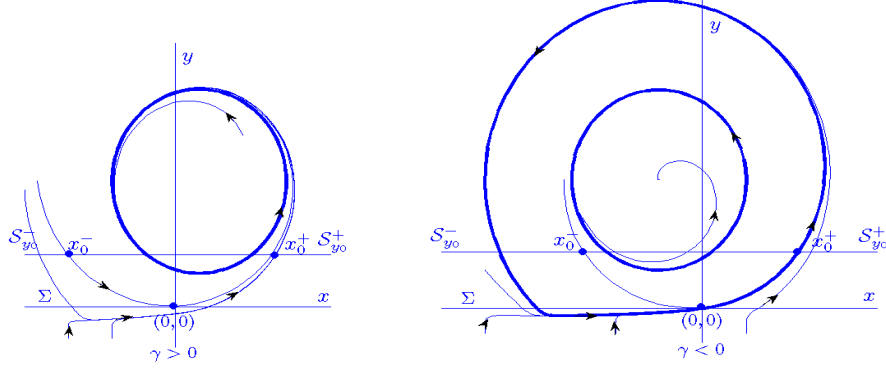


FIGURE 4. Global behavior of the regularized system in the case X^+ has a periodic orbit; the external map P^e sends x_0^+ to $x_0^- + \gamma$. On the left the periodic orbit is attracting ($\gamma > 0$). On the right is repelling ($\gamma < 0$); a periodic orbit of the regularized system Z_ε appears.

Proof. We look for fixed points of the return Poincaré map $P^e \circ P_\varepsilon$. By Theorem 2.1, all the points in the interval \mathcal{I} are sent by P_ε to an interval \mathcal{J} of size, at most, $\mathcal{O}(\varepsilon^{\frac{2p+1}{2p-1}})$ containing the point $x_F = x_0^+ + \alpha^+ \varepsilon + \beta^+ (\eta_p(0))^2 \varepsilon^{2p/(2p-1)}$.

By (20), the map P^e sends this point to:

$$\begin{aligned} & P^e(x_0^+ + \alpha^+ \varepsilon + \beta^+ (\eta_p(0))^2 \varepsilon^{2p/(2p-1)}) \\ &= x_0^- + \gamma + c(\alpha^+ \varepsilon + \beta^+ (\eta_p(0))^2 \varepsilon^{2p/(2p-1)}) + \mathcal{O}(\varepsilon^2) \\ &= x_0^- + \gamma + c\alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{2p/(2p-1)}). \end{aligned}$$

Summarizing, $P^e \circ P_\varepsilon$ sends the whole interval $\mathcal{I} = [L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})]$ to an interval \mathcal{J} of size, at most, $\mathcal{O}(\varepsilon^{2p/(2p-1)})$ containing the point $x_0^- + \gamma + c\alpha^+ \varepsilon$. Moreover, as the external map is independent of ε and smooth, $P^e \circ P_\varepsilon$ restricted to \mathcal{I} is a Lipschitz map with Lipschitz constant exponentially small in ε .

As $\lambda > 1/2$, and γ is fixed, a sufficient condition to ensure that $\mathcal{J} \subset \mathcal{I}$ and therefore that $P^e \circ P_\varepsilon$ is a contraction in \mathcal{I} , is that $\gamma + c\alpha^+ \varepsilon < \alpha^- \varepsilon$, because this implies that the end points of \mathcal{J} and \mathcal{I} satisfy:

$$x_0^- + \gamma + c\alpha^+ \varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) < x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})$$

if ε is small enough. Moreover, this condition is necessary because, as $\beta^- < 0$, if $\gamma + c\alpha^+ \varepsilon = \alpha^- \varepsilon$, one has that

$$x_0^- + \gamma + c\alpha^+ \varepsilon = x_0^- + \alpha^- \varepsilon > x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda}$$

therefore for ε small enough we have

$$x_0^- + \gamma + c\alpha^+ \varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) > x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})$$

and $P^e \circ P_\varepsilon(\mathcal{I}) \cap \mathcal{I} = \emptyset$.

Let us call $\Delta = \alpha^- - c\alpha^+$. Then condition $\gamma + c\alpha^+ \varepsilon < \alpha^- \varepsilon$ reads:

$$\gamma < \Delta \varepsilon. \tag{21}$$

Assume $\gamma > 0$. In this case, taking $\varepsilon > 0$ small enough, it is clear that (21) can not hold. In fact, the end points of J and \mathcal{I} satisfy:

$$x_0^- + \gamma + c\alpha^+\varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) > x_0^- + \gamma/2 > x_0^- + \alpha^-\varepsilon + \beta^-\varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda}).$$

This implies that $P^e \circ P_\varepsilon(\mathcal{I}) \cap \mathcal{I} = \emptyset$.

The same happens for $\gamma = 0$ and $\Delta < 0$: as $\alpha^- > 0$ and $\beta^- < 0$ we have

$$\begin{aligned} x_0^- + \gamma + c\alpha^+\varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) &= x_0^- - \Delta\varepsilon + \alpha^-\varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) \\ &> x_0^- + \alpha^-\varepsilon > x_0^- + \alpha^-\varepsilon + \beta^-\varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda}), \end{aligned}$$

which implies that $P^e \circ P_\varepsilon(\mathcal{I}) \cap \mathcal{I} = \emptyset$.

Assume $\gamma < 0$. Now condition (21) and therefore $\gamma + c\alpha^+\varepsilon < \alpha^-\varepsilon$ is fulfilled if $\varepsilon > 0$ is small enough. Then one can ensure that $P^e \circ P_\varepsilon(\mathcal{I}) \subset J \subset \mathcal{I}$ and the map $P^e \circ P_\varepsilon$ is a contraction. Consequently, there is a unique fixed point $x_p^- = x_0^- + \gamma + c\alpha^+\varepsilon \pm \mathcal{O}(\varepsilon^{2p/(2p-1)}) \in J \subset \mathcal{I}$ which gives rise to a periodic orbit Γ_ε . Observe that the non-smooth system Z has, in this case, a sliding cycle $\Gamma_0 = W_+^u(0,0) \cup \{(x,0), x^* \leq x \leq 0\}$, where $(x^*,0) = W_+^u(0,0) \cap \Sigma$ and $x_\mu = \Gamma_0 \cap \mathcal{S}_{y_0}^- = x_0^- + \gamma$. Therefore, Γ_ε is ε -close to Γ_0 in $\mathcal{S}_{y_0}^-$.

Analogously, if $\gamma = 0$, one can ensure that condition (21) is satisfied if $\Delta > 0$. Observe that, in this case, $\Gamma_0 = W_+^u(0,0)$ is a grazing periodic orbit of X^+ and is ε -close to Γ_ε in $\mathcal{S}_{y_0}^-$.

To finish the proof let us observe that, on one hand, $\Gamma_0 \cap \mathcal{S}_\varepsilon^+ = (\gamma_0^\varepsilon, \varepsilon)$ with $\gamma_0^\varepsilon = \mathcal{O}(\sqrt{\varepsilon})$. On the other hand $\Gamma_\varepsilon \cap \mathcal{S}_\varepsilon^+ = (\gamma_\varepsilon^\varepsilon, \varepsilon)$ and, using (13):

$$\gamma_\varepsilon^\varepsilon = \bar{P}^{-1}(x_0^+ + \alpha^+\varepsilon + \beta^+(\eta(0))^2 \varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^{\frac{2p+1}{2p-1}})) = \eta(0)\varepsilon^{\frac{p}{2p-1}}(1 + o(1)).$$

The ε -closeness in the region $y \geq y_0$ follows from the properties of map \bar{P} (see (13)) sending the points $(\gamma_0^\varepsilon, \varepsilon)$ and $(\gamma_\varepsilon^\varepsilon, \varepsilon)$ to ε -close points in $\mathcal{S}_{y_0}^+$. \square

Remark 1. To give a geometrical interpretation of the condition $\Delta > 0$ let us observe the following. We are assuming that $W_+^s(0,0) \cap \mathcal{S}_{y_0}^- = (x_0^-, y_0)$, but also condition (20) gives that $W_+^u(0,0) \cap \mathcal{S}_{y_0}^- = (x_0^- + \gamma, y_0)$. Therefore, if we consider the Poincaré return map associated to the regular vector field X^+ :

$$\pi^+ : \mathcal{S}_{y_0}^- \rightarrow \mathcal{S}_{y_0}^-$$

and one has that $\pi^+(x_0^-) = x_0^- + \gamma$.

Clearly, the case $\gamma = 0$ corresponds to the case of the vector field X^+ having a grazing periodic orbit Γ_0 . This orbit is hyperbolic attracting when $|(\pi^+)'(x_0^-)| < 1$ and repelling when $|(\pi^+)'(x_0^-)| > 1$.

Let us point out that, by (14), we know that the point $(x_\varepsilon, \varepsilon)$ where the vector field X^+ is tangent to \mathcal{S}_ε satisfies

$$\bar{x}_\varepsilon = P^{-1}(x_\varepsilon) = x_0^- + \alpha^-\varepsilon + \mathcal{O}(\varepsilon^2)$$

but the orbit of this point for the vector field Z_ε coincides with the orbit given by the vector field X^+ , therefore, one has that

$$\pi^+(\bar{x}_\varepsilon) = P^e(P_\varepsilon(\bar{x}_\varepsilon)) = P^e(\bar{P}(x_\varepsilon)) = P^e(x_0^+ + \alpha^+\varepsilon + \mathcal{O}(\varepsilon^2)) = x_0^- + \gamma + c\alpha^+\varepsilon + \mathcal{O}(\varepsilon^2).$$

If we Taylor expand the map π^+ around x_0^- :

$$\pi^+(\bar{x}_\varepsilon) = \pi^+(x_0^-) + (\pi^+)'(x_0^-)(\bar{x}_\varepsilon - x_0^-) + \mathcal{O}(\bar{x}_\varepsilon - x_0^-)^2 = x_0^- + \gamma + (\pi^+)'(x_0^-)\alpha^-\varepsilon + \mathcal{O}(\varepsilon^2)$$

and then we obtain:

$$c\alpha^+ = (\pi^+)'(x_0^-)\alpha^-$$

therefore, $\Delta = \alpha^-(1 - (\pi^+)'(x_0^-))$. As $\alpha^- > 0$, the condition $\Delta > 0$ is equivalent to $0 < (\pi^+)'(x_0^-) < 1$. In the case $\gamma = 0$ this condition is equivalent to ask that the periodic orbit Γ_0 of X^+ is a hyperbolic attracting periodic orbit.

In view of Remark 1, Theorem 2.1 and Proposition 1 do not enable us to analyze the persistence of periodic orbits of the regularized system in the case that X^+ has a center. Naturally, this covers the Hamiltonian case. Nevertheless there exist some results and actual research studying conditions for a planar Dynamical System with an isolated center to have a first integral. See [20, 8]. The case of X^+ having a center is done in next Theorem 2.3. Previously, in next Proposition 2, we give some relations between the map P_ε and P^+ , the Poincaré map associated to the vector field X^+ as a regular vector field in $\mathcal{V}^+ \cup \mathcal{V}^-$:

$$P^+ : D^+ \times \{y_0\} \subset \mathcal{S}_{y_0}^- \rightarrow \mathcal{S}_{y_0}^+, \quad (22)$$

where the domain $D^+ \subset \mathbb{R}$ is a suitable set near x_0^- . Clearly, there exists a suitable constant $k < x_0^-$, which depends of the global properties of X^+ , such that $[k, \bar{x}_\varepsilon] \subset D^+$.

Proposition 2. *Let be $(x_0^-, y_0) = W_+^s(0, 0) \cap \mathcal{S}_{y_0}^-$ and \bar{x}_ε given in (10) and (11). Then, for any $x \in [x_0^-, \bar{x}_\varepsilon]$ one has that*

$$P_\varepsilon(x) < P^+(x).$$

Proof. As the vector fields Z_ε and X^+ are the same in the region $y \geq \varepsilon$ we will take the initial condition at (x, ε) for $x \in [x_\varepsilon^-, x_\varepsilon]$ where $(x_\varepsilon^-, \varepsilon) = W_+^s(0, 0) \cap \mathcal{S}_\varepsilon^-$.

Consider the flow $\phi_{X^+}(t; x, \varepsilon)$. As the vector field X^+ points down in $\mathcal{S}_\varepsilon^-$ and the orbits can not cross the pseudoseparatrix of the fold point, the orbits remain in the region $\{(x, y), 0 \leq y \leq \varepsilon\}$ until they cross $\mathcal{S}_\varepsilon^+$.

Denote by $(x(t), y(t)) = \phi_{X^+}(t; x, \varepsilon)$ and by

$$X_N^+ = X_N^+(x(t), y(t)) = (\dot{y}(t), -\dot{x}(t)) = (X_2^+(x(t), y(t)), -X_1^+(x(t), y(t)))$$

the normal exterior vector to the orbit. Then, we perform the scalar product:

$$\langle X_N^+, Z_\varepsilon \rangle (x(t), y(t)) = \left(\frac{1 + \varphi(\frac{y(t)}{\varepsilon})}{2} \right) (X_2^+ X_1^- - X_1^+ X_2^-) (x(t), y(t)) < 0$$

Last equality is true by taking ε small enough, as (7) is satisfied in this region.

Then, as both vector fields are smooth and, except at $(x_\varepsilon, \varepsilon)$, they are not tangent to $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^- \cup \mathcal{S}_\varepsilon^+$, the orbit of X^+ strictly bounds Z_ε from bellow and therefore, if we denote by t_1 and t_2 the times when $\pi_y(\phi_{X^+}(t_1; x, \varepsilon)) = \pi_y(\phi_{Z_\varepsilon}(t_2; x, \varepsilon)) = \varepsilon$, one has that $\pi_x(\phi_{X^+}(t_1; x, \varepsilon)) > \pi_x(\phi_{Z_\varepsilon}(t_2; x, \varepsilon))$. \square

Theorem 2.3. *Assume the same hypotheses as in Theorem 2.1 and suppose that X^+ has a center in \mathcal{V}^+ surrounded by periodic orbits which intersect the switching surface Σ .*

Then, for ε small enough the unique tangent orbit to $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$ of X^+ is a periodic orbit of Z_ε that is semistable: it is attracting for all the orbits exterior to it but its interior is foliated by periodic orbits.

Proof. Consider the Poincaré map $P_\varepsilon = \bar{P} \circ Q_\varepsilon \circ P$, and the return map $P^e \circ P_\varepsilon$. It is clear that $P^e \circ P_\varepsilon(\bar{x}_\varepsilon) = \bar{x}_\varepsilon$, where \bar{x}_ε is defined in (11), because the orbit through \bar{x}_ε is tangent to \mathcal{S}_ε , and therefore, being a periodic orbit of X^+ , is also a periodic orbit of Z_ε .

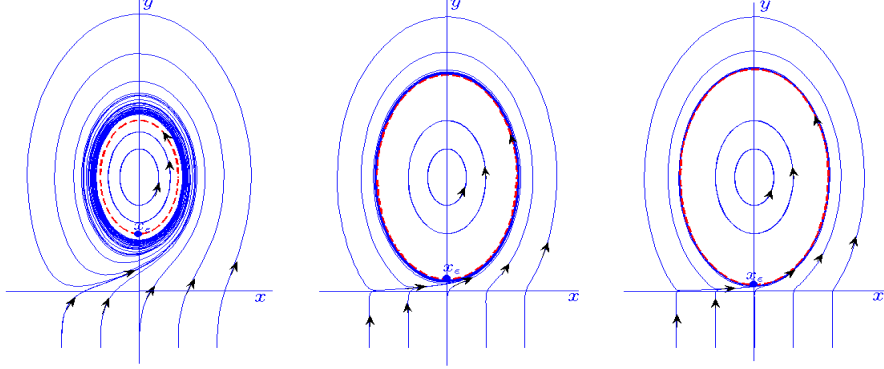


FIGURE 5. Behavior of the regularized system in the case X^+ has a center for $\varepsilon = 0.5$, $\varepsilon = 0.1$ and $\varepsilon = 0.05$, respectively. The dotted orbit is the orbit passing through x_ε , which is semistable.

It is also important to note that $P^e \circ P^+ = \pi^+$, where P^+ is given in (22) and π^+ is the return map in $\mathcal{S}_{y_0}^-$ of X^+ . We know that, as X^+ has a center in \mathcal{V}^+ , $\pi^+(x) = x$ for all the points in its domain.

Now, take $x \in \mathcal{I}$, put $x_0 = x$ and define $x_1 = P^e \circ P_\varepsilon(x_0)$. Clearly $x_1 \in (x_0^-, \bar{x}_\varepsilon)$. In the next step we have $x_2 = P^e \circ P_\varepsilon(x_1) > P^e \circ P^+(x_1) = \pi^+(x_1) = x_1$, by Proposition 2 and the fact that P^e reverses monotony. Going on with this procedure we determine an strictly increasing sequence $x_n = (P^e \circ P_\varepsilon)^n(x)$ whose limit is the fixed point \bar{x}_ε . \square

2.3. The grazing-sliding bifurcation of periodic orbits. Let us now consider some classical bifurcations of periodic orbits in non-smooth systems and see how they behave after the regularization.

Consider a family Z_μ of non-smooth planar systems such that they undergo a grazing sliding bifurcation of a hyperbolic attracting or repelling periodic orbit of the vector field X_μ^+ at $\mu = 0$. Next theorem shows how these bifurcations behave in the corresponding regularized family $Z_{\mu,\varepsilon}$.

Theorem 2.4. *Let Z_μ , $\mu \in \mathbb{R}$ be a family of non-smooth planar systems that undergoes a grazing sliding bifurcation of a hyperbolic periodic orbit Γ_μ of the vector field X_μ^+ at $\mu = 0$. We assume that, for $\mu > 0$ the periodic orbit Γ_μ is entirely contained in \mathcal{V}^+ , it becomes tangent to Σ for $\mu = 0$ and intersects both regions \mathcal{V}^\pm for $\mu < 0$.*

Consider the regularized family $Z_{\mu,\varepsilon}$.

- If Γ_μ is attracting, the regularized system has a periodic orbit $\Gamma_{\mu,\varepsilon}$ for any ε , μ small enough. No bifurcation occurs in the regularized system.
- If Γ_μ is repelling, the regularized system has a periodic orbit $\Gamma_{\mu,\varepsilon}$ for any $\mu > 0$ and $0 < \varepsilon < \varepsilon_0(\mu)$ which co-exists with the periodic orbit Γ_μ contained in $\mathcal{V}^+ \cap \{(x, y), y > \varepsilon\}$. This result is also true for $\mu = \tilde{\mu}\varepsilon$, if $\tilde{\mu} > -\Delta$, where $\Delta < 0$ is the constant given in remark 1. For $\mu \leq 0$ small enough, the system has no periodic orbits near Γ_0 if ε is small enough. Therefore the family $Z_{\mu\varepsilon}$ undergoes a bifurcation of periodic orbits near $\mu = 0$.

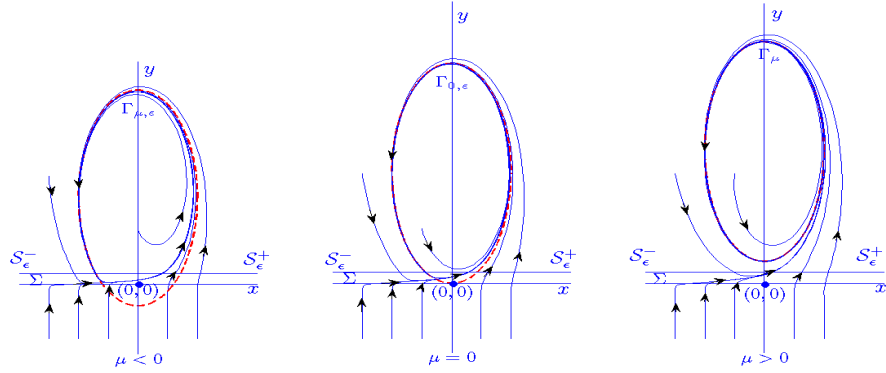


FIGURE 6. Periodic orbit of the regularized family $Z_{\mu,\varepsilon}$ when X_μ^+ has an attracting periodic orbit Γ_μ (the red dotted orbit). For $\mu < 0$ and $\mu = 0$ the regularized system has a periodic orbit $\Gamma_{\mu,\varepsilon}$. Γ_μ is the continuation of $\Gamma_{\mu,\varepsilon}$ for $\mu > 0$.

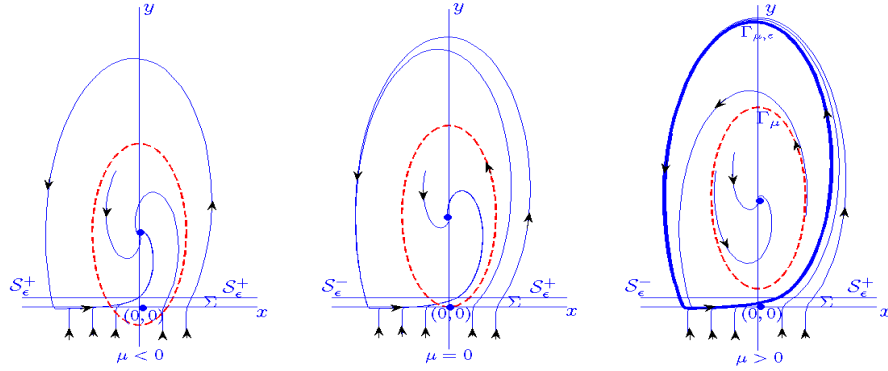


FIGURE 7. Periodic orbit of the regularized family $Z_{\mu,\varepsilon}$ when X_μ^+ has a repelling periodic orbit Γ_μ (the red dotted orbit). For $\mu < 0$ and $\mu = 0$ the regularized system has no periodic orbits. For $\mu > 0$ the regularized system has a periodic orbit $\Gamma_{\mu,\varepsilon}$ which coexists with Γ_μ .

Proof. One can assume that the fold point, which exists for μ small enough, is independent of μ and it is located at $(0,0)$. As usual, we denote by $(x_0^\pm, y_0) = W^{u,s}(0,0) \cap \mathcal{S}_{y_0}^\pm$, the intersection of its stable and unstable pseudo-separatrices with $\mathcal{S}_{y_0}^\pm$ and we also assume that x_0^\pm are independent of μ .

