Singular solutions for a class of traveling wave equations arising in hydrodynamics

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Introduction and objectives.

We study certain weak solutions for ODE of the form

$$\ddot{u}\,u+\frac{1}{2}\dot{u}^2+F'(u)=0,\quad \text{where } F\in\mathcal{C}^{\omega}(\mathbb{R}) \tag{1}$$

This equation appears as a traveling wave equation for certain PDE arising in hydrodynamics. It is *singular at $u=0$*. It can be studied via the planar system

$$\begin{cases}\dot{u}=v\\ u\,\dot{v}=-F'(u)-\frac{1}{2}v^2,\end{cases}$$

with first integral

$$H(u,v)=\frac{u\,v^2}{2}+F(u).$$

Traveling wave solutions ⇔ weak solutions

A bounded function $u\in H^1_{loc}(\mathbb{R})$ is a *TWS* if it satisfies (1) in the sense of distributions, i.e.

$$\int_{\mathbb{R}}(u^2)_t\phi_t+(u_t)^2\phi-2F'(u)\phi\,dt=0, \tag{2}$$

for any $\phi\in\mathcal{C}^{\infty}_c(\mathbb{R})$. But this concept is crude.

The equation (1) admits the order reduction

$$\frac{d}{dt}\Big(\frac{u\dot{u}^2}{2}+F(u)\Big)=\dot{u}\left(\ddot{u}\,u+\frac{1}{2}\dot{u}^2+F'(u)\right),$$

so we look for solutions $u\in H^1_{loc}(\mathbb{R})$ of (2) s.t.

$$\frac{u\dot{u}^2}{2}+F(u)=h,\quad \text{for some constant } h\in\mathbb{R}. \tag{3}$$

except at a countable number of points where the derivative is not defined but the equation is still satisfied in the limit.

Singular Solutions

A solution of (2), $u(t)\in H^1_{loc}(\mathbb{R})$, is a

(a) *strong singular* TWS of (1) if u is a classical solution of (3) on $\mathbb{R}$.

(b) *weak singular* TWS of (1) if u is a classical solution of (3) on $\mathbb{R}$ except on set of countably many t_k such that

$$\lim_{t\to t_k}\frac{u(t)\dot{u}(t)^2}{2}+F(u(t))=h. \tag{4}$$

Objective:

We study how the TWS are obtained from the orbits of the associated system, and how these orbits depend on the qualitative properties of F.

(a) Peaked waves: smooth except at a finite or countable number of points (*peaks*) $\mathcal{S}=\{t_k\in\mathbb{R},\ k\in\mathbb{Z}\}$ where

$$0\neq\lim_{t\to t_k^+}u'(t)=-\lim_{t\to t_k^-}u'(t)\neq\pm\infty,$$

(b) $\mathcal{C}^1$-Compact support and composite waves: a *composite wave* is obtained by gluing together copies of one compact solitary wave in such a way that the supports of each copy do not overlap.

(c) Fronts with finite time-decay and plateau–shaped.

(d) Cusped waves: smooth except at a finite or countable number of points (*cusps*) $\mathcal{S}=\{t_k\in\mathbb{R},\ k\in\mathbb{Z}\}$ where

$$\lim_{t\to t_k^+}u'(t)=-\lim_{t\to t_k^-}u'(t)=\pm\infty.$$

Two examples of the kind of results.

