# APPLICATION OF THE PROXIMAL POINT ALGORITHM TO THE GENERAL ASSYMETRIC TRAFFIC ASSIGNMENT PROBLEM

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## ABSTRACT

An adaptation of the proximal algorithm for the traffic assignment problem under a user equilibrium formulation for a general asymmetric traffic network is presented in this paper, following the recently published results of Pennanen regarding convergence under nonmonotonicity. As is well known the problem can be formulated as a variational inequality and the algorithmic solutions developed uptodate guarantee convergence only under too restrictive conditions which are difficult to appear in practice. In this paper new conditions guaranteing convergence are developed and the possibility of including the algorithm on a bilevel scheme is discussed

**Keywords:** Traffic assignment, variational inequality, proximal point algorithm, Bilevel programming, demand adjustment.

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## 1. Introduction

The traffic assignment problem is one of the core problems in the transportation planning process. Given a transportation network, and certain assumptions of the route choice behaviour of the tripmakers, the problem of traffic assignment is to assign traffic onto the network, so as to fulfill the demands for transportation and to minimize some merit function, based on the behavioural assumption made. Several assumptions on the route choice behaviour have been made. However, the most natural assumption is that each driver chooses the shortest perceived route to his/her destination under prevailing traffic conditions. The result from a generalized decision like that made by all the travelers yields a situation in which no driver can reduce

his/her journey time by choosing another route. This is the *user optimal criteria* for route choice. Wardrop (1952) was the first to state this route choice criteria. The asymmetric traffic assignment problem (i.e. when there are link flow interactions on the link travel costs) was first formulated as a variational inequality problem by Smith (1979b) and a number of algorithms emerged to solve the resulting formulation and basically the convergence of all them required at least the strong monotonicity of the link travel costs. The proximal point method of Martinet (1970) and its application by Rockafellar in order to develop the method of proximal multipliers has been the subject of many researchers that have applied it on a variety of problems. The recent work of Pennanen (2002) has shown that this methods present local convergence under some weaker conditions that the ones stated on previous works. On the other hand, updating an obsolete origin and destination matrix of trips within a transportation area, using available information, is a very common problem in traffic and transportation planning. For traffic networks, a source of information of a relatively moderate cost are traffic counts on a subset of links of the traffic network. The paper is organized as follows. Following this paragraphs in the introduction a subsection for the basic formulations and notation used to described the user equilibrium in an asymmetric traffic network. In section 2, the formulation as a variational inequality is developed reproducing already known results mainly due to Smith (1979b) as well as conditions for existence and uniqueness of solutions. In section 3 the convergence conditions of many algorithms to solve the variational inequality formulations of the traffic assignment problem are revisited (linearization methods, projecton methods and diagonalization methods and simplicial decomposition methods). Section number 4 describes the recently developed convergence conditions for the proximal point algorithm for inclusions due to Pennanen (2002). His results basically state that the convergence can be even if the application map of the inclusion is hypomonotone, while up to date conditions ensured convergence only under maximal monotonicity conditions. In section number 5 the application of the proximal point algorithm is shown for the case of the demand adjustment problem formulated as a bilevel program in which the lower level (the traffic assignment problem) is formulated as a variational inequality in which the link travel costs are strongly monotone.

## 1.1 Formulations

The development of general cost functions was due to the unrealistic assumption that the travel time on a link is independent of the flow on other links. One just has to imagine traffic near a turning intersection or a narrow two-way street to realize this. One reason for the late development of models incorporating more general cost functions may be the difficulty of catching a realistic cost relationship.
Consider the Wardrop's user equilibrium, as it is discussed by authors as Smith and Heydecker in references Smith (1979b) and Heydecker (1986):

> *A traffic distribution is a Wardrop equilibrium when no driver has a less costly alternative route.*

Assume a transportation network $\mathcal{G}$ with a single mode of transit and a fix demand $\mathcal{D}$. Consider a specific origin destination pair $(i, j)$ and $\mathcal{R}_{ij}$ the set of available paths joining OD pair $(i, j)$. The route flow vector $\mathbf{h}$ induces flows $v_a$ on each link $a \in \mathcal{A}$ given by the expression,

$$v_a = \sum_{(i, j) \in \mathcal{C}} \sum_{r \in \mathcal{R}_{ij}} \delta_{ar} \, h_{ijr} \tag{1}$$

where $\delta_{ar} = 1$ if link a belongs to path r; and $\delta_{ar} = 0$ otherwise. In words, $v_a$ is the sum of flows $h_{ijr}$ on all paths $r$, over all OD pairs $(i, j)$.

Let $\mathbf{v}$ denote the vector of arc flows and let $\mathbf{\Delta}$ denote the link path incident matrix. Then in vector form, link flows and path flows are related by the following expression

$$\mathbf{v} = \mathbf{\Delta} \, \mathbf{h}$$

Let each unit of flow on link $a$ incur a travel cost $c_a(\mathbf{v})$ which depends upon the vector $\mathbf{v}$ of link flows in the network. In the separable traffic assignment problem, the cost on a link depends solely upon the flow $v_a$ on that link, in which case it usually increases with increased levels of the link flow.

If we now assume that the cost on any path of the network, as a function of path flows, is the sum of travel costs on the links of that path, then

$$C_{ijr} = \sum_{a \in \mathcal{A}} \delta_{ar} \, c_a(v_a)$$

We refer to this form of route costs as an *additive model*. Stating this relationship more compactly in vector form, we obtain

$$\mathbf{C}(\mathbf{h}) = \mathbf{\Delta}^\top \mathbf{c}(\mathbf{v}),$$

where $^\top$ denotes transposition.

In this expression, $\mathbf{C}(\mathbf{h}) = (\,C_{ijr}(\mathbf{h})\,)$ is a vector-valued function specifying the travel costs on each path $r$ and $\mathbf{c}(\mathbf{v}) = (\,\mathbf{v}\,)$ is a vector valued function whose components specify the link travel costs. The route travel costs $C_{ijr}(\mathbf{h})$ on each route $r$ joining an OD pair $(i, j)$ defines the least travel cost $u_{ij}$ over all paths joining that OD pair. This is

$$u_{ij} \equiv \min_{r \in \mathcal{R}_{ij}} \ C_{ijr}(\mathbf{h})$$

These least travel costs certainly provide a reference point against which to measure any route's ability to attract trips. If different routes are used for a given OD pair $(i, j)$, the travel cost of the routes used is $u_{ij}$, and Wardrop's equilibrium may be rewritten as

$$h_{ijr} > 0 \ \Rightarrow \ C_{ijr} = u_{ij}$$
$$h_{ijr} = 0 \ \Rightarrow \ C_{ijr} \geq u_{ij}$$

