# Multiresolution Families of Indistinguishability Operators

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**Abstract**

Multiresolution is a general mathematical concept that allows us to study a property by means of several changes of resolution. From a fixed resolution, a coarser projection can be calculated and then the changes between a finer resolution and a coarser one can be studied. That information can give a good knowledge about the problem under consideration. Also using multiresolution techniques it is possible to present information with a higher or a lower detail, given a way to get the adequate granularity or abstraction for a context.

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The granularity of a system can be obtained or modeled by the use of indistinguishability operators. In this work the relation between indistinguishability operators and multiresolution theory is studied and different methods to build families of indistinguishability operators with multiresolution capacities are given.

**Keywords:** Indistinguishability Operators, Fuzzy Similarity, Multiresolution Systems, Granular Computing, Fuzzy Set Theory.

# 1 Introduction

Multiresolution Analysis [**?**] is the basic mathematical tool to build wavelet systems in Wavelet Theory [**?**] and it has broadly been used in image processing by applying it in compression, feature selection, denoising and other image processing tasks. However the multiresolution concept is a more general and a very interesting concept and it can be generalized and applied to multiple fields of data processing [**?**]. It also can be considered as an example of a Hierarchical System [**?**].

In its basic and common functional spaces formulation, a Multiresolution System consists of a family of nested functional subspaces that is dense in the whole functional space and has empty intersection. The projections of a function into each subspace give an approximation to this function with different accuracy. Two functions can have the same projection into a subspace but different projection into another one. So it can be possible that two different functions can be indistinguishable into a subspace but distinguishable in another one. An approximation to a function in a functional subspace entails a certain form of making indistinguishable some kind of functions: the approximation considers equal all functions with differences smaller than its accuracy level. In this sense, each level in a multiresolution analysis implicitly has associated an indistinguishability operator on the elements of the functional space.

A multiresolution analysis requires some other, and perhaps more important, properties. In the first place, it must be possible to go from one level to another by means of dyadic expansions or contractions. In the second place, there must exist a function, named scaling function, that generates by integer shifts and dilations a basis of the functional subspaces. There are several such functions, especially B-spline functions [**?**] , that can be used as fuzzy sets and at the same time generating a multiresolution analysis.

A Multiresolution Analysis induces in a natural way a family of distances in the functional space: it is enough to take the Euclidean distance in each functional subspace. This family of distances inherits some multiresolution characteristics.

In Fuzzy Set Theory the concept of indistinguishability operator is a generalization of the equivalence relation concept in the crisp field and includes the concept of similarity relation [**?**]. They have been widely studied and there is a good knowledge of their structure and properties [**?**] [**?**]. Indistinguishability operators allow us to measure the degree of similarity or indistinguishability between elements of a universe of discourse in a coherent way. Different indistinguishability operators will then generate different granularity on the universe of discourse and in this way are good tools to generate a Multiresolution analysis. In this work the possibility to define families of indistinguishability operators with multiresolution properties is studied. Going through the different members of one of such families will allow us to refine or coarsen the granularity of the system as required for a concrete purpose.

# 2 A Basic Scheme for a Multiresolution Analysis

Basically a multiresolution system (MR) allows us to fix an observation system over a universe $X$. If the MR system is good enough we will be able to observe with any level of detail the property of the elements of $X$ that we are considering. But at the same time we can delete, if we need, any difference between elements of $X$, as far as that property is considered, because, for example, we are not interested in those differences. In fact the changes in the resolution level must give us enough information about the characteristics of the elements which we are interested in.

In [**?**] a very general framework for a scheme of a multiresolution representation of data is given. Basically a basic scheme of multiresolution can be defined as follows.

**Definition 2.1.** *A basic scheme of multiresolution is a triplet* ($\mathcal{F}$, $(V_k)_{k=0}^{\infty}$, $\{\mathcal{D}_{k=0}^{\infty}\}$) *where* $\mathcal{F}$ *is a vector space,* $(V_k)_{k=0}^{\infty}$ *a sequence of vector spaces and* $\{\mathcal{D}_{k=0}^{\infty}\}$ *a sequence of linear operators* $D_k : \mathcal{F} \longrightarrow V_k$ *that satisfies two properties:*

*1.* $D_k$ *are onto mappings,*

*2.* $D_k(f) = 0 \Rightarrow D_{k-1}(f) = 0.$

Using the family of operators $D_k$ two sets of operators can be defined:

1. a set of decimation operators $D_k^{k-1} : V_k \longrightarrow V_{k-1}$ that allows us going from a greater resolution level to a lower resolution level,
2. and a set of prediction operators $P_{k-1}^k : V_{k-1} \longrightarrow V_k$ that allows us, using suitable data, going from a lower resolution level to a greater resolution level.

The operators $D_k^{k-1}$ can be defined by means of the expression $D_k^{k-1}(v_k) = D_{k-1}(f)$ where $f$ is any element of $\mathcal{F}$ such as $D_k(f) = v_k$. These operators are well defined thanks to the second property of Definition 2.1.

The family of operators $P_{k-1}^k$ are built thanks to the first property of Definition 2.1. As the operators $D_k$ are onto, there must exists at least a right inverse $R_k : V_k \longrightarrow \mathcal{F}$ for each one. Taking a family $\{R_k\}$ of them it is possible to define the operators $P_{k-1}^k$ as $P_{k-1}^k = D_k \cdot R_{k-1}$. Under these conditions it is easy to prove that $P_{k-1}^k$ is a right-inverse of the decimation operator $D_{k-1}^k$ but not a left-inverse (see [**?**] for details).

Given an element $f \in \mathcal{F}$ it is possible to take their projection at a fixed level $k$. Let $v_k = D_k(f)$ be this projection. Using the decimation operators we can get a sequence of values as $v_{i-1} = D_i^{i-1}(v_i)$. Each value $v_i$ is a coarser representation of the original value $f$. On the other hand from each value $v_{i-1}$ it is possible to predict the value for the level $i$ from the value of level $i-1$ by means of $v_i' = P_{i-1}^i(v_{i-1})$.

The relation between the decimation operators and the prediction operators is essential to get a multiresolution system. These operators must have a certain type of invertible relation between them but it must not be perfect, because if it were perfect, we would have the same information in each level. Usually the prediction operators need some additional information to be able to reconstruct the original projection. This extra information is called the detail information.

In the frame of vectorial spaces the detail coefficients are defined as the difference between the values $v_k$ and $v_k'$, i.e. $d_k = v_k - v_k'$. In this frame the detail coefficients belong to the kernel of the decimation operators. Thanks to the existence of vector basis in the vector spaces $V_k$ it is possible to write the expressions of the elements $v_k, v_k'$ and $d_k$.

Then there is a one-to-one correspondence between $v_k$ and $\{v_{k-1}, d_k\}$ because the decimation and prediction operators allows us to obtain $\{v_{k-1}, d_k\}$ from $v_k$ and the prediction operator allows us to obtain again $v_k$ from $\{v_{k-1}, d_k\}$. The process can be iterated for $k = N \dots 1$ obtaining a one-to-one mapping from $v_N$ to $d_N, d_{N-1}, \dots, d_1, v_0$. Analyzing the values $d_N, d_{N-1}, \dots, d_1$ it is possible to obtain information about the element $v_N$ and as the process can be reversed it is possible to manipulate the detail coefficients to recover a different $v'_N$ with some interesting properties.

Without considering the linear properties of the previous basic scheme of multiresolution associated to vector spaces, a family of nested equivalence relations $E_k$, that is $E_k \subset E_{k-1}$, over a set $X$ allows us to define a multiresolution system in that sense. Taking $V_k$ as the set $X/E_k$ of all equivalence classes under the equivalence relation $E_k$ and the operators $D_k$ as the canonical projections $\pi_k : X \longrightarrow X/E_k$, it is possible to define the decimation operators $D_k^{k-1} : V_k \longrightarrow V_{k-1}$ as the mappings $D_{k-1}^k(\bar{x}_k) := \pi_{k-1}(x)$, where $\bar{x}_k$ represents any equivalence class for the equivalence relation $E_k$. These operations are well defined because if we take another element $y \in \bar{x}_k$ then $(x, y) \in E_{k-1}$ because $E_k \subset E_{k-1}$ and then $\pi_{k-1}(x) = \pi_{k-1}(y)$. As the projections $\pi_k$ are onto they have, at least, a right-inverse. Taking a family of them $\{R_k\}$, we have $\pi_k \circ R_k = Id_k$ for all $k$. Now we define the prediction operators $P_{k-1}^k : V_{k-1} \longrightarrow V_k$ as $P_k^{k-1} = \pi_k \circ R_{k-1}$. These operators are a right-inverse of the decimation operators, i.e. it holds that $D_k^{k-1}(P_{k-1}^k(\bar{x}_{k-1})) = \bar{x}_{k-1}$. To prove it, naming $y = R_{k-1}(\bar{x}_{k-1})$, it is valid that $\pi_{k-1}(y) = \pi_{k-1}(R_{k-1}(\bar{x}_{k-1})) = \bar{x}_{k-1}$ and hence $(y, x) \in E_{k-1}$. Then

$$\begin{aligned} D_k^{k-1}(P_{k-1}^k(\bar{x}_{k-1})) &= D_k^{k-1}(\pi_k(R_{k-1}(\bar{x}_{k-1}))) \\ &= D_k^{k-1}(\pi_k(y)) \\ &= \pi_{k-1}(y) \\ &= \bar{x}_{k-1}. \end{aligned}$$

In general, $P_{k-1}^k$ will not be a left-inverse of $D_k^{k-1}$.

