On The Spectra of Hypertrees ⋆

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Abstract

In this paper we study the spectral properties of a family of trees characterized by two main features: they are spanning subgraphs of the hypercube, and their vertices bear a high degree of (connectedness) hierarchy. Such structures are here called binary hypertrees and they can be recursively defined as the so-called hierarchical product of several complete graphs on two vertices.

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1 Introduction

Many networks associated to real-life complex systems have a hierarchical organization which is useful in their communication processes. This hierarchical structure leads very often to the existence of nodes with a relatively high degree (known as hubs) and to a low average distance in the graph. The characterization of graphs with these properties has therefore attracted much interest in the recent literature, see for example [8] and references therein.

Some classical graphs also display a modular or hierarchical structure. Perhaps the best known example is the hypercube or n-cube which has useful

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communication properties: it is a minimum broadcast graph allowing optimal broadcasting and gossiping under standard communication models. However it has a relatively large number of edges and many of them are not used in these communication schemes as the communication paths usually conform to a spanning tree [6]. It is of interest to have graphs operations allowing the construction of these spanning trees. In [1] the authors introduce the hierarchical product of graphs with produces graphs with a strong (connectedness) hierarchy in their vertices. In fact, the obtained graphs turn out to be subgraphs of the cartesian product of the corresponding factors. Some well-known properties of the cartesian product, such as a reduced mean distance and diameter, simple routing algorithms and some optimal communication protocols are inherited by the hierarchical product. When all the factors are the complete graph K_2, the resulting graph is a spanning tree of the hypercube. Another example of hierarchical product is the deterministic tree obtained by Jung, Kim and Kahng [7] which corresponds to the case when the factors are star graphs.

On the other hand, the study of the spectrum of a graph is relevant for estimating important structural properties, which provide information on the topological and communication properties of the corresponding network [3]. Among these properties, which usually are very hard to obtain by other methods, we have edge expansion and node-expansion, bisection width, diameter, maximum cut, connectivity, and partitions.

In this paper we study the spectral properties of a family of trees, which we call binary hypertrees, characterized by two main features: they are spanning subgraphs of the hypercube, and their vertices bear a high degree of (connectedness) hierarchy. The binary hypertree of dimension m, T_m, is defined as the hierarchical product [1] of m copies of K_2. Among other properties, the hypertrees are shown to be good examples of graphs with distinct eigenvalues. This fact has some structural consequences, such as the Abelianity of its automorphism group [9]. Indeed, we show that the automorphism group of T_m is the symmetric group S_2. This, together with the high degree of hierarchy of our family of trees, results in a number of nice properties of their spectra.

More precisely, because of the recurrence relation satisfied by the characteristic polynomial of T_m, every eigenvalue of a hypertree of a given dimension yields two eigenvalues of the hypertree of the next dimension. Consequently, there is a strong relationship between the eigenvalues and the eigenvectors of the hypertrees of different dimensions.

Concerning the eigenvalues of T_m, we study their asymptotic behavior and how they are distributed with respect to some intervals defined by the eigenvalues of $T_{m'}$, for $m' < m$. Finally, by using the techniques in [4,5], we compute the eigenvectors of T_m. The result is based on obtaining a charge distribution on

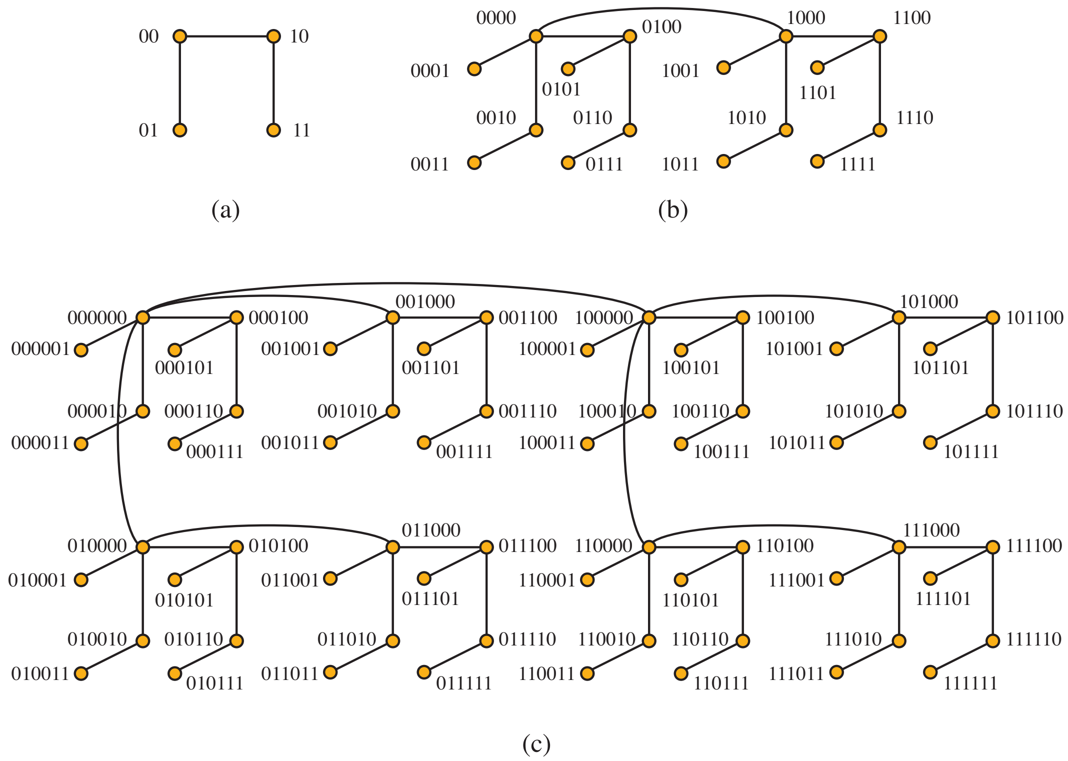

Fig. 1. The hypertrees $T_2 = K_2^2$, $T_4 = K_2^4$ and $T_6 = K_2^6$.

the vertices of T_m from a charge distribution on the vertices of T_{m-1}.