Assume that the periodic orbit Γ_μ of the vector field X_μ^+ is attracting. In this case, for $\mu > 0$, Γ_μ which is contained in \mathcal{V}^+ , it becomes tangent to Σ for $\mu = 0$, and then crosses Σ for $\mu < 0$ but, being Γ_μ attracting, a sliding cycle $\tilde{\Gamma}_\mu$ for the non-smooth system Z appears. Observe that $\tilde{\Gamma}_0 = \Gamma_0$.

The external map P^e satisfies (20). For $\mu = 0$, as the periodic orbit Γ_0 is tangent to Σ , the points (x_0^-, y_0) and (x_0^+, y_0) belong to it and therefore $P^e(x_0^+) = x_0^-$. This gives that, for $\mu = 0$, the parameter γ in (20) can be taken as $\gamma = 0$. We can

assume, without loss of generality, that for μ small enough the map P^e is defined and satisfies $P^e(x_0^+) = x_0^- + \mu$.

For $\mu > 0$, the vector field X^+ has a periodic orbit Γ_μ entirely contained in the region $\{y > \varepsilon\}$ and consequently Γ_μ is also a periodic orbit of $Z_{\mu,\varepsilon}$, because $Z_{\mu,\varepsilon} = X^+$ in this region. Denoting by $(x_\mu, y_0) = (x_0^- + \sigma, y_0) = \Gamma_\mu \cap \mathcal{S}_{y_0}^-$, using that $x_0^- + \sigma$ is a fixed point of π^+ (the return Poincaré map of X^+) and Taylor's formula:

$$x_0^- + \sigma = \pi^+(x_0^- + \sigma) = x_0^- + \mu + \pi'(x_0^-)\sigma + O(\sigma)^2,$$

one obtains, using the definition of Δ in Remark 1: $\sigma = \frac{\mu}{1 - (\pi^+)'(x_0^-)} + O(\mu^2) = \frac{\mu\alpha^-}{\Delta} + O(\mu^2)$ and therefore

$$x_\mu = x_0^- + \frac{\mu\alpha^-}{\Delta} + O(\mu^2).$$

A simple calculation gives that $x_\mu \geq \bar{x}_\varepsilon$ if, and only if, $\mu \geq \mu^* = \mu^*(\varepsilon) = \Delta\varepsilon + O(\varepsilon^2)$, therefore Γ_μ is a periodic orbit of $Z_{\mu,\varepsilon}$ for $\mu \geq \mu^*(\varepsilon)$ and becomes tangent to \mathcal{S}_ε for $\mu = \mu^*$.

Let's now apply the results of Theorem 2.2, using $\gamma = \mu$.

For $\mu < 0$, system $Z_{\mu,\varepsilon}$ has a periodic orbit $\Gamma_{\mu,\varepsilon}$ for ε small enough. The result is also true for $\mu = 0$ because, as Γ_0 is attracting, we have by Remark 1 that $\Delta > 0$. For $\mu > 0$ and fixed, Theorem 2.2 gives that there is no periodic orbit of $Z_{\mu,\varepsilon}$ crossing the section $\mathcal{S}_{y_0}^-$ at points (x, y_0) , with $x \in \mathcal{I}$, for ε small enough.

Moreover, we observe that, in the proof of Theorem 2.2, the condition required to the existence of a periodic orbit of $Z_{\mu,\varepsilon}$ is (21). Therefore, as $\Delta > 0$, for $\mu > 0$, if we write $\mu = \tilde{\mu}\varepsilon$, condition (21) is verified until

$$\tilde{\mu} < \Delta$$

and therefore the periodic orbit $\Gamma_{\mu,\varepsilon}$ which existed for $\mu < 0$ persists for these values of $0 \leq \mu < \Delta\varepsilon$ if ε is small enough.

In conclusion, for $\mu = \tilde{\mu}\varepsilon$, $\tilde{\mu} > \Delta$ the periodic orbit $\Gamma_{\tilde{\mu}\varepsilon}$ of the vector field X^+ does not intersect the region affected by the regularization and is the periodic orbit of the vector field $Z_{\mu,\varepsilon}$. For $\tilde{\mu} = \Delta + O(\varepsilon)$ it becomes tangent to \mathcal{S}_ε and is still a periodic orbit of $Z_{\mu,\varepsilon}$. Therefore the periodic orbit Γ_μ of the vector field X_μ^+ is the continuation of the periodic orbit $\Gamma_{\mu,\varepsilon}$ of $Z_{\mu,\varepsilon}$, for $\mu \geq \Delta\varepsilon$.

Assume now that the periodic orbit Γ_μ of the vector field X_μ^+ is repelling. Then, by Remark 1, one has $\Delta < 0$.

Again, we assume that, for $\mu < 0$, Γ_μ crosses Σ , becomes tangent to Σ for $\mu = 0$ and then is contained in \mathcal{V}^+ for $\mu > 0$. Therefore, in this case, for $\mu > 0$, being Γ_μ repelling, we have the co-existence of this periodic orbit of X_μ^+ and a sliding cycle $\tilde{\Gamma}_\mu$ of the non-smooth system Z_μ . Both collide at $\mu = 0$ and then disappear.

The external map P^e satisfies, for $\mu = 0$, $P^e(x_0^+) = x_0^-$, and we can assume, without loss of generality that for μ small enough the map P^e is defined and $P^e(x_0^+) = x_0^- - \mu$.

In this case we observe, analogously as before:

$$x_\mu = x_0^- - \frac{\mu\alpha^-}{\Delta} + O(\mu^2). \quad (23)$$

Therefore, $x_\mu \geq \bar{x}_\varepsilon$ if, and only if, $\mu \geq \mu^* = \mu^*(\varepsilon) = -\Delta\varepsilon + O(\varepsilon^2)$ and consequently Γ_μ is a periodic orbit of $Z_{\mu,\varepsilon}$ for $\mu \geq \mu^*(\varepsilon)$ and becomes tangent to \mathcal{S}_ε for $\mu = \mu^*$.

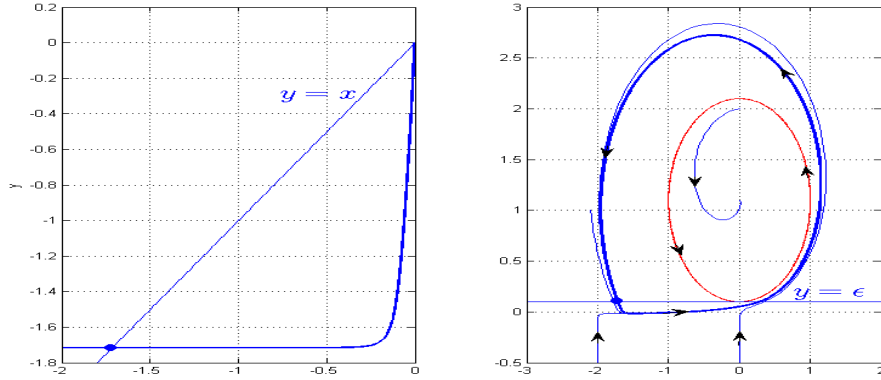


FIGURE 8. Simulations done for system (25) for $\mu = \mu^*$ when the periodic orbit $r = 1$ (in red) is tangent to \mathcal{S}_ε . On the left the return map with two fixed points and on the right one can see both periodic orbits.

We observe that, independently of the sign of Δ , the value

$$\mu^* = \mu^*(\varepsilon) = |\Delta|\varepsilon + \mathcal{O}(\varepsilon^2) \quad (24)$$

is the value of μ where the periodic orbit Γ_μ becomes tangent to \mathcal{S}_ε .

By Theorem 2.2, using $\gamma = -\mu$, we know that, for $\mu > 0$ and fixed, the regularized vector field $Z_{\mu,\varepsilon}$ has a periodic orbit $\Gamma_{\mu,\varepsilon}$ for ε small enough. When $\mu = 0$, as $\Delta < 0$, the same theorem gives that there is no periodic orbit of $Z_{\mu,\varepsilon}$ crossing the section $\Sigma_{y_0}^-$ at points (x, y_0) , with $x \in \mathcal{I}$, for ε small enough. The same happens for $\mu < 0$.

We can improve the range of μ for which the periodic orbit still exists. We observe that, in the proof of Theorem 2.2, the condition required to the existence of a periodic orbit of $Z_{\mu,\varepsilon}$ is (21). Therefore, as $\Delta < 0$, for $\mu > 0$, if we write $\mu = \tilde{\mu}\varepsilon$, condition (21) is verified until

$$-\tilde{\mu} < \Delta$$

and therefore the periodic orbit $\Gamma_{\mu,\varepsilon}$ which existed for $\mu > 0$ and fixed persists for these values of $\tilde{\mu} > -\Delta$ if ε is small enough.

Moreover, the periodic orbit $\Gamma_{\mu,\varepsilon}$ intersects $\mathcal{S}_{y_0}^-$ in a point (x_p, y_0) , with $x_p = x_0^- - \mu + c\alpha^+\varepsilon + \mathcal{O}(\varepsilon^{2p/(2p-1)})$. Therefore, as $\Delta < 0$, we have that $x_p < x_0^- < x_\mu$, and both periodic orbits coexist if ε is small enough.

For $\tilde{\mu} < -\Delta$, Γ_μ is no longer an orbit of the $Z_{\mu,\varepsilon}$ as it has entered the regularization zone and, as we know that for $\mu \leq 0$ there is not periodic orbit of $Z_{\mu,\varepsilon}$, in the range $0 \leq \mu \leq -\Delta\varepsilon$ a bifurcation of periodic orbits takes place (see Remark 2). \square

Remark 2. In Theorem 2.4 we have seen that, for $\Delta < 0$, when the parameter $\mu \geq 0$ approaches to zero, there is a bifurcation of periodic orbits. To see that this is a saddle node bifurcation we would need to make a detailed analysis of the map $P^\varepsilon \circ P_\varepsilon$ outside the interval \mathcal{I} .

Let us observe the following. The interval \mathcal{I} is sent, by the return map $P^\varepsilon \circ P_\varepsilon$ to a small interval J on the right of the point corresponding to the Fenichel manifold (see Theorem 3.1): $x_F = x_0^- - \mu + c\alpha^+\varepsilon + c\beta^+\eta(0)^2\varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^2)$. The condition

for this point being in \mathcal{I} is $x_F < x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})$ which gives

$$\mu > \bar{\mu} = \bar{\mu}(\varepsilon) = -\Delta\varepsilon - \beta^- \varepsilon^{2\lambda} + \varepsilon + c\beta^+ \eta(0)^2 \varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^{1+\lambda}).$$

Observe that $\bar{\mu} > \mu^* > 0$, where μ^* (see (24)) is the value of μ where the periodic orbit Γ_μ is tangent to \mathcal{S}_ε . Therefore the proof of the existence of a fixed point of the return map near x_F based on the contraction property fails before the periodic orbit Γ_μ is tangent to \mathcal{S}_ε .

Nevertheless, we observe that, for $\mu^* < \mu < \bar{\mu}$, the return map sends the interval \mathcal{I} to an interval J on the right of x_F which is on the right of \mathcal{I} . As we are dealing with orbits in the plane we obtain that the sequence $x_n = (P^e \circ P_\varepsilon)^n(x_F)$ is an increasing sequence bounded from above by $x_\mu = \Gamma_\mu \cap \mathcal{S}_{y_0}^-$, which is a fixed point. Therefore this sequence has a limit $x_p = \lim_{n \rightarrow \infty} x_n$ which is a fixed point of the return map $P^e \circ P_\varepsilon$ outside \mathcal{I} and gives rise to a periodic orbit to which the Fenichel manifold spirals.

Moreover, as x_μ is a repelling fixed point, x_n can not converge to x_μ and therefore $x_p \neq x_\mu$ corresponds to the periodic orbit $\Gamma_{\mu,\varepsilon}$ that still exists and is different of Γ_μ for these values of μ .

Using the formulas for x_F and x_μ (see (23)) one can guess that both orbits collide at $\mu_{sn} = -\Delta\varepsilon - \frac{\beta^+}{\alpha^+} \Delta\eta(0)^2 \varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^2)$. To prove that for $\mu > \mu_{sn}$ the points x_μ and x_p are the only fixed points of the return map and that collapse in a saddle node bifurcation one needs to see, for instance, that the map is convex between them.

We show, in Figure 8 a simulation of the return map from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^-$ (which corresponds to $P \circ P^e \circ \mathcal{Q}_\varepsilon \circ P^{-1}$) for the regularization of the family of vector fields $Z_\mu = (X_\mu^+, X_\mu^-)$ where X^+ is given by:

$$\left. \begin{aligned} \dot{x} &= -y + \mu + 1 + x(1-r)(r-2) \\ \dot{y} &= x + (y - \mu - 1)(1-r)(r-2) \end{aligned} \right\} r = \sqrt{x^2 + (y - \mu - 1)^2} \quad (25)$$

and $X^- = (0, 1)$. The repelling periodic orbit Γ_μ is given by $r = 1$ and $\Delta = -1$.

One can see on the left the graph of the return map from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^-$, which is convex and has two fixed points corresponding to two periodic orbits showed on the right. The thick point is the intersection of the periodic orbit with $y = \varepsilon$.

Moreover, numerical simulations give that the bifurcation occurs at

$$\mu_{sc} = \varepsilon - 0.5914\varepsilon^{4/3}(1 + o(\varepsilon)),$$

but we leave the rigorous study of the possible saddle node bifurcation as a future work.

2.4. Application to dry friction systems in a single degree of freedom.

Let us consider a mass m attached to a spring with a constant of recovery \mathcal{K} . The mass is on a moving belt with constant velocity v_d .

If x denotes the displacement of m with respect to the equilibrium position of the spring \mathcal{K} , on m act two forces: a force of resistance of the spring $-\mathcal{K}x$ (assuming the spring linear), and a friction force between the mass and the belt.

If we start from the equilibrium position $x = 0$, the mass will begin to move in stick with the belt (stick phase) at velocity v_d till the recovery force of the spring $-\mathcal{K}x$ compensate the static friction force and produce on m a damped harmonic motion (slip phase) until that, by energy dissipation, the mass will be once more in sticking with the belt, and so on.

So the equations are divided according to whether or not the relative speed between the mass and the belt, $v_r = \dot{x} - v_d$, is zero in two phases:

- Stick phase ($v_r = 0$), the equations are:

$$m\ddot{x} = -\mathcal{K}x + \mathcal{F}_s(x),$$

where the friction static force is $\mathcal{F}_s(x) = \min(|\mathcal{K}x|, F_s)\text{sgn}(\mathcal{K}x)$, and F_s is its maximum value.

Note that if $|\mathcal{K}x| < F_s$, then $\ddot{x} = 0$ and $\dot{x} = v_d$, ie, m moves in sticking with the belt until the force of the spring recovery reaches F_s . From this moment on, m begins to oscillate on the belt. But now it enters into a state where $v_r \neq 0$ and there the frictional force depends on v_r . The system is now in slip phase.

- Slip phase ($v_r \neq 0$), the equations of motion are

$$m\ddot{x} = -\mathcal{K}x + \mathcal{F}_d(v_r),$$

where $\mathcal{F}_d(v_r)$, represents the dynamic friction which has opposite sign to v_r .

Following R.I. Leine [19, 18] one considers three basic models of friction related to three different types of $\mathcal{F}_d(v_r)$.

- *Stribeck model*. It incorporates the experimental evidence that the force of static friction is larger than the dynamic one, and there is a continuous transition from one state to other.
- *Coulomb model*. It assumes that the dynamic and static friction are constant and equal.
- *Stiction model*. It assumes that there is not a regular transition between static and dynamic friction. When the spring reaches the value of static friction, the frictional force falls instantaneously and discontinuously to a strictly less value. Note that in this model, unlike the other two, the dynamic friction has no lateral limits, but tends to whole intervals $[F_d, F_s]$ and $[-F_s, -F_d]$, respectively.

2.4.1. *The Stribeck model*. In [18], a possible function which describes the Stribeck and Coulomb models, putting $v_d = m = \mathcal{K} = 1$ is formulated:

$$\mathcal{F}_d(v_r) = -\left(\frac{F_s - F_d}{1 + \delta|v_r|} + F_d\right)\text{sign}(v_r), \quad 0 < \delta \ll 1$$

where $v_r = \dot{x} - 1$ and δ is a parameter. $\delta = 0$ gives the Coulomb model, whereas $\delta > 0$ gives the Stribeck one.

The stick and slip systems are:

$$\left. \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \min(|x|, F_s)\text{sgn}(x), \end{aligned} \right\} y = 1 \quad (\text{stick}) \quad (26)$$

and

$$\left. \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \left(\frac{F_s - F_d}{1 + \delta|y-1|} + F_d\right)\text{sign}(y-1), \end{aligned} \right\} y \neq 1 \quad (\text{slip}). \quad (27)$$

The slip system can be written as a Filippov system $Z = (X^+, X^-)$, with

$$X^+ : \left. \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \left(\frac{F_s - F_d}{1 + \delta(y-1)} + F_d\right), \end{aligned} \right\} y > 1 \quad (28)$$

$$X^- : \left. \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \left(\frac{F_s - F_d}{1 + \delta(1-y)} + F_d\right), \end{aligned} \right\} y < 1 \quad (29)$$

and switching surface $\Sigma = \{(x, 1), x \in \mathbb{R}\}$.

The region $\{(x, 1), |x| < F_s\}$ is a sliding region and the sliding Filippov vector field is: $\dot{x} = 1$, which coincides with the stick field.

The vector field X^+ has a invisible fold at $(-F_s, 1)$ and points toward Σ for $x > -F_s$.

It turns out that X^- has a repeller focus at the point $(\frac{F_s - F_d}{1 + \delta} + F_d, 0)$ for δ small enough with eigenvalues: $\Lambda_{\pm} = \frac{\nu}{2} \pm \frac{i}{2}\sqrt{4 - \nu^2}$, where $\nu = \frac{\delta(F_s - F_d)}{(1 + \delta)^2} > 0$, therefore this is an unstable focus. Calling $\alpha = F_s - F_d$ and $\beta = F_d$, it is easy to see that the function

$$V(x, y) = (x - \beta - \frac{\alpha}{1 + \delta})^2 + y^2$$

satisfies $\frac{dV}{dt}(x, y) = 2\alpha\delta y^2 \frac{1}{1 + \delta} \frac{1}{1 - \delta(y - 1)} > 0$ if $y < \frac{1 + \delta}{\delta}$. Therefore, is strictly growing over the solutions of X^- .

Note that X^- has a visible tangency point at $(F_s, 1)$ and its unstable pseudoseparatrix $W_-^u(F_s, 1)$ intersects the switching manifold at a point $(x^*, 1)$ between the two fold points if δ is small enough. Therefore the Stribeck model has a sliding periodic orbit:

$$\Gamma_0 = W_-^u(F_s, 1) \cup \{(x, 1), x^* \leq x \leq F_s\}.$$

Changing the roles of X^+ and X^- hypotheses (2), (3) and (4) are satisfied. Moreover, the exterior Poincaré map satisfies (20) with $\gamma < 0$. We can then apply Theorem 2.2 to this system and ensure that the corresponding regularized system Z_ε has a periodic orbit $\Gamma_\varepsilon \rightarrow \Gamma_0$ as $\varepsilon \rightarrow 0$ (see figure 9).

2.4.2. The Coulomb model. If for simplicity we take $m = \mathcal{K} = v_d = 1$, the equations of motion for the Coulomb model are (26) and (27) with $\delta = 0$ and the slip equations give the Filippov system (28) and (29) with $\delta = 0$.

As in the Stribeck case, the region $\{(x, 1), |x| < F_s\}$ is a sliding region and the sliding Filippov vector field is: $\dot{x} = 1$, which coincides with the stick field.

The points $(-F_s, 1)$ and $(F_s, 1)$ are, respectively, invisible and visible tangency points. The difference in this model is that the point $(F_s, 0)$ is a center surrounded by periodic orbits of the vector field X^- . Therefore, one can apply Theorem 2.3 and we obtain, in the regularized system, a periodic orbit tangent to the section $y = 1 - \varepsilon$ which persists in the regularized system and becomes a semi-stable periodic orbit (see figure 9).

This coincidence between the stick equations and the Filippov sliding vector field does not occur in the Stiction model. This model has the same slip equations, and therefore gives the same non-smooth vector field outside the switching manifold $y = 1$, but different stick ones (see [19, 18]). The resulting system does not follow the Filippov convention, so it is outside the scope of this paper. A study of different conventions and its regularizations will be the main goal of a forthcoming paper.

2.5. Bifurcation of a sliding homoclinic to a saddle. In this section we will study how the regularized vector field Z_μ behaves when the non-smooth vector field Z has a sliding homoclinic orbit.

Let's consider the non-smooth vector field Z with the same conditions (2), (3) and (4) but now assume that the fold point $(0, 0)$ has a separatrix connection with a saddle $(x_h, y_h) \in \mathcal{V}^+$ (see Figure 10).

Generically, this can happen in one parameter families Z_μ undergoing a sliding homoclinic bifurcation to a saddle [17]. That is, Z_μ has a saddle (x_h, y_h) in \mathcal{V}^+ and, without loss of generality, we suppose independent of μ . We suppose that, for $\mu < 0$

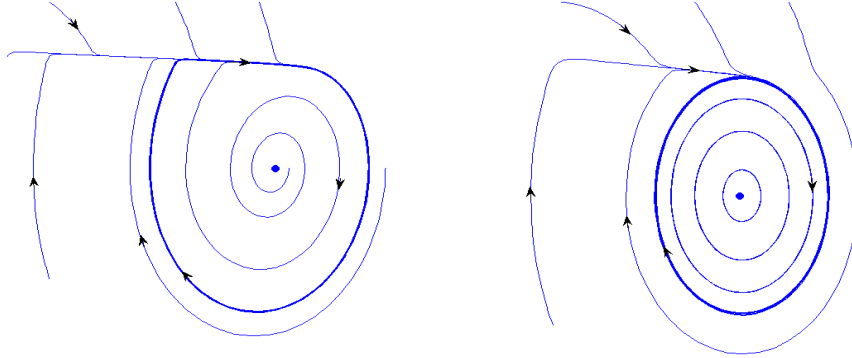


FIGURE 9. The regularization of the dry friction oscillator following Stribeck and Coulomb models. On the left the Stribeck model leading to an attracting periodic orbit of the regularized system. On the right the Coulomb model leading to a semistable periodic orbit of the regularized system.