In this general frame the definition of the detail coefficients is a problem dependent issue. It will be necessary to obtain an operator $D$ that gives the detail coefficients in function of the projections and predictions $d_k = D(v_k, P_{k-1}^k(v_{k-1}))$ and an operator $R$ that allows us to rebuild $v_k$ from $\{d_k, v_{k-1}\}$, i.e. it must satisfy

$v_k = R(d_k, v_{k-1})$. In order to get an efficient process, the faster the operators $D_k^{k-1}, P_{k-1}^k, D, R$ can be, the better.

It is interesting to note that diverse properties of the operators $D_k$ determine different properties of a MR basic scheme, for example:

- If $\forall x, y\ \exists k$ such that $D_k(x) \neq D_k(y)$, then for any couple of values it exists a resolution level that distinguish them. A MR scheme with this property will be named dense.
- If $\forall x, y\ \exists k$ such that $D_k(x) = D_k(y)$, then any couple of values can be considered as indistinguishable in a certain resolution level. In this case the MR scheme will be named completely abstract or just complete.

Since indistinguishability operators are the fuzzy generalizations of equivalence relations in the crisp case, it is interesting to study how families of nested indistinguishability operators which can define a basic multiresolution scheme can be built. Previous to start this task a basic example in the real numbers set can show how a basic scheme of multiresolution can be useful to process information.

# 3 An Example in the Real Numbers Set

For any $n \in \mathbb{Z}$, let $P_n : \mathbb{R}^+ \longrightarrow \mathbb{N}$ be the operator defined by

$$P_n r = \sup\{m \in \mathbb{N} : 10^n m \leq r\} = \lfloor 10^n r \rfloor.$$

These operators are onto because for any $m \in \mathbb{N}$, $m = P_n 10^{-n} m$. They also satisfy that if $P_n r = 0$, then $P_{n-1} r = 0$ because $P_n r = 0$ if and only if $r < 10^{-n}$ and $r < 10^{-n} < 10^{-(n-1)}$. These operators allow us to define a basic multiresolution scheme. It is interesting to observe that these operators are non-linear, they are superadditive because the floor function satisfies $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$. So as it can be seen from this example, the linear property of operators $D_n$ is not essential to obtain a basic scheme of multiresolution.

In classical set theory, to have an onto map on a set is equivalent to have an equivalence relation defined on this set and also equivalent to have a partition defined on it. In this example, the operators $P_n$ define the equivalence relations

$$r \sim_n r' \Longleftrightarrow P_n r = P_n r'.$$

It is easy to prove that these equivalence relations satisfy:

$$r \sim_n r' \Longrightarrow r \sim_{n-1} r',$$

and a nested family or equivalence relations is defined.

Also a nested family of partitions can be defined, because the equivalent classes of each relation $\sim_n$ are $C_n^i = [10^{-n}i, 10^{-n}(i+1))$ and it holds that

$$C_{n-1}^i = C_n^{10i} \cup C_n^{10i+1} \cup \ldots \cup C_n^{10i+9}.$$

The family of operators $P_n$ with $n \in \mathbb{Z}$ defines a basic scheme of multiresolution taking the triplet $(\mathbb{R}^+, C_n, D_n)$ where $C_n = \cup_{i\in\mathbb{N}} C_n^i$ and $D_n : \mathbb{R}^+ \longrightarrow C_n$ is defined by

$$D_n(r) = C_n^{P_n r.}$$

The operators $D_n$ inherit the properties of operators $P_n$.

The operators of decimation and prediction can be defined by:

$$D_n^{n-1}(C_n^i) = D_{n-1}(r)$$

where $P_n r = i$. In fact it can be calculated that $D_n^{n-1}(C_n^i) = C_{n-1}(\lfloor 10^{-1} \rfloor)$, because if $P_n r = i$ then $r = 10^{-n}i + \theta(r,n)$ where $\theta(r,n) < 10^{-n}$. Calculating $D_{n-1}(r)$ it results that $P_n r = 10^{n-1}(10^n i + \theta(r,n)) = 10^{-1}i + 10^{n-1}\theta(r,n)$ and $10^{n-1}\theta(r,n) < 10^{-1} < 1$ and it will be a fractional part; hence $P_n$ depends only on $i$. The operator of decimation basically consists on eliminating a significant digit of the expression of $r$ with respect to the significant digits obtained with $P_n$.

Several projection operators can be built because the operators $P_n$ have several different right inverse functions. In fact any mapping of the form $R_n(C_n^m) = 10^{-n}m + \theta(m,n)$ where $\theta(m,n)$ can be any real number such as $\theta(m,n) < 10^{-n}$. For example we can define the family $R_n(C_n^m) = 10^{-n}m + 5 \cdot 10^{-(n+1)}\theta(m,n)$. In that case the next prediction operators $P_{n-1}^n$ are obtained:

$$\begin{aligned} P_{n-1}^n(C_{n-1}^m) &= P_n \cdot R_{n-1}(C_{n-1}^m) = P_n(10^{-(n-1)}m + 5 \cdot 10^{-n}) \\ &= \lfloor 10^n 10^{-(n-1)}m + 10^n 5 \cdot 10^{-n} \rfloor \\ &= \lfloor 10m + 5 \rfloor \\ &= 10m + 5. \end{aligned}$$

This operation $P_{n-1}^n$ corresponds to put the digit 5 as a new significant digit to a number. Clearly other prediction operators are possible having different properties corresponding in a certain sense to the inverse of operations like rounding a number or truncating it. If we have a projection of a positive real number $r$ given by a class in level $n$, named it as $\bar{r}_n = C_n^{P_n r}$, we can obtain a projection into a coarser level using the decimation operator $\bar{r}_{n-1} = D_n^{n-1}(\bar{r}_n)$. If we apply the prediction operator to $\bar{r}_{n-1}$ we will obtain $\bar{r}'_n = P_{n-1}^n(\bar{r}_{n-1})$. In general we can not recover the original projection without extra information. In our example the detail coefficients can be calculated as

$$d_n = D(\bar{r}_n, \bar{r}'_{n-1}) = \bar{r}_n - \bar{r}_{n-1}.$$

These details belong to the set of numbers $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$ and they tell us what information is lost when we go from a finer to a coarser resolution. The reconstruction operator will be in this case

$$\bar{r}_n = R(\bar{r}_{n-1}, d_n) = P_{n-1}^n(\bar{r}_{n-1}) + d_n.$$

Clearly the basic scheme of multiresolution given by this example allows us to study the significant digits of a real number. When a first projection of a positive real number is taking we perform a truncation operation of this number. Successive decimation operations carry out more truncations of significant digits. The detail coefficients tell us how the information is lost with respect to our prediction operator.

In the traditional form of presenting multiresolution systems to build wavelets systems, the multiresolution analysis is based on a scaling function. In our example the function $f(x) = 10^{-1}x$ plays this role. It is also interesting to note that the next properties are satisfied by our example:

1. $r \sim_n r'$ if and only if $f^{-1}(r) \sim_{n-1} f^{-1}(r')$. In this sense each relation $\sim_n$ is a scale version of the another one thanks to the scaling function $f$.
2. The next equalities are also satisfied.

$$\begin{aligned} \lim_{n \to \infty} f^n |P_n x - P_n y| &= |x - y|, \\ \lim_{n \to -\infty} f^n |P_n x - P_n y| &= 0. \end{aligned}$$

# 4 $T$-Indistinguishability Operators

Indistinguishability operators are fuzzy generalizations of equivalence relations. They satisfy fuzzified properties for reflexivity, symmetry and transitivity. This last property is modeled using a given t-norm and depends therefore on it, meaning that a fuzzy relation can be an indistinguishability operator for a particular t-norm and may not be such a relation for another given t-norm. They are also known as fuzzy equivalence relations [2], fuzzy equalities [4] and similarity relations [**?**]. They generate granularity on the universe of discourse so that refining the indistinguishability operator also the granularity is refined. In this sense, a family of equivalence relations can be used to generate a multiresolution schema. This will be developed in the next sections. Before showing several ways to obtain multiresolution families of indistinguishability operators, some known results on t-norms and indistinguishability operators that will be needed later on in the paper are presented in this section.