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or equivalently as

$$
\begin{align}
h_{ijr} \cdot (C_{ijr} - u_{ij}) &= 0 && \forall\, r \in \mathcal{R}_{ij} \;\; ; \forall\, (i,j) \in \mathcal{C} && (2) \\
C_{ijr} - u_{ij} &\geq 0 && \forall\, r \in \mathcal{R}_{ij} \;\; ; \forall\, (i,j) \in \mathcal{C} && (3) \\
\sum_{r \in \mathcal{R}_{ij}} h_{ijr} &= d_{ij} && \forall\, (i,j) \in \mathcal{C} && (4) \\
h_{ijr} &\geq 0 && \forall\, r \in \mathcal{R}_{ij} \;\; ; \forall\, (i,j) \in \mathcal{C} && (5) \\
u_{ij} &\geq 0 && \forall\, (i,j) \in \mathcal{C} && (6)
\end{align}
$$

# 2.  Variational Inequality Formulation [TAP-VI]

Wardrop conditions for the general traffic assignment model can be reformulated into three main types of equivalent problems: variational inequality formulations, (Smith (1979b), Dafermos (1980) (1982), Florian and Spiess (1982)), nonlinear complementary formulations (Aashtiani and Magnanti (1981)) and fixed point formulations ( Asmuth 1978 and Fisk and Nguyen (1980)). The variational inequality formulations have become very popular, due to the large literature devoted to **VIP** , compacteness for writing and flexibility in choosing the ground set **X**.

**Theorem 1.1** *Wardrop conditions for user equilibrium are equivalent to the following variational inequality formulation, defining the general traffic assignment problem* [**TAP-VI**] *,*

$$
Find\ \boldsymbol{h}^* \in \mathcal{H}\ s.\,t.\ \boldsymbol{C}(\boldsymbol{h}^*)^{\top}(\boldsymbol{h} - \boldsymbol{h}^*) \geq 0\ \ \forall\, \boldsymbol{h} \in \mathcal{H} \tag{7}
$$

*and*

$$
\mathcal{H} \overset{\text{def}}{=} \{\, \mathbf{h} \mid \sum_{r \in \mathcal{R}_{ij}} h_{ijr} = d_{ij},\ h_{ijr} \geq 0,\ \ \forall\, r \in \mathcal{R}_{ij}\ ,\ \forall\, (i,j) \in \mathcal{C} \,\} \tag{8}
$$

It should be noted that the variational inequality formulation [**TAP-VI**] is independent of the form of the cost functions $C_{ijr}(\mathbf{h})$, this is, the travel cost on a route need not be additive. The additivity property is used in our formulations and is assumed to hold when developing algorithmic approaches. There are two main reasons for this:

1. Under the non additive assumption there does not exist a transformation of the above arc route formulation of [**TAP-VI**] in terms of arc flows.

2. On working with large networks, as our case, the arc route formulations of traffic assignment models involves very large scale problems, and as far as, it is unknown if an efficient algorithm could be designed for solving large general problems, it has been preferred to remove the extra size due to the route flow model and concentrate efforts on solving the aggregated arc flow model.

We state the arc flow variational inequality formulation for the traffic assignment problem when travel costs on a route satisfy the additivity property. This formulation is studied below and serves as the basis for developing algorithms to compute equilibrium solutions to [**TAP-VI**].

$$\text{Find } \mathbf{v}^* \in \mathcal{V} \text{ s. t. } \mathbf{c}(\mathbf{v}^*)^\top \cdot (\mathbf{v} - \mathbf{v}^*) \geq 0 \ \ \forall \mathbf{v} \in \mathcal{V} \tag{9}$$

and

$$\mathcal{V} \stackrel{\text{def}}{=} \{ \mathbf{v} \mid v_a = \sum_{(i,j) \in \mathcal{C}} \sum_{r \in \mathcal{R}_{ij}} \delta_{ar} \, h_{ijr} \ \ \forall a \in \mathcal{A} , \ \ \mathbf{h} \in \mathcal{H} \} \tag{10}$$

Smith (1979b) discusses the properties of equilibrium solutions and formulates the problem [**TAP-VI**] for both the arc node case and the arc route case for additive route costs. Dafermos (1980) develops an equivalent finite dimensional variational inequality formulation. Formulations for the elastic case have been proposed by Dafermos and Florian. Dafermos and Nagurney develop an arc flow formulation, equivalent to the above model, and study the stability and sensitivity of equilibria, based on that formulation.

In this point we introduce the primal and dual gap functions $\mathbf{G_P}(\mathbf{x})$ and $\mathbf{G_D}(\mathbf{x})$ for [**TAP-VI**].

$$\mathbf{G_P}(\mathbf{v}) = - \min_{w \in \mathcal{V}} \mathbf{c}(\mathbf{v})^\top \cdot (\mathbf{w} - \mathbf{v}) \tag{11}$$

$$\mathbf{G_D}(\mathbf{w}) = - \max_{v \in \mathcal{V}} \mathbf{c}(\mathbf{v})^\top \cdot (\mathbf{w} - \mathbf{v}) \tag{12}$$

The primal gap function may be interpreted in this context as the difference of total travel costs between the current flow $\mathbf{v}$ and the shortest route flow $\mathbf{z}$. Thus a positive gap function corresponds to a potential benefit for some travelers in adjusting their route choices. On the other hand, $\mathbf{G}(\mathbf{v}) = 0$ precisely when no traveler has an incentive to change route, that is, when the flow satisfies Wardrop conditions of equilibria. Viewing the gap function in terms of an error in Wardrop conditions, Hearn suggests the use of this formulation even in the separable case, since the objective function of its mathematical formulation is rather artificial.

Algorithms based on primal gap function are given Hearn (1982); methods based on simplicial descomposition are given by Lawphongpanich and Hearn (1984), Marcotte and Guelat (1988) and Pang and Yu (1984). Cutting plane methods are given by Nguyen and Dupuis (1984) and Hearn and Lawphongpanich (1989).

Now, we present conditions for the existence and uniqueness of the variational inequality formulation [**TAP-VI**]. Although they are, in the traffic equilibrium context, essentially equivalent, the existence results presented differ. It seems that too strong assumptions have sometimes been put on the model when establishing the existence of a solution, mainly because these conditions serves to validate the proposed algorithms as well.

We recall that sufficient conditions for the existence of a solution to **VIP**, where based on:

- Boundedness of the feasible set.

- Sufficient monotonicity of the mapping defining the problem.

The feasible set in the traffic assignment problem is, in general, not bounded, due to the existence of cycles in the network. Anyway, we can ensure the existence of a solution to [**TAP**], by restricting the cost functions $\mathbf{c}(\mathbf{v})$ or $C_{ijr}(\mathbf{h})$ to be strictly positive for all feasible flows. The implication of this is that an optimal solution cannot include cyclic flows and the variables can therefore be restricted as follows,

$$0 \leq v_a \leq \sum_{(i,j) \in \mathcal{C}} d_{ij}, \ \ \forall\, a \in \mathcal{A}$$

Lawphongpanich and Hearn (1984) consider for [**TAP-VI**] how the original problem can be replaced by restricted problems over sets of simple or loop-free routes. In convergence results or existence and uniqueness results, boundness it is always assumed.