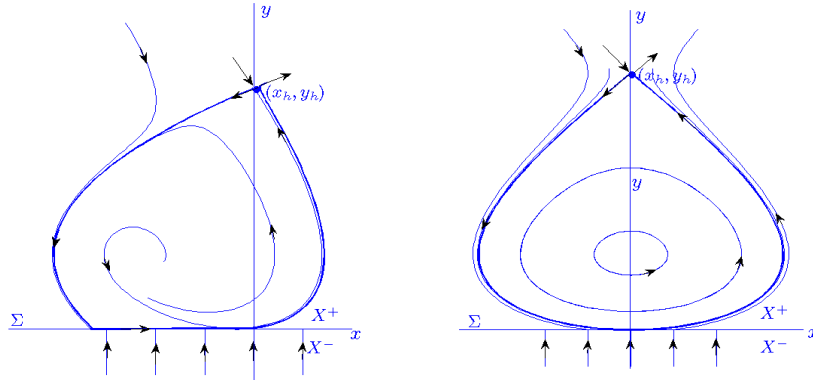


FIGURE 10. A sliding homoclinic orbit to a saddle. On the left, a generic vector field X^+ . On the right, a conservative vector field X^+ .

both stable and unstable curves of the saddle $W^{s,u}(x_h, y_h)$ intersect transversally the switching manifold Σ . For $\mu = 0$ the unstable manifold $W^u(x_h, y_h)$ remains transversal to Σ , but the stable $W^s(x_h, y_h)$ touches Σ tangentially in a visible fold point, that we assume at $(0, 0)$, producing a pseudo-separatrix connection between the stable manifold of the saddle and the unstable pseudo-separatrix of the fold, in \mathcal{V}^+ (see Figure 10):

$$W^s(x_h, y_h) = W_+^u(0, 0).$$

For $\mu > 0$ the unstable manifold of the saddle $W^u(x_h, y_h)$ remains transversal to Σ , but the stable $W^s(x_h, y_h)$ moves away from Σ inside \mathcal{V}^+ , and the unstable pseudo-separatrix of the fold does not intersect Σ anymore. We assume, without loss of generality, that we use a coordinate system such that the fold point remains at $(0, 0)$ for μ small enough.

The analysis of the regularization of this bifurcation follows closely theorems 2.2 and 2.4, provided we control the exterior map P^e . In order to do it, suppose without loss of generality that the eigenvalues of the saddle point (x_h, y_h) , $\lambda_2 < 0 < \lambda_1$, are independent of μ . It is well known that there exists a local change of variables, $(x, y) \rightarrow (u, v)$, in a neighborhood of the saddle point $|x - x_h| < \delta_1$, $|y - y_h| < \delta_1$, such that, in the new coordinates, that we denote by (u, v) , the system X^+ reads:

$$\left. \begin{aligned} \dot{u} &= \lambda_1 u + uf(u, v) \\ \dot{v} &= \lambda_2 v + vg(u, v) \end{aligned} \right\} |u| \leq \delta_2, \quad |v| \leq \delta_2, \quad f, g = \mathcal{O}(u, v) \quad (30)$$

with $\lambda_2 < 0 < \lambda_1$.

Clearly, one has that given any $K > 0$ one can choose $\delta_2 > 0$ such that $|f(u, v)| < K$ and $|g(u, v)| < K$ if $|u| < \delta_2$ and $|v| < \delta_2$.

In the coordinates (u, v) the saddle is at $(0, 0)$ and the stable and unstable manifolds are given, respectively, by $u = 0$, and $v = 0$. The following lemma is an straightforward application of Gronwall Lemma ([10] p. 7) to system (30).

Lemma 2.5. *Let $K > 0$ such that $\lambda_2 + K < 0$. Then, there exists $\delta_2 = \delta_2(K) > 0$ small enough such that: given any solution of system (30) with initial conditions $(u_0, -\delta_2)$, with $u_0 \in [-\delta_2, 0)$, there exists $T \geq 0$ such that $u(T) = -\delta_2$, moreover,*

$$|v(T)| \leq \delta_2^{1 + \frac{\lambda_2 + K}{\lambda_1 + K}} |u_0|^{-\frac{\lambda_2 + K}{\lambda_1 + K}}.$$

Now, we can study the regularization $Z_{\mu, \varepsilon}$ of the sliding homoclinic to a saddle bifurcation in Z_μ . Next lemma gives the behavior of the external map P^e :

Lemma 2.6. *Denote by $x_0^u = W^u(x_h, y_h) \cap \mathcal{S}_{y_0}^-$ and by $x_0^\pm = W^{s,u}(0, 0) \cap \mathcal{S}_{y_0}^\pm$. Assume that x_0^u, x_0^\pm are independent of μ and that $x_0^u < x_0^-$. Denote by $x_0^s = W^s(x_h, y_h) \cap \mathcal{S}_{y_0}^+$ and assume $x_0^s = x_0^+ - \mu$, with μ small enough.*

Consider the exterior map

$$P^e : D^e \times \{y_0\} \subset \mathcal{S}_{y_0}^+ \rightarrow \mathcal{S}_{y_0}^-$$

where $D^e = (d^e, x_0^s)$ is a (left) neighborhood of $x = x_0^s$.

- If $\mu < 0$, one has that $x_0^+ \in D^e$ and $P^e(x_0^+) = x_0^- + \gamma$ with $\gamma = \gamma(\mu) < 0$.
- If $\mu > 0$, the exterior map is not defined at x_0^+ .

Proof. The exterior map P^e follows the orbits which pass close to the saddle (x_h, y_h) , therefore, using Lemma 2.5, we have that if $x \in D^e$, then $x < x_0^s = x_0^+ - \mu$ and:

$$|P^e(x) - x_0^u| \leq \Omega |x - x_0^+ + \mu|^{-\frac{\lambda_2 + K}{\lambda_1 + K}} \quad (31)$$

where $\Omega > 0$ is a suitable constant independent of μ . In particular, if $\mu < 0$ small enough, we have that $x_0^+ < x_0^+ - \mu = x_0^s$, and therefore:

$$P^e(x_0^+) - x_0^u = \mathcal{O}(|\mu|^{-\frac{\lambda_2 + K}{\lambda_1 + K}}).$$

Now we have:

$$P^e(x_0^+) - x_0^- = P^e(x_0^+) - x_0^u + x_0^u - x_0^- = x_0^u - x_0^- + \mathcal{O}(|\mu|^{-\frac{\lambda_2 + K}{\lambda_1 + K}}).$$

Now, as $-\frac{\lambda_2 + K}{\lambda_1 + K} > 0$, if we take μ small enough, using that $x_0^u - x_0^- < 0$, one has that

$$P^e(x_0^+) - x_0^- < 0,$$

therefore $P^e(x_0^+) = x_0^- + \gamma$, with $\gamma = \gamma(\mu) < 0$.

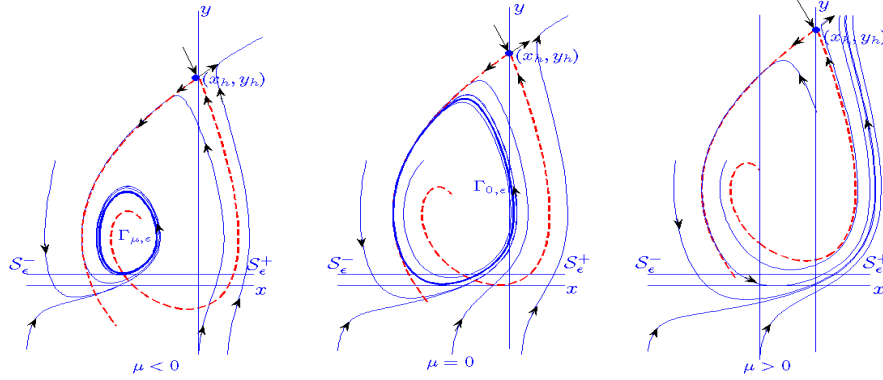


FIGURE 11. Bifurcation in the regularized system corresponding with the sliding homoclinic bifurcation in the Filippov system. The red dotted curves are the branches of the stable and unstable manifolds of the saddle point (x_h, y_h) of X^+ . The attracting periodic orbit exists for $\mu \leq 0$.

When $\mu > 0$, one has that $x_0^+ > x_0^s = x_0^+ - \mu$. As the unstable pseudo-separatrix of the fold can not intersect the stable manifold on the saddle, it can not intersect again the section $S_{y_0}^-$. \square

Now, we can give the result about periodic orbits in the regularized system.

Theorem 2.7. *Let $Z_\mu = (X_\mu^+, X_\mu^-)$ be a family of non-smooth vector fields that undergoes a sliding homoclinic bifurcation generated by a generic tangency between the stable manifold of a saddle point $(x_h, y_h) \in \mathcal{V}^+$ of X^+ and the switching manifold Σ , which occurs for $\mu = 0$, while the unstable manifold of the saddle is transversal to Σ . Assume that for $\mu < 0$ both stable and unstable curves of the saddle $W^{u,s}(x_h, y_h)$ intersect transversally the switching manifold Σ and for $\mu > 0$ the unstable manifold of the saddle $W^u(x_h, y_h)$ remains transversal to Σ , but the stable $W^s(x_h, y_h)$ moves away from Σ inside \mathcal{V}^+ , creating a visible fold point (of X^+) whose unstable pseudo-separatrix in \mathcal{V}^+ does not intersect Σ anymore. Assume also that X_μ^- is transversal to Σ and points towards Σ for any μ small enough. Consider the regularized family $Z_{\mu,\varepsilon}$, $\varepsilon > 0$, then:*

- If $\mu < 0$, the non-smooth system Z_μ has a sliding periodic orbit Γ_μ , and the regularized system $Z_{\mu,\varepsilon}$ has an attracting periodic orbit $\Gamma_{\mu,\varepsilon}$ for ε small enough uniformly in μ which approaches, when $\varepsilon \rightarrow 0$ the sliding periodic orbit Γ_μ .
- If $\mu = 0$, the system Z_μ has an sliding homoclinic orbit Γ_0 , and the regularized system has an attracting periodic orbit $\Gamma_{0,\varepsilon}$ for ε small enough which approaches, when $\varepsilon \rightarrow 0$, the sliding homoclinic orbit Γ_0 .
- If $\mu > 0$, for ε small enough, the vector field $Z_{\mu,\varepsilon}$ has no periodic orbits in a region close to the stable separatrix of the saddle point.
- If $\mu = \tilde{\mu}\varepsilon$, with $0 \leq \tilde{\mu} < -\alpha^+$, where α^+ is given in Proposition 1, the family $Z_{\tilde{\mu}\varepsilon,\varepsilon}$ has an attracting periodic orbit for ε small enough, and has a homoclinic orbit to (x_h, y_h) for $\tilde{\mu} = -\alpha^+ + \mathcal{O}(\varepsilon)$.

Proof. One can assume that the fold point, which exists for μ small enough, is independent of μ and it is located at $(0, 0)$. As usual, we denote by $(x_0^\pm, y_0) =$

$W^{u,s}(0,0) \cap \mathcal{S}_{y_0}^\pm$, the intersection of its stable and unstable pseudo-separatrices with $\mathcal{S}_{y_0}^\pm$ and by $W^{s,u}(x_h, y_h) \cap \mathcal{S}_{y_0}^\pm = (x_0^{s,u}, y_0)$. We also assume that x_0^\pm and x_0^u are independent of μ with $x_0^u < x_0^-$, and that $x_0^s = x_0^+ - \mu$.

For $\mu < 0$, we can apply Lemma 2.6 and we obtain that $P^e(x_0^+) = x_0^- + \gamma$ with $\gamma = \gamma(\mu) < 0$. This implies that the non-smooth vector field Z_μ has a sliding cycle $\Gamma_\mu = W_+^u(0,0) \cup \{(x,0), x^* \leq x \leq 0\}$, where $(x^*,0) = W_+^u(0,0) \cap \Sigma$. As $\gamma < 0$, one can apply Theorem 2.2 and we obtain the existence of a periodic orbit $\Gamma_{\mu,\varepsilon}$ for $Z_{\mu,\varepsilon}$.

In the case $\mu = 0$ one can not directly apply Theorem 2.2 because $x_0^+ = x_0^s$ and therefore the exterior Poincaré map P^e is not regular at $x = x_0^+$. Then, we use the results about P_ε in Theorem 2.1, and we have that, on one hand $x_0^u \in \mathcal{I} = [L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})]$, and on the other hand the return map $P^e \circ P_\varepsilon$ is defined in the interval \mathcal{I} and, if $x \in \mathcal{I}$:

$$P^e \circ P_\varepsilon(x) = P^e(x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}}))$$

and then, as $\alpha^+ < 0$, one has that $x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}}) < x_0^+ = x_0^s$ and, applying inequality (31) for $\mu = 0$:

$$|P^e(x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})) - x_0^u| < \Omega(\alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}}))^{-\frac{\lambda_2+K}{\lambda_1+K}} \leq \bar{\Omega} \varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}}.$$

Therefore, we can ensure that, as $-\frac{\lambda_2+K}{\lambda_1+K} > 0$, for any $x \in \mathcal{I}$:

$$P^e \circ P_\varepsilon(x) = P^e(x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})) = x_0^u + \mathcal{O}(\varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}}) \in \mathcal{I}$$

and therefore $P^e \circ P_\varepsilon$ sends the interval \mathcal{I} to an interval $J \subset \mathcal{I}$ containing the point $x_0^u < x_0^-$ and of size $\mathcal{O}(\varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}})$, and is a contraction. This gives the existence of a periodic orbit $\Gamma_{0,\varepsilon}$ of the regularized vector field $Z_{0,\varepsilon}$.

Let us observe that, for $\mu = 0$ the non-smooth system Z_0 has not a periodic cycle but a homoclinic one $\Gamma_0 = W_+^u(0,0) \cup W^u(x_s, y_s) \cup \{(x,0), x^* \leq x \leq 0\}$, where $(x^*,0) = W^u(x_s, y_s) \cap \Sigma$. Clearly $\Gamma_{0,\varepsilon} \rightarrow \Gamma_0$ as $\varepsilon \rightarrow 0$. Nevertheless, it is straightforward to see that, for $\mu < 0$ we also have that $P^e \circ P_\varepsilon$ send the interval \mathcal{I} to an interval $J \subset \mathcal{I}$ and is a contraction. This gives the uniformity in ε for $\mu \leq 0$.

If $\mu > 0$, we use again that the interval \mathcal{I} is sent by P_ε to an interval \mathcal{J} containing $x_0^+ + \alpha^+ \varepsilon$ of size $\mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$. On the other hand the intersection of $W^s(x_h, y_h) \cap \mathcal{S}_{y_0}^+ = (x_0^s, y_0)$, with $x_0^s = x_0^+ - \mu$, therefore if

$$x_0^s < x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$$

the exterior map is not defined in this interval.

Observe that this happens if ε is small enough and $-\alpha^+ \varepsilon < \mu$. As $\alpha^+ < 0$, this condition is satisfied if ε is small enough for $\mu > 0$. One then conclude that if $\mu > 0$ there is no return of the whole interval \mathcal{I} to itself and therefore the system has no periodic orbits in this neighborhood of the saddle if ε is small enough.

If $\mu = \tilde{\mu}\varepsilon$, with $\tilde{\mu} > 0$, we use again that $x_0^u \in \mathcal{I} = [L^-, x_0^- + \alpha^- \varepsilon + \mu^- \varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})]$, and $P_\varepsilon(\mathcal{I})$ is an interval \mathcal{J} containing $x_0^+ + \alpha^+ \varepsilon$ of size $\mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$. Then, the first condition one needs to ensure that P^e is defined in this interval is

$$x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}}) < x_0^s = x_0^+ - \tilde{\mu}\varepsilon$$

which is fulfilled if $\tilde{\mu} < -\alpha^+$. Under this condition, we have again that

$$|P^e(x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})) - x_0^u| < \Omega((\alpha^+ + \tilde{\mu})\varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}}))^{-\frac{\lambda_2+K}{\lambda_1+K}} \leq \bar{\Omega} \varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}}$$

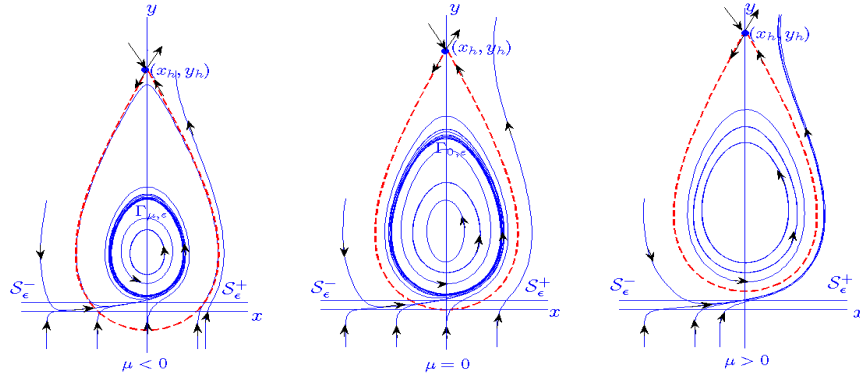


FIGURE 12. Bifurcation in the regularized system corresponding with the sliding homoclinic bifurcation in the Filippov system: Hamiltonian case. The dotted orbit represents the branches of the stable and unstable manifolds of the saddle point (x_h, y_h) of X^+ . For $\mu \leq 0$, the periodic orbit is semistable. For $\mu > 0$ the semistable periodic orbit disappears.

and therefore

$$P^\varepsilon(x_0^+ + \alpha^+\varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})) = x_0^u + \mathcal{O}(\varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}}) \in \mathcal{I}$$

and consequently $P^\varepsilon \circ P_\varepsilon$ sends the interval \mathcal{I} to an interval $J \subset \mathcal{I}$ containing $x_0^u < x_0^-$ of size $\mathcal{O}(\varepsilon^{-\frac{\lambda_2+K}{\lambda_1+K}})$ and is a contraction. This gives the existence of a periodic orbit $\Gamma_{\tilde{\mu}, \varepsilon}$ of the regularized vector field $Z_{\tilde{\mu}, \varepsilon}$ for $\tilde{\mu} < -\alpha^+$ and ε small enough.

We want to emphasize that, as $x_0^u \in \mathcal{I}$ one has that $P_\varepsilon(x_0^u) = x_0^+ + \alpha^+\varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$. Therefore, for $\tilde{\mu} < -\alpha^+$ one has that $P_\varepsilon(x_0^u) < x_0^s = x_0^+ - \tilde{\mu}\varepsilon$ and, if $\tilde{\mu} > -\alpha^+$ one has that $P_\varepsilon(x_0^u) > x_0^s = x_0^+ - \tilde{\mu}\varepsilon$. Therefore the value $\tilde{\mu} = -\alpha^+$ corresponds, in first order, to the value where the regularized vector field has a homoclinic orbit associated to the saddle (x_h, y_h) . We have then that the periodic orbits which existed for $\mu < -\alpha^+\varepsilon$ disappear and an homoclinic orbit appears. Analogous arguments as the ones given in Remark 2 can be considered to the question if this is an “homoclinic” bifurcation of $Z_{\mu, \varepsilon}$ (see Figure 11). \square

Another situation where this phenomenon occurs is when X_μ^+ is a Hamiltonian system. In this case, generically, the stable and unstable manifolds of (x_h, y_h) coincide along a homoclinic orbit which surrounds a collection of subharmonic orbits. In this case, for $\mu < 0$ both stable and unstable curves of the saddle intersect transversally the switching manifold Σ and therefore the homoclinic connection disappears. Then for $\mu = 0$ the homoclinic orbit is tangent to Σ , producing a pseudo-separatrix connection between the saddle and the visible fold (see Figure 10). For $\mu > 0$ the homoclinic orbit is contained in \mathcal{V}^+ and the unstable pseudoseparatrix of the visible fold, does not intersect Σ anymore.

Theorem 2.8. *Let Z_μ be a family of non-smooth vector fields such that X_μ^+ is a Hamiltonian vector field and has an homoclinic orbit to a saddle point $(x_h, y_h) \in \mathcal{V}^+$ of X^+ , that undergoes a sliding homoclinic bifurcation generated by a generic tangency between the homoclinic orbit of the saddle and the switching manifold Σ*

which occurs for $\mu = 0$. Assume that for $\mu < 0$ both stable and unstable curves of the saddle intersect transversally the switching manifold Σ and for $\mu > 0$ the homoclinic orbit is contained in \mathcal{V}^+ . Assume also that X_μ^- is transversal to Σ and points towards Σ for μ small enough. Consider the regularized family $Z_{\mu,\varepsilon}$, then:

- If $\mu < 0$, the system Z_μ has a grazing periodic orbit Γ_μ and the regularized system $Z_{\mu,\varepsilon}$ has a semistable periodic orbit $\Gamma_{\mu,\varepsilon}$ for ε small enough uniformly in μ , which approaches, when $\varepsilon \rightarrow 0$, the grazing periodic orbit Γ_μ .
- If $\mu = 0$, the system Z_μ has an sliding homoclinic orbit Γ_0 , and $Z_{0,\varepsilon}$ has a semistable periodic orbit $\Gamma_{0,\varepsilon}$ which approaches, when $\varepsilon \rightarrow 0$, the sliding homoclinic orbit Γ_0 .
- If $\mu > 0$ the only periodic orbits of $Z_{\mu,\varepsilon}$ near the stable separatrix of the saddle are the subharmonic orbits of Z_μ .
- The periodic orbit $\Gamma_{\mu,\varepsilon}$ exists until $\mu = \tilde{\mu}\varepsilon$, with $\tilde{\mu} < -\alpha^+$, where α^+ is given in Proposition 1. When $\tilde{\mu} = -\alpha^+$ there is an homoclinic orbit of X^+ .