**Definition 4.1.** *[**?**] A continuous t-norm is a continuous map* $T : [0,1] \times [0,1] \to [0,1]$ *satisfying for all* $x, y, z, x', y' \in [0,1]$

1. $T(x, T(y,z)) = T(T(x,y), z)$ *(Associativity)*
2. $T(x,y) = T(y,x)$ *(Commutativity)*
3. *If* $x \leq x'$ *and* $y \leq y'$*, then* $T(x,y) \leq T(x', y')$ *(Monotonicity)*
4. $T(1,x) = x$

Since a t-norm $T$ is associative, we can extend it to an $n$-ary operation in the standard way:

$$\begin{aligned} T(x) &= x \\ T(x_1, x_2, ...x_n) &= T(x_1, T(x_2, ..x_n)). \end{aligned}$$

In particular, $T(\overbrace{x, x, ...x}^{n\ times})$ will be denoted by $x^{(n)}$.

The $n$-th root $x^{(\frac{1}{n})}$ of $x$ with respect to $T$ is defined by

$$x^{(\frac{1}{n})} = sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in \mathbb{N}$, $x^{(\frac{m}{n})} = \left(x^{(\frac{1}{n})}\right)^{(m)}$.

Let $E(T) = \{x \in [0,1] \mid x^{(2)} = x\}$ be the set of idempotent elements of $T$ and $NIL(T) = \{x \in [0,1] \mid x^{(n)} = 0 \text{ for some } n \in \mathbb{N}\}$ the set of nilpotent elements of $T$.

**Definition 4.2.** *A continuous t-norm $T$ is Archimedean if and only if $E(T) = \{0, 1\}$. $T$ is called strict when $NIL(T) = \{0\}$. Otherwise it is called non-strict and $NIL(T) = [0, 1)$.*

**Theorem 4.3 (Ling [?]).** *A continuous t-norm $T$ is Archimedean if and only if there exists a continuous decreasing map $t : [0, 1] \to [0, \infty]$ with $t(1) = 0$ such that*

$$T(x, y) = t^{[-1]}(t(x) + t(y)),$$

*where $t^{[-1]}$ stands for the pseudo-inverse of $t$ defined by*

$$t^{[-1]}(x) = \begin{cases} 1 & \text{if } x < 0 \\ t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

*$T$ is strict if $t(0) = \infty$ and non-strict otherwise.*

$t$ is called an additive generator of $T$ and two additive generators of the same t-norm differ only by a multiplicative constant.

**Definition 4.4.** *The residuation $\overrightarrow{T}$ of a t-norm $T$ is defined by*

$$\overrightarrow{T}(x|y) = sup\{\alpha \in [0, 1] \mid T(x, \alpha) \leq y\}.$$

**Definition 4.5.** *The biresiduation or natural $T$-indistinguishability operator $E_T$ associated to a given t-norm $T$ is the fuzzy relation on* [0,1] *defined by*

$$E_T(x, y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)) = \min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

**Example 4.6.**

1. *If $T$ is a continuous Archimedean t-norm with additive generator $t$, then $E_T(x, y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0, 1]$.*
2. *If $Ł$ is the Łukasiewicz t-norm ($Ł(x, y) = \max(x+y-1, 0)$), then $E_Ł(x, y) = 1 - |x - y|$ for all $x, y \in [0, 1]$.*
3. *If $P$ is the Product t-norm, then $E_P(x, y) = \min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0, 1]$, where $\frac{z}{0} = 1$.*
4. *If $\min$ is the Minimum t-norm, then*

$$E_{\min}(x, y) = \begin{cases} \min(x, y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$$

5. *If* $T_D$ *is the drastic t-norm*

$$T_D(x,y) = \begin{cases} 0 & \text{if } x,y \neq 1 \\ x & \text{if } y = 1 \\ y & \text{if } x = 1 \end{cases},$$

*then*

$$E_{T_D}(x,y) = \begin{cases} 1 & \text{if } x,y \neq 1 \\ x & \text{if } y = 1 \\ y & \text{if } x = 1 \end{cases}.$$

$E_T$ is indeed a special kind of (one-dimensional) $T$-indistinguishability operator [**?**] [**?**] (Definition 4.7) and, in a logical context where $T$ plays the role of the conjunction, $E_T$ is interpreted as the bi-implication associated to $T$ [**?**].

The general definition of $T$-indistinguishability operator is

**Definition 4.7.** *[**?**][**?**] Given a t-norm $T$, a $T$-indistinguishability operator $E$ on a set $X$ is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x,y,z \in X$*

1. $E(x,x) = 1$ *(Reflexivity)*
2. $E(x,y) = E(y,x)$ *(Symmetry)*
3. $T(E(x,y), E(y,z)) \leq E(x,z)$ *($T$-transitivity).*

*$E$ separates points if and only if $E(x,y) = 1$ implies $x = y$.*

For the sake of simplicity, all $T$-indistinguishability operators in the paper will separate points.

# 5 Families of Nested $T$-Indistinguishability Operators

In this section several methods to obtain a nested family of $T$-indistinguishability operators which can generate Multiresolution Schemes are provided.

The properties that are considered in this paper for a family of $T$-indistinguishability operators to generate a Multiresolution Scheme are given in the next definition.

**Definition 5.1.** *A family $(E_n)_{n \in \mathbb{Z}}$ of $T$-indistinguishability operators on a subset $X$ of $\mathbb{R}$ is a multiresolution family if and only if*

1. $E_n \leq E_m$ *if* $n \geq m$
2. $\lim_{n \to -\infty} E_n(x, y) = 1$
3. $\lim_{n \to \infty} E_n(x, y) = 0$ *if* $x \neq y$
4. $E_n(x, y) = E_{n-1}(2x, 2y)$.

From the first condition the family is nested, meaning that the granularity of $X$ generated by the indistinguishability operator is refined when $n$ increases.

The second property, named the complete property, says that when $n$ approaches $-\infty$ the indistinguishability operators approach the greatest equivalence relation, in which all elements are equivalent or indistinguishable, and for which the granularity is the coarsest possible.

The third property, named the density property, is dual to the previous one in the sense that when $n$ tends to $\infty$, then the operators approach the classical equality for which every pair of elements of $X$ are completely distinguishable.

Property 4 allows us to go from one level to the following one in a coherent way, so that at each step the granularity is doubled or halved.

The granularity in a multiresolution family allows us then to go from the coarsest to the more refined granularity on $X$.

## 5.1 Using Scales on $\mathbb{R}$

A scale on a subset $X$ of $\mathbb{R}$ is defined as a monotonic map $\varphi : X \to X$. The possibility to distinguish real numbers in a fuzzy environment is based on the use of a scale. Different scales give different degrees of accuracy and determine different fuzzy equivalence relations (indistinguishability operators) on $X$.

In order to prove that relations generated by scales in Propositions 5.4 and 5.5 are indistinguishability operators the following two proposition will be used.