2 Definition and basic properties of the hypertree

A simple definition of the binary hypertree of dimension m, T_m, is as follows.

Definition 1 *Given an integer $m > 0$, T_m is the rooted tree with vertex set $\mathbb{Z}_2^m$ and the adjacencies defined by the following rule: two vertices are adjacent if and only if their labels differ in exactly one position and the maximum common suffix is either empty or it contains only zeroes.*

The root of T_m is $\mathbf{0} = 00\ldots0$. (We naturally take $T_0 = K_1$.)

In fact, this definition is equivalent to consider the hierarchical product of m copies of K_2. That is, $T_m = K_2^m = K_2 \sqcap \overset{m}{\cdots} \sqcap K_2$, where the operator "$\sqcap$" indicates this kind of product [1]. As an example, Fig. 1 shows the hierarchical products of two, four and six copies of the complete graph K_2.

Since the graphs obtained from the hierarchical product are spanning subgraphs of the corresponding cartesian products, we have that T_m is a spanning subgraph of the hypercube Q_m. (Recall that Q_m has set of vertices $\mathbb{Z}_2^m$, and two vertices are adjacent if and only if they differ in exactly one position.)

Every $\boldsymbol{i} = i_{m-1} \ldots i_1 i_0 \in \mathbb{Z}_2^m$ can be viewed as the expression in base two, with fixed length m, of $i = \sum_{k=0}^{m-1} i_k 2^k$, with $i \in [0, 2^m - 1]$. In particular, we will consider the vertices of T_m labeled $\{0, 1, \ldots, 2^m - 1\}$. In this sense, we use $\boldsymbol{i} = i$, provided that m is fixed. By convention, we take $\mathbb{Z}_2^0 = \{\varnothing\}$.

Let us now recall some basic properties of the hypertree T_m, which are drawn from a previous study of the authors [1] dealing with the hierarchical product.

- For every $m \geq 0$, the hypertree T_m has order $n = 2^m$ and size $2^m - 1$.
- $T_m = T_{m-1} \sqcap K_2$ (as the hierarchical product has the associative property).
- $T_m^* = T_m - \mathbf{0} = \bigcup_{k=0}^{m-1} T_k$.
- $T_m - e$, where e is the edge $\{\mathbf{0}, 10 \overset{m-1}{\ldots} 0\}$, is isomorphic to the disjoint union of two copies of T_{m-1}. In fact, such copies of T_{m-1} are the subgraphs induced by the sets of vertices $V_0 = \{0\boldsymbol{w} | \boldsymbol{w} \in \mathbb{Z}_2^{m-1}\}$ and $V_1 = \{1\boldsymbol{w} | \boldsymbol{w} \in \mathbb{Z}_2^{m-1}\}$.
- T_m has 2 vertices of degree m and 2^{m-j} vertices of degree j, for $1 \leq j \leq m - 1$. Namely,
  - · $\delta(\mathbf{0}) = \delta(10 \overset{m-1}{\ldots} 0) = m$;
  - · $\delta(\boldsymbol{w}100 \overset{j-1}{\ldots} 0) = j$, for every $\boldsymbol{w} \in \mathbb{Z}_2^{m-j}$, for $1 \leq j \leq m - 1$.

With respect to the symmetries of the hypertrees, we have the following result:

Proposition 2 *For every $m \geq 1$ the automorphism group of T_m is S_2.*

PROOF. Let $\phi : T_m \to T_m$ be defined by $\phi(0\boldsymbol{i}) = 1\boldsymbol{i}$ and $\phi(1\boldsymbol{i}) = 0\boldsymbol{i}$, for every $\boldsymbol{i} \in \mathbb{Z}_2^{m-1}$. We claim that $Aut(T_m) = \{Id, \phi\}$. Thus, we have to prove that ϕ is the only non-trivial automorphism of T_m.

Let us first show that ϕ is a T_m-automorphism. From its definition, it is clear that ϕ is an involutive bijection, that is, $\phi(\phi(v)) = v$ for every vertex v of T_m. Now, let u and v be two vertices of T_m. We have to show that if $u \sim v$ then $\phi(u) \sim \phi(v)$. With this aim, assume without loss of generality that u starts by a zero. If $u = \mathbf{0}$ and $v = 10 \overset{m-1}{\ldots} 0$, then $\phi(u) = v$ and $\phi(v) = u$. Otherwise, v also starts by a zero. By symmetry, we can take $u = 0\boldsymbol{w}0 \overset{j}{\ldots} 0$ and $v = 0\boldsymbol{w}10 \overset{j-1}{\ldots} 0$. In this case, $\phi(u) = 1\boldsymbol{w}0 \overset{j}{\ldots} 0$ and $\phi(v) = 1\boldsymbol{w}10 \overset{j-1}{\ldots} 0$, which are clearly adjacent in T_m.