Proof. As X^+ is Hamiltonian, the homoclinic orbit $W^u(x_h, y_h) = W^s(x_h, y_h)$ surrounds a family of subharmonic periodic orbits. As in Theorem 2.7, one can assume that the fold point, which exists for μ small enough, is located at $(0, 0)$. Again, we denote by $(x_0^\pm, y_0) = W^{u,s}(0, 0) \cap \mathcal{S}_{y_0}^\pm$, the intersection of its stable and unstable pseudo-separatrices with $\mathcal{S}_{y_0}^\pm$ and by $W^{s,u}(x_h, y_h) \cap \mathcal{S}_{y_0}^\pm = (x_0^{s,u}, y_0)$. We also assume that x_0^\pm are independent of μ , and $x_0^s = x_0^+ - \mu$ and $x_0^u = x_0^- + \mu$.

Is $\mu < 0$ small but fixed, and ε is small enough, one has that $x_0^+ < x_0^s$. Therefore, in this case, the stable and unstable pseudo-separatrices of the fold coincide in a grazing periodic orbit Γ_μ whose interior is full of periodic orbits surrounding a center. We are then in the hypotheses of Theorem 2.3 and we obtain, in the regularized system, that the periodic orbit $\Gamma_{\mu,\varepsilon}$ of X_μ^+ which is tangent to the section $y = \varepsilon$ persists in the regularized system becoming a semistable periodic orbit.

When $\mu = 0$ one has that $x_0^+ = x_0^u$ and $x_0^- = x_0^s$, therefore, we have two heteroclinic connections between the fold and the saddle forming an homoclinic orbit of the saddle (x_h, y_h) tangent to Σ . By Theorem 2.1, we have that the map P_ε sends the interval $\mathcal{I} = [L^-, x_0^- + \alpha^-\varepsilon + \beta^-\varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})]$, to an interval \mathcal{J} containing $x_0^+ + \alpha^+\varepsilon$ of size $\mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$. As $\alpha^+ < 0$, one has that $x_0^+ + \alpha^+\varepsilon < x_0^s$ and one can apply inequality (31) obtaining

$$|P^e(x_0^+ + \alpha^+\varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})) - x_0^u| \leq \Omega|\alpha^+\varepsilon + \mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})|^{-\frac{\lambda_2+K}{\lambda_1+K}}$$

and then $P^e(\mathcal{J})$ is an interval J containing the point $x_0^u = x_0^-$ and of size $|\varepsilon|^{-\frac{\lambda_2+K}{\lambda_1+K}}$. An important observation is that, being the interval \mathcal{J} in the left of the point x_0^s we know that $P^e(\mathcal{J}) = J \subset [x_0^-, x_0^- + \mathcal{O}(|\varepsilon|^{-\frac{\lambda_2+K}{\lambda_1+K}})]$.

Once we have this interval J contained in the interior of the homoclinic loop, we can use the reasoning of Theorem 2.3 to obtain that the successive iterates of the return map for any point of this interval form an increasing sequence which converges to the point \bar{x}_ε , which is the intersection of the periodic orbit $\Gamma_{\mu,\varepsilon}$ of X^+ tangent to $\{y = \varepsilon\}$ with \mathcal{S}_{y_0} .

If $\mu > 0$, then one has that $x_0^+ > x_0^s = x_0^+ - \mu$. By Theorem 2.1, we have that the map P_ε sends the interval $\mathcal{I} = [L^-, x_0^- + \alpha^-\varepsilon + \beta^-\varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})]$, to an interval \mathcal{J} containing $x_0^+ + \alpha^+\varepsilon$ of size $\mathcal{O}(\varepsilon^{\frac{2p}{2p-1}})$. But now one has that, if ε is small enough, $x_0^+ + \alpha^+\varepsilon > x_0^+ - \mu = x_0^s$ and therefore the exterior map P^e is not defined

in this interval. As a consequence, all the orbits beginning at \mathcal{I} do not intersect $\mathcal{S}_{y_0}^-$ anymore and there is no possibility of existence of periodic orbits near the fold.

When $\mu = \tilde{\mu}\varepsilon$, $0 \leq \tilde{\mu}$, one has that the exterior map P^e is still defined in the interval \mathcal{J} if $x_0^+ + \alpha^+\varepsilon < x_0^+ - \tilde{\mu}\varepsilon$ and this occurs, again, while $\tilde{\mu} < -\alpha^+$. Therefore, for these range of parameters, we still have a semistable periodic orbit in the system. Let us observe that the value $\tilde{\mu} = -\alpha^+$ gives, in first order, the value of μ such that the homoclinic orbit of X^+ is tangent to \mathcal{S}_ε and, therefore, this tangent semistable periodic orbit disappears (see figure 12).

Similarly as in Remark 2, to see that for $-\alpha^+\varepsilon \leq \mu$ a unique family of periodic orbits of growing period collapse to the homoclinic orbit, an accurate study of the return map would be needed. \square

2.6. Conclusions. In this paper we have considered the Sotomayor-Teixeira regularization of a Filippov system near a visible fold-regular point. This regularized vector field is a slow-fast system that we study using geometric singular perturbation theory. The main result is given in Theorem 2.1 where we prove that all the orbits beginning in a region close to the fold point leave it exponentially close to the Fenichel manifold of the system, which can be studied using asymptotic methods.

The behavior of the Fenichel manifold is leaded by the solution of an equation that, for \mathcal{C}^1 regularizations, is a well known Riccati equation. In the general \mathcal{C}^{p-1} case, we have proved the existence of a distinguished solution of the equation which leads the Fenichel manifold.

Once we know the behavior of the orbits near the tangency point, we devote the rest of the paper to study how codimension one global bifurcations of Filippov systems evolve when one regularizes the family undergoing them.

In Theorem 2.2 we prove that if the Filippov vector field has a sliding periodic orbit or a grazing periodic orbit which is attracting, the regularized system also has a periodic orbit nearby.

Even if the result of Theorem 2.2 can not be applied when the periodic orbit of the Filippov system is not hyperbolic, in Theorem 2.3 we study the case of a Filippov system having a center surrounded by periodic orbits. We prove that in the regularized system the orbit tangent to the regularizing zone is semistable. This case is generic in Hamiltonian systems.

In Section 2.4 we apply theorems 2.3 and 2.2 to Coulomb and Stribeck models of the dry friction oscillator.

In Theorem 2.4 we study the grazing-sliding bifurcation of periodic orbits. We prove that the regularized system has a bifurcation of periodic orbits when the Filippov system has a repelling periodic orbit. No bifurcation in the regularized system corresponds to the attracting case.

In Theorem 2.7 we study the sliding homoclinic bifurcation, that is, one of the vector fields of the Filippov system has a saddle equilibrium point whose stable manifold is tangent to the switching surface. We prove that the corresponding regularized system has a bifurcation: for some values of the parameter one has periodic orbits and a homoclinic orbit appears at a critical value of the parameter. An analogous result is given in Theorem 2.8 for the Hamiltonian case.

Summarizing, in all the cases considered, roughly speaking, the regularization preserves the topological features of the corresponding Filippov system. However, to complete the study, a more precise determination of the character of the bifurcations must be made. We intend to carry out this study soon.

3. Proof of Theorem 2.1.

3.1. Sketch of the proof. In this section we will study the Poincaré map $P_\varepsilon = \bar{P} \circ \mathcal{Q}_\varepsilon \circ P : \mathcal{S}_{y_0}^- \rightarrow \mathcal{S}_{y_0}^+$. By Proposition 1 we know the behavior of the maps P and \bar{P} , which only depend of the vector field X^+ and are the same for Z and for its regularization Z_ε . That's not the case for the map \mathcal{Q}_ε that, for the non-smooth system, is:

$$\mathcal{Q}_0(x) = \gamma\sqrt{\varepsilon} + O(\varepsilon), \quad \gamma > 0, \text{ if } x < -\gamma\sqrt{\varepsilon}$$

where, $W_+^{u,s}(0,0) \cap \mathcal{S}_\varepsilon^\pm = (\pm\gamma\sqrt{\varepsilon} + O(\varepsilon), \varepsilon)$

To study the map \mathcal{Q}_ε we need to control the behavior of solutions of Z_ε near $(0,0)$, which is the visible fold point of Z . This is done in next sections using geometric singular perturbation theory and matching asymptotic expansions.

The first step of the proof is the classical Fenichel Theorem 3.1, which gives the existence of an invariant manifold Λ_ε in compact sets of the form $\{(x,v), |v| \leq 1, -L \leq x \leq -N\}$, where $v = \varepsilon y$, $N > 0$, $L > 0$, and which is ε -close to the so called critical manifold. We are able to see that, in our case, it is exponentially attracting of all solutions of the vector field Z_ε in any compact. The main difficulties of the proof are twofold: to follow the Fenichel manifold until it reaches $v = 1$ and to see that its region of attraction can be extended to x near zero.

Both results depend of the regularity of the function φ used in the regularization. For this reason, they are separated in three sections. Section 3.3 deals with the case that φ is piecewise linear and continuous. Experts in the field can go directly to Section 3.4, where one studies in detail the \mathcal{C}^1 case. Finally, in Section 3.5 we generalize the results to the \mathcal{C}^{p-1} case, $p \geq 3$.

In the linear case, Fenichel theory can be applied to compact sets containing $x = 0$ and therefore it is easy to obtain the behavior of the manifold Λ_ε when it crosses $v = 1$. In Proposition 4 we see that the region of attraction of Λ_ε arrives to points $(x,1)$ with $x = \mathcal{O}(\varepsilon^\lambda) < 0$, $0 < \lambda < 1$. This gives the behavior of Poincaré map in Section 3.3.1.

In the \mathcal{C}^1 case, Fenichel theory is only valid in compacts sets avoiding $x = 0$, $v = 1$. In Proposition 5, using asymptotic methods, we extend this manifold until it reaches $v = 1 - \mathcal{O}(\varepsilon^{\lambda_1})$, for any $0 < \lambda_1 < \frac{1}{3}$ and we see that it is still close to the critical manifold. To go further we need to introduce new variables, called the inner variables (58), and we obtain a new approximation of the Fenichel manifold as the solution of an equation independent of parameters, usually called inner equation. This equation is a well known Riccati equation already studied in [21] and a distinguished solution will lead the Fenichel manifold until it reaches $v = 1$ as it is seen in Proposition 6.

Once we know the behavior of the Fenichel manifold Λ_ε at $v = 1$ it only remains to ensure in Proposition 8 that its region of attraction arrives to points of the form $(x,1)$ with $x = \mathcal{O}(\varepsilon^\lambda) < 0$, $\lambda < 2/3$. This gives the behavior of Poincaré map in Section 3.4.4.

An analogous proof for the \mathcal{C}^{p-1} case $p \geq 3$ is done in Proposition 9, to extend the invariant manifold until it reaches $v = 1 - \mathcal{O}(\varepsilon^{\lambda_1})$, for any $0 < \lambda_1 < \frac{1}{2p-1}$. The study of the inner equation is done in Proposition 10, obtaining a distinguished solution that will lead the Fenichel manifold until it reaches $v = 1$ as it is seen in Proposition 11. Finally, in Proposition 13 we see that the region of attraction arrives to points of the form $(x,1)$ with $x = \mathcal{O}(\varepsilon^\lambda) < 0$, $0 < \lambda < \frac{p}{2p-1}$. This gives the behavior of Poincaré map in Section 3.5.2.

As we want to perform a local analysis near $(0, 0)$, which is a fold-regular point for Z , following [9], we assume that, locally, near Σ , the system can be written as:

$$X^+(x, y) = \begin{pmatrix} 1 \\ 2x \end{pmatrix} \quad (32)$$

and

$$X^-(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (33)$$

In Section 3.6 we extend the results to a general vector field Z near a visible fold-regular point.

For the vector fields (32) and (33) we have explicit expressions for the maps P , \bar{P} given in (13):

$$P(x) = -\sqrt{x^2 + \varepsilon - y_0}, \quad \bar{P}(x) = \sqrt{x^2 - \varepsilon + y_0}. \quad (34)$$

Therefore, in this case, one has $x_0^\pm = \pm\sqrt{y_0}$, and the constants given in Proposition 1 are $\alpha^\pm = \mp\frac{1}{2\sqrt{y_0}}$, and $\beta^\pm = \pm\frac{1}{2\sqrt{y_0}}$.

3.2. The slow invariant manifold. The regularized system Z_ε in (9) leads to the differential equations:

$$\begin{aligned} \dot{x} &= \frac{1}{2}(1 + \varphi(\frac{y}{\varepsilon})) \\ \dot{y} &= \frac{1+2x}{2} + \frac{1}{2}\varphi(\frac{y}{\varepsilon})(2x-1). \end{aligned} \quad (35)$$

System (35) can be written, with the change of variable $y = \varepsilon v$ as:

$$\begin{aligned} \dot{x} &= \frac{1+\varphi(v)}{2} \\ \varepsilon\dot{v} &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1). \end{aligned} \quad (36)$$

This system is usually called slow system. If we now perform the change of time $t = \varepsilon\tau$ we get the so called fast system, corresponding to a vector field \tilde{Z}_ε which depends regularly on ε :

$$\begin{aligned} x' &= \varepsilon\frac{1+\varphi(v)}{2} \\ v' &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1). \end{aligned} \quad (37)$$

If we put $\varepsilon = 0$ in system (36) we get a differential equation in a manifold:

$$\begin{aligned} \dot{x} &= \frac{1+\varphi(v)}{2} \\ 0 &= 1 + 2x + \varphi(v)(2x-1). \end{aligned}$$

This manifold is usually called the *critical manifold*, and, for our system, is a curve:

$$\Lambda_0 = \{(x, v), \varphi(v) = \frac{1+2x}{1-2x}, x \leq 0\}. \quad (38)$$

Observe that, for the functions φ considered in (16), Λ_0 only exists for negative values of x , because for these values one has that $-1 \leq \frac{1+2x}{1-2x} \leq 1$.

Λ_0 is a manifold of critical points of the fast system (37) for $\varepsilon = 0$. Moreover, for $(x, v) \in \Lambda_0$:

$$D\tilde{Z}_0(x, v) = \begin{pmatrix} 0 & 0 \\ 1 + \varphi(v) & \frac{\varphi'(v)}{2}(2x-1) \end{pmatrix} \quad (39)$$

As $\varphi'(v)(2x-1) \leq 0$ for all the points in Λ_0 , the manifold Λ_0 is a normally hyperbolic attracting manifold for the vector field \tilde{Z}_0 . Except in the linear case, for the functions φ we consider, it is clear that $\varphi'(1) = 0$, and therefore, as $(0, 1) \in \Lambda_0$, we will have that Λ_0 loses its hyperbolic character when $x \rightarrow 0$. In any compact subset of the region $x < 0$, we can apply Fenichel Theorem [6, 11], which ensures

the existence of a normally hyperbolic attracting invariant manifold Λ_ε for ε small enough of system (37) (and (36)):

Theorem 3.1. *Consider any numbers $L, N > 0$. Then, there exists $\varepsilon_0 > 0$ and constants $K, C > 0$, such that for $|\varepsilon| < \varepsilon_0$ system (36) has a normally hyperbolic invariant manifold Λ_ε such that in the region $-L \leq x \leq -N$ is ε -close to Λ_0 . That is, there exists a smooth function $m(x; \varepsilon)$ such that*

- $\Lambda_\varepsilon = \{(x, v), -L \leq x \leq -N, v = m(x; \varepsilon)\}$ is a normally hyperbolic attracting locally invariant manifold of system (37).
- If $-L \leq x \leq -N$ we have that $|m(x; \varepsilon) - m_0(x)| \leq K\varepsilon$, where $m_0(x) = \varphi^{-1}\left(\frac{1+2x}{1-2x}\right)$.
- There exists a neighborhood U of Λ_ε such that for any $z_0 \in U$ there exists $z^* \in \Lambda_\varepsilon$ such that

$$|\phi(t, z_0) - \phi(t, z^*)| \leq Ke^{-C\frac{t}{\varepsilon}}, \quad t \geq 0$$

where ϕ is the flow of system (36).

- The set $\{(x_0, 1), -L \leq x_0 \leq -N\}$ is contained in U .

Proof. The proof of this theorem can be found in [6]. We only need to prove the last item. By Fenichel Theorem, we know that there exists a neighborhood U of the manifold Λ_ε where it is exponentially attracting for the slow system (36). Consider now a subset U' such that $\Lambda_\varepsilon \subset U' \subset U$. Fix $-3N \leq x_0 \leq -2N$, and consider the solution $z(t; \varepsilon)$ of system (37) with initial condition $z_0 = (x_0, 1)$. For any $T > 0$, there exists $\varepsilon_0 = \varepsilon_0(T)$ such that for $|\varepsilon| \leq \varepsilon_0$:

$$z(\tau; \varepsilon) = z(\tau; 0) + O(\varepsilon), \quad 0 \leq \tau \leq T$$

where $z(\tau; 0) = (x_0, v(\tau))$, and $v(\tau)$ is the solution of the second equation of system (37) for $\varepsilon = 0$ with initial condition $v(0) = 1$. On the other hand the second component of this vector field is zero in the critical manifold Λ_0 and is negative for points (x, v) such that $v > m_0(x)$. Moreover, for (x, v) such that $-3N \leq x \leq -2N$, and $m(x; \varepsilon) \leq v \leq 1$, such that $(x, v) \notin U'$, there exists $M > 0$ such that

$$\frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1) \leq -M$$

and therefore, we know that there exists a time $T = T(x_0)$ such that $z(T; 0) \in U \setminus U'$, and consequently, there exists $\varepsilon_0 = \varepsilon_0(x_0)$ such that for $|\varepsilon| \leq \varepsilon_0$ one has that $z(T; \varepsilon) \in U \setminus U'$. Now, as x_0 is in a compact set, there exists ε_0 such that the result is true for any point $(x_0, 1)$, with $-L \leq x_0 \leq -2N$. Now, we rename N as $2N$ and we obtain the result. \square

This theorem gives us the existence of the slow invariant manifold Λ_ε and its property of being attracting for points of the form $(x, 1)$ for $-L \leq x \leq -N$, for fixed $N > 0$. Later, in propositions 4, 8 and 13, we will see that this manifold is also attracting for points closer to the point $(0, 1)$.

Remark 3. By Theorem 3.1 we know that, for any $N > 0$, in $-L \leq x \leq -N < 0$ the Fenichel invariant manifold can be described by

$$v = m(x; \varepsilon), \quad -L \leq x \leq -N, \quad 0 \leq \varepsilon \leq 0$$

where $m(x; \varepsilon)$ is a differentiable function, even for $\varepsilon = 0$. Moreover, the invariant character and the fact that $m(x; 0) = m_0(x)$ implies that $m(x; \varepsilon)$ has a unique expansion on $-L \leq x < 0$:

$$m(x; \varepsilon) = m_0(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2).$$

This expansion is only valid on $-L \leq x < -N$. When $N \rightarrow 0$, the range of ε -validity of the expansion tends to zero.

Nevertheless, as $m'_0(x) \rightarrow \infty$ as $x \rightarrow 0$, if we fix $L > 0$ small enough (but independent of ε), one can guarantee that $m'_0(x) > M > 0$. Therefore, we can express the critical manifold Λ_0 as $x = n_0(v)$ for $m_0(-L) \leq v \leq 1$, and due to the unicity of the asymptotic expansion and the uniform validity in $-L \leq x \leq -N$, the invariant manifold $v = m(x; \varepsilon)$ can also be expressed, inverting m as $x = n(v; \varepsilon)$, with

$$n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \mathcal{O}(\varepsilon^2)$$

where the functions n_i are uniquely determined for $m_0(-L) \leq v \leq 1$ by the invariance condition. Naturally, the asymptotic validity can only take place for $m_0(-L) \leq v \leq m_0(-N)$.

Then, if $m_1(x) < 0$ for $-L \leq x < 0$, we will have:

$$m(x; \varepsilon) < m_0(x), \quad -L \leq x \leq -N < 0,$$

equivalently, if $n_1(v) > 0$, we will have:

$$n(v; \varepsilon) > n_0(v), \quad m_0(-L) \leq v \leq m_0(-N) < 0.$$

Once we know that the orbit of all the points in U gets exponentially close to Λ_ε and that Λ_ε is ε -close to Λ_0 until (x, v) enter the region $x \geq -N$, now we want to follow the orbits when they get closer to the point $(0, 1)$. In this region Fenichel Theorem is no valid so we will use some asymptotic expansions to get the main terms in the asymptotic series of the invariant manifold Λ_ε . As all the orbits are exponentially small close to Λ_ε , these terms will be valid for the asymptotic expansion of any solution of the system (37).

As we will see in next sections, the way the manifold Λ_ε , and therefore all the orbits in U , behave near $(0, 1)$ strongly depends of the regularity of function φ .

3.3. The slow manifold close to $(0, 1)$: Linear case. We first consider the linear case where φ is defined in (15). In this case, system (36) reads:

$$\left. \begin{aligned} \dot{x} &= \frac{1+v}{2} \\ \varepsilon \dot{v} &= \frac{1+2x}{2} + \frac{v}{2}(2x-1) \end{aligned} \right\}, \quad \text{for } -1 \leq v \leq 1, \quad (40)$$

and is given by the vector fields X^+ in (32) for $v \geq 1$ and by X^- in (33) for $v \leq -1$. If one considers system (40) for any $(x, v) \in \mathbb{R}^2$, it has a critical manifold $\Lambda_0 = \{(x, v) \mid x < \frac{1}{2}, v = \frac{1+2x}{1-2x}\}$ and it is a normally hyperbolic attracting invariant manifold for $x \leq N$, if we fix $N < \frac{1}{2}$. Applying Fenichel Theorem for $0 < N < \frac{1}{2}$ we get a normally hyperbolic invariant manifold Λ_ε for ε small enough given by $v = m(x; \varepsilon)$ satisfying:

$$1 + 2x + m(x; \varepsilon)(2x - 1) = \varepsilon(1 + m(x; \varepsilon))m'(x; \varepsilon), \quad (41)$$

and

$$m(x; \varepsilon) = m_0(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2) \quad (42)$$

with

$$m_0(x) = \frac{1+2x}{1-2x}, \quad m_1(x) = \frac{1+m_0(x)}{2x-1} m'_0(x) = -\frac{8}{(1-2x)^4}.$$

The manifold Λ_ε is the invariant manifold of the regularized system (40) until it reaches $v = 1$ at a point $(x_1, 1)$, with $1 = m_0(x_1) + \varepsilon m_1(x_1) + \mathcal{O}(\varepsilon^2)$, which, using (42), gives $x_1 = 2\varepsilon + \mathcal{O}(\varepsilon^2)$.