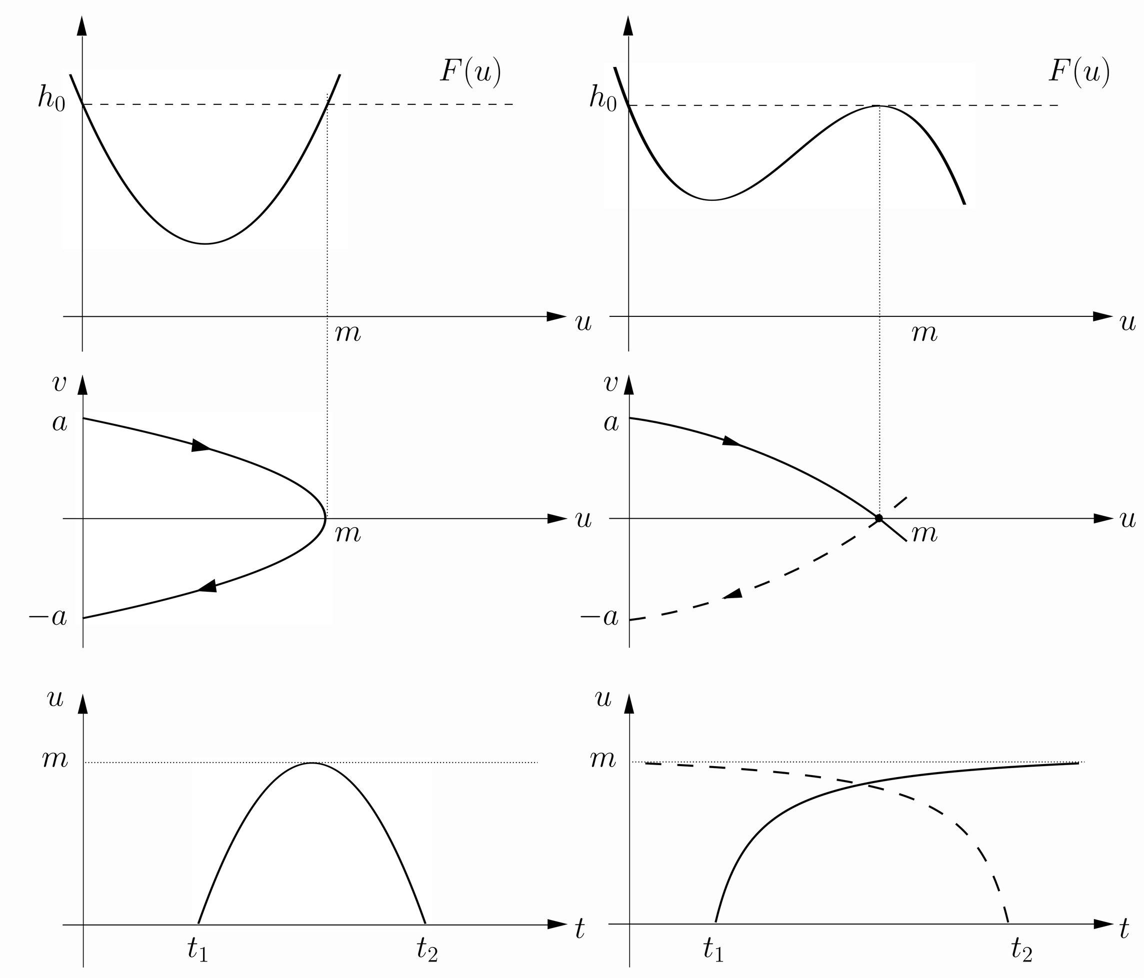
Theorem (peaked TWS)

The equation (1) has peaked TWS if and only if $F'(0)<0$ and there exists $m>0$ such that $F(m)=F(0)$ and $F(u)<F(0)$ for $u\in(0,m)$. These solutions are either

(a) peaked periodic, with period $T=\frac{2}{\sqrt{2}}\int_0^m\sqrt{\frac{u}{F(0)-F(u)}}du$, if and only if in addition $F'(m)\neq 0$, or

(b) peaked solitary if and only if in addition $F'(m)=0$.

These solutions are *weak singular TWS*.



Elementary forms for peaked waves: $F'(0)<0$

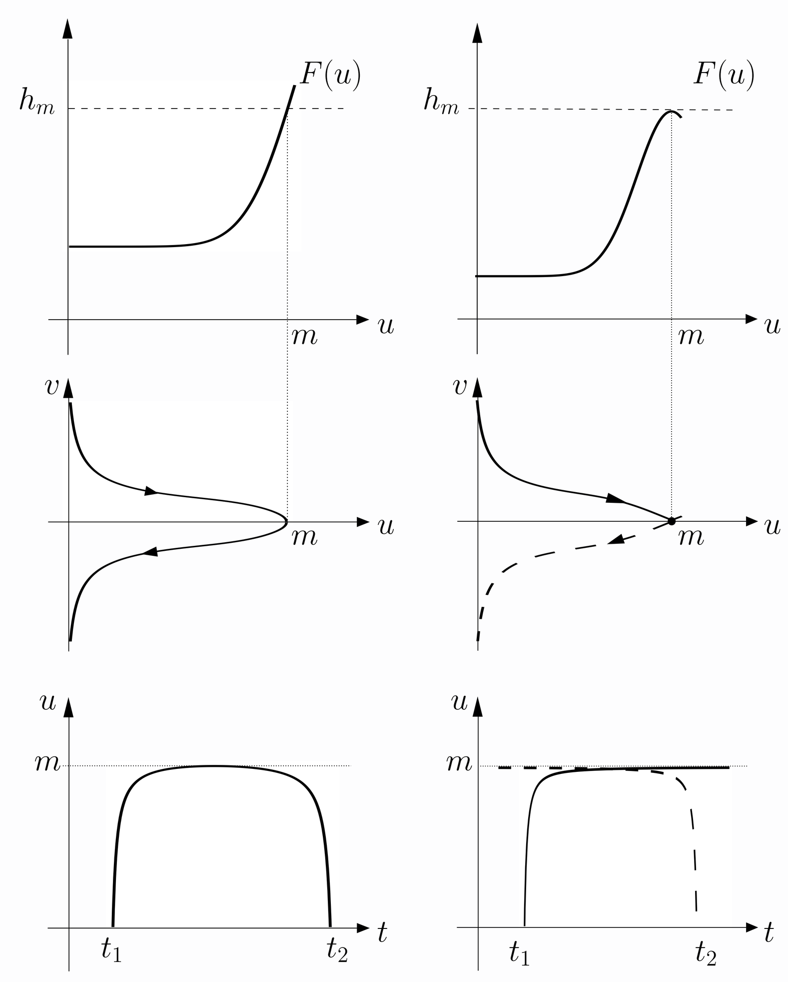
Cusped waves

The equation (1) has cusped TWS if and only if there exists $m>0$ such that $F(m)-F(u)>0$ for all $u\in[0,m)$. These solutions are either:

(a) cusped periodic with period $T=\frac{2}{\sqrt{2}}\int_0^m\sqrt{\frac{u}{F(m)-F(u)}}du$ if and only if in addition $F'(m)\neq 0$, or

(b) cusped solitary if and only if in addition $F'(m)=0$.