**Proposition 5.2.** *[?] Let $T$ be a continuous Archimedean t-norm and $t$ an additive generator of $T$.*

1. *If $d$ is a pseudo distance on a set $X$, then $E = t^{[-1]} \circ d$ is a $T$-indistinguishability operator on $X$.*

2. *If $E$ is a $T$-indistinguishability on $X$, then $d = t \circ E$ is a pseudo distance on $X$.*

*$d$ is a distance on $X$ if and only if $E$ separates points.*

**Proposition 5.3.** *[5] Let $f$ be a map $f : X \to \mathbb{R}$. Then $d(x, y) = |f(x) - f(y)|$ for all $x, y \in X$ is a pseudodistance on $X$. Moreover, if $f$ is a one-to-one map, then $d$ is a distance.*

**Proposition 5.4.** *Every monotonic map (a scale) on $\mathbb{R}$ generates a $L$-indistinguishability operator $E_\varphi$ on $\mathbb{R}$ in the following way:*

$$E_\varphi(x, y) = \max(1 - |\varphi(x) - \varphi(y)|, 0) \text{ for all } x, y \in \mathbb{R}.$$

*Proof.* Considering the additive generator $t(x) = 1 - x$ of Ł, according to Proposition 5.2 we must prove that $t \circ E_\varphi$ is a pseudodistance. But

$$t \circ E_\varphi(x, y) = \max(1 - |\varphi(x) - \varphi(y)|, 0) = \min(|\varphi(x) - \varphi(y)|, 1)$$

which is a distance on $\mathbb{R}$ thanks to Proposition 5.3. □

**Proposition 5.5.** *Every monotonic map (a scale) $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ generates a $P$-indistinguishability operator $E_\varphi$ on $\mathbb{R}^+$ in the following way:*

$$E_\varphi(x, y) = \min\left(\frac{\varphi(x)}{\varphi(y)}, \frac{\varphi(y)}{\varphi(x)}\right) \text{ for all } x, y \in \mathbb{R}^+.$$

*Proof.* Considering the additive generator $t(x) = -\ln(x)$ of $P$, according to Proposition 5.2 we must prove that $t \circ E_\varphi$ is a pseudodistance. But

$$t \circ E_\varphi(x, y) = -\ln\left(\min\left(\frac{\varphi(x)}{\varphi(y)}, \frac{\varphi(y)}{\varphi(x)}\right)\right) = |\ln(\varphi(x)) - |\ln(\varphi(y))|$$

which is a distance on $\mathbb{R}^+$ thanks to Proposition 5.3. □

From the following results on isomorphic t-norms, every scale on $\mathbb{R}$ will generate a $T$-indistinguishability operator on $\mathbb{R}$ for every non-strict continuous Archimedean t-norm $T$, and every scale on $\mathbb{R}^+$ will generate a $T$-indistinguishability operator on $\mathbb{R}^+$ for every strict continuous Archimedean t-norm $T$ (Corollary 5.10).

**Definition 5.6.** *[?] An automorphism of* $[0,1]$ *is a bijective map* $f : [0,1] \to [0,1]$ *with* $f(0) = 0$ *and* $f(1) = 1$.

**Definition 5.7.** *[?] Two continuous t-norms* $T$ *and* $T'$ *are isomorphic if and only if there exists a bijection* $f : [0,1] \to [0,1]$ *such that* $\forall x, y \in [0,1]$

$$f(T(x,y)) = T'(f(x), f(y)).$$

$f$ *is called an isomorphism between* $T$ *and* $T'$.

**Proposition 5.8.** *[?] Let* $T$ *and* $T'$ *be two isomorphic continuous t-norms and* $f$ *an isomorphism between* $T$ *and* $T'$. *Then* $f$ *is an automorphism of* $[0,1]$.

**Proposition 5.9.** *If* $f$ *is an isomorphism between two t-norms* $T$ *and* $T'$, *then* $f$ *is an isomorphism between the corresponding residuations and natural* $T$*-indistinguishability operators.*

*Proof.* If $f \circ T = T' \circ (f \times f)$, then for all $x, y \in [0,1]$,

$$\begin{aligned} f \circ \overrightarrow{T}(x|y) &= f(\sup\{\beta \in [0,1] \,|\, T(\beta, x) \le y\}) \\ &= \sup\{f(\beta) \in [0,1] \,|\, f^{-1} \circ T'(f(\beta), f(x)) \le y\} \\ &= \sup\{f(\beta) \in [0,1] \,|\, T'(f(\beta), f(x)) \le f(y)\}) \\ &= \overrightarrow{T'}(f(x)|f(y)). \end{aligned}$$

The corresponding natural $T$-indistinguishability operators are also isomorphic since they are the symmetrized fuzzy relations obtained from the residuation $\overrightarrow{T}$ of $T$. □

**Corollary 5.10.**

1. *Every scale on* $\mathbb{R}$ *defines a* $T$*-indistinguishability operator on* $\mathbb{R}$ *for every continuous non-strict Archimedean t-norm* $T$.
2. *Every scale on* $\mathbb{R}^+$ *defines a* $T$*-indistinguishability operator on* $\mathbb{R}^+$ *for every continuous strict Archimedean t-norm* $T$.

*Proof.*

1. For the scale $\varphi$ on $\mathbb{R}$, the fuzzy relation $E_\varphi$ on $\mathbb{R}$ defined for all $x, y \in \mathbb{R}$ by

$$E_\varphi(x,y) = t^{-1}(|\varphi(x) - \varphi(y)|)$$

where $t$ is an additive generator of $T$ is a $T$-indistinguishability operator on $\mathbb{R}$.

2. For the scale $\varphi$ on $\mathbb{R}^+$, the fuzzy relation $E_\varphi$ on $\mathbb{R}^+$ defined for all $x, y \in \mathbb{R}^+$ by

$$E_\varphi(x,y) = t^{-1}(|\varphi(x) - \varphi(y)|)$$

where $t$ is an additive generator of $T$ is a $T$-indistinguishability operator on $\mathbb{R}^+$.

□

The next two propositions provide a multiresolution family for the Łukasiewicz t-norm and another one for the Product t-norm starting from families of scales.

**Proposition 5.11.** *For every $n \in \mathbb{Z}$ we can define the scale $\varphi_n$ on $\mathbb{R}$ by $\varphi_n(x) = 2^n x$ that generates the Ł-indistinguishability operator $E_n(x,y) = \max(0, 1 - 2^n |x-y|)$ on $\mathbb{R}$.*

*The family $(E_n)_{n\in\mathbb{Z}}$ generated by the scales $\varphi_n$ is a multiresolution family of $\mathbb{R}$.*

*Proof.*

- If $n \geq m$, then $2^n \geq 2^m$ and $1 - 2^n|x-y| \leq 1 - 2^m|x-y|$.
- $\lim_{n\to-\infty} E_n(x,y) = \lim_{n\to-\infty} \max(0, 1-2^n|x-y|) = 1$.
- If $x \neq y$, then $\lim_{n\to\infty} E_n(x,y) = \lim_{n\to\infty} \max(0, 1-2^n|x-y|) = 0$.
- $E_{n-1}(2x, 2y) = \max(0, 1-2^{n-1}|2x-2y|) = \max(0, 1-2^{n-1}2|x-y|) = E_n(x,y)$.

□

**Proposition 5.12.** *For every $n \in \mathbb{Z}$ let us consider the scale $\varphi_n(x) = 2^{2^n x}$ on $\mathbb{R}^+$. These scales generate the $P$-indistinguishability operators $E_n(x,y) = \min\left(\frac{2^{2^n x}}{2^{2^n y}}, \frac{2^{2^n y}}{2^{2^n x}}\right)$ on $\mathbb{R}^+$. The family $(E_n)_{n\in\mathbb{Z}}$ is a multiresolution family of $\mathbb{R}^+$.*

*Proof.* Without loss of generality we can assume $x \leq y$.

- If $n \geq m$, then $2^n \geq 2^m$ and $\frac{2^{2^n x}}{2^{2^n y}} \leq \frac{2^{2^m x}}{2^{2^m y}}$.
- $\lim_{n\to-\infty} E_n(x,y) = \lim_{n\to-\infty} \frac{2^{2^n x}}{2^{2^n y}} = 1$.
- If $x \neq y$, then $\lim_{n\to\infty} E_n(x,y) = \lim_{n\to\infty} \frac{2^{2^n x}}{2^{2^n y}} = 0$.
- $E_{n-1}(2x, 2y) = \frac{2^{2^{n-1}2x}}{2^{2^{n-1}2y}} = \frac{2^{2^n x}}{2^{2^n y}} = E_n(x,y)$.

□

## 5.2 Powers of a $T$-Indistinguishability Operator

The powers of a $T$-indistinguishability operator generate a nested family of such operators. Selecting appropriate subfamilies we obtain multiresolution families of $T$-indistinguishability operators as can be seen in this section.

**Lemma 5.13.** *[3] Let $E$ be a $T$-indistinguishability operator on a set $X$ and $n$ a positive real number. The powers $E^{(n)}$ defined by $E^{(n)}(x,y) = (E(x,y))^{(n)}$ are $T$-indistinguishability operators on $X$.*

Given a t-norm $T$ and a $T$-indistinguishability operator $E$ on a set $X$, consider the family $(E_n)_{n\in\mathbb{Z}}$ defined by

$$E_n(x,y) = \begin{cases} (E(x,y))^{(n)} & \text{for all positive integer } n \\ (E(x,y))^{(-\frac{1}{n})} & \text{for all negative integer } n \\ E(x,y) & \text{if } n = 0 \end{cases}$$

It is obvious that $(E_n)_{n\in\mathbb{Z}}$ is a nested family of $T$-indistinguishability operators. For the Minimum t-norm the family is trivial ($E_n = E_m$ for all $n \in \mathbb{Z}$).