Existence and uniqueness results listed below are extracted from Aashtiani and Magnanti (1981) and Smith (1979b). The graph $\mathcal{G}$ is assumed to be *strongly connected,* i.e., that there exists at least one route connecting each origin and destination. The problem [**TAP-VI**] has an equilibrium solution $\mathbf{v}^{(*)}$ under one of the following conditions:

1. Under additive cost model and $\mathbf{c}(\mathbf{v})$ continuous, positive and $d_{ij} \geq 0$ (Aashtiani and Magnanti).

2. Under additive cost model and $\mathbf{c}(\mathbf{v})$ continuous, nonnegative and the feasible set closed and convex (Smith).

3. Under nonadditive cost model and $\mathbf{C}(\mathbf{h})$ continuous, positive and $d_{ij} \geq 0$ (Aashtiani and Magnanti).

Uniqueness results are similar to those obtained for the separable case, where strictly increasing condition on $c_a(v_a)$ is substituted by the condition of strictly monotonicity of $\mathbf{c}(\mathbf{v})$. Results listed below proceed from the same sources mentioned before:

1. Under additive cost model and $\mathbf{c}(\mathbf{v})$ continuous, nonnegative, strictly monotone and the feasible set closed and convex, the equilibrium link flow solution $\mathbf{v}^{(*)}$ is unique (Smith.)

2. Under additive cost model and $\mathbf{c}(\mathbf{v})$ continuous, positive, strictly monotone and $d_{ij} \geq 0$, the link flow volumes $\mathbf{v}^{(*)}$ and the accessibility vector $\boldsymbol{u}$ are unique (Aashtiani and Magnanti.)

Observe that the results require that the vector $\mathbf{c}(\mathbf{v})$ of cost functions be strictly monotone in terms of link volumes $\mathbf{v}$, not in terms of path flows $\mathbf{h}$. Path flows need not to be unique, since two collections of path flows might correspond to the same link flows.

# 3. Convergence conditions for [TAP - VI] algorithms

The first methods applied to the general finite dimensional variational inequality problem were based on fixed point problem reformulations, but for large scale problems, these algorithms are impractical, due to the large memory requirements, and their failure to utilize problem structure. Efficient methods for **VIP** may be grouped into the following methodology classes:

- Linearization methods. They are based on iterative approximation of the mapping defining **VIP** by affine mappings at the current point.

- Diagonalization methods. Similar to the algorithms proposed for solving equations: Extensions of Jacobi, Gauss-Seidel and Newton methods.

- Simplicial Decomposition methods.

- Dual cutting plane methods. These methods apply to solving the maximization of the dual gap function, which it is equivalent to solving [**TAP-VI**] . Convergence results require strict monotonicity and the set of feasible arc flows to be a compact polyhedron.

- Gap descent Newton method. Essentially, the method generates a sequence such that each iterate is the solution to a variational inequality problem involving a linear approximation of the functional cost at the previous iterate, whose solution is used to define a descent direction to the gap function. Monotonicity and continuous differentiability of the functional cost are required for global convergence results.

## 3.1 Linearization Methods

Several variants are described in reports, but all of them can be cast as special cases of the following scheme:

> **Algorithm.** Given $\mathbf{x}^{(t)} \in X$, let $\mathbf{x}^{(t+1)}$ solve the variational inequality subproblem $\mathbf{VI}(\mathbf{f}^{(t)}, \mathbf{X})$, where $\mathbf{f}^{(t)}(\mathbf{x})$ is some mapping approximating the original $\mathbf{f}(\mathbf{x})$ at the point $\mathbf{x}^{(t)}$.

Presumably, each subproblem $\mathbf{VI}(\mathbf{f}^{(t)}, \mathbf{X})$ is numerically easier to solve than the original problem $\mathbf{VI}(\mathbf{f}, \mathbf{X})$.

A linearization method is based on an iterative approximation of $\mathbf{f}(\mathbf{x})$ on the current iteration point $\mathbf{x}^{(t)}$, called $\mathbf{f}^{(t)}(\mathbf{x})$

The method is classified as a *linear approximation method* if $\mathbf{f}^{(t)}(\mathbf{x})$ is of the form

$$\mathbf{f}(\mathbf{x}) \sim \mathbf{f}^{(t)}(\mathbf{x}) = \mathbf{c} + \mathbf{A}^{(t)}\mathbf{x}$$

where $\mathbf{A}^{(t)}$ is a constant matrix at iteration $t$ and $\mathbf{c} = \rho\mathbf{f}(\hat{\mathbf{x}}) - \mathbf{A}^{(\mathbf{t})}\hat{\mathbf{x}}$ for $\hat{x} \in X$ and $\rho > 0$.

Included in the family of linear approximation methods are:

**Newton method.** The mapping $\mathbf{f}(\mathbf{x})$ is assumed to be differentiable and $\mathbf{A}^{(t)} = \nabla\mathbf{f}(\mathbf{x}^{(t)})$. The linear approximation is in this case the first order Taylor expansion of $\mathbf{f}(\mathbf{x})$ around $\mathbf{x}^{(t)}$.

**Quasi-Newton methods.** $\mathbf{A}^{(t)}$ is taken to be an approximation to the Jacobian matrix $\nabla\mathbf{f}(\mathbf{x}^{(t)})$.

**Linear SOR methods.** Successive overrelaxation methods in which the matrix $\mathbf{A}^{(t)}$ is taken to be

$$\mathbf{A}^{(t)} = \left\{ \begin{array}{ll} \mathbf{L}^{(t)} + \mathbf{D}^{(t)}/\omega^* & \\ \mathbf{U}^{(t)} + \mathbf{D}^{(t)}/\omega^* & 0 < \omega^* < 2 \end{array} \right.$$

Assuming $\mathbf{L}^{(t)}$, $\mathbf{U}^{(t)}$ and $\mathbf{D}^{(t)}$ are respectively the strictly lower and upper triangular parts and diagonal of $\nabla\mathbf{f}(\mathbf{x}^{(t)})$. When $\omega^*$ is taken to be 1, the scheme becomes *the linearized Gauss-Seidel method*.

**Linear Jacobi method.** The matrix $\mathbf{A}^{(t)}$ is taken to be the diagonal of the Jacobian matrix at the current point, i.e., $\mathbf{A}^{(t)} = \mathrm{Diag}(\nabla\mathbf{f}(\mathbf{x}^{(t)}))$.

**Projection methods.** The matrix $\mathbf{A}^{(t)}$ is taken to be a fixed symmetric and positive definite matrix $\mathbf{G}$ equal for all iterations $t$.