Finally, we prove that ϕ is the only non-trivial T_m-automorphism by using induction on m. For $m = 1$, $T_1 = K_2$, and $Aut(K_2) = S_2$. Let $m > 1$. As mentioned above, $V_0 = \{0\boldsymbol{w} | \boldsymbol{w} \in \mathbb{Z}_2^{m-1}\}$ and $V_1 = \{1\boldsymbol{w} | \boldsymbol{w} \in \mathbb{Z}_2^{m-1}\}$ induce two disjoint subgraphs of T_m, isomorphic to T_{m-1}. We denote these subgraphs by $G_0 = G[V_0]$ and $G_1 = G[V_1]$. Assume that $Aut(T_{m-1}) = \{Id, \phi\}$, and let γ be a T_m-automorphism. Because of the degree sequence of T_m, either

- $\gamma(\mathbf{0}) = \mathbf{0}$ and $\gamma(10 \overset{m-1}{\cdots} 0) = 10 \overset{m-1}{\cdots} 0$, or
- $\gamma(\mathbf{0}) = 10 \overset{m-1}{\cdots} 0$ and $\gamma(10 \overset{m-1}{\cdots} 0) = \mathbf{0}$.

In the first case, γ maps G_0 and G_1 onto themselves. Moreover, the induced automorphisms let the root fixed. Hence, by induction hypothesis, $\gamma = Id$. In the second case, γ maps G_0 onto G_1 and G_1 onto G_0. For every $\boldsymbol{i} \in \mathbb{Z}_2^{m-1}$, we define γ_0 and γ_1 in the following way:

- if $\gamma(0\boldsymbol{i}) = 1\boldsymbol{v}$ then $\gamma_0(\boldsymbol{i}) = \boldsymbol{v}$;
- if $\gamma(1\boldsymbol{i}) = 0\boldsymbol{w}$ then $\gamma_1(\boldsymbol{i}) = \boldsymbol{w}$.

It can be easily checked that γ_0 and γ_1 are both T_{m-1}-automorphisms that let the root fixed. By induction hypothesis, $\gamma_0 = \gamma_1 = Id$, which implies that $\gamma = \phi$. This completes the proof.

3 Spectral properties

The adjacency matrix of T_m is

$$\boldsymbol{A}_m = \begin{pmatrix} \boldsymbol{A}_{m-1} & \boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix}, \tag{1}$$

where the dimensions of each block are $2^{m-1} \times 2^{m-1}$, and its characteristic polynomial

$$\phi_m(x) = \det(x\boldsymbol{I} - \boldsymbol{A}_m) = \det \begin{pmatrix} x\boldsymbol{I} - \boldsymbol{A}_{m-1} & -\boldsymbol{I} \\ -\boldsymbol{I} & x\boldsymbol{I} \end{pmatrix}$$

satisfies the recurrence

$$\begin{aligned} \phi_m(x) &= \det((x^2-1)\boldsymbol{I} - x\boldsymbol{A}_{m-1}) \\ &= \det(x[(x - \tfrac{1}{x})\boldsymbol{I} - \boldsymbol{A}_{m-1}]) \\ &= x^{\frac{n}{2}} \phi_{m-1}(x - \tfrac{1}{x}), \end{aligned} \tag{2}$$

where we have used the result in [10]. (Recall that $n = 2^m$.)

3.1 Eigenvalues

Given a graph G on n vertices, we denote by $\operatorname{ev} G$ the set of its eigenvalues in increasing order, say $\operatorname{ev} G = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ (notice that this notation presumes that all eigenvalues are distinct). The recurrence equation satisfied by the characteristic polynomial of T_m gives rise to a number of spectral properties.

From (2) we have that, if $\lambda_{\boldsymbol{i}} \in \operatorname{ev} T_{m-1}$, then both solutions of $x - \frac{1}{x} = \lambda_{\boldsymbol{i}}$ are in $\operatorname{ev} T_m$. This equation is equivalent to

$$x^2 - \lambda_{\boldsymbol{i}} x - 1 = 0. \tag{3}$$

A useful notation for these solutions is $\lambda_{0\boldsymbol{i}}$ and $\lambda_{1\boldsymbol{i}}$ since, by using the functions

$$f_0(\lambda) := \frac{1}{2}(\lambda - \sqrt{\lambda^2 + 4}), \quad f_1(\lambda) := \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4}), \tag{4}$$

they can be computed as $\lambda_{0\boldsymbol{i}} = f_0(\lambda_{\boldsymbol{i}})$ and $\lambda_{1\boldsymbol{i}} = f_1(\lambda_{\boldsymbol{i}})$.

Moreover, by recursively applying these functions starting from $\lambda_\varnothing := 0$ (here $\varnothing$ represents the empty sequence), we obtain the whole set of eigenvalues $\operatorname{ev} T_m = \{\lambda_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^m\}$ where, if $\boldsymbol{i} = i_{m-1} i_{m-2} \ldots i_0$, then

$$\lambda_{\boldsymbol{i}} = (f_{i_{m-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0})(0). \tag{5}$$

This presentation provides a natural ordering, in such a way that the higher the number i (whose binary representation of length m is $\boldsymbol{i}$), the larger the eigenvalue $\lambda_{\boldsymbol{i}}$ [1]. As a direct consequence, all the 2^m eigenvalues are different, a property which has some far-reaching consequences. In particular, any automorphism of T_m is involutive [9]. With this regard, we have shown in Proposition 2 that $Aut(T_m) = S_2$.