**Proposition 5.14.** *Let $T$ be a continuous Archimedean t-norm and $(E_n)_{n\in\mathbb{Z}}$ the family defined as above from a $T$-indistinguishability operator $E$ on a set $X$. Then*

1. $\lim_{n\to-\infty} E_n(x,y) = 1$ *if* $E_n(x,y) \neq 0$.
2. $\lim_{n\to\infty} E_n(x,y) = 0$ *if* $x \neq y$.

*Proof.*

1. Let For $x \neq y$, let $a = E(x,y) \neq 0$. If $n < 0$, then

$$E_n(x,y) = a^{(-\frac{1}{n})} = t^{[-1]}(-\frac{1}{n}t(a)).$$

Since $t(a) \neq \infty$, $-\frac{1}{n}t(a)$ tends to 0 when $n$ tends to $-\infty$ and hence

$$\lim_{n\to-\infty} t^{[-1]}(-\frac{1}{n}t(a)) = 1.$$

2. Let $a = E(x,y) \neq 1$ since $E$ separates points. If $n > 0$, then

$$E_n(x,y) = a^{(n)} = t^{[-1]}(nt(a)).$$

Since $t(a) \neq 0$, $nt(a)$ tends to $\infty$ when $n$ tends to $\infty$ and hence

$$\lim_{n\to\infty} t^{[-1]}(nt(a)) = 0.$$

□

**Proposition 5.15.** *Let $E$ be the natural Ł-indistinguishability operator on $[0,1]$. The following family of indistinguishability operators is a multiresolution family.*

$$E_n(x,y) = \begin{cases} (E(x,y))^{(2^n)} & \text{for all positive integer } n \\ (E(x,y))^{(-\frac{1}{2^n})} & \text{for all negative integer } n \\ E(x,y) & \text{if } n = 0 \end{cases}$$

*Proof.* In fact this family coincides with the family from Proposition 5.11. □

## 5.3 $T$-Indistinguishability Operators and Distances

Using the fact that if $d$ is a pseudometric on a set $X$ and $T$ a continuous Archimedean t-norm with additive generator $t$, then $t^{[-1]} \circ d$ is a $T$-indistinguishability operator on $X$ and, reciprocally, if $E$ is a $T$-indistinguishability operator on $X$, then $t \circ E$ is a pseudodistance on $X$ (Proposition5.2), from a multiresolution family $(d_n)_{n\in\mathbb{Z}}$ of pseudodistances, we can obtain multiresolution families of $T$-indistinguishability operators for continuous Archimedean t-norms.

**Example 5.16.** *From the family of distances $(d_n)_{n\in\mathbb{Z}}$ on $\mathbb{R}$ defined by $d_n(x,y) = 2^n |x-y|$ we obtain the multiresolution families of examples 5.11 and 5.12 using the additive generators $t(x) = 1-x$ and $t(x) = -\log_2(x)$ respectively.*

Bounded distances can be used for non-strict continuous Archimedean t-norms.

**Example 5.17.** *We can consider the family of distances $d_n(x,y) = \frac{|x-y|}{|x-y|+2^{-n}}$ on $\mathbb{R}$ that satisfy $d_n(x,y) \leq 1$ for all $x,y \in \mathbb{R}$ and $n \in \mathbb{Z}$.*

*The family $(d_n)_{n\in\mathbb{Z}}$ satisfies*

- $d_n \leq d_m$ *if* $n \leq m$
- $\lim_{n\to\infty} d_n = 0$,
- $\lim_{n\to-\infty} d_n = 1$,
- $d_n(x, y) = d_{n-1}(2x, 2y)$.

*Given a non-strict continuous Archimedean t-norm* $T$ *with normalized additive generator* $t$ *(i.e.* $t(0) = 1$*), we obtain the multiresolution family of* $T$*-indistinguishability operators* $(E_n)_{n\in\mathbb{Z}}$ *on* $\mathbb{R}$*, where* $E_n(x, y) = t^{-1} \circ d_n(x, y)$.

## 5.4 Compatible Partitions with a Family of $T$-Indistinguishability Operators

In the crisp case, there is a bijection between the set of partitions and the set of equivalence relations on a set $X$. This meaning that the granularity of $X$ can be expressed by the corresponding equivalence relations. Refining the relation a refined partition is obtained. In the fuzzy case, there are different ways to associate a partition to a $T$-indistinguishability operator. For non-strict continuous Archimedean t-norms the Bezdek approach is useful for applied purposes [**?**]. The importance of multiresolution families of $T$-indistinguishability operators on a set $X$ lies in the fact that they can refine the precision or granularity of $X$ which can be translated to the corresponding associate partitions in the sense of this subsection.

**Definition 5.18.** *[**?**] Let* $P = \{\mu_1, \mu_2, ..., \mu_n\}$ *be a finite set of fuzzy sets on* $X$*,* $T$ *a t-norm and* $S$ *a t-conorm.* $P$ *is a partition of* $X$ *with respect to* $(T, S)$ *or a* $T, S$*-partition if and only if*

- $T(\mu_i(x), \mu_j(x)) = 0$ *for all* $i, j = 1, 2, ..., n$ $i \neq j$ *and for all* $x \in X$.
- $S(\mu_1(x), \mu_2(x), ..., \mu_n(x)) = 1$ *for all* $x \in X$.

**Definition 5.19.** *[**?**] A* $T, S$*-partition* $P = \{\mu_1, \mu_2, ..., \mu_n\}$ *on a set* $X$ *is said to be compatible with a* $T$*-indistinguishability operator* $E$ *on* $X$ *if and only if for all* $i = 1, ..., n$ *and all* $x, y \in X$*,*

1. $T(\mu_i(x), \mu_i(y)) \leq E(x, y)$.
2. *There exists* $a_i \in X$ *with* $E(x, a_i) \leq \mu_i(x)$.

The first property implies that every pair of elements $x$ and $y$ belonging to the same class $\mu_i$ are equivalent, while the second one assures the existence of a prototype $a_i$ for each class of the partition.

**Proposition 5.20.** *[?] For every class $\mu_i$ of a compatible $T,S$-partition there exists $a_i \in X$ with $\mu_i(a_i) = 1$.*

**Proposition 5.21.** *[?] Let $ai \in X$ be such that $\mu_i(a_i) = 1$ where $\mu_i$ is a fuzzy subset of a compatible $T,S$-partition. Then $\mu_i(x) = E(x, a_i)$ for all $x \in X$.*

**Definition 5.22.** *Given a $T$-indistinguishability operator $E$ on a set $X$ and $x \in X$, the fuzzy set $\mu_x$ defined for all $y \in X$ by $\mu_x(y) = E(x, y)$ is called a column of $E$.*

From the last two propositions it is straightforward to prove that the fuzzy subsets of a compatible $T,S$-partition of $X$ with respect to a $T$-indistinguishability operator on $X$ are columns of $E$.

Let $\mu_{ij}(x) = \max(0, 1 - j|x - \frac{i}{j}|)$ be a family of fuzzy subsets with $i, j \in \mathbb{N}$ and $0 \le i \le j$. We will prove that this family of fuzzy subsets includes a multiresolution analysis when $j = 2^k$.

Consider the natural Ł-indistinguishability operator on $[0, 1]$, $E(x, y) = 1-|x-y|$ on $[0, 1]$ and its powers $E^{(j)}(x, y) = Ł(E, \overset{j}{\ldots}, E) = \max(0, 1-j|x - y|)$. Each relation $E^{(j)}$ has as a unique compatible partition $P_j = \{\mu_{ij}\}$[?]. It means that $\mu_{ij}$ are columns of $E^{(j)}$.

Consider now the next family $\{E_j\}$ of indistinguishability operators built from the fuzzy subsets $\mu_{ij}$ in the following way:

$$E_j(x, y) = \inf_{\substack{i \in \mathbb{N} \\ 0 \le i \le j}} (1 - |\mu_{ij}(x) - \mu_{ij}(y)|),$$

for all $j \in \mathbb{N}$ and $[0, 1]$ as the universe of discourse.

**Proposition 5.23.** *Each indistinguishability operator $E_j$ has a column $\mu_{ij}$ in $\frac{i}{j}$. Moreover, the partition $P_j$ is a compatible partition with $E_j$.*

*Proof.* The proof requires to prove two points:

1. $Ł(\mu_{ij}(x), \mu_{ij}(y)) \le E_j(x, y)$,
2. There exists $a_i \in X$ such that $E_j(x, a_i) \le \mu_{ij}(x)$.