Although a fast local convergence is ensured for Newton's method, the problem $\mathbf{VI}(\mathbf{f}^{(t)}, \mathbf{X})$ is not trivial, and the affine variational inequality defined is not equivalent to a mathematical program, since $\nabla\mathbf{f}(\mathbf{x})$ is, in general, assymetric. Quasi-Newton methods have been proposed with an approximation matrix chosen to be symmetric, to ensure that the variational inequality defined is equivalent to a mathematical programming program. SOR methods lead to decomposition of the original subproblem into independent problems.

The reason why methods in the last category are termed projection methods is due to the following geometrical interpretation of the iterates $\{\mathbf{x}^{(t)}\}$. Indeed, it is easy to show that if the set $\mathbf{X}$ is closed and convex and if $\mathbf{A}^{(t)} = \frac{1}{\gamma}\mathbf{G}$ is symmetric and positive definite, then the vector $\mathbf{x}^{(t+1)}$ that solves the subproblem $\mathbf{VI}(\mathbf{f}^{(t)}, \mathbf{X})$ is the projection of the point $\mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)})$ onto the set $\mathbf{X}$ where the projection is defined with respect to the G-norm, i. e.,

$$\mathbf{x}^{(t+1)} = \mathbf{Pr}_{\mathbf{X}}^{\mathbf{G}} \left( \mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)}) \right)$$

where for a given vector $\mathbf{z}$, $\mathbf{Pr}_{\mathbf{X}}^{\mathbf{G}}(\mathbf{z})$ is the unique vector solving the mathematical programming program

$$\begin{array}{l} \min \| \mathbf{z} - \mathbf{y} \|_{\mathbf{G}} \\ \text{s. t. } \mathbf{y} \in \mathbf{X} \end{array}$$

and

$$\| \mathbf{x} \|_{\mathbf{G}} = (\mathbf{x}^\top \mathbf{G}\, \mathbf{x})^{\frac{1}{2}}$$

is the G-norm of the vector $\mathbf{x}$.

## 3.2 Projection methods

Projection methods are easily converted to mathematical programming equivalents. A projection algorithm for solving **VIP** sets $\mathbf{A}^{(t)} = \frac{1}{\gamma}\mathbf{G}$, for all $t$, where $\mathbf{G}$ is an arbitrary symmetric and positive definite matrix. The projection algorithm is defined by the iterative formula

$$\mathbf{x}^{(t+1)} = \mathbf{Pr}_{\mathbf{X}}^{\mathbf{G}}\left(\mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)})\right) \quad t = 1, 2, \cdots \tag{13}$$

Each iteration $t$ is equivalent to a quadratic mathematical programming program stated as,

$$\mathbf{x}^{(t+1)} = \min_{\mathbf{y} \in \mathbf{X}} \frac{1}{2} \parallel \mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)}) - \mathbf{y} \parallel_{\mathbf{G}}^{2} \tag{14}$$

defining the projection of the point $\mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)})$ onto the feasible set $\mathbf{X}$ according to the metric $\parallel \cdot \parallel_{\mathbf{G}}$.

The quadratic program is a special case of affine variational inequalities. Let us define the function $H(\cdot)$ as

$$H(\mathbf{y}) = \frac{1}{2} \parallel \mathbf{x}^{(t)} - \gamma\mathbf{G}^{-1}\mathbf{f}(\mathbf{x}^{(t)}) - \mathbf{y} \parallel_{\mathbf{G}}^{2} \quad \forall \mathbf{y} \in \mathbf{X}$$

Then, according to the correspondence between a convex program (derived from the positive definiteness of matrix $\mathbf{G}$) and the optimality conditions, $\mathbf{x}^{(t+1)}$ solves the quadratic program if and only if $\mathbf{x}^{(t+1)}$ satisfies local minima condition (that are sufficient conditions for global minimum in convex programming)

$$\nabla H(\mathbf{x}^{(t+1)})^{\top}(\mathbf{y} - \mathbf{x}^{(t+1)}) \geq 0 \,\, \forall \mathbf{y} \in \mathbf{X}$$

or equivalently,

$$\left(\mathbf{f}(\mathbf{x}^{(t)}) + \frac{1}{\gamma}\mathbf{G}(\mathbf{x}^{(t+1)} - \mathbf{x}^{(t)})\right)^{\top}(\mathbf{y} - \mathbf{x}^{(t+1)}) \geq 0 \,\, \forall \mathbf{y} \in \mathbf{X}$$

or equivalently,

$$\mathbf{f}^{(t)}(\mathbf{x}^{(t+1)})^{\top}(\mathbf{y} - \mathbf{x}^{(t+1)}) \geq 0 \,\, \forall \mathbf{y} \in \mathbf{X}$$

At this point the equivalence between a mathematical program and an specific affine variational inequality has been shown. It is interesting to note, that any linearization method formerly described can be viewed as a projection algorithm with variable metric; the metric is defined by the approximation matrix $\mathbf{A}^{(t)}$ defined at each iteration $t$, provided the matrices are symmetric and positive definite. The projection algorithm with variable metric is equivalent to the following iterative statement:

$$\mathbf{x}^{(t+1)} = \mathbf{Pr}_{\mathbf{X}}^{\mathbf{A}^{(t)}}\left(\mathbf{x}^{(t)} - (\mathbf{A}^{(t)})^{-1}\mathbf{f}(\mathbf{x}^{(t)})\right) \quad t = 1, 2, \cdots \tag{15}$$

In fact, it is not easy to solve the quadratic program equivalent to the affine variational inequality for [**TAP-VI**] and the convergence of a pure fix projection method is very poor. Most linearization algorithms applied to [**TAP-VI**] have been proposed

in the context of simplicial decomposition methods, because an affine variational inequalities defined over a convex hull of extreme flows might be converted into a quadratic program with simple constraints.

Dafermos was the first to apply general projection methods to [**TAP-VI**]. She also gave convergence properties in the context of traffic equilibrium.

Convergence results for projection algorithms are complex and it seems that conditions are rather restrictive: it is required to $\mathbf{f}(\mathbf{x})$ be continuously differentiable and strongly monotone, provided $\rho > 0$ to be sufficiently small.

## 3.3   Diagonalization Methods

Diagonalization methods have in common that the interaction among blocks of variables are removed. These algorithms, as applied to variational inequalities, extend the Jacobi and Gauss-Seidel methods for linear and nonlinear equations. They are also called *relaxation methods*, due to the relaxation of the cost interactions. Algorithms of this type were developed for in the 70's in the analysis of multiclass user networks.

Dafermos (1982) presents a unified description of diagonalization methods. She introduces a smooth function

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) : \mathbf{X} \times \mathbf{X} \mapsto \mathbb{R}^n$$

with the following properties

1. $\mathbf{g}(\mathbf{x}, \mathbf{x}) = \mathbf{f}(\mathbf{x})$, for all $\mathbf{x} \in \mathbf{X}$.