As T_m is trivially bipartite, its eigenvalue mesh is symmetric [2] and hence, for every $\boldsymbol{i} \in \mathbb{Z}_2^m$,

$$\lambda_{\boldsymbol{i}} = -\lambda_{\overline{\boldsymbol{\imath}}}, \tag{6}$$

where $\overline{\boldsymbol{\imath}} (= n - i)$ denotes the ones'complement of $\boldsymbol{i}$ (that is, the bitwise NOT operation). For instance, Fig. 2 shows the spectra of the hypertrees T_m for the cases $0 \leq m \leq 6$ and how every eigenvalue of T_{m-1} gives rise to two eigenvalues of T_m.

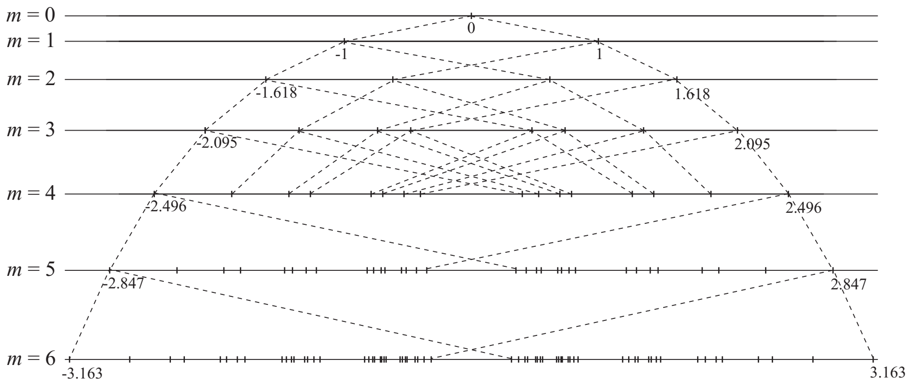

Fig. 2. The eigenvalue meshes of T_m for $0 \le m \le 6$.

It is shown in [1] that the asymptotic behaviors of the maximum eigenvalue (spectral radius), $\rho_m = \max_{0\le i\le n-1}\{|\lambda_{\boldsymbol{i}}|\} = \lambda_{111\ldots1}$, and the minimum positive eigenvalue, $\sigma_m = \min_{0\le i\le n-1}\{|\lambda_{\boldsymbol{i}}|\} = \lambda_{100\ldots0}$, of the hypertree T_m are:

$$\rho_m \sim \sqrt{2m}, \quad \sigma_m \sim 1/\sqrt{2m}. \tag{7}$$

Let us now consider the eigenvalues of T_m in decreasing order $\lambda_m^{(0)} > \lambda_m^{(1)} > \lambda_m^{(2)} > \cdots$. That is, the first (0-th) eigenvalue is $\lambda_m^{(0)} = \lambda_{111\ldots1}(= \rho_m)$, the second (1-st) one is $\lambda_m^{(1)} = \lambda_{111\ldots10}$ and so on. In general, for any fixed integer $r > 0$, consider the r-th largest eigenvalue $\lambda_m^{(r)}$ of T_m, with $m \ge k = \lceil \log_2 r \rceil$. Let $\boldsymbol{r}$ be the binary representation of r. Then k is the lenght of $\boldsymbol{r}$ and $\lambda_m^{(r)} = \lambda_{111\ldots1\overline{\boldsymbol{r}}}$. In this context, as in the case of the spectral radius, a natural question is to ask about the (asymptotic) behavior of the sequence $\{\lambda_m^{(r)} = \lambda_{111\ldots1\overline{\boldsymbol{r}}}\}_{m\ge k}$.

Proposition 3 *For every fixed $r > 1$, let γ_m denote the r-th largest eigenvalue of T_m, that is, $\lambda_m^{(r)} = \gamma_m$. Then, the asymptotic behavior of γ_m is:*

$$\gamma_m \sim \sqrt{2m}.$$

PROOF. For $m \ge k$,

$$\gamma_{m+1} = f_1(\gamma_m) = \frac{1}{2}\Big(\gamma_m + \sqrt{\gamma_m^2 + 4}\Big).$$

This function tends to a power law, $\gamma_m \sim \alpha\, m^\beta$ for $m \to \infty$, for some constants α and β. Indeed, if we put this expression of γ_m in the equation, we get:

$$\alpha(m+1)^\beta \sim \frac{\alpha m^\beta + \sqrt{\alpha^2 m^{2\beta} + 4}}{2} \quad \Rightarrow \quad \alpha^2(m+1)^\beta[(m+1)^\beta - m^\beta] \sim 1.$$

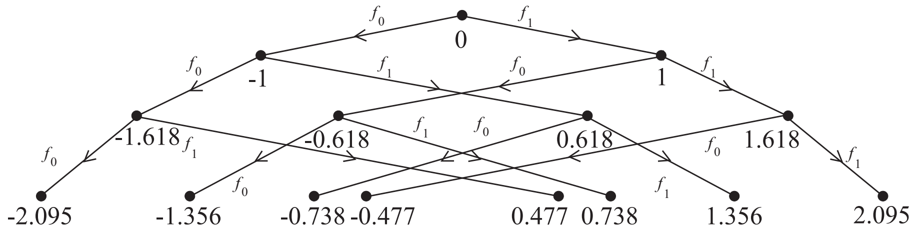

Fig. 3. The digraph of the eigenvalues of T_m.