To prove that the invariant manifold Λ_ε is attracting for points closer to the fold $(0, 0)$, we need some extra information of it. This is done in next proposition.

Proposition 3. *There exists $K > 0$ and $\varepsilon_0 > 0$, such that, if $0 < \varepsilon < \varepsilon_0$ the invariant manifold $v = m(x; \varepsilon)$ satisfies, for $-L \leq x \leq \frac{1}{4}$:*

$$m_0(x) - \varepsilon K \leq m(x; \varepsilon) \leq m_0(x) \quad (43)$$

Proof. As $x \leq \frac{1}{4} < \frac{1}{2}$ we can apply Theorem 3.1 in this region. Moreover $-K < m_1(x) < 0$, if $K > 2^7$. The result follows by taking $\varepsilon_0 > 0$ small enough. \square

Next proposition shows that all the solutions with initial conditions at $(x_0, 1)$, with $-L \leq x_0 \leq -\varepsilon^\lambda$, and $\lambda < 1$ are attracted by the Fenichel manifold Λ_ε . Let's introduce the equations for the orbits of system (40):

$$\varepsilon \frac{dv}{dx} = \frac{1 + 2x + v(2x - 1)}{1 + v}. \quad (44)$$

Proposition 4. *Fix $0 < \lambda < 1$ and take any point $(x_0, 1)$, with $-L \leq x_0 \leq -\varepsilon^\lambda$. Then, the orbit of system (44) with initial condition $v(x_0) = 1$ stays exponentially close to the invariant manifold $v = m(x; \varepsilon)$ in the region $x \geq 0$.*

Proof. We perform the change of variables $w = v - m(x; \varepsilon)$ in equation (44) obtaining:

$$\varepsilon \frac{dw}{dx} = -g(x, \varepsilon)w \quad (45)$$

where $g(x; \varepsilon) = \frac{-2x+1+\varepsilon m'(x; \varepsilon)}{1+m(x; \varepsilon)+w(x; \varepsilon)} \geq 0$. Note that we already know the existence of the solution $w(x; \varepsilon)$ for $x \leq 0$, in fact we know it is bounded by:

$$0 \leq w(x; \varepsilon) \leq 1 - m(x; \varepsilon).$$

Clearly, the solution of (45) with initial condition $w(x_0) = 1 - m(x_0; \varepsilon)$ can be written as:

$$w(x) = e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(s; \varepsilon) ds} w(x_0).$$

Using that for $x \leq 0$ we have that $g(x; \varepsilon) \geq \frac{1}{2}$ we can bound $w(x)$:

$$|w(x; \varepsilon)| \leq |w(x_0)| e^{-\frac{1}{2\varepsilon}(x-x_0)},$$

therefore if $x_0 \leq -\varepsilon^\lambda$ with $\lambda < 1$, any solution gets exponentially closer to the invariant manifold $v = m(x; \varepsilon)$ for $x \geq 0$. \square

3.3.1. Asymptotics for the Poincaré map P_ε . After Theorem 3.1, propositions 3 and 4, and the fact that the Fenichel manifold reaches $v = 1$ for $x_1 = 2\varepsilon + \mathcal{O}(\varepsilon^2)$, we know that any solution of the system arrives to $v = 1$ exponentially close to it, therefore it also cuts $v = 1$ at $x_1 = 2\varepsilon + \mathcal{O}(\varepsilon^2)$. This gives the behavior of the Poincaré map \mathcal{Q}_ε :

$$\forall x \in [-L, -\varepsilon^\lambda], \quad \mathcal{Q}_\varepsilon(x) = 2\varepsilon + \mathcal{O}(\varepsilon^2). \quad (46)$$

Moreover, as all the orbits evolve exponentially close to the Fenichel manifold, studying the variational equations around it one obtains the classical result that its Lipschitz constant is exponentially small with respect to ε (see [14, 1]).

Fix $0 < \lambda < 1$. Taking into account that, by (13)

$$P^{-1}(-\varepsilon^\lambda) = x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})$$

we have that

$$P([L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})]) \subset [-L, -\varepsilon^\lambda]$$

for a suitable constant L^- .

Using formulas (13) for \bar{P} we conclude that the map $P_\varepsilon = \bar{P} \circ \mathcal{Q}_\varepsilon \circ P$ satisfies:

$$\forall x \in [L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})], \quad P_\varepsilon(x) = \bar{P}(2\varepsilon + \mathcal{O}(\varepsilon^2)) = x_0^+ + \alpha^+ \varepsilon + \mathcal{O}(\varepsilon^2). \quad (47)$$

Therefore, all the points in the set $\mathcal{I} = [-L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})] \times \{y_0\}$, $0 < \lambda < 1$, are sent by P_ε to a set $\mathcal{J} \times \{y_0\}$ and the interval \mathcal{J} has, at most, size ε^2 containing the point $x_0^+ + \alpha^+ \varepsilon$. Moreover, as P and \bar{P} are independent of ε , the Lipschitz constant of P_ε is also exponentially small in ε .

3.4. The slow manifold close to $(0, 1)$: Smooth \mathbb{C}^1 case.

3.4.1. *Extending the outer domain.* When the regularizing function φ in (16) is \mathbb{C}^{p-1} , with $p \geq 2$, the critical manifold Λ_0 given in (38), bends near $(0, 1)$:

$$m_0(0) = 1, \quad m'_0(x) \rightarrow \infty \text{ as } x \rightarrow 0^-.$$

As a consequence, the Fenichel manifold can not be expressed as a graph over the x variable when x is near 0. In scope of Remark 3, we can consider that the solutions we deal with begin exponentially close to the Fenichel manifold, which, if x is near 0, is already in the region $1 + 2x + \varphi(v)(2x - 1) > 0$ and can't leave it as the flow of (37) points inwards to this region through Λ_0 . So, from now on, we look for the Fenichel manifold and also for all the orbits of system (37) inside this region as graphs over the v variable satisfying:

$$\frac{dx}{dv} = \varepsilon \frac{1 + \varphi(v)}{1 + 2x + \varphi(v)(2x - 1)}. \quad (48)$$

The formal expansion of the Fenichel manifold is

$$x = n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \dots + \mathcal{O}(\varepsilon^n)$$

where the critical manifold is $\Lambda_0 = \{(x, v), x = n_0(v), v \leq 1\}$ and $n_0(v) = \frac{1}{2} \frac{\varphi(v) - 1}{\varphi(v) + 1}$.

As the function $n(v; \varepsilon)$ is a solution of the equation (48), it satisfies:

$$(1 + 2n + \varphi(v)(2n - 1))n' = \varepsilon(1 + \varphi(v)) \quad (49)$$

where $' = \frac{d}{dv}$. Solving this invariance equation for n formally one obtains:

$$n_0(v) = \frac{1}{2} \frac{\varphi(v) - 1}{\varphi(v) + 1}, \quad (50)$$

$$n_1(v) = \frac{1}{2} \frac{1}{n'_0(v)}, \quad (51)$$

$$n_2(v) = -2n'_1(v)n_1^2(v) = \frac{1}{2} \frac{n''_0(v)}{(n'_0(v))^2},$$

It will be enough for our purposes to keep the two first terms in this expansion. By (17), the behavior of these functions near $v = 1$ depends on the value p .

From now on in this section we will deal with the \mathbb{C}^1 case, which corresponds to $p = 2$. In this case, expanding near $v = 1$ one has:

$$n_0(v) = \frac{\varphi''(1)}{8}(v - 1)^2 + \mathcal{O}(v - 1)^3 \quad (52)$$

$$n_1(v) = \frac{2}{\varphi''(1)} \frac{1}{(v - 1)} + \mathcal{O}(1) \quad (53)$$

$$n_2(v) = \mathcal{O}\left(\frac{1}{(v - 1)^4}\right)$$

where we use the notation

$$\varphi''(1) := \lim_{v \rightarrow 1^-} \varphi''(v). \quad (54)$$

Therefore, close to $v = 1$, $n(v; \varepsilon)$ behaves as:

$$n(v; \varepsilon) = \frac{\varphi''(1)}{8}(v-1)^2(1+\mathcal{O}(v-1)) + \frac{2}{\varphi''(1)} \frac{\varepsilon}{v-1}(1+\mathcal{O}(v-1)) + O\left(\frac{\varepsilon^2}{(v-1)^4}\right) + \dots \quad (55)$$

This asymptotic expansion fails for $(v-1)^3 = O(\varepsilon)$, which indicates that the invariant manifold remains close to $x = n_0(v)$ until $v = 1 - O(\varepsilon^{\frac{1}{3}})$, as it is shown in next proposition:

Proposition 5. *Take any $0 < \lambda_1 < \frac{1}{3}$. Then, there exists $M > 0$ big enough, $\delta = \delta(M) > 0$ small enough, and $\varepsilon_0 = \varepsilon_0(M, \delta) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, any solution of system (37) which enters the set*

$$\mathbf{B} = \left\{ (x, v), -\delta < v-1 < -\varepsilon^{\lambda_1}, n_0(v) \leq x \leq n_0(v) + \frac{M\varepsilon}{|v-1|} \right\}$$

leaves it through the boundary $v = 1 - \varepsilon^{\lambda_1}$.

Proof. We will see that the vector field \tilde{Z}_ε points inwards in three of the four boundaries of \mathbf{B} .

The exterior vector to $\mathbf{B}^+ = \left\{ (x, v), -\delta < v-1 < -\varepsilon^{\lambda_1}, x = n_0(v) + \frac{M\varepsilon}{1-v} \right\}$ is:

$$n^* = \left(1, -n'_0(v) - \frac{M\varepsilon}{(1-v)^2} \right)$$

and we will prove that $\langle \tilde{Z}_\varepsilon, n^* \rangle_{\mathbf{B}^+} < 0$. Computing this scalar product, using the definition of n_0 in (50), and the fact that $1 + \varphi(v) \geq 0$, we get the equivalent inequality:

$$1 + \frac{2Mn'_0}{v-1} + \frac{2M^2\varepsilon}{(v-1)^3} < 0. \quad (56)$$

Now we need to check that, taking M big enough and δ small enough, there exists $\varepsilon_0 = \varepsilon_0(M, \delta)$, such that for $0 < \varepsilon \leq \varepsilon_0$, this inequality holds if $-\delta \leq v-1 \leq -\varepsilon^{\lambda_1}$ for $0 < \lambda_1 < 1/3$.

Using (52) one has that there exists a constant C independent of δ and M such that:

$$\left| \frac{n'_0(v)}{v-1} - \frac{\varphi''(1)}{4} \right| \leq C\delta.$$

In \mathbf{B} , one has that $\frac{\varepsilon}{\delta^3} \leq \frac{\varepsilon}{|v-1|^3} \leq \varepsilon^{1-3\lambda_1} < 1$. Therefore one can write (56) as

$$M \frac{\varphi''(1)}{2} + 1 + g(v; \varepsilon) < 0 \quad (57)$$

where the function $g(v)$ satisfies: $|g(v; \varepsilon)| \leq 2M [C\delta + M\varepsilon^{(1-3\lambda_1)}]$. As $\varphi''(1)$ is negative, one can choose M big enough, for instance $M \frac{\varphi''(1)}{2} + 1 < -2C$, and then take ε_0 and δ small enough such that $|g(v; \varepsilon)| < C$ if $\varepsilon < \varepsilon_0$, to have that (57) holds.

On $\mathbf{B}^- = \{(x, v), x = n_0(v)\}$ the vector field (37) is $\tilde{Z}_\varepsilon(n_0(v), v) = (\frac{1+\varphi(v)}{2}, 0)$ and therefore, as $1 + \varphi(v) > 0$ the flow points inward in this boundary.

When $v = 1 - \delta$ and $n_0(v) \leq x \leq n_0(v) + \frac{M\varepsilon}{|v-1|}$ we also have that $\dot{v} > 0$ and therefore the flow also points inward in this boundary. To conclude the proof we

observe that once the orbits enter the set \mathbf{B} as $\dot{v} > 0$ in \mathbf{B} , they can only leave it through the boundary $v = 1 - \varepsilon^{\lambda_1}$. \square

By Fenichel Theorem 3.1 and Remark 3, the invariant manifold Λ_ε is a smooth manifold ε -close to Λ_0 , which is given by $v = m_0(x)$, until it arrives to $v = 1 - \delta$. Moreover, $m_0(x)$ is an invertible function whose inverse is $n_0(v)$. Therefore, in this region the Fenichel manifold can be written as:

$$x = n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \mathcal{O}(\varepsilon^2), \text{ for } v = 1 - \delta$$

and, as $n_1(v) > 0$ for $-1 \leq v \leq 1$ (see (51)), redefining the constants M big enough and δ small enough in Proposition 5, the manifold enters in the domain \mathbf{B} for $v^* = 1 - \delta$. Then, it satisfies:

$$n_0(v) < n(v; \varepsilon) < n_0(v) + \frac{M\varepsilon}{1-v}, \text{ if } 1 - \delta \leq v \leq 1 - \varepsilon^{\lambda_1}.$$

Moreover, using Theorem 3.1, as the manifold Λ_ε attracts exponentially any other solution, all the solutions of system (36) with initial conditions in U satisfy the same inequality.

Furthermore, as, for any $\lambda > 0$, one has that $n_0(1 - \varepsilon^\lambda) = \frac{\varphi''(1)}{8}\varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{3\lambda})$, one concludes that

$$n(1 - \varepsilon^\lambda; \varepsilon) = \frac{\varphi''(1)}{8}\varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1-\lambda}, \varepsilon^{3\lambda})$$

for any $0 < \lambda \leq \lambda_1 < \frac{1}{3}$ and λ_1 is the value given in Proposition 5.

As all the solutions enter in the block exponentially closer to Λ_ε , any solution $x(v)$ with initial condition $x(1) = x_0$ with $-L \leq x_0 \leq -N$ satisfies the same asymptotics:

$$x(1 - \varepsilon^\lambda) = \frac{\varphi''(1)}{8}\varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1-\lambda}, \varepsilon^{3\lambda}).$$

3.4.2. *The inner domain.* To reach $v = 1$ we need to change our strategy. The expansion of $n(v; \varepsilon)$ (55) loses its asymptoticity for $v = 1 - O(\varepsilon^{1/3})$. Moreover, $n(v; \varepsilon)$ has order $\varepsilon^{2/3}$ for these values of v . To study this range of values of v we perform the change:

$$\begin{aligned} x &= \varepsilon^{2/3}\eta \\ v &= 1 + \varepsilon^{1/3}u \end{aligned} \tag{58}$$

to system (36) obtaining:

$$\begin{aligned} \eta' &= \varepsilon^{1/3} \frac{1 + \varphi(1 + \varepsilon^{1/3}u)}{2} \\ u' &= \frac{\varepsilon^{-1/3}}{2} \left(1 + 2\varepsilon^{2/3}\eta + \varphi(1 + \varepsilon^{1/3}u)(2\varepsilon^{2/3}\eta - 1) \right) \end{aligned} \tag{59}$$

The equation for the orbits (48) in these new variables, calling $\mu = \varepsilon^{1/3}$, becomes:

$$\frac{d\eta}{du} = \frac{\mu^2(1 + \varphi(1 + \mu u))}{(1 + 2\mu^2\eta(u) + \varphi(1 + \mu u)(2\mu^2\eta(u) - 1))}. \tag{60}$$

We need to study the extension of a solution of this equation $\eta(u; \varepsilon)$, with initial condition $\eta(u^*; \varepsilon)$, with

$$u^* = \frac{v^* - 1}{\varepsilon^{1/3}} = -\varepsilon^{\lambda_2 - 1/3}, \text{ with } 0 < \lambda_2 \leq \lambda_1, \tag{61}$$

where λ_1 is given in Proposition 5, satisfying

$$|\varepsilon^{2/3}\eta(u^*; \varepsilon) - n_0(v^*)| \leq M\varepsilon^{1-\lambda_2} \quad (62)$$

where

$$v^* = 1 + \varepsilon^{1/3}u^* = 1 - \varepsilon^{\lambda_2}, \quad (63)$$

to the domain:

$$u^* \leq u \leq 0, \quad u^* = -\varepsilon^{\lambda_2-1/3} \quad (64)$$

which corresponds to $v^* \leq v \leq 1$.

Formally expanding the solution $\eta(u; \varepsilon)$ of equation (60) in powers of $\mu = \varepsilon^{1/3}$

$$\eta(u; \varepsilon) = \eta_0(u) + \mu\eta_1(u) + \mathcal{O}(\mu^2) \quad (65)$$

one can see that η_0 is the solution of the so called *inner equation*:

$$\eta_0' = \frac{d\eta_0}{du} = \frac{4}{8\eta_0 - \varphi''(1)u^2} \quad (66)$$

which, with the changes $\bar{\eta} = \alpha\eta$, $\bar{u} = \mu u$, where

$$\alpha = -\left(\frac{\varphi''(1)}{2}\right)^{1/3}, \quad \beta = \left(\frac{(\varphi''(1))^2}{32}\right)^{1/3}$$

becomes

$$\frac{d\bar{\eta}}{d\bar{u}} = \frac{1}{\bar{\eta} + \bar{u}^2}. \quad (67)$$

It is known [21] that this equation has a unique solution $\bar{\eta}_0(\bar{u})$ satisfying:

$$\bar{\eta}_0(\bar{u}) = -\bar{u}^2 - \frac{1}{2\bar{u}} - \frac{1}{8\bar{u}^4} + \mathcal{O}\left(\frac{1}{\bar{u}^7}\right), \quad \bar{u} \rightarrow -\infty \quad (68)$$

$$\bar{\eta}_0(\bar{u}) = \Omega_0 - \frac{1}{\bar{u}} + \mathcal{O}\left(\frac{1}{\bar{u}^3}\right), \quad \bar{u} \rightarrow \infty. \quad (69)$$

Going back to our variables one has that equation (66) has a solution $\eta_0(u)$ satisfying:

$$\begin{aligned} \eta_0(u) &= \frac{\varphi''(1)}{8}u^2 + \frac{2}{\varphi''(1)u} + \frac{16}{(\varphi''(1))^3 u^4} + \mathcal{O}\left(\frac{1}{u^7}\right), \quad u \rightarrow -\infty \\ \eta_0(u) &= -\frac{2^{1/3}\Omega_0}{(\varphi''(1))^{1/3}} + \frac{4}{\varphi''(1)u} + \mathcal{O}\left(\frac{1}{u^3}\right), \quad u \rightarrow \infty. \end{aligned} \quad (70)$$

If one considers the next term in the expansion (65) of $\eta(u, \varepsilon)$, one has that $\eta_1(u)$ is the solution of the equation:

$$\eta_1'(u) = -\frac{8}{(4\eta_0(u) - \frac{\varphi''(1)}{2}u^2)^2}\eta_1 + \frac{2\frac{\varphi'''(1)}{6}u^3}{(4\eta_0 - \frac{\varphi''(1)}{2}u^2)^2}$$

which is a linear equation. It is straightforward to see that there is a solution η_1 of this equation that, near $-\infty$, behaves as:

$$\eta_1(u) \simeq \frac{\varphi'''(1)}{24}u^3 + \mathcal{O}(u^2). \quad (71)$$

This suggests to consider the isolating block defined by a condition of the type

$$|\eta(u) - \eta_0(u)| \leq K\mu|u|^3. \quad (72)$$

As a consequence of the expansion of η_0 near $-\infty$ in (70) and the asymptotic expansion of $n_0(v)$ near $v = 1$ (52), one has that there exist constants K_1, K_2 , such that

$$|\varepsilon^{2/3}\eta_0(u^*) - n_0(v^*)| \leq K_1\varepsilon^{3\lambda_2} + K_2\varepsilon^{1-\lambda_2},$$

where v^* , u^* are given in (63) and (61) and therefore, by (62) and (72) one has:

$$|\varepsilon^{2/3}\eta(u^*; \varepsilon) - \varepsilon^{2/3}\eta_0(u^*)| \leq M\varepsilon^{1-\lambda_2} + K_1\varepsilon^{3\lambda_2} + K_2\varepsilon^{1-\lambda_2}, \quad (73)$$

therefore the solution given by Proposition 5 satisfies (73) at $u = u^*$ if $\lambda_2 < \min(1/4, \lambda_1)$.

Next proposition proves that any solution satisfying (72) at $u = u^*$, stays close to $\eta_0(u)$ until $u = 0$ which corresponds to $v = 1$.

Proposition 6. *Take any $0 < \lambda_2 < \frac{1}{4}$. Then, there exists $u_0 > 0$, $K > 0$, and $\varepsilon_0 = \varepsilon_0(u_0, K)$, such that for $|\varepsilon| \leq \varepsilon_0$, any solution of system (59) which enters the set*

$$\mathbf{B}_2 = \{(u, \eta), u^* \leq u \leq 0, |\eta(u) - \eta_0(u)| \leq K\mu M(u)\}$$

where $u^* = -\varepsilon^{\lambda_2 - \frac{1}{3}}$, $\mu = \varepsilon^{1/3}$, and the function $M(u)$ is defined by:

$$M(u) = \begin{cases} -u^3 & -\infty \leq u \leq -u_0 < 0 \\ u_0^3 & -u_0 \leq u \leq 0, \end{cases}$$

leaves it through the boundary $u = 0$.