2. $\nabla_{\mathbf{x}} \mathbf{g}(\mathbf{x}, \mathbf{y})$ is positive definite and symmetric.

The latter properties ensure a unique solution of the variational inequality

$$\mathbf{g}(\mathbf{x}^{(t+1)}, \mathbf{x}^{(t)})^\top (\mathbf{x} - \mathbf{x}^{(t+1)}) \geq 0 \ \ \forall \mathbf{x} \in \mathbf{X}$$

that, by symmetry, reduces to the strictly convex program

$$\mathbf{x}^{(t+1)} = \min_{\mathbf{x} \in \mathbf{X}} H(\mathbf{x}, \mathbf{x}^{(t)})$$

where $\nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{y}) = \mathbf{g}(\mathbf{x}, \mathbf{y})$ and $\mathbf{x}^{(t)}$ solves **VIP** if $\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)}$.

Some methods that fit into Dafermos general framework are

| | | | | |
|---|---|---|---|---|
| $\mathbf{g}(\mathbf{x}, \mathbf{y})$ | $=$ | $\mathbf{f}(\mathbf{y}) + \mathbf{A}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$ | | Linearization |
| $(g_i(\mathbf{x}, \mathbf{y}))$ | $=$ | $\mathbf{f}_i(y_1, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_n)$ | $i = 1, \cdots, n$ | Nonlinear Jacobi |
| $(g_i(\mathbf{x}, \mathbf{y}))$ | $=$ | $\mathbf{f}_i(x_1, \cdots, x_i, y_{i+1}, \cdots, y_n)$ | $i = 1, \cdots, n$ | Nonlinear Gauss-Seidel |

Dafermos' convergence conditions are tested practically to be too restrictive and much more complex but less restrictive sufficient conditions are proposed by some authors, these conditions have an intuitive interpretation that the Jacobian of the cost function should have weak asymmetries and can be expressed as,

$$\left\| \nabla_{\mathbf{x}} \mathbf{g}^{-\frac{1}{2}}(\mathbf{x}^1, \mathbf{y}^1) \dot{\nabla}_{\mathbf{y}} \mathbf{g}(\mathbf{x}^2, \mathbf{y}^2) \dot{\nabla}_{\mathbf{x}} \mathbf{g}^{-\frac{1}{2}}(\mathbf{x}^3, \mathbf{y}^3) < 1, \right\|$$

for $\forall \mathbf{x}^i, \mathbf{y}^i \in \mathbf{X}$ and $i = 1, 2, 3$.

## 3.4 Simplicial Decomposition Methods

Simplicial decomposition techniques have been applied to [**TAP-VI**] by Smith (1983), Pang and Yu (1984), Bertsekas and Gafni (1982) and Lawphongpanich and Hearn (1984).

Bertsekas and Gafni utilize the arc route formulation of [**TAP-VI**] and define a master problem as the variational inequality over the current restricted set, i. e., $\mathbf{VI}(\mathbf{f}(\mathbf{x}), \mathbf{X}^{(t)})$, and apply a projection algorithm for solving the master problem. The vast number of routes makes the application of any algorithm directly to the arc route formulation intractable, but this drawback is avoided by embedding the algorithm in a simplicial decomposition scheme. The algorithm is a linearized method, where new routes are added after each iteration of the linearization scheme. No column dropping is used. Algorithms with fixed and variable projection metric are proposed for solving the master problem. The global convergence of the algorithm is shown in the fixed projection metric case, under the following assumptions :

1. $\mathbf{f}(\mathbf{x})$ strongly monotone.

2. $\mathbf{f}(\mathbf{x})$ Lipschitz continuous.

and a linear convergence rate is obtained. For the variable projection method case, a safeguard on the selection of the variable projection matrix at each iteration has to be forced in order to show a similar convergence result.

Lawphongpanich and Hearn propose an algorithm where dropping extremes is considered for extremes with zero weight in the current point if a sufficient gap function descent has been encountered in the current iteration. Two levels of aggregation are proposed:

1. Link flow extremes.

2. Commodity link flow extremes, where a commodity groups OD pairs with the same origin, we refer to them as *origin link flow* extremes.

The first option is shown to be the most efficient. The master problem is defined as the variational inequality over the current restricted set, i.e., $\mathbf{VI}(\mathbf{f}(\mathbf{x}), \mathbf{X}^{(t)})$ and the resolution of the master problem is proposed by means of a fixed metric projection method derived from that proposed by Bertsekas and Gafni.

Pang and Yu propose a linearized simplicial decompositon method similar to that of Lawphongpanich and Hearn. The aggregation level is set to link flow extremes. The master problem is defined as a linear approximation to the functional defining the variational inequality at the current point and restricted to the subset of link flow extremes, i.e.,$\mathbf{VI}(\mathbf{f}^{(t)}(\mathbf{x}), \mathbf{X}^{(t)})$. Extreme points with weight coefficients greater that zero in the current iteration are maintained to the next. The method for solving the master problem is a modified version of the Dantzig, van de Panne and Whinston algorithm for quadratic programming that operates on the simplicial representation of $\mathbf{X}^{(t)}$ in terms of its extreme points. The linear approximations proposed for defining the master problem are of the type:

1. For symmetric problems, the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ at the current point.

2. For asymmetric problems, the diagonal part of the Jacobian of $\mathbf{f}(\mathbf{x})$ at the current point.

Some remarks related to the RSD algorithm are:

- A small positive tolerance $\delta$ is required for ensuring convergence when extreme dropping is used.

- A convergent monotone sequence $\{\epsilon^{(t)}\}$ is required, i.e.,

$$\epsilon^{(t)} > \epsilon^{(t+1)} > 0, \text{ and } \epsilon^{(t)} \to 0 \text{ as } t \to \infty$$

- $\mathbf{c}(\mathbf{v})$ is assumed to be strictly monotone and nonnegative and the demand feasible set of total link flows $\mathcal{V}$ is assummed to be compact (i.e., closed and bounded) and convex.

- Any convergent method for solving variational inequalities may be applied to the master problem in Step 1, and this problem need only be solved approximately. The authors use a variant of Bertsekas and Gafni projection method.

- The subproblem in Step 2 can be decomposed into minimum cost commodity routes.

- The gap function value is a natural byproduct of solving the subproblem and provides a stopping criteria, as well as monitoring the dropping extreme process. A deep study on gap functions may be found in Hearn (1982).

- For large scale problem a *relative gap* is suggested as more suitable and defined as

$$\bar{G}(\mathbf{v}^{(t)}) = \frac{\mathbf{c}(\mathbf{v}^{(t)}) \cdot (\mathbf{v}^{(t)} - \mathbf{f}^{(t)})}{\mathbf{c}(\mathbf{v}^{(t)}) \cdot \mathbf{f}^{(t)}}$$

and a certain $\epsilon$-solution determined more practical for implementation purposes.