It is easy to check that $\gamma_m = \sqrt{2m}$ is a solution when $m \to \infty$ since,

$$2(m+1)^{\frac{1}{2}}[(m+1)^{\frac{1}{2}} - m^{\frac{1}{2}}] = \frac{2(m+1)^{\frac{1}{2}}}{(m+1)^{\frac{1}{2}} + m^{\frac{1}{2}}} \to 1.$$

This solution corresponds to $\alpha = \sqrt{2}$ and $\beta = \frac{1}{2}$.

The following result is derived from the fact that $\lambda_{0\boldsymbol{i}}$ and $\lambda_{1\boldsymbol{i}}$ are the roots of the quadratic polynomial in (3), $x^2 - \lambda_{\boldsymbol{i}} x - 1 = 0$.

Lemma 4 *For every $\alpha \in \mathbb{Z}_2$ and $\boldsymbol{i} \in \mathbb{Z}_2^{m-1}$,*

$$\lambda_{0\boldsymbol{i}} + \lambda_{1\boldsymbol{i}} = \lambda_{\boldsymbol{i}}, \tag{8}$$

$$\lambda_{0\boldsymbol{i}}\lambda_{1\boldsymbol{i}} = -1, \tag{9}$$

$$\lambda_{\alpha\boldsymbol{i}}\lambda_{\alpha\overline{\boldsymbol{i}}} = 1, \tag{10}$$

$$\lambda_{\alpha\boldsymbol{i}} = \lambda_{\alpha\overline{\boldsymbol{i}}} + \lambda_{\boldsymbol{i}}. \tag{11}$$

PROOF. The two equalities (8) and (9) come from (3). Equality (10) is a consequence of (9) and the "symmetry property" (6).

From (8)–(10) we get

$$\lambda_{\boldsymbol{i}} = \lambda_{0\boldsymbol{i}} - \frac{1}{\lambda_{0\boldsymbol{i}}} = \lambda_{0\boldsymbol{i}} - \lambda_{0\overline{\boldsymbol{i}}} \quad \Rightarrow \quad \lambda_{0\boldsymbol{i}} = \lambda_{0\overline{\boldsymbol{i}}} + \lambda_{\boldsymbol{i}},$$

$$\lambda_{\boldsymbol{i}} = \lambda_{1\boldsymbol{i}} - \frac{1}{\lambda_{1\boldsymbol{i}}} = \lambda_{1\boldsymbol{i}} - \lambda_{1\overline{\boldsymbol{i}}} \quad \Rightarrow \quad \lambda_{1\boldsymbol{i}} = \lambda_{1\overline{\boldsymbol{i}}} + \lambda_{\boldsymbol{i}}.$$

As an illustration of the equality (8), see Fig. 3 (to be compared with Fig. 2).

In particular, by (11) the maximum $\rho_m = \lambda_{11\ldots1}$ and the minimum $\sigma_m = \lambda_{10\ldots0}$ absolute values in $\operatorname{ev} T_m$ are inverse of each other, that is, $\rho_m\sigma_m = 1$, in agreement with (7).

Moreover, by applying recursively (11) we have that the sum of the first m minimum positive eigenvalues yields the spectral radius of T_m:

$$\rho_m = \lambda_{111\ldots1} = \lambda_{100\ldots0} + \lambda_{11\ldots1} = \cdots = \sigma_m + \sigma_{m-1} + \cdots + \sigma_1.$$

A more particular example could be the following:

$$\begin{aligned}\lambda_{10110} &= \lambda_{11001} + \lambda_{0110}\\ &= \lambda_{11001} + \lambda_{0001} + \lambda_{110}\\ &= \lambda_{11001} + \lambda_{0001} + \lambda_{101} + \lambda_{10}\\ &= \lambda_{11001} + \lambda_{0001} + \lambda_{101} + \lambda_{11} + \lambda_{0}.\end{aligned}$$

Now we concentrate on the distribution of the eigenvalues of T_m with respect to some intervals defined by the eigenvalues of $T_{m'}$, for $m' < m$. Let us first prove that all the eigenvalues are distinct, even if they belong to hypertrees of different dimensions.

Lemma 5 *For any pair of binary sequences* $\boldsymbol{i} \in \mathbb{Z}_2^r$, $\boldsymbol{j} \in \mathbb{Z}_2^s$,

$$\boldsymbol{i} = \boldsymbol{j} \quad \Longleftrightarrow \quad \lambda_{\boldsymbol{i}} = \lambda_{\boldsymbol{j}}\,. \tag{12}$$

PROOF. The sufficiency is trivial by (5). With respect to the necessity, assume that we have $\lambda_{\boldsymbol{i}} = \lambda_{\boldsymbol{j}}$, for $\boldsymbol{i} = i_{r-1}i_{r-2}\ldots i_1 i_0$ and $\boldsymbol{j} = j_{s-1}j_{s-2}\ldots j_1 j_0$. Hence,

$$f_{i_{r-1}}(f_{i_{r-2}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0)) = f_{j_{s-1}}(f_{j_{s-2}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0)). \tag{13}$$

Then, as $f_0(x) = \frac{1}{2}(x - \sqrt{x^2+4}) < 0$ and $f_1(x) = \frac{1}{2}(x + \sqrt{x^2+4}) > 0$ for any x, it must be $i_{r-1} = j_{s-1}$ and hence, $f_{i_{r-2}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = f_{j_{s-2}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0)$. Following the same reasoning we get $i_{r-2} = j_{s-2}$ and $f_{i_{r-3}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = f_{j_{s-3}} \circ \cdots \circ f_{j_1} \circ f_{j_0}(0)$, and so on.