The point 2 is trivial because taking $a_i = \frac{i}{j}$ it holds that

$$\begin{aligned} E_j(\frac{i}{j}, y) &= \inf_{\substack{l\in\mathbb{N} \\ 0\leq l\leq j}} (1 - |\mu_{lj}(\frac{i}{j}) - \mu_{lj}(y)|) \\ &\leq 1 - |\mu_{ij}(\frac{i}{j}) - \mu_{lj}(y)| \\ &= \mu_{ij}(y) \end{aligned}$$

The point 1 is verified because it is true that

$$Ł(r, s) \leq \overrightarrow{Ł}(\max(r, s)| \min(r, s)) \ \forall r, s \in [0, 1],$$

since it is equivalent to

$$Ł(Ł(r, s), \max(r, s)) \leq \min(r, s),$$

and by the associativity property of Ł, that is equivalent to

$$Ł(r, Ł(s, \max(r, s))) \leq \min(r, s),$$

but this is true because

$Ł(r, Ł(s, \max(r, s))) \leq r$ and $Ł(r, Ł(s, \max(r, s))) \leq Ł(s, \max(r, s)) \leq s$.

Taking this in consideration we have

$$Ł(\mu_{ij}(x), \mu_{ij}(y)) \leq \overrightarrow{Ł}(\max(\mu_{ij}(x), \mu_{ij}(y))| \min(\mu_{ij}(y), \mu_{ij}(x)),$$

and then

$$\begin{aligned} Ł(\mu_{ij}(x), \mu_{ij}(y)) &\leq \inf_i \overrightarrow{Ł}(\max(\mu_{ij}(x)), \mu_{ij}(y))| \min(\mu_{ij}(y)), \mu_{ij}(x)), \\ &= \inf_i (1 - |\mu_{ij}(x) - \mu_{ij}(y)|) \\ &= E_j(x, y) \ \ \forall i \in \mathbb{N} \ 0 \leq i \leq j. \end{aligned}$$

□

So we have that $E^{(j)}$ is a family of indistinguishability operators on $[0, 1]$ that has $P_j$ as the unique compatible partition. It is easy to prove that

1. $E^{(j)} \leq E^{(j-1)} \leq \ldots \leq E(x, y) = 1 - |x - y| = E_{Ł}(x, y)$.
2. $\lim_{j\to\infty} E^j(x, y) = 0$ if $x \neq y$.

3. $E^{(j)}(2x, 2y) = E^{(2j)}(x, y)$.

It is worth noticing that $E_j$ also has $P_j$ as a compatible partition but $E_j$ is different from $E^{(j)}$. To prove this last statement it is enough to take $j = 2, x = \frac{1}{5}, y = \frac{3}{4}$ and to calculate $E^{(2)}(x, y) = 0$ and $E_2(x, y) = \frac{2}{5}$. As conclusion we can say that the concept of compatible partition given in [**?**] do not determine in a univocal way a $T$-indistinguishability operator.

# 6 Graded Families of $T$-Indistinguishability Operators

Until here, we have considered the t-norm $T$ fixed. Another possibility of generating multiresolution families of indistinguishability operators is allowing to vary the t-norm. Then not only the granularity will change with the different indistinguishability operators but the logics associated to the corresponding t-norms will also change.

**Lemma 6.1.** *Let $f$ be an automorphism of $[0,1]$ and $n \in \mathbb{N}$. Then $f^n$ and $f^{-n}$ are automorphisms of $[0,1]$, where $f^n = \overbrace{f \circ ... \circ f}^{n \text{ times}}$, $f^{-n} = \left(f^{-1}\right)^n$ and $f^0 = Id$.*

*Proof.* Trivial. □

**Definition 6.2.** *Let $T$ be a t-norm and $f$ an automorphism of $[0,1]$. The family of t-norms generated by $T$ and $f$ is $(T_n)_{n\in\mathbb{Z}}$ where*

$$T_n(x, y) = f^{-n}(T(f^n(x), f^n(y)),$$

*for all $x, y \in [0,1]$.*

**Proposition 6.3.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$, $f$ an automorphism of $[0,1]$ and $(T_n)_{n\in\mathbb{Z}}$ the family of t-norms generated by $T$ and $f$. Then $t \circ f^n$ is an additive generator of $T_n$ $\forall n \in \mathbb{Z}$.*

*Proof.*

$$T_n(x, y) = f^{-n}(T(f^n(x), f^n(y)) = f^{-n} \circ t^{[-1]}(t \circ f^n(x) + t \circ f^n(y)).$$

□

In a similar way we can define families of residuations and natural indistinguishability operators.

**Definition 6.4.** *Let $T$ be a t-norm and $f$ an automorphism of $[0,1]$. The family of natural $T_n$-indistinguishability operators generated by $T$ and $f$ is $(E_n)_{n\in\mathbb{Z}}$ where*

$$E_n(x,y) = f^{-n}(E_T(f^n(x), f^n(y))$$

*for all $x, y \in [0,1]$.*

**Proposition 6.5.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$, $f$ an automorphism of $[0,1]$ and $(E_n)_{n\in\mathbb{Z}}$ the family of natural $T_n$-indistinguishability operators generated by $T$ and $f$. Then $E_n(x,y) = f^{-n} \circ t^{[-1]} \left(|t \circ f^n(x) - t \circ f^n(y)|\right) \; \forall n \in \mathbb{Z}$.*

*Proof.* From Proposition 6, $t \circ f^n$ is a generator of $T_n$ and the natural $T_n$-indistinguishability operator $E_n$ is

$$E_n(x,y) = (t\circ f^n)^{[-1]}(|t\circ f^n(x) - t\circ f^n(y)| = f^{-n}\circ t^{[-1]} \left(|t \circ f^n(x) - t \circ f^n(y)|\right)$$

□

**Example 6.6.** *Let $T$ be the Łukasiewicz t-norm, $f(x) = x^2$ and $(T_n)_{n\in\mathbb{Z}}$ and $(E_n)_{n\in\mathbb{Z}}$ the corresponding families of t-norms and natural indistinguishability operators generated by $T$ and $f$ respectively. then for all $n \in \mathbb{Z}$ $T_n$ belong to the Schweizer and Sklar family of t-norms [**?**]:*

*For $n \geq 0$,*

$$T_n(x,y) = Max((x^{2n} + y^{2n} - 1, 0))^{\frac{1}{2n}},$$

*$t_n(x) = 1 - x^{2n}$ is an additive generator of $T_n$ and*

$$E_n(x,y) = \left(1 - |x^{2n} - y^{2n}|\right)^{\frac{1}{2n}}.$$

*For $n \leq 0$,*

$$T_n(x,y) = Max((x^{-\frac{1}{2n}} + y^{-\frac{1}{2n}} - 1, 0))^{-2n},$$

*$t_n(x) = 1 - x^{-\frac{1}{2n}}$ is an additive generator of $T_n$ and*

$$E_n(x,y) = \left(1 - |x^{-\frac{1}{2n}} - y^{-\frac{1}{2n}}|\right)^{-2n}.$$

Since the inverse map $f^{-1}$ of an automorphism $f$ is an automorphism as well, we can relate the families generated by $f$ and $f^{-1}$.

**Proposition 6.7.** *Let $T$ be a t-norm and $f$ an automorphism of $[0,1]$, $(T_n)_{n\in\mathbb{Z}}$ and $(E_n)_{n\in\mathbb{Z}}$ the families of t-norms and natural $T_n$-indistinguishability operators generated by $T$ and $f$. The families of t-norms and natural $T_n$-indistinguishability operators generated by $T$ and $f^{-1}$ are $(T_{-n})_{n\in\mathbb{Z}}$ and $(E_{-n})_{n\in\mathbb{Z}}$.*

*Proof.* Trivial. □

Families generated by $f$ and $f^{-1}$ will be called reversed families.

**Example 6.8.** *Let $T$ be the Łukasiewicz t-norm and $f(x) = \sqrt{x}$ and $(T_n)_{n\in\mathbb{Z}}$ and $(E_n)_{n\in\mathbb{Z}}$ the corresponding families of t-norms and natural indistinguishability operators generated by $T$ and $f$ respectively. $(T_n)_{n\in\mathbb{Z}}$ and $(E_n)_{n\in\mathbb{Z}}$ are the reversed families of Example 6.6.*

In order to get a *graded* family of t-norms or indistinguishability operators, we need to impose monotonicity.

Let us study which conditions must $f$ fulfill to obtain monotonicity on the families of t-norms and natural indistinguishability operators.