Lawphongpanich and Hearn results demonstrate the potential of the simplicial decomposition algorithm with regard to computer time. They also show that the method obtains good solutions while retaining only a small number of extreme flow patterns.

# 4. Proximal point methods for VI's

The proximal point algorithm has been analyzed for inclusions of the form

$$0 \in \mathbf{T}(\mathbf{x}) \tag{16}$$

in Rockafellar (1976a) for the case of $\mathbf{T}$ being a monotone point-to-set map on a Hilbert space as a generalization of the algorithm of Martinet (1970) for convex minimization problems. The results of Rockafellar (1976a) apply when the mapping

is maximal monotone. In this case the proximal operator $\mathbf{P}_c(\mathbf{x}) \triangleq (\mathbf{I} + c\mathbf{T})^{-1}(\mathbf{x})$ is also maximal monotone and single valued (see for instance, Rockafellar and Wets (1998), theorem 12.12). The iterations of the proximal point method for the problem (16) can be simply described as,

$$\text{Find } x_{k+1} \text{ so that } x_{x+1} \approx P_{c_k}(x_k) \tag{17}$$

where $c_k$ is a sequence so that $inf\ c_k > 0$ and the tolerances $\epsilon_k \to 0+$. If exactly $x_{x+1} = P_{c_k}(x_k)$, then this is equivalent to solve the problem

$$\text{Find } \mathbf{x}_{k+1} \text{ so that } 0 \in c_k^{-1}(\mathbf{x}_{k+1} - \mathbf{x}_k) + \mathbf{T}(\mathbf{x}_{k+1}) \tag{18}$$

or for a variational inequality on $\mathbb{R}^n$

$$\text{Find } \mathbf{x}^* \in \mathbf{X} \text{ so that } \mathbf{F}(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) \geq \mathbf{0},\ \forall \mathbf{x} \in \mathbf{X} \tag{19}$$

iteration (17) expresses as:

$$\text{Find } \mathbf{x}_{k+1} \in \mathbf{X} \text{ so that } (\mathbf{F}(\mathbf{x}_{k+1}) + c_k^{-1}(\mathbf{x} - \mathbf{x}_{k+1}))^\top (\mathbf{x} - \mathbf{x}_{k+1}) \geq 0,\ \forall \mathbf{x} \in \mathbf{X} \tag{20}$$

Attaining to the general methods for VI's descreibed in previous sections, iteration (20) seems to be well posed for its resolution provided that $\nabla \mathbf{F}(\cdot) + c_k^{-1}\mathbf{I}$ is positive definite.

Eckstein and Bertsekas (1992) develope a relaxed algorithm that allows for the proximal iteration to be solved only approximately for a sequence of tolerances $\epsilon_k \to 0+$ :

$$\text{Find } x_{k+1} \text{ so that } \| x_{x+1} - P_{c_k}(x_k) \| \leq \epsilon_k \tag{21}$$

If iteration (20) is carried out approximately, then, accordingly to (21) this is equivalent to choose a sequence of $\delta_k \to 0+$ and solve the proximal subproblems (20) approximately,

$$\text{Find } \mathbf{x}_{k+1} \in \mathbf{X},\ (\mathbf{F}(\mathbf{x}_{k+1}) + c_k^{-1}(\mathbf{x} - \mathbf{x}_{k+1}))^\top (\mathbf{x} - \mathbf{x}_{k+1}) \geq -\delta_k,\ \forall \mathbf{x} \in \mathbf{X} \tag{22}$$

suggesting clearly a stopping criterion based on the primal gap. The algorithm of Eckstein and Bertsekas can be summarized as follows. At iteration $k$:

$$\mathbf{x}_{k+1} = \sigma_k \bar{\mathbf{x}}_k + (1 - \sigma_k)\mathbf{x}_k \tag{23}$$

being $\bar{\mathbf{x}}_k$ a solution of (21) (or (22) in the context of VI's), and the parameters $\sigma_k$, $\epsilon_k$ and $c_k$ taken so that, $inf\ \sigma_k > 0$, $sup\ \sigma_k < 2$, $inf\ c_k = \bar{c} > 0$ and $\sum_k \epsilon_k < +\infty$. They prove the convergence of the algorithm under the maximal monotonicity of the operator $\mathbf{T}$ and provided that a solution of the inclusion (16) exists (otherwise the algorithm may not converge).

The work of Pennanen (2002) shows that, in fact the so stringent condition of maximal monotonicity required up to now for the convergence of the proximal point algorithm, can be substituted by the hypomonotonicity condition. (a mapping $\mathbf{T}$ is

called hypomonotone if the $\mathbf{T} + \rho\mathbf{I}$ is monotone for some $\rho > 0$). He uses the Yosida regularization $\mathbf{T}_\rho$ of $\mathbf{T}$, $\mathbf{T}_\rho \triangleq (\mathbf{T}^{-1} + \rho\mathbf{I})^{-1}$ and shows that, in fact:

$$(\mathbf{I} + c\mathbf{T}_\rho)^{-1} = \frac{1}{c + \rho}\,(c\mathbf{P}_{c+\rho} + \rho\mathbf{I})$$

for $c \neq 0$. This is equivalent to say that the application of the relaxed proximal point of Eckstein-Bertsekas to the Yosida regularization $\mathbf{T}_\rho$ of $\mathbf{T}$ is equivalent to the application of the proximal point algorithm of Rockafellar to $\mathbf{T}$. This is stated if the following theorem due to Pennanen:

**Theorem 1.2** *(Pennanen (2002), theorem 9) Let $\mathbf{T} : X \mapsto Y$, with $0 \in Y$ be an hypomonoton point-to-set map for some $\rho > 0$ and let $\bar{\mathbf{x}} \in \mathbf{T}^{-1}(0) \cap X \neq \emptyset$ and $\mathbf{T}^{-1}(0) \cap X$ a closed set. Then, there exists $\epsilon > 0$ so that if the starting point $\mathbf{x}_0 \in B_\epsilon(0)$. Then, the iteration:*

$$\mathbf{x}_{k+1} = B_\epsilon(0) \cap (\mathbf{I} + c_k\mathbf{T})^{-1}(\mathbf{x}_k) \tag{24}$$

*converges weakly and Fejér monotonically to a point $\bar{\mathbf{x}} \in \mathbf{T}^{-1}(0) \cap X$ if $c_k \nearrow \bar{c} \leq +\infty$ and $\inf c_k > 2\rho$. Further if the map $\mathbf{y} \mapsto \mathbf{T}_\rho^{-1}(\mathbf{y}) \cap X$ satisfies for some positive constant $L_T$ the calming condition at $\bar{\mathbf{x}}$, $\mathbf{x} \in \mathbf{T}_\rho(\mathbf{y}) \Rightarrow \|\mathbf{x} - \bar{\mathbf{x}}\| \leq L_T\|\mathbf{y}\|$, then the sequence of points generated by (24) converges to $\mathbf{T}^{-1}(0) \cap X$ linearly with rate*

$$\left(1 - \frac{\bar{c}}{\bar{c} - \rho}\left(2 - \frac{\bar{c}}{\bar{c} - \rho}\right)\frac{\bar{c}^2}{(L_T + \rho)^2 + \bar{c}^2}\right)^{\frac{1}{2}} < 1$$