These solutions are *weak singular TWS*.

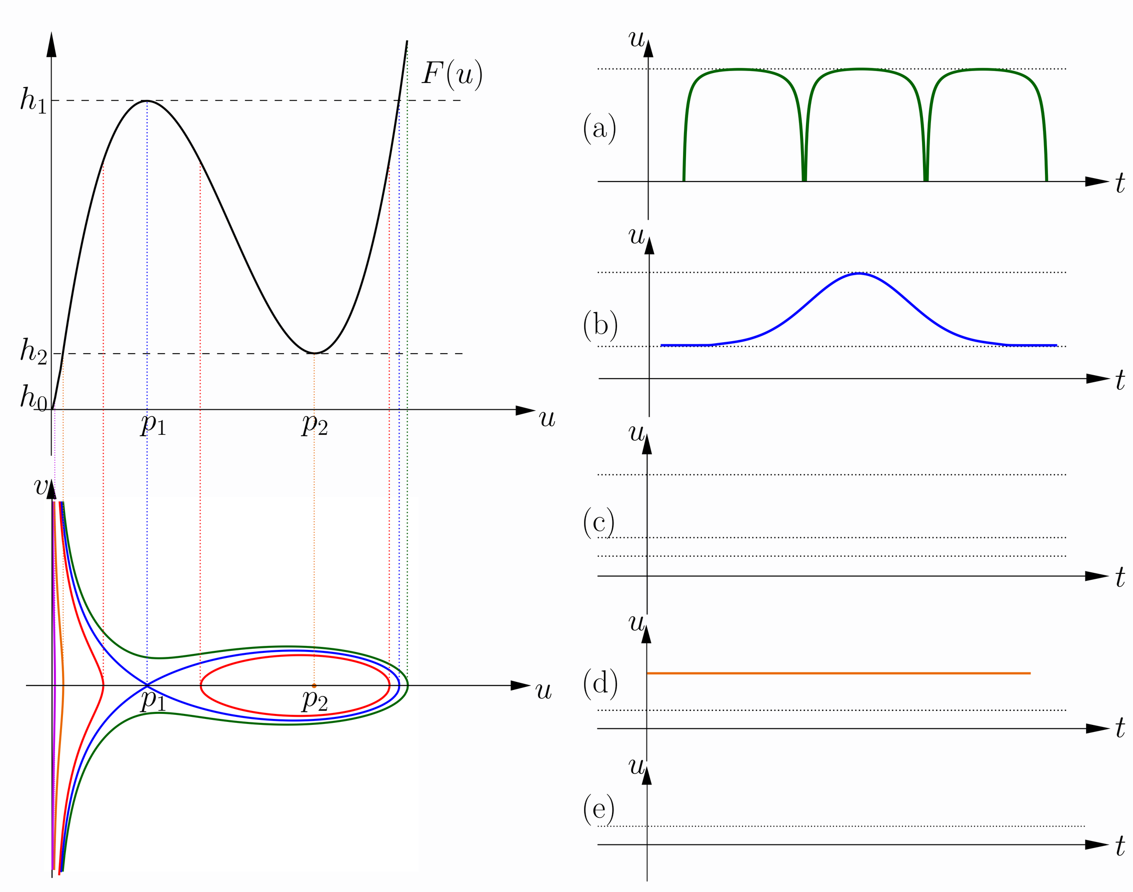


Elementary forms cusped waves: $F(m)-F(u)>0$ for all $u\in[0,m)$

An application: The equation for surface waves of moderate amplitude in shallow water,

$$u_t+u_x+6uu_x-6u^2u_x+12u^3u_x+u_{xxx}-u_{xxt}+14uu_{xxx}+28u_xu_{xx}=0,$$

was derived by Johnson (2002). Geyer (2012) and Gasull and Geyer (2014) gave a characterization of its TWS. Taking $u(x,t)=u(x-c\,t)$, integrating once, and after the change $u\mapsto u-1+c/14$ we get an equation of the type (1). For different levels of the energy in a certain case we have:



Energy	Singular TWS	Energy	Singular TWS
$h>h_1$	cusped periodic	$h=h_1$	cusped & smooth solitary
$h_1>h>h_2$	cusped & smooth periodic	$h=h_2$	cusped periodic & constant
$h_2>h>h_0$	cusped periodic	$h\leq h_0$	

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