The following result is well known:

**Proposition 6.9.** *[?] Let $T$ and $T'$ be two continuous Archimedean t-norms with additive generators $t$ and $u$ respectively. $T \leq T'$ if and only if $t \circ u^{[-1]}$ is a subadditive map.*

**Definition 6.10.** *Let $T$ be a t-norm and $f$ an automorphism of $[0,1]$. The family $(T_n)_{n\in\mathbb{Z}}$ of t-norms generated by $T$ and $f$, $(T_n)_{n\in\mathbb{Z}}$, is an increasing (decreasing) graded family if and only if $T_n(x,y) \leq T_m(x,y)$ ($T_n(x,y) \geq T_m(x,y)$) $\forall n \leq m$.*

**Proposition 6.11.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$ and $f$ an automorphism of $[0,1]$. $(T_n)_{n\in\mathbb{Z}}$ is an increasing graded family of t-norms generated by $T$ and $f$ if and only if $t \circ f^{-1} \circ t^{[-1]}$ is a subadditive map.*

*Proof.* $(T_n)_{n\in\mathbb{Z}}$ is an increasing family of t-norms if and only if $T_n \leq T_{n+1}$, $n \in \mathbb{Z}$.

$t \circ f^n$ and $t \circ f^{n+1}$ are additive generators of $T_n$ and $T_{n+1}$ respectively. Then $T_n \leq T_{n+1}$ if and only if $t \circ f^n \circ (t \circ f^{n+1})^{[-1]}$ is a subadditive map. But

$$t \circ f^n \circ (t \circ f^{n+1})^{[-1]} = t \circ f^n \circ f^{-n-1} \circ t^{[-1]} = t \circ f^{-1} \circ t^{[-1]}$$

□

Similarly,

**Proposition 6.12.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$ and $f$ an automorphism of $[0,1]$. $(T_n)_{n\in\mathbb{Z}}$ is a decreasing graded family of t-norms generated by $T$ and $f$ if and only if $t \circ f \circ t^{[-1]}$ is a subadditive map.*

**Proposition 6.13.** *Let $T$ and $T'$ be two continuous Archimedean t-norms with additive generators $t$ and $u$ respectively. $E_T \leq E_{T'}$ if and only if $u \circ t^{-1}$ is a subadditive map.*

*Proof.*

$$t^{-1}\left(|t(x) - t(y)|\right) \leq u^{-1}\left(|u(x) - u(y)|\right)$$
$$u \circ t^{-1}\left(|t(x) - t(y)|\right) \geq |u(x) - u(y)|$$

If $t(x) = a$ and $t(y) = b$, then

$$u \circ t^{-1}\left(|(a-b)|\right) \geq \left|u \circ t^{-1}(a) - u \circ t^{-1}(b)\right|$$

If $a \geq b$, let $c = a - b$. Then, since $f = u \circ t^{-1}$ is a non-decreasing map, the last inequality reads

$$f(c) \geq f(b+c) - f(b)$$

or

$$f(b+c) \leq f(b) + f(c).$$

□

**Corollary 6.14.** *Let $T$ and $T'$ be two continuous Archimedean t-norms. $T \leq T'$ if and only if $E_T \geq E_{T'}$ and $T \geq T'$ if and only if $E_T \leq E_{T'}$.*

**Definition 6.15.** *Let $T$ be a t-norm and $f$ an automorphism of $[0,1]$. The family $(E_n)_{n\in\mathbb{Z}}$ of $T$-indistinguishability operators generated by $T$ and $f$ is an increasing (decreasing) graded family if and only if $E_n(x,y) \leq E_m(x,y)$ $(E_n(x,y) \geq E_m(x,y))$ $\forall n \leq m$.*

**Corollary 6.16.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$ and $f$ an automorphism of $[0,1]$. $(E_n)_{n\in\mathbb{Z}}$ is an increasing (decreasing) graded family of $T_n$-indistinguishability operators generated by $T$ and $f$ if and only if $t \circ f \circ t^{[-1]}$ $(t \circ f^{-1} \circ t^{[-1]})$ is a subadditive map.*

**Example 6.17.** *Since $h(x) = 1 - (1-x)^2$ is subadditive in $[0,1]$, the families of Examples 6.6 and 6.8 are graded ones.*

**Corollary 6.18.** *Let $T$ be a non-strict continuous Archimedean t-norm, $t$ the additive generator of $T$ with $t(0) = 1$ and $(T_n)_{n\in\mathbb{N}}$ the graded family of t-norms generated by $T$ and $t^2$. Then $t^2$ or $t^{-2}$ is a subadditive map.*

**Corollary 6.19.** *Let $T$ be a non-strict continuous Archimedean t-norm, $t$ the additive generator of $T$ with $t(0) = 1$ and $(E_{T_n})_{n\in\mathbb{N}}$ the graded family of $T_n$-indistinguishability operators by $T$ and $t^2$. Then $t^2$ or $t^{-2}$ is a subadditive map.*

A natural way to generate families associated to a non-strict continuous Archimedean t-norm $T$ would be considering that if $t$ is the normalized additive generator of $T$ (i.e.: $t(0) = 1$), then $t^2$ is an automorphism of $[0,1]$. Unfortunately, if we impose monotonicity to the families they turn out to be trivial. This will be stated in Proposition 6.24 as a consequence of the following results.

**Proposition 6.20.** *Let $T$ be a non-strict continuous Archimedean t-norm and $t$ the additive generator of $T$ with $t(0) = 1$. Then the family of t-norms generated by $T$ and $t^2$ is $(T_n)_{n\in\mathbb{Z}}$ with*

$$T_n(x,y) = t^{[-(2n+1)]}\left(t^{2n+1}(x) + t^{2n+1}(y)\right).$$

*Proof.* $t^{2n+1}$ is the normalized additive generator of $T_n$ $\forall n \in \mathbb{Z}$. □

**Proposition 6.21.** *Let $T$ be a non-strict continuous Archimedean t-norm and $t$ the additive generator of $T$ with $t(0) = 1$. Then the family $(E_n)_{n\in\mathbb{Z}}$ of natural $T_n$-indistinguishability operators generated by $T$ and $t^2$ is*

$$E_n(x,y) = t^{[-2n+1]}\left(\left|t^{2n+1}(x) - t^{2n+1}(y)\right|\right)$$

*Proof.* $t^{2n+1}$ is the normalized additive generator of $T_n$ $\forall n \in \mathbb{Z}$. □

**Definition 6.22.** *[?] A decreasing map $n : [0,1] \to [0,1]$ with $n(0) = 1$ and $n(1) = 0$ is called a negation.*

- *If $n^2(x) \geq x$ $\forall x \in [0,1]$, $n$ is called a weak negation.*
- *If $n^2(x) = x$ $\forall x \in [0,1]$, $n$ is called a strong negation.*

**Proposition 6.23.** *[?] A continuous weak negation is a strong negation.*

Corollaries 6.18 and 6.19 restrict the use of $t^2$ dramatically, since they imply that $t^2 = id$ and therefore graded families generated by $T$ and $t^2$ are constant ($T_n = T$ and $E_{T_n} = E_T$ for all $n \in \mathbb{N}$):

**Proposition 6.24.** *Let $T$ be a non-strict continuous Archimedean t-norm, $t$ the additive generator of $T$ with $t(0) = 1$. $(T_n)_{n\in\mathbb{N}}$ and $(E_{T_n})_{n\in\mathbb{N}}$ are graded families generated by $T$ and $t^2$ if and only if $t$ is a strong negation. In this case, $T_n = T$ and $E_{T_n} = E_T$ for all $n \in \mathbb{Z}$.*

*Proof.* Being $t^2$ a continuous increasing map with $t^2(0) = 0$, subadditivity implies $t^2(x) \geq x \ \forall x \in [0, 1]$ and due to Proposition 6.23 $t$ is a strong negation and $t^2 = identity$. □

Let us finish this section with an example of an increasing graded family.

**Example 6.25.** *Let $(E_n)_{n\in\mathbb{Z}}$ be the graded family of indistinguishability operators generated by the Lukasiewicz t-norm and the automorphism $f(x) = \sqrt{x}$ for all $x \in [0, 1]$. Then*

1. $E_n \leq E_m$ *if* $n \leq m$.
2. $\lim_{n\to\infty} E_n(x, y) = 1$.
3. *If* $x \neq y$, *then* $\lim_{n\to-\infty} E_n(x, y) = \min(x, y)$.

*Proof.* $E_n(x, y) = (1 - |x^{2n} - y^{2n}|)^{\frac{1}{2n}}$.

1. Without loss of generality we can assume that $x > y$. Then $E_n(x, y) = (1 - x^{2n} + y^{2n})^{\frac{1}{2n}}$. The derivative of $E_n(x, y)$ with respect to $n$ is

$$-\frac{(-x^{2n} + y^{2n} + 1)^{\frac{1}{2n}} \ln(-x^{2n} + y^{2n} + 1))}{2n^2} - \frac{x^{2n} \ln x(-x^{2n} + y^{2n} + 1)^{\frac{1}{2n}-1}}{n} + \frac{y^{2n} \ln y(-x^{2n} + y^{2n} + 1)^{\frac{1}{2n}-1}}{n}$$

which is positive since $h(x) = x^{2n} \ln x$ is a decreasing function in $[0, 1]$.