It is proved that applying the proximal point iteration to $\mathbf{T}$ with the parameters stated in the above theorem is equivalent to the application of the over-relaxed proximal point of Eckstein-Bertsekas with parameters $c_k' = c_k - \rho$ and $\sigma_k = c_k/(c_k - \rho)$ and alternatively that the application of the over-relaxed iteration of Eckstein-Bertsekas on $\mathbf{T}$ with $\sigma_k = (c_k' - \rho)/c_k'$, with $\inf c_k' > 2\rho$ is equivalent to the proximal point iteration $\mathbf{x}_{k+1} = B_\epsilon(0) \cap (\mathbf{I} + (c_k' - \rho)\mathbf{T})^{-1}(\mathbf{x}_k)$.

# 5. Applications to the demand adjustment problem

The trip matrix adjustment problem on a traffic network can be formulated as the following bilevel programming problem under a general asymmetric traffic assignment scheme,

$$\begin{aligned}
Min_{g \geq 0} \quad & F(g) = z_1 F_1(v) + z_2 F_2(g) \\
s.t. \quad & v \in \mathcal{V}^*(g)
\end{aligned} \tag{25}$$

where by $v$ it is denoted the vector of total link flows and by $g$ it is denoted the origin-destination trip matrix so that, when assigned to the traffic network reproduces some traffic observed conditions. Also in (25), $\mathcal{V}(g)$ is the set of feasible link flows

corresponding to the O-D matrix $g$ and $\mathcal{V}^*(g)$ is the set of solutions in terms of the total link flows of the variational inequality (26) that formulates the traffic assignment problem on an asymmetric traffic network.

$$\text{Find } v \in \mathcal{V}(g), \text{ so that } c(v)^\top(v' - v) \geq 0, \ \forall v' \in \mathcal{V}(g) \tag{26}$$

It will be assumed that the solutions set is a singleton, i.e. $\mathcal{V}^*(g) = \{v^*(g)\}$.

The functions $F_1$ and $F_2$ can be viewed as distances between the response of the lower level problem and the data input that is used for the adjustment, i.e., the link flow observations and an obsolete or reference trip matrix. Usually $F_1$ is adopted as $F_1(v) = \frac{1}{2}(v_1 - \hat{v}_1)^\top U (v_1 - \hat{v}_1)$ and $F_2$ as $F_2(g) = \frac{1}{2}(g - \bar{g})^\top B (g - \bar{g})$ or an entropy function and the subscript 1 on the link flows $v_1$ stands for a subset of components where there are observations available, $\hat{v}_1$. $U$ and $W$ are weighting matrices. Usually $\bar{g}$ is referred to as the target matrix. The coefficients $z_1$ and $z_2$ are nonnegative weight factors which express the confidence of the model's user with regard to the link flow observations $\hat{v}_1$ versus the reliability of the outdated matrix $\bar{g}$.

For a non negative O-D matrix $g$, let $\boldsymbol{\Omega}$ denote the feasible cone of flows-demands or the set of pairs $(v, g)$ so that $g \geq 0$ and $v \in \mathcal{V}(g)$. For additive and separable TAP, let $V(g)$ the optimal value of the objective function $T(v) = \sum_{a \in \mathcal{A}} \int_0^{v_a} c_a(x) dx$, of the fixed demand additive and separable TAP for a given O-D matrix $g$. The function $V(g) = T(v^*(g))$ is convex and differentiable.

In general if the assignment map $\mathcal{V}^*(g)$ is a singleton, then the bilevel programming problem (25) for the demand adjustment problem can be reformulated as:

$$Min_{g \geq 0} \ F(g) = z_1 F_1(v^*(g)) + z_2 F_2(g) \tag{27}$$

It must be remarked that the upper level function $F(g)$ is non-differentiable.

The proximal point algorithm for minimizing a (continuously differentiable) function $F(g)$ on a convex set $X$ consists of solving a sequence of problems of the type:

$$Min_{g \in X} \ F(g) + \frac{\mu_\ell}{2} \left\| g - g^\ell \right\|_2^2 \tag{28}$$

for a sequence $\{\mu_\ell\}$ of parameters, $\mu_\ell \leq \bar{\mu}$. It must be also noted that, if $\mu_\ell$ is high enough, a solution $g^*(\mu_\ell)$ of the above problem (28) provides a good approximation to an element of the generalized gradient of Clarke $\partial F(g^*(\mu_\ell))$ for the function $F(g)$ at $g^*(\mu_\ell)$, because the Fermat rule applied to problem (28) results in:

$$-\mu_\ell(g^*(\mu_\ell) - g^\ell) \in \partial F(g^*(\mu_\ell)). \tag{29}$$

(See, for instance Codina and Barceló (2000) or Codina and Montero (2001). In our case $F(g) = H(v^*(g), g)$, where, as already mentioned:

$$H(v, g) = \frac{z_1}{2}(v_1 - \hat{v}_1)^\top U (v_1 - \hat{v}_1) + \frac{z_2}{2}(g - \bar{g})^\top B (g - \bar{g}) \tag{30}$$

and the set $X$ is the nonnegative orthant. In order to apply the proximal point iteration for the demand adjustment problem, the previous problem (28) needs to

be transformed in the following one:

$$Min_{(v,g)\in\Omega} \quad H(v,g) + \frac{\mu_\ell}{2} \|g - g^\ell\|_2^2 \tag{31}$$
$$s.t. \quad G(v,g) = 0$$

Being $G(v,g)$ a gap function for the variational inequality (26) parametrized by $g$. This function must verify the conditions:

$$\begin{array}{ll} G(v,g) = 0 & \text{if } v \in \mathcal{V}^*(g), \text{ for } g \geq 0 \\ G(v,g) > 0 & \text{if } v \in \Omega - \mathcal{V}^*(g), \text{ for } g \geq 0 \end{array} \tag{32}$$

The primal and dual gap functions $G_P(v)$ and $G_D(v)$, have already been mentioned in section 2. As the methods to be used are primal feasible it is immaterial the value of $G(v,g)$ outside $\Omega$. The function $T(v) - V(g)$, that applies for the case of diagonal and separable inelastic demand traffic assignemnt problem, is also a gap function accordingly to (32) because it is always nonnegative on the cone $\Omega$. Also, at the equilibrium points, $T(v^*(g)) - V(g) = 0$ and finally that $T(v) - V(g) > 0$ if $(v,g) \in \Omega$ but $v \neq v^*(g)$.