Proof. We need to see that the vector field (59) points inwards in the three boundaries of \mathbf{B}_2 :

$$\mathbf{B}_2^\pm = \{(u, \eta), u^* \leq u \leq 0, \eta(u) = \eta_0(u) \pm K\mu M(u)\}, \text{ and } u = u^*.$$

The exterior normal vector to \mathbf{B}_2^+ is $n^+ = (1, -\eta'_0(u) - K\mu M'(u))$, and we have to check that:

$$E := \langle v, n^+ \rangle < 0, \text{ where } v = (\mu^2(1 + \varphi(1 + \mu u)), 1 + 2\mu^2\eta + \varphi(1 + \mu u)(2\mu^2\eta - 1)).$$

First observation is that

$$\begin{aligned} E &= \mu^2(1 + \varphi(1 + \mu u)) - E_2 \\ E_2 &= E_1(\eta'_0(u) + K\mu M'(u)) \\ E_1 &= 1 + 2\mu^2\eta + \varphi(1 + \mu u)(2\mu^2\eta - 1) = 1 - \varphi(1 + \mu u) + 2\mu^2(1 + \varphi(1 + \mu u))\eta \end{aligned}$$

We can develop E_1 by using the Taylor series of the function φ :

$$\varphi(1 + \mu u) = 1 + \frac{\varphi''(1)}{2}(\mu u)^2 + O((\mu u)^3)$$

and the fact that, in \mathbf{B}_2^+ , one has that $\eta(u) = \eta_0(u) + K\mu M(u)$, and (66), obtaining

$$E_1 = \frac{2\mu^2}{\eta'_0(u)} + 4K\mu^3 M(u) + g(u; \mu)$$

where $g(u; \mu)$ is exactly given by

$$g(u; \mu) = -\varphi(1 + \mu u) + 1 + \frac{\varphi''(1)}{2}(\mu u)^2 + 2\mu^2(\varphi(1 + \mu u) - 1)(\eta_0(u) + K\mu M(u)) \quad (74)$$

From the asymptotics of E_1 one easily obtains:

$$E_2 = 2\mu^2 + 4K\mu^3 M(u)\eta'_0(u) + \tilde{g}(u; \mu)$$

where

$$\tilde{g}(u; \mu) = \left(\frac{2\mu^2}{\eta'_0} + 4K\mu^3 M(u) + g(u; \mu) \right) K\mu M'(u) + g(u; \mu)\eta'_0(u) \quad (75)$$

and finally:

$$E = -4K\mu^3\eta'_0(u)M(u) + \bar{g}(u; \mu)$$

and

$$\bar{g}(u; \mu) = -\tilde{g}(u; \mu) + \mu^2(\varphi(1 + \mu u) - 1) \quad (76)$$

We need to bound the remainder $\bar{g}(u; \mu)$. Using the asymptotics for η_0 given in (70), we know that there exists $a > 1$, $C > 0$ such that:

$$\begin{aligned} |\eta_0(u)| &\leq Cu^2, & |\eta'_0(u)| &\leq Cu, & \text{if } u &\leq -a \\ |\eta_0(u)| &\leq C, & |\eta'_0(u)| &\leq C, & \text{if } -a &\leq u \leq 0. \end{aligned}$$

In the sequel we will take $u_0 > a$ and we denote by the letter C to any constant independent of u_0 , K . Also, we will use that, in the considered domain, $|\mu u| < 1$ and we can assume that $K > 1$.

Using these bounds for η_0 and (17) with $p = 2$, we can bound $g(u; \mu)$ as

$$\begin{aligned} u^* \leq u \leq -u_0 & \quad , & |g(u; \mu)| &\leq C(|\mu u|^3(1 + K|\mu u|^2)) \\ -u_0 \leq u \leq 0 & \quad , & |g(u; \mu)| &\leq C(\mu u_0)^3(1 + K|\mu u_0|^2) \end{aligned}$$

From this bound we obtain:

$$\begin{aligned} u^* \leq u \leq -u_0 & \quad , & |\tilde{g}(u; \mu)| &\leq C(\mu^3|u|^4 + \mu^3K|u| + \mu^4K^2|u|^5 + \mu^5K|u|^6 + \mu^6K^2|u|^7) \\ -u_0 \leq u \leq 0 & \quad , & |\tilde{g}(u; \mu)| &\leq C(\mu u_0)^3(1 + K|\mu u_0|^2) \end{aligned}$$

and for $\bar{g}(u; \mu)$:

$$\begin{aligned} u^* \leq u \leq -u_0 & \quad , & |\bar{g}(u; \mu)| &\leq C(\mu^3|u|^4 + \mu^3K|u| + \mu^4K^2|u|^5 + \mu^5K|u|^6 + \mu^6K^2|u|^7) \\ -u_0 \leq u \leq 0 & \quad , & |\bar{g}(u; \mu)| &\leq C(\mu u_0)^3(1 + K|\mu u_0|^2) + \mu^4u_0^2 \end{aligned}$$

Finally, one can write:

$$E = 4K\mu^3M(u)\eta'_0(u)(-1 + G(u; \mu))$$

where G is the function

$$G(u; \mu) = \frac{\bar{g}(u; \mu)}{4K\mu^3M(u)\eta'_0(u)} = \frac{\bar{g}(u; \mu)}{4K\mu^3M(u)}(8\eta_0(u) - \varphi''(1)u^2).$$

Using that

$$\begin{aligned} |4K\mu^3M(u)\eta'_0(u)| &\geq CK\mu^3u^4 & \text{if } u &\leq -u_0 \\ |4K\mu^3M(u)\eta'_0(u)| &\geq CK\mu^3u_0^3 & \text{if } -u_0 &\leq u \leq 0, \end{aligned}$$

the function G has the following bounds:

$$\begin{aligned} u^* \leq u \leq -u_0 & \quad , & |G(u; \mu)| &\leq C\left(\frac{1}{K} + \frac{1}{|u|^3} + \mu K|u| + \mu^2|u|^2 + \mu^3K|u|^3\right) \\ -u_0 \leq u \leq 0 & \quad , & |G(u; \mu)| &\leq C\left(\frac{1}{K} + \mu^2u_0^2 + \frac{\mu}{Ku_0}\right) \end{aligned}$$

and therefore, using that $\eta'_0(u)$ is a positive function for any $u \leq 0$ one can choose K and u_0 big enough in such a way that

$$\left|\frac{1}{K} + \frac{1}{u_0^3}\right| \leq \frac{1}{4C}$$

and then, using that $|\mu u| \leq |\mu u^*| = \varepsilon^{\lambda_2}$, E is negative if ε , and therefore $\mu = \varepsilon^{\frac{1}{3}}$, is small enough.

The proof for \mathbf{B}_2^- is analogous.

When $u = u^*$ one has that the flow of (59) satisfies $u' > 0$, therefore it also points inwards \mathbf{B}_2 . \square

As in \mathbf{B}_2 one has that $u' > 0$, the solutions which enter \mathbf{B}_2 leave it at $u = 0$. By (73), the invariant manifold $n(v; \varepsilon) = \varepsilon^{2/3} \eta(\frac{v-1}{\varepsilon^{1/3}})$, and therefore any solution $x(v; \varepsilon)$, enters in it at $v = v^*$ and it crosses the line $v = 1$ at a point satisfying:

$$x(1; \varepsilon) = \mu^2 \eta_0(0) + O(\mu^3) = \varepsilon^{2/3} \eta(0) + O(\varepsilon)$$

3.4.3. *Exponential attraction of the whole neighborhood of the fold.* Once we have a complete control of the Fenichel invariant manifold until it reaches the boundary $v = 1$ of our regularized system (36), now it is necessary to prove that this manifold attracts all the points in the section $\{(x, v), v = 1, -L \leq x \leq -\varepsilon^\lambda\}$ for $0 < \lambda < \frac{2}{3}$. This is an extension of the last item of Theorem 3.1. To this end, we need a better control of the manifold $\Lambda_\varepsilon = \{(x, v), v = m(x; \varepsilon)\}$ in this region.

As Λ_ε is invariant for (37), the $m(x; \varepsilon)$ satisfies:

$$1 + 2x + \varphi(m(x; \varepsilon))(2x - 1) = \varepsilon(1 + \varphi(m(x; \varepsilon)))m'(x; \varepsilon).$$

Writing: $m(x; \varepsilon) = m_0(x) + \varepsilon m_1(x) + \varepsilon^2 m_2(x) + \dots$ one gets:

$$\begin{aligned} m_0(x) &= \varphi^{-1}\left(\frac{1+2x}{1-2x}\right) \\ m_1(x) &= \frac{(1 + \varphi(m_0(x)))m'_0(x)}{\varphi'(m_0(x))(2x - 1)} = -\frac{1}{2}(m'_0(x))^2 \end{aligned} \quad (77)$$

where we have used the relation

$$\varphi'(m_0(x)) = \frac{4}{m'_0(x)(1-2x)^2}. \quad (78)$$

Observe that:

$$m_0(x) = \varphi^{-1}\left(\frac{1+2x}{1-2x}\right) = \varphi^{-1}(1 + 2x + 4x^2 + \dots).$$

Using that $\varphi(v) = 1 + \frac{\varphi''(1)}{2}(v-1)^2 + \mathcal{O}(v-1)^3$, $v \leq 1$, we obtain:

$$m_0(x) = 1 - \frac{2}{\sqrt{-\varphi''(1)}}\sqrt{|x|} + \mathcal{O}(x), \quad m_1(x) = \mathcal{O}\left(\frac{1}{x}\right), \quad x \leq 0. \quad (79)$$

Looking at these terms one can guess that the asymptotic expansion for $m(x; \varepsilon)$ will fail at $x = \mathcal{O}(\varepsilon^{2/3})$, which corresponds to $v = m_0(x) \simeq 1 + \mathcal{O}(\varepsilon^{1/3})$ as we saw in Proposition 5. This is given in next proposition.

Proposition 7. *Let $L > 0$ the constant given in Theorem 3.1 and $0 < \lambda < 2/3$. Then, there exists $K > 0$ and $\varepsilon_0 > 0$, such that, if $0 \leq \varepsilon < \varepsilon_0$ the invariant manifold $\Lambda_\varepsilon = \{(x, v), v = m(x; \varepsilon)\}$ satisfies, for $-L \leq x \leq -\varepsilon^\lambda$:*

$$m_0(x) + \frac{\varepsilon K}{x} \leq m(x; \varepsilon) \leq m_0(x). \quad (80)$$

Proof. We will see that the set

$$\tilde{B} = \{(x, v), -L \leq x \leq -\varepsilon^\lambda, \quad m_0(x) + \frac{\varepsilon K}{x} \leq v \leq m_0(x)\} \quad (81)$$

is positively invariant for system (37) checking that the flow points inwards in three of the borders of \tilde{B} .

In the border $\tilde{B}^+ = \{(x, v), -L \leq x \leq -\varepsilon^\lambda, \quad v = m_0(x)\}$, the vector field \tilde{Z}_ε in (37) is of the form $(\varepsilon(1 + \varphi(m_0(x)))/2, 0)$, and $1 + \varphi(m_0(x)) > 0$ therefore the flow points inward \tilde{B} along this border.

The exterior vector to the border $\tilde{B}^- = \{(x, v), -L \leq x \leq -\varepsilon^\lambda, v = m_0(x) + \frac{K\varepsilon}{x}\}$, is $n = (m'_0(x) - \frac{K\varepsilon}{x^2}, -1)$. It is enough to see that

$$\langle n, X \rangle_{|\tilde{B}^-} < 0$$

for $X = (\varepsilon(1 + \varphi(m_0(x) + \frac{K\varepsilon}{x})), 1 + 2x + \varphi(m_0(x) + \frac{K\varepsilon}{x})(2x - 1))$, which becomes:

$$(m'_0(x) - \frac{K\varepsilon}{x^2})\varepsilon[1 + \varphi(m_0(x) + \frac{K\varepsilon}{x})] - [1 + 2x + \varphi(m_0(x) + \frac{K\varepsilon}{x})(2x - 1)] < 0.$$

Taylor expanding the function φ one has that:

$$\varphi(m_0(x) + \frac{K\varepsilon}{x}) = \varphi(m_0(x)) + \varphi'(m_0(x))\frac{K\varepsilon}{x} + h(x; \varepsilon) \quad (82)$$

and our condition reads:

$$\varepsilon \left[m'_0(x)(1 + \varphi(m_0(x))) - \varphi'(m_0(x))\frac{K}{x}(2x - 1) \right] + M(x; \varepsilon) < 0 \quad (83)$$

where

$$\begin{aligned} M(x; \varepsilon) &= \varepsilon^2 m'_0(x) \varphi'(m_0(x)) \frac{K}{x} - \varepsilon^2 \frac{K}{x^2} [1 + \varphi(m_0(x))] \\ &\quad - \varepsilon^3 \frac{K^2}{x^3} \varphi'(m_0(x)) + \left(\varepsilon m'_0(x) - \frac{K\varepsilon^2}{x^2} - (2x - 1) \right) h(x; \varepsilon). \end{aligned}$$

Using (78) and that, by (79), there exist C_1, C_2 :

$$\begin{aligned} \frac{C_1}{\sqrt{|x|}} &\leq m'_0(x) \leq \frac{C_2}{\sqrt{|x|}}, \quad \text{for } -L \leq x < 0 \\ C_1 \sqrt{|x|} &\leq 1 - m_0(x) \leq C_2 \sqrt{|x|}, \end{aligned} \quad (84)$$

we obtain that, the $\mathcal{O}(\varepsilon)$ terms of (83) can be bounded, choosing K big enough depending on C_1, C_2 , and therefore on L :

$$\begin{aligned} &\varepsilon \left[m'_0(x)(1 + \varphi(m_0(x))) - \varphi'(m_0(x))\frac{K}{x}(2x - 1) \right] \\ &= \varepsilon \frac{2(m'_0(x))^2 x - 4K}{m'_0(x)(1 - 2x)x} \leq \frac{2C_2^2 - 4K}{C_1} \frac{\varepsilon}{\sqrt{|x|}} \leq -2 \frac{\varepsilon}{\sqrt{|x|}}. \end{aligned}$$

To end the proof we need to bound the higher order terms of (83) contained in $M(x; \varepsilon)$. Using again (78) and bounds (84), we obtain:

$$\begin{aligned} |\varepsilon^2 m'_0(x) \varphi'(m_0(x)) \frac{K}{x}| &\leq \varepsilon^2 \frac{4K}{(1 - 2x)^2 x} \leq 4K \varepsilon^{1 - \frac{1}{2}\lambda} \frac{\varepsilon}{\sqrt{|x|}}, \\ |\varepsilon^2 \frac{K}{x^2} [1 + \varphi(m_0(x))]| &\leq \frac{\varepsilon^2 2K}{x^2(1 - 2x)} \leq 2K \varepsilon^{1 - \frac{3}{2}\lambda} \frac{\varepsilon}{\sqrt{|x|}}, \\ |\varepsilon^3 \frac{K^2}{x^3} \varphi'(m_0(x))| &\leq \frac{4\varepsilon^3 K^2}{C_1(1 - 2x)^2 |x|^{5/2}} \leq \frac{4K^2 \varepsilon^{2 - 2\lambda}}{C_1} \frac{\varepsilon}{\sqrt{|x|}}. \end{aligned}$$

Finally, using that, for any $0 < \delta < 1$, there exists $C_3 > 0$ such that

$$|\varphi''(v)| \leq C_3 \quad \text{for } 0 < v \leq 1 - \delta$$

and using that, for ε small enough $|m_0(x) + \frac{K\varepsilon}{x} - 1| \leq \delta$ if $-L \leq x \leq -\varepsilon^\lambda$ and also (82), one has:

$$\left| \left(\varepsilon m'_0(x) - \frac{K\varepsilon^2}{x^2} - (2x-1) \right) h(x, \varepsilon) \right| \leq \left(\varepsilon^{1-\frac{1}{2}\lambda} C_2 + K\varepsilon^{2-2\lambda} + (2L+1) \right) C_3 K^2 \varepsilon^{1-\frac{3}{2}\lambda} \frac{\varepsilon}{\sqrt{|x|}}.$$

Finally, putting all these bounds together, one has that, if ε is small enough, we get

$$|M(x, \varepsilon)| \leq \frac{1}{2} \frac{\varepsilon}{\sqrt{|x|}}$$

and therefore

$$\langle n, X \rangle_{\tilde{B}^-} \leq \left(-2 + \frac{1}{2}\right) \frac{\varepsilon}{\sqrt{|x|}} \ll 0.$$

At the boundary $x = -L$ one has that $\dot{x} > 0$ therefore the flow points inward also in this border.

Now, we know that any orbit entering \tilde{B} stays in it until it reaches $x = -\varepsilon^\lambda$. But, by Theorem 3.1 and Remark 3 we know that the invariant manifold Λ_ε at $x = -L$ is given by

$$v = m(x; \varepsilon) = \overline{m_0}(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2)$$

and $m_1(x) < 0$. Therefore, adjusting the constants to have $K > Lm_1(-L)$, the manifold enters \tilde{B} and satisfies (80) for $-L \leq x \leq -\varepsilon^\lambda$. \square

Next step is to see that the manifold Λ_ε attracts all the solutions with initial conditions at points $(x_0, 1)$, if $-L \leq x_0 \leq -\varepsilon^\lambda$. Let's introduce the equation for the orbits of system (37):

$$\frac{\varepsilon dv}{dx} = \frac{1 + 2x + \varphi(v)(2x-1)}{1 + \varphi(v)} \quad (85)$$

Then, one has:

Proposition 8. *Fix $0 < \lambda < \frac{2}{3}$ and take any point $(x_0, 1)$, with $-L \leq x_0 \leq -\varepsilon^\lambda$. Then, the orbit of system (85) with initial condition $v(x_0) = 1$ stays exponentially close to the invariant manifold $v = m(x; \varepsilon)$ in the region $x_0 \leq x < -\varepsilon^{2/3}$.*

Proof. We perform the change of variables $w = v - m(x; \varepsilon)$ in equation (85) obtaining:

$$\varepsilon \frac{dw}{dx} = -g(x; \varepsilon) \varphi'(m(x; \varepsilon)) w - g(x; \varepsilon) F(x, w; \varepsilon) \quad (86)$$

where

$$F(x, w; \varepsilon) = \varphi(m(x; \varepsilon) + w) - \varphi(m(x; \varepsilon)) - \varphi'(m(x; \varepsilon))w.$$

and where $g(x; \varepsilon)$ is the positive function $g(x; \varepsilon) = \frac{-2x+1+\varepsilon m'(x; \varepsilon)}{1+\varphi(m(x; \varepsilon)+w(x; \varepsilon))}$.

Note that we already know the existence of the solution $w(x; \varepsilon) = 1 - m(x; \varepsilon)$ for $x_0 \leq x$, satisfying:

$$0 \leq w(x; \varepsilon) \leq 1 - m(x; \varepsilon).$$

For this reason we use the notation $g(x; \varepsilon)$ even if this function depends on $w(x; \varepsilon)$.

In the sequel, we will use the following expression for the function $F(x, w; \varepsilon)$:

$$F(x, w; \varepsilon) = A(x; \varepsilon)w, \quad A(x; \varepsilon) = \int_0^1 \varphi'(m(x; \varepsilon) + sw(x; \varepsilon)) ds - \varphi'(m(x; \varepsilon)) ds. \quad (87)$$

It is important to stress that as $w(x; \varepsilon) \geq 0$ and φ' is decreasing in the considered domain, the function $A(x; \varepsilon)$ is negative.

Clearly, the solution of (86) with initial condition $w(x_0) = 1 - m(x_0; \varepsilon)$ can be written as:

$$w(x) = e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(s; \varepsilon) \varphi'(m(s; \varepsilon)) ds} \tilde{w}(x; \varepsilon)$$

where:

$$\tilde{w}(x; \varepsilon) = \left[w(x_0) - \frac{1}{\varepsilon} \int_{x_0}^x e^{\frac{1}{\varepsilon} \int_{x_0}^\nu g(s; \varepsilon) \varphi'(m(s; \varepsilon)) ds} g(\nu; \varepsilon) F(\nu, w(\nu; \varepsilon)) d\nu \right].$$

Using (87) we obtain:

$$\begin{aligned} |\tilde{w}(x; \varepsilon)| &\leq |w(x_0)| + \frac{1}{\varepsilon} \int_{x_0}^x |g(\nu; \varepsilon) A(\nu; \varepsilon) \tilde{w}(\nu; \varepsilon)| d\nu \\ &= |w(x_0)| - \frac{1}{\varepsilon} \int_{x_0}^x g(\nu; \varepsilon) A(\nu; \varepsilon) |\tilde{w}(\nu; \varepsilon)| d\nu. \end{aligned}$$

Applying Gronwall's Lemma we get:

$$|\tilde{w}(x; \varepsilon)| \leq |w(x_0)| e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(\nu; \varepsilon) A(\nu; \varepsilon) d\nu}$$

and therefore

$$\begin{aligned} |w(x; \varepsilon)| &\leq |w(x_0)| e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(\nu; \varepsilon) (A(\nu; \varepsilon) + \varphi'(m(\nu; \varepsilon))) d\nu} \\ &= |w(x_0)| e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(\nu; \varepsilon) (\int_0^1 \varphi'(m(\nu; \varepsilon) + s w(\nu; \varepsilon)) ds) d\nu}. \end{aligned}$$

To bound this last expression we use the following facts:

- For $x \leq 0$ we have that $g(x; \varepsilon) \geq \frac{1}{2}$
- Given $0 < \delta < 1$, there exist constants c_1, c_2 , such that for $0 < v \leq 1 - \delta$ one has:

$$c_1(1 - v) \leq \varphi'(v) \leq c_2(1 - v).$$

Using (80) and (79), one has that $|m + sw - 1| \leq \delta$ and therefore we have, for $x \leq -\varepsilon^\lambda$:

$$\begin{aligned} |w(x; \varepsilon)| &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (\int_0^1 (1 - m(\nu; \varepsilon) + s w(\nu; \varepsilon)) ds) d\nu} \\ &= |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x ((1 - m(\nu; \varepsilon) + \frac{w(\nu; \varepsilon)}{2})) d\nu} \\ &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (1 - m(\nu; \varepsilon)) d\nu} \\ &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (1 - m_0(\nu)) d\nu} \leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} (|x_0|^{3/2} - |x|^{3/2})} \end{aligned}$$

where we have used (84). Now, these bounds guarantee that the solution $w(x; \varepsilon)$ exists for $x_0 < x \leq -\varepsilon^\lambda$ and has the same bounds. \square

3.4.4. Asymptotics for the Poincaré map P_ε . Fix $0 < \lambda < 2/3$. After Theorem 3.1 and propositions 5, 6 and 8, we can conclude that the Poincaré map \mathcal{Q}_ε is defined in the set $[-L, -\varepsilon^\lambda] \times \{\varepsilon\}$. Moreover, its Lipschitz constant is exponentially small and

$$\forall x \in [-L, -\varepsilon^\lambda], \quad \mathcal{Q}_\varepsilon(x) = \varepsilon^{2/3} \eta_0(0) + O(\varepsilon). \quad (88)$$

Taking into account that, by (13)

$$P^{-1}(-\varepsilon^\lambda) = x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})$$

we have that, calling $\mathcal{I} = [L^-, x_0^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + O(\varepsilon^{1+\lambda})]$,

$$P(\mathcal{I}) \subset [-L, -\varepsilon^\lambda]$$

where $P(L^-) = -L$.