Now, we only need to show that $r = s$. Assume without loss of generality that $r > s$. Then, repeating s times the above process we will have $i_{r-1} = j_{s-1}$, $i_{r-2} = j_{s-2}, \ldots, i_{r-s} = j_0$ and $f_{i_{r-s-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0}(0) = 0$. This contradicts the fact that, for every x and $i = 0, 1$, $f_i(x) \neq 0$. This completes the proof.

To consider the eigenvalues of the hypertrees of all possible dimensions, we need to consider $\mathbb{Z}_2^*$, that is, the union of $\mathbb{Z}_2^m$ for all m (or the set of all the sequences over $\mathbb{Z}_2$).

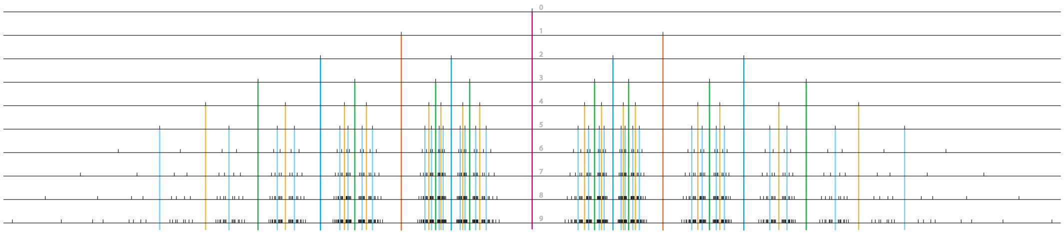

Fig. 4. The distribution of the eigenvalues of T_m, for $m = 0, \ldots, 9$.

Definition 6 *Let $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_2^*$ and let $\boldsymbol{w}$ be their (possibly void) maximum common prefix. We say that $\boldsymbol{i} <_T \boldsymbol{j}$ if and only if one of the following condition holds:*

(1) $\boldsymbol{i} = \boldsymbol{w}0\boldsymbol{i}_1$ and $\boldsymbol{j} = \boldsymbol{w}1\boldsymbol{j}_1$;
(2) $\boldsymbol{i} = \boldsymbol{w}$ and $\boldsymbol{j} = \boldsymbol{w}1\boldsymbol{j}_1$;
(3) $\boldsymbol{i} = \boldsymbol{w}0\boldsymbol{i}_1$ and $\boldsymbol{j} = \boldsymbol{w}$.

We say that $\boldsymbol{i} \leq_T \boldsymbol{j}$ if and only if $\boldsymbol{i} <_T \boldsymbol{j}$ or $\boldsymbol{i} = \boldsymbol{j}$.

Note that two different binary sequences could represent the same natural number. Hence, the relation $<_T$, which turns out to be a total ordering of $\mathbb{Z}_2^*$, is not equivalent to the natural order.

Definition 7 *For any $\ell \geq 0$ and $\boldsymbol{w} \in \mathbb{Z}_2^\ell$, the $\boldsymbol{w}$-*translation*, $\tau_{\boldsymbol{w}}$, is the function*

$$\tau_{\boldsymbol{w}} = f_{w_{\ell-1}} \circ \cdots \circ f_{w_1} \circ f_{w_0},$$

where $\boldsymbol{w} = w_{\ell-1} \ldots w_1 w_0$.

The functions f_0 and f_1 are both monotone increasing. This implies that, for every $\boldsymbol{w}$, $\tau_{\boldsymbol{w}}$ is monotone increasing. On the other hand, it is worth mentioning that, since $\tau_{\boldsymbol{w}}$ does not preserves distances, it is not a translation in a geometric sense.

In addition, we have the following lemma.

Lemma 8 *For every $i = 0, 1$ and any arbitrary x, y,*

$$|f_i(x) - f_i(y)| < |x - y|. \tag{14}$$

PROOF. For $i = 1$, we can assume without loss of generality that $x < y$. Then,

$$f_1(x) < f_1(y) \quad \text{and} \quad f_1(y) - f_1(x) = \frac{1}{2}\left(y + \sqrt{y^2+4} - x - \sqrt{x^2+4}\right).$$

Now, we only need to notice that $\sqrt{y^2+4} - \sqrt{x^2+4} < y - x$, because

$$
\begin{aligned}
(\sqrt{y^2+4}-\sqrt{x^2+4})(\sqrt{y^2+4}+\sqrt{x^2+4}) &= (y-x)(y+x) \\
&< (y-x)(\sqrt{y^2+4}+\sqrt{x^2+4})
\end{aligned}
$$

This yields $|f_1(x) - f_1(y)| < |x-y|$. By a similar reasoning we get $|f_0(x) - f_0(y)| < |x-y|$.

We use the $\boldsymbol{w}$-translations and Lemma 8 to prove the following result. (See Fig. 4.)

Theorem 9 *The set of all the eigenvalues,*

$$\bigcup_{m=0}^{\infty} \operatorname{ev} T_m = \{\lambda_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^*\}$$

satisfies the following properties.