In this paper we shall deal only with the primal gap function $G_P(v)$ parametrized by $g$. If the constraint $G_P(v,g) = 0$ is penalized with parameter $1/\alpha_\ell$, then (31) can be approximated by:

$$Min_{(v,g)\in\Omega} \quad \psi_\ell(v,g) = G_P(v,g) + \alpha_\ell H(v,g) + \frac{\alpha_\ell\mu_\ell}{2} \|g - g^\ell\|_2^2 \tag{33}$$

where $\alpha_\ell \cdot \mu_\ell \to 0$.

For the case of the gap function $T(v) - V(g)$, valid for the case of diagonal and separable inelastic demand traffic assignment problem, it can be shown that problem (33) consistently approximates problem (31). This is shown in Codina (2001) and in Codina and Montero (2001) where a very similar scheme is followed but for the purpose of approximating the gradient of the upper level objective function $H(v^*(g),g)$ at a point $g$. In order to solve the non-convex problem (33), the partial linearization of Patriksson (1993) can be applied for the case of the gap function being $T(v) - V(g)$ as the perturbation function $V(g)$ is differentiable. It must be remarked that the partial linearization algorithm of Patriksson is intended primarily for pseudoconvex functions. The convergence for the problem (33) is also proved in Codina and Montero (2001). An efficient way to solve the resulting subproblems in the partial linearization scheme can be the application of the excess demand Gartner's transformation (Gartner 1980)) and to use a restricted simplicial decomposition method, such as the classical one of Hearn et al. (1987) on the resulting network.

The optimality conditions for problem (33) can be stated by means of the Fermat rule as:

$$0 \in \bar{\partial}G_P(v,g) + \alpha_\ell\nabla H(v,g) + \alpha_\ell\mu_\ell(g - g^\ell) + N_\Omega(v,g) = \mathbf{T}(v,g) \tag{34}$$

The calculation of the Clarke's subgradient of the primal gap function is done in Codina (2002) and it is shown there that:

$$\bar{\partial}G_P(\tilde{v},\tilde{g}) = \begin{pmatrix} \dfrac{c(\tilde{v}) + \left(\dfrac{\partial c}{\partial v}\right)_{\tilde{v}}^{\top}(w-\tilde{v})}{t(\tilde{g})} \end{pmatrix} \tag{35}$$

where $w \in argmin\{ G_P(\tilde{v}) = -Min_{w \in \mathcal{V}(\tilde{g})} c(\tilde{v})^{\top}(w-\tilde{v}) \}$.

The generalized equation (34) can be difficult to solve and we will consider now the case in which the travel costs $c(v)$ are strongly monotone with modulus $m_c$. Now it is easy to see that

$$\begin{pmatrix} \dfrac{c(v^*(g))}{-t(g) + \alpha_\ell \mu_\ell(g-g^\ell)} \end{pmatrix} + \alpha_\ell \nabla H(v^*(g),g) + N_{\boldsymbol{\Omega}}(v^*(g),g) \in \mathbf{T}(v^*(g),g) \tag{36}$$

If $c_\alpha(v,g) = c(v) + \alpha \nabla_v H(v,g)$ and $t_{\alpha,\mu}(v,g) = t(g) - \mu\alpha(g-g^\ell) - \alpha\nabla_g H(v,g)$, then (36) can be rewritten as:

$$\begin{pmatrix} \dfrac{c_\alpha(v^*(g),g)}{-t_{\alpha,\mu}(v^*(g),g)} \end{pmatrix} + N_{\boldsymbol{\Omega}}(v^*(g),g) \in \mathbf{T}(v^*(g),g) \tag{37}$$

This suggests that a good approximation for the inclusion (34) can be the variational inequality:

$$c_\alpha(v,g)^{\top}(v'-v) - t_{\alpha,\mu}(v,g)^{\top}(g'-g) \geq 0, \quad \forall (v',g') \in \boldsymbol{\Omega} \tag{38}$$

presenting the structure of an elastic demand traffic assignment problem that can be converted to a fixed demand traffic assignment problem by means of the Gartner's transformation.

The good approximation of (38) to (34) is justified because the primal gap function $\Gamma(v,g)$ for (38) verifies,

$$|\Gamma(v^*(g),g)| \leq L_\Gamma \|v^*(g) - v\|_2 \overset{(1)}{\leq} \frac{L_\Gamma}{m_c} \|c_{\alpha,\mu}(v^*(g)) - c(v)\|_2 \leq \\ \leq \frac{\alpha L_\Gamma}{m_c} \|\nabla_v H(v^*(g),g)\|_2 \tag{39}$$

being $m_c > 0$ the modulus of strong monotonicity of the function $c(v)$ (The inequality (1) is based on theorem 5.1 in Dafermos and Nagurney (1984)).

The cost approximation algorithms in Patriksson (1999) can be used for the case of the gap function $G_P(v,g)$ and $H = H(v)$. At iteration $k$, of the cost approximation algorithm, the resulting variational inequality subproblem at point $g^k$ is:

$$c_\alpha(v)^{\top}(w-v) + (-t(g^k) + \mu\alpha(g-g^\ell))^{\top}(g'-g) \geq 0, \quad \forall(w,g') \in \boldsymbol{\Omega} \tag{40}$$

where $t(g_\ell)$ are the equilibrium O-D travel times resulting from the assignment of the O-D matrix $g^k$ with costs $c(v)$. The algorithm then can be summarized as follows:

**Algorithm:** (Start with $\mu_1$ small.)

At iteration $\ell$-th:

1. Carry out just few iterations $k = 1, ..., L$ of the cost approximation scheme in order to approximately solve problem (38):

   - Solve subproblem (40)
     - Evaluate $t(g^k) =$ O-D travel times for $g^k$; $\quad k \leftarrow k + 1$

2. Let $g' \leftarrow g^L$ and perform a line search along $g' - g^\ell$. Let $g^{\ell+1} = g^\ell + \beta(g' - g^\ell)$ be the next iterate.

3. Take $\mu_{\ell+1} < \mu_\ell$ ( $\mu_\ell \to \bar{\mu}$ )

The line search step is optional and not necessary for the convergence. In this case simply $g^{\ell+1} = g'$. For upper level objective functions of the form (30) a line search consisting of a simple quadratic fit can be used very advantageously. Moreover, for the direction $g' - g_\ell$ provided by the algorithm at the end of step 1, the simple quadratic fit seems to provide always a better point $g_{\ell+1}$ (i.e. with lower objective function value).

# 6. Conclusions and further work

This paper reviews firstly the conditions for convergence of the more used algorithms to solve variational inequalities with special incidence in the traffic assignment problem and summarizes the recently new conditions developed by Pennanen (2002) for the proximal point algorithm which do not require strong monotonicity. The application of the proximal point algorithm to a problem of bilevel programming that formulates the adjustment of origin-destination trip matrices is also shown. A set of numerical experiments will be required in order to evaluate practically and in comparison with other methods, the performance of the proximal point algorithm to the problems examined in this paper.

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