On the other hand we know that the map \bar{P} is given by formulas (13).

Therefore we conclude that the map $P_\varepsilon = \bar{P} \circ \mathcal{Q}_\varepsilon \circ P$

$$\begin{aligned} P_\varepsilon : \mathcal{I} \times \{y = y_0\} &\rightarrow \mathcal{S}_{y_0}^+ \\ (x, y_0) &\mapsto (P_\varepsilon(x), y_0) \end{aligned} \quad (89)$$

is given by

$$P_\varepsilon(x) = \bar{P}(\varepsilon^{2/3}\eta_0(0) + O(\varepsilon)) = x_0^+ + \alpha^+\varepsilon + \beta^+(\eta_0(0))^2\varepsilon^{4/3} + O(\varepsilon^{5/3}).$$

Therefore, all the points in the interval \mathcal{I} are sent by P_ε to an interval \mathcal{J} which has, at most, size $\mathcal{O}(\varepsilon^{5/3})$ containing the point $x_0^+ + \alpha^+\varepsilon + \beta^+(\eta_0(0))^2\varepsilon^{4/3}$. Moreover, the Lipschitz constant of P_ε is exponentially small in ε .

3.5. The slow manifold close to $(0, 1)$: Smooth \mathbb{C}^{p-1} case. When the regularizing function φ is \mathbb{C}^{p-1} with $p \geq 3$, the critical manifold has the same qualitative behavior explained in the previous section. In this section we will stress the main quantitative differences between the \mathbb{C}^1 case and the general \mathbb{C}^{p-1} case.

By the same reasons explained to transform system (37) in equation (48), we deal with this equation and look for the Fenichel manifold and also for all the orbits close to it as graphs over the v variable. Then, the expansion of the solution

$$x = n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \dots + \mathcal{O}(\varepsilon^n)$$

is exactly the same as in (50) and (51) but now, the local behavior of the terms in this expansion is different. Without loss of generality we assume in this section that p is even and that $\varphi^{(p)}(1) < 0$. The case p odd is identically treated with $\varphi^{(p)}(1) > 0$.

We will have that, near $v = 1$, using that

$$\varphi(v) = 1 + \frac{\varphi^{(p)}(1)}{p!}(v-1)^p + \mathcal{O}((v-1)^{p+1}), \quad v \leq 1$$

one has

$$n_0(v) = \frac{\varphi^{(p)}(1)}{4p!}(v-1)^p + \mathcal{O}((v-1)^{p+1}) \quad (90)$$

$$n_1(v) = \mathcal{O}\left(\frac{1}{(1-v)^{p-1}}\right)$$

$$n_2(v) = \mathcal{O}\left(\frac{1}{(1-v)^{3p-2}}\right)$$

$$n_3(v) = \mathcal{O}\left(\frac{1}{(1-v)^{5p-3}}\right)$$

(91)

in general we have:

$$n_l(v) = \mathcal{O}\left(\frac{1}{(v-1)^{(2l-1)p-l}}\right)$$

therefore, the asymptotic expansion for $n(v; \varepsilon)$ close to $v = 1$ behaves as

$$n(v; \varepsilon) = \frac{\varphi^{(p)}(1)}{4p!}(v-1)^p + \mathcal{O}\left(\frac{\varepsilon}{(v-1)^{p-1}} + \frac{\varepsilon^2}{(v-1)^{3p-2}} + \dots + \frac{\varepsilon^l}{(v-1)^{(2l-1)p-l}}\right)$$

and this expansion loses its asymptotic character for

$$(v-1)^{2p-1} = \mathcal{O}(\varepsilon)$$

which indicates that the invariant manifold is close to $x = n_0(v)$ until $v = 1 - \mathcal{O}(\varepsilon^{\frac{1}{2p-1}})$. Next proposition, whose proof is completely analogous to Proposition 5, gives rigorously this behavior

Proposition 9. *Take any $0 < \lambda_1 < \frac{1}{2p-1}$. Then, there exists $M > 0$ big enough, $\delta = \delta(M) > 0$ small enough and $\varepsilon_0 = \varepsilon_0(M, \delta) > 0$, such that, for $0 \leq \varepsilon \leq \varepsilon_0$, any solution of system (36) which enters the set*

$$\mathbf{B}^{\mathbf{P}} = \left\{ (x, v), -\delta < v - 1 < -\varepsilon^{\lambda_1}, n_0(v) \leq x \leq n_0(v) + \frac{M\varepsilon}{|v-1|^{p-1}} \right\}$$

leaves it through the boundary $v = 1 - \varepsilon^{\lambda_1}$.

Then the invariant manifold Λ_ε , which is given by

$$x = n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \mathcal{O}(\varepsilon^2),$$

with $n_1(1 - \delta) > 0$, enters in the domain $\mathbf{B}^{\mathbf{P}}$ and it stays there at least until $v^* = 1 - \varepsilon^{\lambda_1}$ satisfying:

$$n_0(1 - \varepsilon^{\lambda_1}) < n(1 - \varepsilon^{\lambda_1}; \varepsilon) < n_0(1 - \varepsilon^{\lambda_1}) + M\varepsilon^{1-(p-1)\lambda_1}.$$

As the manifold attracts exponentially any solution beginning in U (see Theorem 3.1), all the solutions of the system satisfy the same inequality.

Moreover, as $n_0(1 - \varepsilon^\lambda) = \frac{\varphi^{(p)}(1)}{4p!} \varepsilon^{p\lambda} + \mathcal{O}(\varepsilon^{(p+1)\lambda})$ one has that, for any solution $x(v)$ beginning in U :

$$x(1 - \varepsilon^\lambda) = \frac{\varphi^{(p)}(1)}{4p!} \varepsilon^{p\lambda} + \mathcal{O}(\varepsilon^{(p+1)\lambda}, \varepsilon^{1-(p-1)\lambda})$$

for any $0 < \lambda \leq \lambda_1 < 1/(2p-1)$.

For $v = 1 - \mathcal{O}(\varepsilon^{1/(2p-1)})$, $n(v; \varepsilon) = \mathcal{O}(\varepsilon^{p/(2p-1)})$. Therefore, in this case, we perform the change:

$$\begin{aligned} v &= 1 + \varepsilon^{1/(2p-1)} u \\ x &= \varepsilon^{p/(2p-1)} \eta. \end{aligned}$$

The equation for the orbits (48) in these new variables is:

$$\frac{d\eta}{du} = \frac{\varepsilon^{p/(2p-1)}(1 + \varphi(1 + \varepsilon^{1/(2p-1)}u))}{(1 + 2\varepsilon^{p/(2p-1)}\eta(u) + \varphi(1 + \varepsilon^{1/(2p-1)}u)(2\varepsilon^{p/(2p-1)}\eta(u) - 1))}. \quad (92)$$

Calling $\mu = \varepsilon^{1/(2p-1)}$, one can write this equation as:

$$\frac{d\eta}{du} = \mu^p \frac{(1 + \varphi(1 + \mu u))}{(1 + 2\mu^p \eta(u) + \varphi(1 + \mu u)(2\mu^p \eta(u) - 1))}, \quad (93)$$

and we need to study the extension of a solution of this equation $\eta(u; \varepsilon)$, with initial condition $\eta(u^*; \varepsilon)$, with $u^* = \frac{v^*-1}{\varepsilon^{1/(2p-1)}} = -\varepsilon^{\lambda_2-1/(2p-1)}$, for $0 < \lambda_2 \leq \lambda_1 < 1/(2p-1)$, satisfying

$$|\varepsilon^{p/(2p-1)}\eta(u^*; \varepsilon) - n_0(v^*)| \leq M\varepsilon^{1-(p-1)\lambda_2} \quad (94)$$

where $v^* = 1 + \varepsilon^{1/(2p-1)}u^* = 1 - \varepsilon^{\lambda_2}$, to the domain:

$$u^* \leq u \leq 0. \quad (95)$$

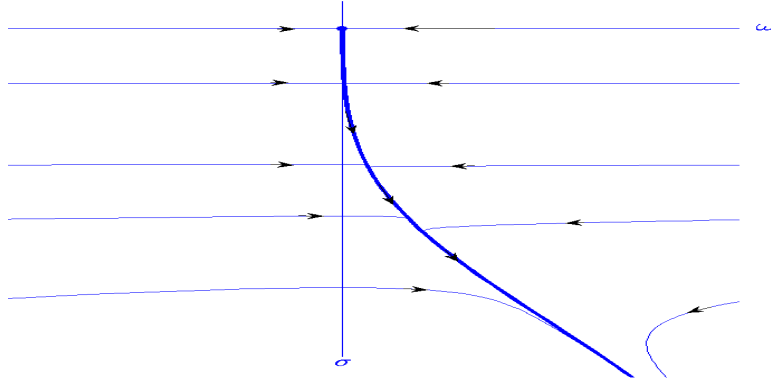


FIGURE 13. The branch in $\sigma < 0, \omega > 0$ of the central invariant manifold of system (100), which determines the behaviour of the p-Riccati equation (97) as $\bar{u} \rightarrow -\infty$.

Expanding the solution $\eta(u; \varepsilon)$ of equation (93) in powers of $\mu = \varepsilon^{1/(2p-1)}$, one can see that η_0 is the solution of the equation:

$$\eta_0' = \frac{d\eta_0}{du} = \frac{2}{4\eta_0 - \frac{\varphi^{(p)}(1)}{p!}u^p}. \quad (96)$$

We need to study equation (96) to obtain an analogous result as the one given in [21] for equation (66). With the changes of variables: $\bar{\eta} = \alpha\eta$, $\bar{u} = \beta u$, where

$$\alpha = 2^{\frac{p-2}{2p-1}} \left(-\frac{\varphi^{(p)}(1)}{p!} \right)^{\frac{1}{2p-1}}, \quad \beta = 2^{-\frac{3}{2p-1}} \left(-\frac{\varphi^{(p)}(1)}{p!} \right)^{\frac{2}{2p-1}}$$

it becomes

$$\frac{d\bar{\eta}}{d\bar{u}} = \frac{1}{\bar{\eta} + \bar{u}^p}. \quad (97)$$

Proposition 10. Equation (97) has a unique solution $\bar{\eta}_0(\bar{u})$ satisfying:

$$\bar{\eta}_0(\bar{u}) = -\bar{u}^p - \frac{1}{p} \frac{1}{\bar{u}^{p-1}} + O\left(\frac{1}{\bar{u}^{3p-2}}\right), \quad \bar{u} \rightarrow -\infty \quad (98)$$

Moreover, there exists a constant $K > \frac{1}{p}$ such that:

$$-\bar{u}^p < \bar{\eta}_0(\bar{u}) < -\bar{u}^p - \frac{K}{\bar{u}^{p-1}}, \quad \bar{u} \leq 0. \quad (99)$$

Proof. To prove this proposition we consider the vector field whose orbits are solutions of (97):

$$\begin{aligned} \dot{\bar{\eta}} &= 1 \\ \dot{\bar{u}} &= \bar{\eta} + \bar{u}^p \end{aligned}$$

for $u \leq 0$ and $\eta \leq 0$.

As the curve $\bar{\eta} + \bar{u}^p = 0$ is a isocline of slope zero, we will see that the region

$$\mathcal{B} = \{(\bar{u}, \bar{\eta}), -\bar{u}^p \leq \bar{\eta} \leq -\bar{u}^p - \frac{K}{\bar{u}^{p-1}}, \bar{u} < 0\}$$

is an isolating block in the region $\bar{u} < 0$ as $\bar{u} \rightarrow -\infty$. The boundary

$$\mathcal{B}^- = \{(\bar{u}, \bar{\eta}), \bar{\eta} = -\bar{u}^p, \bar{u} < 0\}$$

is positively invariant because the vector field is given by $(1, 0)$ and it points inwards \mathcal{B} . To see that \mathcal{B}^+ is also positively invariant we take the exterior normal vector $(1, p\bar{u}^{p-1} + K(1-p)\bar{u}^p)$ and we need to check that

$$\langle (1, \bar{\eta} + \bar{u}^p), (1, p\bar{u}^{p-1} + \frac{K(1-p)}{\bar{u}^p}) \rangle_{|\mathcal{B}^+} < 0,$$

that is:

$$1 - Kp - \frac{K^2(1-p)}{\bar{u}^{2p-1}} < 0.$$

As we are assuming that p is even, the term $\frac{K^2(1-p)}{\bar{u}^{2p-1}}$ is positive, therefore the expression above is negative if we take $K > \frac{1}{p}$.

To prove the existence of the solution $\bar{\eta}_0(\bar{u})$ we perform the changes:

$$w = \bar{\eta} + \bar{u}^p, \text{ and } \sigma = \frac{1}{\bar{u}}$$

obtaining:

$$\begin{aligned} w' &= 1 + \frac{p}{\sigma^{p-1}}w \\ \sigma' &= -\sigma^2 w \end{aligned}$$

for $w \geq 0$ and $\sigma \leq 0$. After a change of time (multiplying the equations by $-\sigma^{p-1}$) one obtains an equivalent system whose orbits are the same:

$$\begin{aligned} \frac{dw}{d\tau} &= -pw - \sigma^{p-1} \\ \frac{d\sigma}{d\tau} &= \sigma^{p+1}w \end{aligned} \quad (100)$$

whose equilibrium point $(0, 0)$ corresponds to the null-cline $\bar{\eta} + \bar{u}^p = 0$ at $\bar{u} = -\infty$. This equilibrium point is partially hyperbolic and the linearization of the vector field at $(0, 0)$ is given by

$$\begin{aligned} \frac{dw}{d\tau} &= -pw \\ \frac{d\sigma}{d\tau} &= 0. \end{aligned}$$

whose matrix has eigenvectors $(1, 0)$ and $(0, 1)$ associated to the eigenvalues $-p$ and 0 . One can apply to this point the Central Manifold Theorem [3] and we know that there exists a local invariant manifold which can be described by $\Lambda_c = \{(w, \sigma), w = g(\sigma)\}$ with $g(\sigma)$ a \mathbb{C}^∞ function, in a neighborhood of $\sigma = 0$ with $g(0) = g'(0) = 0$ and which satisfies:

$$0 = pg(\sigma) + \sigma^{p-1} + g(\sigma)g'(\sigma)\sigma^{p+1}, \quad \forall \sigma$$

which gives:

$$g(\sigma) = -\frac{1}{p}\sigma^{p-1} + \mathcal{O}(\sigma^{3p-2}).$$

On the central manifold Λ_c we have that

$$\sigma' = g(\sigma)\sigma^{p+1} = -\frac{1}{p}\sigma^{2p} + \mathcal{O}(\sigma^{4p-1}).$$

We see that, for $\sigma < 0$, the central manifold Λ_c is overflowing ($\sigma' < 0$) and therefore it is unique [22]. We conclude that there is a unique solution $(w_0(\tau), \sigma_0(\tau))$ in $\sigma < 0$ such that

$$(w_0(\tau), \sigma_0(\tau)) \rightarrow (0, 0) \text{ as } \tau \rightarrow -\infty.$$

The situation is summarized in figure 13. Going back to the original variables $(\bar{\eta}, \bar{u})$, we get that the unique central manifold enters the region $\{(\bar{\eta}, \bar{u}), \bar{\eta} + \bar{u}^p > 0, \bar{u} < 0\}$.

Moreover, it has the asymptotic expression:

$$\bar{\eta}_0 = -\bar{u}^p - \frac{1}{p}\bar{u}^{1-p} + \mathcal{O}(\bar{u}^{2-3p})$$

but this solution for \bar{u} near $-\infty$ is inside the block \mathcal{B} , and we have seen that this block is positively invariant for the flow if $K > \frac{1}{p}$. Therefore, if K is big enough, the central manifold remains \mathcal{B} until $\bar{u} \leq 0$. \square

Remark 4. If $p \geq 3$ is odd, the equivalent equation to (97) is:

$$\frac{d\bar{\eta}}{d\bar{u}} = \frac{1}{\bar{\eta} - \bar{u}^p},$$

and the block is given by

$$\mathcal{B} = \{(\bar{u}, \bar{\eta}), \bar{u}^p \leq \bar{\eta} \leq \bar{u}^p - \frac{K}{\bar{u}^{p+1}}, \bar{u} < 0\}$$

and is isolating for $K > \frac{1}{p}$. The obtained solution satisfies:

$$\bar{\eta}(\bar{u}) = \bar{u}^p + \frac{1}{p} \frac{1}{\bar{u}^{p-1}} + \mathcal{O}\left(\frac{1}{\bar{u}^{3p-2}}\right), \quad \bar{u} \rightarrow -\infty.$$

From Proposition 10, and using that: $\frac{2\alpha}{\beta} = \frac{4}{\alpha} = -\frac{\varphi^{(p)}(1)}{p!\beta^p}$, we obtain that

$$\frac{4\beta^p}{\alpha} = 2\alpha\beta^{p-1} = \frac{\varphi^{(p)}(1)}{p!},$$

going back to our variables one has that $\eta_0(u)$ satisfies:

$$\begin{aligned} \eta_0(u) &= \frac{\varphi^{(p)}(1)}{4p!}u^p + \frac{2(p-1)!}{\varphi^{(p)}(1)}u^{1-p} + \mathcal{O}(u^{2-3p}), \quad u \rightarrow -\infty \\ \frac{\varphi^{(p)}(1)}{4p!}u^p &< \eta_0(u) < \frac{\varphi^{(p)}(1)}{4p!}u^p + \frac{2K(p-1)!}{\varphi^{(p)}(1)}u^{1-p}, \quad u \leq 0 \end{aligned}$$

with $k > \frac{1}{p}$. As a consequence of this expansion and the asymptotic expansion of $n_0(v)$ near $v = 1$ given in (90), one has that there exist constants K_1, K_2 , such that

$$|\varepsilon^{p/(2p-1)}\eta_0(u^*) - n_0(v^*)| \leq K_1\varepsilon^{\lambda_2(p+1)} + K_2\varepsilon^{1-\lambda_2(p-1)},$$

and therefore, by (94) one has, as in (73):

$$|\varepsilon^{p/(2p-1)}\eta(u^*; \varepsilon) - \varepsilon^{p/(2p-1)}\eta_0(u^*)| \leq M\varepsilon^{1-\lambda_2(p-1)} + K_1\varepsilon^{\lambda_2(p+1)} + K_2\varepsilon^{1-\lambda_2(p-1)}, \quad (101)$$

and we can conclude that the solution given by Proposition 9 satisfies (101) at $u = u^*$ if we take $\lambda_2 < \min(1/2p, \lambda_1)$.

On the other hand, if one consider the next term in the expansion of $\eta(u, \varepsilon)$, one has:

$$\eta(u; \varepsilon) = \eta_0(u) + \mu\eta_1(u) + \mathcal{O}(\mu^2)$$

where $\eta_1(u)$ is the solution of the equation:

$$\eta_1'(u) = -\frac{8}{\left(4\eta_0(u) - \frac{\varphi^{(p)}(1)}{(p)!}u^p\right)^2}\eta_1 + \frac{2\frac{\varphi^{(p+1)}(1)}{(p+1)!}u^{p+1}}{\left(4\eta_0(u) - 2\frac{\varphi^{(p)}(1)}{(p)!}u^p\right)^2}$$

and one can see that the solution η_1 near $-\infty$ behaves as:

$$\eta_1(u) \simeq \frac{\varphi^{(p+1)}(1)}{4(p+1)!}u^{p+1}, \quad u \rightarrow -\infty$$

and one can see the next proposition, whose proof is analogous to Proposition 6:

Proposition 11. *Take any $0 < \lambda_2 < \frac{1}{2p}$. Then, there exists $u_0 > 0$, $K > 0$, and $\varepsilon_0 = \varepsilon_0(u_0, K)$, such that for $0 < \varepsilon \leq \varepsilon_0$, the set*

$$\mathbf{B}_2^p = \{(u, \eta), u^* \leq u \leq 0, \quad |\eta(u) - \eta_0(u)| \leq \bar{K}\mu M(u)\}$$

where $u^* = -\varepsilon^{\lambda_2 - \frac{1}{2p-1}}$, and $M(u)$ is the function defined by:

$$M(u) = \begin{cases} -u^{p+1} & -\infty \leq u \leq -u_0 < 0 \\ u_0^{p+1} & -u_0 \leq u \leq 0 \end{cases}$$

and $\mu = \varepsilon^{1/(2p-1)}$, is an isolating block for equation (92).

Once we have that \mathbf{B}_2^p is an isolating block and that, by (101), the solution $x(v, \varepsilon)$ enters in it at $v = v^*$ we have that our solution crosses the line $v = 1$ at a point satisfying:

$$x(1; \varepsilon) = \mu^p \eta_0(0) + O(\mu^{p+1}) = \varepsilon^{p/(2p-1)} \eta_0(0) + O(\varepsilon^{(p+1)/(2p-1)})$$

3.5.1. *Exponential attraction of the whole neighborhood of the fold.* As we did in Section 3.4.3 we now see that the invariant manifold attracts all the points in the section $\{(x, v), v = 1, -L \leq x \leq -\varepsilon^\lambda\}$ for $0 < \lambda < \frac{p}{2p-1}$. We point out the main differences in this case. The expansion

$$m(x; \varepsilon) = m_0(x) + \varepsilon m_1(x) + \varepsilon^2 m_2(x) + \dots$$

behaves now as

$$m_0(x) = \varphi^{-1}\left(\frac{1+2x}{1-2x}\right) = \varphi^{-1}(1+2x+4x^2+\dots)$$

and using that $\varphi(v) = 1 + \frac{\varphi^{(p)}(1)}{p!}(v-1)^p + \mathcal{O}(v-1)^{p+1}$ we obtain:

$$m_0(x) = 1 + \mathcal{O}(|x|^{\frac{1}{p}}), \quad m_1(x) = \mathcal{O}(|x|^{\frac{2(1-p)}{p}})$$

Looking at these terms one can guess that the asymptotic expansion for $m(x; \varepsilon)$ will fail at $x = \mathcal{O}(\varepsilon^{\frac{p}{2p-1}})$.