- (a) *For every* $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_2^*$, $\boldsymbol{i} <_T \boldsymbol{j}$ *if and only if* $\lambda_{\boldsymbol{i}} < \lambda_{\boldsymbol{j}}$.
- (b) *The interval determined by two consecutive eigenvalues of dimension* m, *contains exactly* 2^k *consecutive eigenvalues of* T_{m+k}, *for* $k \geq 1$.
- (c) *The two successions* $\{\lambda_{\boldsymbol{w}10\overset{k}{\ldots}0}\}_{k>0}$ *and* $\{\lambda_{\boldsymbol{w}01\overset{k}{\ldots}1}\}_{k>0}$ *have both limit* $\lambda_{\boldsymbol{w}}$.

PROOF. The translation $\tau_{\boldsymbol{w}}$ maps the set

$$\{\lambda_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^*\} = \{\lambda_\varnothing, \lambda_0, \lambda_1, \lambda_{00}, \ldots\}$$

onto the set

$$\{\lambda_{\boldsymbol{w}\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^*\} = \{\lambda_{\boldsymbol{w}}, \lambda_{\boldsymbol{w}0}, \lambda_{\boldsymbol{w}1}, \lambda_{\boldsymbol{w}00}, \ldots\}$$

preserving the order. This implies that in (a) we can assume that $\boldsymbol{i}$ and $\boldsymbol{j}$ have no common prefix. And this together with Lemma 8 implies that in (c) we can assume that $\boldsymbol{w} = \varnothing$.

(a) If $\boldsymbol{i}$ and $\boldsymbol{j}$ have no common prefix, then $\boldsymbol{i} <_T \boldsymbol{j}$ if and only if one of the following conditions hold:

- $\boldsymbol{i} = 0\boldsymbol{i}_1$ and $\boldsymbol{j} = 1\boldsymbol{j}_1$;
- $\boldsymbol{i} = \varnothing$ and $\boldsymbol{j} = 1\boldsymbol{j}_1$;
- $\boldsymbol{i} = 0\boldsymbol{i}_1$ and $\boldsymbol{j} = \varnothing$.

The first condition is equivalent to $\lambda_{\boldsymbol{i}} < 0$ and $\lambda_{\boldsymbol{j}} > 0$. The second condition is equivalent to $\lambda_{\boldsymbol{i}} = 0$ and $\lambda_{\boldsymbol{j}} > 0$. Finally, the third condition is equivalent to $\lambda_{\boldsymbol{i}} < 0$ and $\lambda_{\boldsymbol{j}} = 0$. Hence, we have that if $\boldsymbol{i}$ and $\boldsymbol{j}$ have no common prefix, then $\boldsymbol{i} <_T \boldsymbol{j}$ if and only if $\lambda_{\boldsymbol{i}} < \lambda_{\boldsymbol{j}}$.

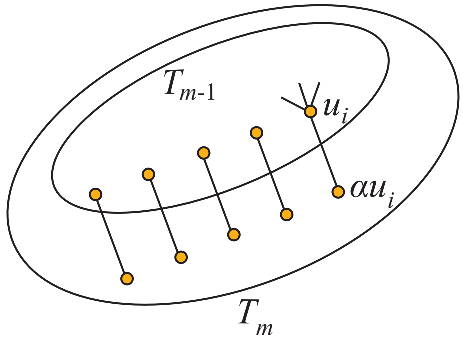

Fig. 5. The construction of the eigenvectors of T_m from the eigenvectors of T_{m-1}.

(b) Let $\boldsymbol{i} \in \mathbb{Z}_2^m$ and $\boldsymbol{j} = \boldsymbol{i} + 1 \in \mathbb{Z}_2^m$. Using (a) we only have to prove that

$$|\{\boldsymbol{w} \in \mathbb{Z}_2^{m+k} \,|\, \boldsymbol{i} <_T \boldsymbol{w} <_T \boldsymbol{j}\}| = 2^k.$$

By definition of $<_T$,

$$\{\boldsymbol{w} \in \mathbb{Z}_2^{m+k} \,|\, \boldsymbol{i} <_T \boldsymbol{w} <_T \boldsymbol{j}\} = \{\boldsymbol{i}1\boldsymbol{i}_1 \,|\, \boldsymbol{i}_1 \in \mathbb{Z}_2^{k-1}\} \cup \{\boldsymbol{j}0\boldsymbol{j}_1 \,|\, \boldsymbol{j}_1 \in \mathbb{Z}_2^{k-1}\}$$

whose cardinality is $2^{k-1} + 2^{k-1} = 2^k$.

(c) The result is a direct consequence of (7) and the fact that $\lambda_{100\overset{k}{\ldots}0} = \sigma_k$, the minimum positive eigenvalue of T_k, and $\lambda_{011\overset{k}{\ldots}1} = -\sigma_k$.

3.2 Eigenvectors

For any (di)graph, it is known that the components of its eigenvalues can be seen as charges on each vertex (see [4,5]). More precisely, suppose that $G = (V, A)$ is a digraph (a graph can be seen as a symmetric digraph where every edge $\{i, j\}$ is represented by two opposite arcs (i, j), (j, i)) with adjacency matrix $\boldsymbol{A}$ and λ-eigenvector $\boldsymbol{v}$. Then the charge of a vertex $i \in V$ is the corresponding entry v_i of $\boldsymbol{v}$, and the equation $\boldsymbol{Av} = \lambda\boldsymbol{v}$ means that

$$\sum_{i\to j} v_j = \lambda v_i \quad \text{for every} \quad i \in V. \tag{15}$$

That is, each vertex "absorbs" the charges of its out-neighbors to get a final charge λ times the one it had originally.