2. $\lim_{n\to\infty} E_n(x, y) = \lim_{n\to\infty}(1 - |x^{2n} - y^{2n}|)^{\frac{1}{2n}} = 1^0 = 1$.

3. Without loss of generality we can assume that $x > y$. Then $E_n(x,y) = (1 - x^{2n} + y^{2n})^{\frac{1}{2n}}$. Writing $X = \frac{1}{x}$ and $Y = \frac{1}{y}$,

$$\begin{aligned}\lim_{n\to-\infty}(1-|x^{2n}-y^{2n}|)^{\frac{1}{2n}} &= \lim_{n\to\infty}(1-\frac{1}{x^{2n}}+\frac{1}{y^{2n}})^{-\frac{1}{2n}}\\ &= \lim_{n\to\infty}(1-X^{2n}+Y^{2n})^{-\frac{1}{2n}} \quad (1).\end{aligned}$$

The logarithm of this expression is

$$\begin{aligned}\ln(\lim_{n\to\infty}(1-X^{2n}+Y^{2n})^{-\frac{1}{2n}}) &= \lim_{n\to\infty}\ln(1-X^{2n}+Y^{2n})^{-\frac{1}{2n}}\\ &= \lim_{n\to\infty} -\frac{\ln(1-X^{2n}+Y^{2n})}{2n}\end{aligned}$$

Applying L'Hôpital rule to the last limit, we get

$$\begin{aligned}\lim_{n\to\infty} -\frac{\ln(1-X^{2n}+Y^{2n})}{2n} &= \lim_{n\to\infty}\frac{X^{2n}\ln X - Y^{2n}\ln Y}{1-X^{2n}+Y^{2n}}\\ &= \lim_{n\to\infty}\frac{(\frac{X}{Y})^{2n}\ln X - (\frac{X}{Y})^{2n}\ln Y}{(\frac{1}{Y})^{2n} - (\frac{X}{Y})^{2n} + (\frac{Y}{Y})^{2n}}\end{aligned}$$

Since $Y > X \geq 1$, $\lim_{n\to\infty}(\frac{X}{Y})^{2n} = \lim_{n\to\infty}(\frac{1}{Y})^{2n} = 0$ and hence

$$\lim_{n\to\infty}\frac{(\frac{X}{Y})^{2n}\ln X - (\frac{X}{Y})^{2n}\ln Y}{(\frac{1}{Y})^{2n} - (\frac{X}{Y})^{2n} + (\frac{Y}{Y})^{2n}} = -\ln Y.$$

(1) is therefore $e^{-\ln Y} = \frac{1}{Y} = y$.

□

*Properties 6.25.2 and 6.25.3 mean that when $n \to \infty$, $E_n$ tends to the natural indistinguishability operator associated to the smallest t-norm $T_D$, while when $n \to -\infty$, $E_n$ tends to the natural indistinguishability operator associated to the greatest t-norm Minimum.*

*From the results of this section we obtain the following result.*

*Let $(T_n)_{n\in\mathbb{Z}}$ be the corresponding graded family of indistinguishability operators. Then*

*1.* $\lim_{n\to\infty} T_n(x,y) = T_D(x,y)$.

*2.* $\lim_{n\to-\infty} T_n(x,y) = \min(x,y)$.

# 7 Families of $T$-Indistinguishability Operators and Extensionality

Extensionality is the most important property that a fuzzy subset can satisfy with respect to a given indistinguishability operator $E$ on a set $X$. The granularity generated by $E$ restrict the fuzzy subsets that can be observed in $X$ to the ones that are extensional with respect to it. Since we are interested in the different precisions in which we can observe the set $X$ with respect to the different elements of a nested family $(E_i)$ of indistinguishability operators, it is interesting to study the relationship between the extensional fuzzy subsets of the different $E_i$.

Let us recall the definition of extensional fuzzy subset.

**Definition 7.1.** *Let $E$ be a $T$-indistinguishability operator on a set $X$. A fuzzy subset $\mu$ of $X$ is extensional with respect to $E$ if and only if for all $x, y \in X$*

$$T(E(x, y), \mu(y)) \leq \mu(x).$$

*$H_E$ will denote the set of all fuzzy subsets of $X$ that are extensional with respect to $E$.*

**Proposition 7.2.** [?] *Let $\mu$ be a fuzzy subset of $X$ and $T$ a continuous t-norm. The fuzzy relation $E_\mu$ on $X$ defined for all $x, y \in X$ by*

$$E_\mu(x, y) = E_T(\mu(x), \mu(y))$$

*is a $T$-indistinguishability operator on $X$.*

**Proposition 7.3.** *[?] Let $E$ be a $T$-indistinguishability operator on a set $X$. A fuzzy subset $\mu$ of $X$ is extensional with respect to $E$ if and only if for all $x, y \in X$*

$$E(x, y) \leq E_\mu(x, y).$$

It is clear that if $E$ and $F$ two $T$-indistinguishability operators on a set $X$ with $E \leq F$, then an extensional fuzzy subset of $F$ is also extensional with respect to $E$ (i.e., $E \leq F$ if and only if $H_E \supseteq H_F$). If $(E_n)_{n \in \mathbb{Z}}$ is a multiresolution family (in the sense of Definition 5.1.), then the relationship between the different sets $H_{E_n}$ is very strict as stated in the following proposition.

**Proposition 7.4.** *Let $(E_n)_{n \in \mathbb{Z}}$ be a multiresolution family of $T$-indistinguishability operators on a subset $X$ of $\mathbb{R}$. Then*

1. $H_{E_n} \leq H_{E_m}$ *if* $n \leq m$
2. $H_{Id} = [0,1]^X$
3. *Let* $\mathbf{1}$ *be the universal indistinguishability operator on* $X$ *(*$\mathbf{1}(x,y) = 1$ *for all* $x, y \in X$*).* $H_{\mathbf{1}}$ *is the set of constant fuzzy subsets of* $X$*.*
4. *For a fuzzy subset* $\mu$ *of* $X$*, let* $\mu_2$ *be the fuzzy subset of* $X$ *defined for all* $x \in X$ *by* $\mu_2(x) = \mu(2x)$*. Then* $\mu \in H_{E_{n-1}}$ *if and only if* $\mu_2 \in H_{E_n}$*.*

*Proof.*

1. It is a consequence of the monotonicity of the t-norm.
2. Let $x, y \in X$ and $\mu$ a fuzzy subset of $X$.

$$T(Id(x,y), \mu(x)) = \begin{cases} 0 & \text{if } x \neq y \\ \mu(y) & \text{if } x = y. \end{cases}$$

3. Let $x, y \in X$ and $\mu$ a fuzzy subset of $X$.

$$T(\mathbf{1}(x,y), \mu(x)) = \mu(y)$$

and hence $\mu$ is a constant fuzzy subset of $X$.

4. Let $\mu \in H_{E_{n-1}}$. Then

$$T(E_{n-1}(2x, 2y), \mu(2x)) \leq \mu(2y)$$

that is equivalent to

$$T(E_n(x,y), \mu_2(x)) \leq \mu_2(y)$$

that states that $\mu \in H_{E_n}$.

□

Hence, given a multiresolution family $(E_n)_{n\in\mathbb{Z}}$ of indistinguishability operators on $X$, its corresponding family $(H_{E_n})_{n\in\mathbb{Z}}$ of sets of extensional fuzzy subsets goes from the greatest possible set ($[0,1]^X$) corresponding to the Identity or smallest indistinguishability operator to the smallest one (the set of constant fuzzy subsets of $X$) corresponding to the greatest or coarsest indistinguishability operator on $X$. At each step we can obtain the corresponding extensional fuzzy subsets $\mu$ from the ones of the previous one thanks to Property 4 of Proposition 7.4.

# 8 Concluding Remarks

In this work, several methods to build multiresolution families of indistinguishability operators have been presented. These methods use different techniques as scaling functions, powers of $T$-indistinguishability operators, families of t-norms or the fundamental theorem of representation for indistinguishabilitiy operators. A member of this kind of families is a complete and dense collection of a nested set of indistinguishability operators which are related in a dyadic way from one level to the next one.

The definition given here directly tries to capture the basic properties of a classical multiresolution system but other definitions based on a different set of axioms can be possible and it will be interesting to study new properties with the goal of building systems that allow to analyze and to present information with a different granularity.

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