Proposition 12. *Consider $-L < -N < 0$ and $0 < \lambda < \frac{p}{2p-1}$. Then, there exists $K > 0$ and $\varepsilon_0 > 0$, such that, if $0 \leq \varepsilon < \varepsilon_0$ the invariant manifold $v = m(x; \varepsilon)$ satisfies, for $-L \leq x \leq -\varepsilon^\lambda$:*

$$m_0(x) + \frac{\varepsilon K}{x^{\frac{2p-2}{p}}} \leq m(x; \varepsilon) \leq m_0(x) \quad (102)$$

Proof. The proof follows the same lines that Proposition 7, proving that the set

$$\tilde{B} = \{(x, v), -L \leq x \leq -\varepsilon^\lambda, \quad m_0(x) + \frac{\varepsilon K}{x^{\frac{2p-2}{p}}} \leq m(x; \varepsilon) \leq m_0(x)\} \quad (103)$$

is positively invariant for system (36). Now, instead of (84), we will use (78), which gives that there exist C_1, C_2 :

$$\frac{C_1}{|x|^{\frac{1}{p}}} \leq m'_0(x) \leq \frac{C_2}{|x|^{\frac{1}{p}}}, \quad \text{for } -L \leq x < 0. \quad (104)$$

□

Next step is to see that the manifold Λ_ε attracts all the solutions with initial conditions at points $(x_0, 1)$, if $-L \leq x_0 \leq -\varepsilon^\lambda$.

Proposition 13. Fix $0 < \lambda < \frac{p}{2p-1}$ and take any point $(x_0, 1)$, with $-L \leq x_0 \leq -\varepsilon^\lambda$. Then, the orbit of system (85) with initial condition $v(x_0) = 1$ stays exponentially close to the invariant manifold $v = m(x; \varepsilon)$ in the region $x_0 \leq x < -\varepsilon^{\frac{p}{2p-1}}$.

Proof. The proof of this proposition is also similar to Proposition 8, performing the change of variables $w = v - m(x; \varepsilon)$ in equation (48) and using Gronwall's Lemma to bound w we arrive to:

$$|w(x; \varepsilon)| \leq |w(x_0)| e^{-\frac{1}{\varepsilon} \int_{x_0}^x g(\nu; \varepsilon) (f_0^1 \varphi'(m(\nu; \varepsilon) + sw(\nu; \varepsilon)) ds) d\nu}$$

To bound this last expression we use the following facts:

- For $x \leq 0$ we have that $g(x; \varepsilon) \geq \frac{1}{2}$
- Given $0 < \delta < 1$, there exist constants c_1, c_2 , such that for $|v - 1| \leq \delta$ one has:

$$c_1(1 - v)^{p-1} \leq \varphi'(v) \leq c_2(1 - v)^{p-1}$$

Obtaining:

$$\begin{aligned} |w(x; \varepsilon)| &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (f_0^1 (1 - m(\nu; \varepsilon) + sw(\nu; \varepsilon))^{p-1} ds) d\nu} \\ &= |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x ((1 - m(\nu; \varepsilon) + \frac{w(\nu; \varepsilon)}{2})^{p-1} d\nu} \\ &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (1 - m(\nu; \varepsilon))^{p-1} d\nu} \\ &\leq |w(x_0)| e^{-\frac{c_1}{2\varepsilon} \int_{x_0}^x (1 - m_0(\nu))^{p-1} d\nu} |w(x_0)| e^{-\frac{c_1}{2\varepsilon} (|x_0|^{\frac{2p-1}{p}} - |x|^{\frac{2p-1}{p}})} \end{aligned}$$

□

And then, if $x_0 < x \leq -\varepsilon^{\frac{p}{2p-1}}$ the orbits gets exponentially close to the invariant manifold.

3.5.2. *Asymptotics for the Poincaré map P_ε .* Fix $0 < \lambda < \frac{p}{2p-1}$. After Theorem 3.1 and propositions 9, 11 and 13, we can conclude that the Poincaré map \mathcal{Q}_ε is defined in the set $[-L, -\varepsilon^\lambda] \times \{\varepsilon\}$. Moreover, its Lipschitz constant is exponentially small in ε and

$$\forall x \in [-L, -\varepsilon^\lambda], \quad \mathcal{Q}_\varepsilon(x) = \varepsilon^{\frac{p}{2p-1}} \eta_0(0) + \mathcal{O}(\varepsilon^{\frac{p+1}{2p-1}}). \quad (105)$$

Taking into account that, by (13)

$$P^{-1}(-\varepsilon^\lambda) = x^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})$$

we have that

$$P(\mathcal{I}) \subset [-L, -\varepsilon^\lambda]$$

where $\mathcal{I} = [L^-, x^- + \alpha^- \varepsilon + \beta^- \varepsilon^{2\lambda} + \mathcal{O}(\varepsilon^{1+\lambda})]$ and $L^- = P^{-1}(-L)$.

On the other hand we know that the map \bar{P} is given by formulas (13).

Therefore we conclude that the map $P_\varepsilon = \bar{P} \circ \mathcal{Q}_\varepsilon \circ P$

$$\begin{aligned} P_\varepsilon : \mathcal{I} \times \{y = y_0\} &\rightarrow \mathcal{S}_{y_0}^+ \\ (x, y_0) &\mapsto (P_\varepsilon(x), y_0) \end{aligned} \quad (106)$$

is given by

$$P_\varepsilon(x) = \bar{P}(\varepsilon^{\frac{p}{2p-1}} \eta_0(0) + \mathcal{O}(\varepsilon^{\frac{p+1}{2p-1}})) = x^+ + \alpha^+ \varepsilon + \beta^+ (\eta_0(0))^2 \varepsilon^{\frac{2p}{2p-1}} + \mathcal{O}(\varepsilon^{\frac{2p+1}{2p-1}}).$$

Therefore, all the points in the set $\mathcal{I} \times \{y_0\}$ are send by P_ε to a set $\mathcal{J} \times \{y_0\}$ and the interval \mathcal{J} has, at most, size $\mathcal{O}(\varepsilon^{\frac{2p+1}{2p-1}})$ containing the point $x^+ + \alpha^+ \varepsilon + \beta^+ (\eta_0(0))^2 \varepsilon^{\frac{2p}{2p-1}}$.

Moreover, the Lipschitz constant of P_ε is exponentially small in ε .

Remark 5. The results of Theorem 2.1 and Proposition 1 lead to the classical result that, independently of the regularity of φ , the Poincaré map P_ε is Lipschitz with exponentially small Lipschitz constant. But we also obtain that the Poincaré map P_ε has a domain of attraction which includes a region at distance $\mathcal{O}(\varepsilon^\lambda)$ to the stable pseudoseparatrix $W_\mp^s(0,0)$ of the fold. This is an improvement of previous results and will be crucial to obtain the statements of Section 2.2.

3.6. The general fold. In the previous sections we have rigorously computed the Poincaré map P_ε on the sections $\mathcal{S}_{y_0}^\pm$ as a composition of three maps:

$$P_\varepsilon = P \circ Q_\varepsilon \circ \bar{P}$$

The maps P and \bar{P} were studied for a generic vector field X^+ having a tangency point at $(0,0)$ in Proposition 1 giving formulas (13), but the singular map Q_ε was computed using singular perturbation theory in a simplified vector $Z = (X^+, X^-)$ in (32), (33), coming from a normal form in [9]. Nevertheless, as our method needs differentiability properties, we can not claim that the results obtained are automatically valid for any Filippov vector field with a regular-fold visible point. For this reason, in this section we will consider the case of a general vector field and we will point out the main technicalities to obtain the same result as in (88).

So, let us assume that we have the non smooth system (1), and we assume that X^+ has a visible fold at $(0,0)$ and X^- is pointing towards Σ . Assume also that conditions (2),(3), (4) are verified. The first simplification of the vector field Z is provided by the classical Flow-box Theorem applied to the vector X^- . Applying the change of variables to both vector fields defining Z , we obtain:

Proposition 14. *There exists a smooth change of variables $(x, y) = \hat{\psi}(\hat{x}, \hat{y})$, where $\hat{\psi} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying $\hat{\psi}(\hat{x}, 0) = (\hat{x}, 0)$, such that, if we call $\hat{Z}(\hat{x}, \hat{y}) = \hat{\psi}^* Z(\hat{x}, \hat{y}) = (D\hat{\psi}(\hat{x}, \hat{y}))^{-1} Z \circ \hat{\psi}(\hat{x}, \hat{y})$ to the transformed vector field, one has $\hat{Z} = (\hat{X}^+, \hat{X}^-)$, and*

- $\hat{X}^- = (0, 1)^t$
- $\hat{X}^+ = (1 + O_1(\hat{x}, \hat{y}), 2\hat{x} + \hat{b}\hat{y} + O_2(\hat{x}, \hat{y}))^t$, and $O_2(\hat{x}, 0) = 0$.

Proof. The first part of the proof consists in applying the Flow-box Theorem to the vector field X^- . This theorem provides a smooth change of variables $(x, y) = \psi(\tilde{x}, \tilde{y})$, where $\psi = (\psi_1, \psi_2)$, that transforms the vector field X^- into $\tilde{X}^- = (0, 1)^t$. One can also ask the function ψ to leave invariant a transversal manifold of the flow, that we choose to be Σ . Therefore this map satisfies $\psi(\tilde{x}, 0) = (\tilde{x}, 0)$ and, consequently, $\frac{\partial \psi_1}{\partial \tilde{x}}(\tilde{x}, 0) = 1$, and $\frac{\partial \psi_2}{\partial \tilde{x}}(\tilde{x}, 0) = 0$. Moreover, as

$$D\psi(0, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} X_1^-(0, 0) \\ X_2^-(0, 0) \end{pmatrix}$$

one has that $\frac{\partial \psi_2}{\partial \tilde{y}}(0, 0) = X_2^-(0, 0) > 0$. Now using that $D\psi(\tilde{x}, 0)\tilde{X}^+(\tilde{x}, 0) = X^+(\tilde{x}, 0)$, one obtains that

$$\tilde{X}^+(0, 0) = \begin{pmatrix} c \\ 0 \end{pmatrix},$$

with $c = X_1^+(0, 0) \neq 0$. Moreover,

$$\tilde{X}_2^+(\tilde{x}, 0) = \left(\frac{\partial \psi_2}{\partial \tilde{y}}\right)^{-1}(\tilde{x}, 0)X_2^+(\tilde{x}, 0)$$

therefore the tangency at $(0, 0)$ is preserved and visible. Once we have applied the Flow box Theorem, the new vector field \tilde{X}^+ has the form

$$\tilde{X}^+ = \begin{pmatrix} c + O_1(\tilde{x}, \tilde{y}) \\ a\tilde{x} + b\tilde{y} + O_2(\tilde{x}, \tilde{y}) \end{pmatrix},$$

where $a = \frac{\partial_x X_2^+(0,0)}{X_2^-(0,0)} > 0$ and $c = X_1^+(0,0) \neq 0$. Now, the change of variables and time:

$$\bar{x} = \frac{a}{2}\tilde{x}, \quad \bar{y} = \frac{ac}{2}\tilde{y}, \quad \tau = \frac{ac}{2}t$$

transforms the vector field \tilde{Z} into \bar{Z} with $\bar{X}^- = \tilde{X}^-$ and:

$$\bar{X}^+ = \begin{pmatrix} 1 + O_1(\bar{x}, \bar{y}) \\ 2\bar{x} + b\bar{y} + O_2(\bar{x}, \bar{y}) \end{pmatrix}.$$

To perform the last change, we observe that the second order terms in the second component of \bar{X}^+ can be separated:

$$O_2(\bar{x}, \bar{y}) = f_2(\bar{x}) + g_2(\bar{x}, \bar{y}), \quad g_2(\bar{x}, 0) = 0$$

then, our last change is

$$\hat{x} = \bar{x} + \frac{1}{2}f_2(\bar{x}).$$

This change is well defined in a neighborhood of zero and leaves the vector field \bar{X}^- invariant but changes \bar{X}^+ into:

$$\bar{X}^+ = \begin{pmatrix} 1 + O_1(\hat{x}, \hat{y}) \\ 2\hat{x} + b\hat{y} + O_2(\hat{x}, \hat{y}) \end{pmatrix},$$

but the term $O_2(\hat{x}, \hat{y})$ vanishes at $y = 0$ for any value of \hat{x} . \square

This proposition allows us to assume that we have a Filippov vector field $Z = (X^+, X^-)$ where:

$$X^+(x, y) = \begin{pmatrix} 1 + f_1(x, y) \\ 2x + by + f_2(x, y) \end{pmatrix} \quad (107)$$

where $f_i(x, y) = O_i(x, y)$ and $f_2(x, 0) = 0$, and

$$X^-(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (108)$$

The system given by X^+ has a visible fold at $(0, 0)$ and X^- is regular at this point. Therefore $(0, 0)$ is a fold-regular point for Z . Moreover, it satisfies conditions (2), (3), (4).

The regularized system (35) will be in the general case:

$$\begin{aligned} \dot{x} &= \frac{1}{2}(1 + \varphi(\frac{y}{\varepsilon}))(1 + f_1(x, y)) \\ \dot{y} &= \frac{1+2x+by+f_2(x,y)}{2} + \frac{1}{2}\varphi(\frac{y}{\varepsilon})(2x + by - 1 + f_2(x, y)), \end{aligned} \quad (109)$$

and, in the variable $v = \frac{y}{\varepsilon}$ we obtain:

$$\begin{aligned} \dot{x} &= \frac{1+\varphi(v)}{2}(1 + f_1(x, \varepsilon v)) \\ \varepsilon \dot{v} &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x - 1) + \frac{1+\varphi(v)}{2}(b\varepsilon v + f_2(x, \varepsilon v)). \end{aligned} \quad (110)$$

Observe that the slow system for $\varepsilon = 0$ is given by:

$$\begin{aligned} \dot{x} &= \frac{1+\varphi(v)}{2}(1 + f_1(x, 0)) \\ 0 &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x - 1) \end{aligned} \quad (111)$$

and therefore the critical manifold Λ_0 is given in the general case by the same equation (38) and the $D\tilde{Z}_0$ (see (37)) is exactly given by (39). Consequently it has the same hyperbolicity properties and Fenichel Theorem 3.1 can also be applied in the general case giving the existence of the invariant manifold given by $\Lambda_\varepsilon = \{(x, v), v = m(x; \varepsilon)\}$ and also by $\Lambda_\varepsilon = \{(x, v), x = n(v; \varepsilon)\}$ in the corresponding domains.

To study the invariant manifold near $(0, 1)$ we can proceed as we did in Section 3.4 and equation (48), regarding also the system (110) as an equation in the v variable. The only thing is to assure that the Fenichel manifold and nearby orbits are in the region $\dot{v} > 0$. This is performed just down. Expanding the orbits $x = n(v; \varepsilon) = n_0(v) + \varepsilon n_1(v) + \dots$ and using that $n_0(v)$ is given by (38), some easy computations give that

$$n_1(v) = \frac{1}{2} \left(\frac{1 + f_1(n_0(v), 0)}{n_0'(v)} - bv - \frac{\partial f_2}{\partial y}(n_0(v), 0)v \right).$$

Even if, in the general case, the term $n_1(v)$ is different from (51), the behavior near $v = 1$ is the same as in (53). Therefore the behavior of the slow manifold near $v = 1$ is also given in (52), (53) and one can easily prove Proposition 5 in the general fold case. But now $x = n_0(v)$ is no longer a isocline of zero slope. Nevertheless, the flow in \mathbf{B}^- also points inward \mathbf{B} . Moreover, to ensure that the Fenichel manifold not only enters in the block \mathbf{B} when $v = 1 - \delta$ but exits it for $v = 1 - \varepsilon^\lambda$, $0 < \lambda < \frac{1}{3}$, it is enough to see that $n(1 - \delta; \varepsilon) > \bar{n}(1 - \delta; \varepsilon)$ where $x = \bar{n}(v; \varepsilon)$ is the expression of the isocline of slope zero given by:

$$\frac{1 + 2x + b\varepsilon v + f_2(x, \varepsilon v)}{2} + \frac{1}{2}\varphi(v)(2x - 1 + b\varepsilon v + f_2(x, \varepsilon v)) = 0.$$

To see this, we observe that

$$\bar{n}(v; \varepsilon) = n_0(v) + \varepsilon \bar{n}_1(v) + \mathcal{O}(\varepsilon^2)$$

with $\bar{n}_1(v) = -\frac{v}{2}(b + \frac{\partial f_2}{\partial y}(n_0(v), 0))$, therefore:

$$n_1(v) - \bar{n}_1(v) = \frac{1}{2} \frac{1 + f_1(n_0(v), 0)}{n_0'(v)}.$$

Now, using that $f_1(x, y) = \mathcal{O}(x, y)$ and that $n_0(v) = \frac{\varphi''(1)}{8}(v-1)^2 + \mathcal{O}((v-1)^3)$ near $v = 1$, in a neighborhood of $(0, 1)$ and that $n_0'(v) > 0$ (see (50)) we have that

$$n_1(v) - \bar{n}_1(v) > 0$$

and then $n(v; \varepsilon) > \bar{n}(v; \varepsilon)$.

Therefore the Fenichel manifold enters the region $\dot{v} > 0$ and can not leave it. Also $n_1(v) > 0$, and the Fenichel manifold enters inside the block \mathbf{B} by $v = 1 - \delta$ and exits it at $v = 1 - \varepsilon^\lambda$, with $0 < \lambda < 1/3$.

When $v = 1 - \mathcal{O}(\varepsilon^{1/3})$ we proceed as usual, and the change (58) transforms equations (110) into:

$$\begin{aligned} \varepsilon^{-\frac{1}{3}}\dot{\eta} &= \frac{1 + \varphi(1 + \varepsilon^{\frac{1}{3}}u)}{2} (1 + f_1(\varepsilon^{\frac{2}{3}}\eta, \varepsilon(1 + \varepsilon^{\frac{1}{3}}u))) \\ \varepsilon^{\frac{1}{3}}\dot{u} &= \frac{1 + 2\varepsilon^{\frac{2}{3}}\eta}{2} + \frac{1}{2}\varphi(1 + \varepsilon^{\frac{1}{3}}u)(2\varepsilon^{\frac{2}{3}}\eta - 1) \\ &+ \frac{1 + \varphi(1 + \varepsilon^{\frac{1}{3}}u)}{2} (b\varepsilon(1 + \varepsilon^{\frac{1}{3}}u) + f_2(\varepsilon^{\frac{2}{3}}\eta, \varepsilon(1 + \varepsilon^{\frac{1}{3}}u))). \end{aligned} \quad (112)$$

Considering the equation for the orbits, calling $\mu = \varepsilon^{\frac{1}{3}}$ and expanding $\eta(u) = \eta_0(u) + \mu\eta_1(u) + \mathcal{O}(\mu^2)$ one obtains, for η_0 the same equation (66). For η_1 , it

appears a new term instead:

$$\eta_1' = -\frac{8}{(4\eta_0 - \frac{\varphi''(1)}{2}u^2)^2}\eta_1 + \frac{\varphi''(1)u^3}{3(4\eta_0 - \frac{\varphi''(1)}{2}u^2)^2} + \frac{2(b + \frac{\partial f_2}{\partial y}(0,0))}{(4\eta_0 - \frac{\varphi''(1)}{2}u^2)^2}$$

Nevertheless the asymptotic behavior at $-\infty$ is the same as (71):

$$\eta_1 \simeq \frac{\varphi''(1)}{24}u^3 + \mathcal{O}(u^4), \quad u \rightarrow -\infty,$$

then, Proposition 6 also works, and we will arrive at $v = 1$ having:

$$x(1; \varepsilon) = \varepsilon^{2/3}\eta_0(0) + \mathcal{O}(\varepsilon).$$

To see that the Fenichel manifold attracts points near $(0, 1)$, concretely points of the section $\{(x, v), v = 1, -L \leq x \leq -\varepsilon^\lambda\}$, $0 < \lambda < 2/3$, we proceed as we did in Section 3.4.3 proving propositions 7 and 8. The only thing to bear in mind, as Remark 3 does, is that, in spite $v = m_0(x)$ is no longer a isocline of slope zero, the inequality

$$m(x; \varepsilon) < m_0(x)$$

also is satisfied if the constant L appearing in Fenichel Theorem 3.1 is small enough, but fixed. The reason is that the term $m_1(x)$ in the expansion of the Fenichel manifold:

$$m(x; \varepsilon) = m_0(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2)$$

is

$$m_1(x) = -\frac{2(1 + \varphi(m_0(x)))^2(1 + f_1(x, 0))}{(\varphi'(m_0(x))(2x - 1))^2} - \frac{1}{2} \frac{(1 + \varphi(m_0(x)))m_0(x)}{\varphi'(m_0(x))(2x - 1)} \left(b + \frac{\partial f_2}{\partial y}(x, 0)\right)$$

and we know that $f_1(x, y) = \mathcal{O}(x, y)$, therefore, for x near zero, the dominant term in this expression is

$$-\frac{2(1 + \varphi(m_0(x)))^2}{(\varphi'(m_0(x))(2x - 1))^2} < 0$$

in this region. So we can ensure that $m(x; \varepsilon) < m_0(x)$. On the other hand, if we consider the isocline of zero slope $v = \bar{m}(x; \varepsilon)$ defined by:

$$1 + 2x + \varphi(v)(2x - 1) + (1 + \varphi(v))(b\varepsilon v + f_2(x, \varepsilon v)) = 0$$

one obtains that

$$\bar{m}(x; \varepsilon) = m_0(x) - \frac{\varepsilon}{2} \frac{(1 + \varphi(m_0(x)))m_0(x)}{\varphi'(m_0(x))(2x - 1)} \left(b + \frac{\partial f_2}{\partial y}(x, 0)\right) + \mathcal{O}(\varepsilon^2),$$

and therefore we also have $m(x, \varepsilon) < \bar{m}(x, \varepsilon)$. With all these considerations, one can prove propositions 7 and 8 for the general fold case, obtaining the same formulas (89) for the Poincaré map P_ε in this case.

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