This approach allows us to compute the eigenvectors of T_m from the eigenvectors of T_{m-1}, as the next result shows.

Proposition 10 *Every $\lambda_{\boldsymbol{i}}$-eigenvector $\boldsymbol{u}_{\boldsymbol{i}}$ of the hypertree T_{m-1} gives rise to*

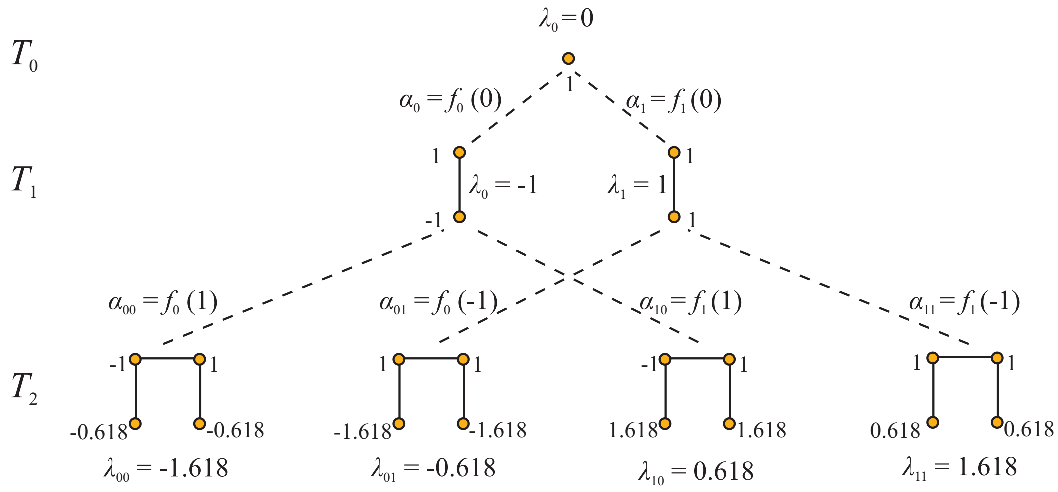

Fig. 6. The eigenvectors of the hypertrees T_0, T_1 and T_2.

the following eigenvectors of T_m*:*

$$\boldsymbol{u}_{0\boldsymbol{i}} = (\boldsymbol{u}_{\boldsymbol{i}}, \alpha_{0\boldsymbol{i}}\boldsymbol{u}_{\boldsymbol{i}})^\top, \quad \boldsymbol{u}_{1\boldsymbol{i}} = (\boldsymbol{u}_{\boldsymbol{i}}, \alpha_{1\boldsymbol{i}}\boldsymbol{u}_{\boldsymbol{i}})^\top, \tag{16}$$

where $\alpha_{0\boldsymbol{i}} = f_0(-\lambda_{\boldsymbol{i}})$ *and* $\alpha_{1\boldsymbol{i}} = f_1(-\lambda_{\boldsymbol{i}})$*, with corresponding eigenvalues* $\lambda_{0\boldsymbol{i}} = \alpha_{0\boldsymbol{i}}^{-1}$ *and* $\lambda_{1\boldsymbol{i}} = \alpha_{1\boldsymbol{i}}^{-1}$*.*

PROOF. The basic idea of the proof is shown in Fig. 5. From the eigenvector $\boldsymbol{u}$ of T_{m-1}, we construct the eigenvector $\boldsymbol{u}' = (\boldsymbol{u}, \alpha\boldsymbol{u})^\top$ of T_m, for some α to be determined. Formally, let $\boldsymbol{A}$ be the adjacency matrix of the binary hypertree T_{m-1}, such that $\boldsymbol{A}\boldsymbol{u} = \lambda_{\boldsymbol{i}}\boldsymbol{u}$. Then, the eigenvalue $\lambda'_{\boldsymbol{i}}$ of T_m the corresponding to the eigenvector $\boldsymbol{u}'$ satisfy

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \alpha\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} + \alpha\boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} (\lambda_{\boldsymbol{i}} + \alpha)\boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix} = \lambda'_{\boldsymbol{i}} \begin{pmatrix} \boldsymbol{u} \\ \alpha\boldsymbol{u} \end{pmatrix} \tag{17}$$

whence

$$\lambda_{\boldsymbol{i}} + \alpha = \frac{1}{\alpha} \quad \Rightarrow \quad \alpha - \frac{1}{\alpha} = -\lambda_{\boldsymbol{i}}. \tag{18}$$

Notice that the last equation in (18) and the first one in (3) coincide except for the sign of $\lambda_{\boldsymbol{i}}$. Consequently, the possible values of α, denoted $\alpha_{0\boldsymbol{i}}$ and $\alpha_{1\boldsymbol{i}}$, are obtained by applying, respectively, the functions f_0 and f_1 in (4) to $-\lambda_{\boldsymbol{i}}$.

Observe that the above proof is based on obtaining a charge distribution in T_m from a charge distribution in T_{m-1}. Thus, the first equation in (18) corresponds to the two ways (depending on the type of vertex considered) of computing the new eigenvalue $\lambda'_{\boldsymbol{i}}$ by using (15).

By way of example, Fig. 6 shows how to obtain the eigenvectors of the (binary) hypertree T_m for $m = 0, 1, 2$.

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