Decisiveness indices are semiindices

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Abstract

In this note we prove that any decisiveness index, defined for any voter as the probability of him/her being decisive, is a semiindex when the probability distribution over coalitions is anonymous, and it is a semiindex with binomial coefficients when the probability over coalitions is anonymous and independent.

Key words: game theory; voting; simple games; power indices; decisiveness.
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1 Preliminaries

1.1 Simple game context

Assume that a proposal is put to the vote, that it must be either approved or rejected and that each voter can only vote "yes" or "no". A model for this binary voting scenario is a *simple game* $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of *voters* and $\mathcal{W}$ denotes the set of *winning coalitions*. A *coalition* is any subset $S \subseteq N$, and is interpreted as the set of voters which vote "yes". A coalition is winning when its occurrence causes the proposal to be accepted. For any coalition S we denote $s = |S|$. When the set of voters, N, is fixed we will write $\mathcal{W}$ instead of $(N, \mathcal{W})$.

Let $\mathcal{S}_N$ be the set of simple games on N. A *power index* on $\mathcal{S}_N$ is a map $\psi : \mathcal{S}_N \to \mathbb{R}^n$ that assigns to every simple game $\mathcal{W}$ a vector $\psi(\mathcal{W})$ with components $\psi_i(\mathcal{W})$ for all $i \in N$.

Many power indices have been considered in the literature. Most of them are based on the notion of cruciality. If $\mathcal{W}$ is a simple game, a player $i \in N$ is said to be *crucial* for a coalition $S \subseteq N \setminus \{i\}$ if $S \notin \mathcal{W}$ but $S \cup \{i\} \in \mathcal{W}$. We then write $S \in \mathcal{C}_i(\mathcal{W})$. This is equivalent to say that the marginal contribution of player i to $S \cup \{i\}$ in $\mathcal{W}$ is 1. Intuitively, any measure of power should take account of the times each player is crucial in a game. Let's recall a family of power indices of this type, named semiindices. These power indices were first introduced by Weber [8] on simple games and extended to all cooperative games by Dubey et al [4].

A semiindex is a power index on $\mathcal{S}_N$ that satisfies the well-known axioms of: symmetry, positivity, dummy player property and transfer. The following characterization of semiindices by means of coefficients was established in Carreras et al. [3], following the characterization given in Dubey et al. [4]:

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(a) for every $\mathbf{p} = (p_1, \ldots, p_n)$ such that $\sum_{s=1}^{n} p_s \binom{n-1}{s-1} = 1$ and $p_s \geq 0$ for all s, the expression

$$\mathfrak{S}_i^{\mathbf{p}}(\mathcal{W}) = \sum_{S \in \mathcal{C}_i} p_s, \tag{1}$$

where $s = |S|$, defines a semiindex $\mathfrak{S}^{\mathbf{p}}$ on $\mathcal{S}_N$;

(b) conversely, every semiindex on $\mathcal{S}_N$ can be obtained in this way;

$c)$ the correspondence given by $\{p_k\} \mapsto \mathfrak{S}^{\mathbf{p}}$ is bijective.

Thus, the payoff that a semiindex allocates to every player in any game is a weighted sum of his/her marginal contributions in the game. Well known examples of semiindices are: the Shapley-Shubik power index φ (Shapley and Shubik [7]), for which $p_s = 1/\left[n\binom{n-1}{s-1}\right]$ for any s $(1 \leq s \leq n)$, and the Banzhaf power index β (Banzhaf [1] and Penrose [6]) for which $p_s = 1/2^{n-1}$. The Shapley and Shubik power index is the only efficient semiindex in the sense that $\sum_{i \in N} \varphi_i(\mathcal{W}) = 1$ for every game $\mathcal{W}$. The Banzhaf power index is the only semiindex having constant coefficients. A subclass of semiindices, that contains the Banzhaf index, is formed by those with binomial coefficients: $p_s = t^{s-1}(1-t)^{n-s}$ for some $t \in [0,1]$, introduced by Carreras and Freixas [2].

From now on we will denote by $\mathfrak{S}^{\mathbf{p}}$ the semiindex uniquely defined by the vector $\mathbf{p}$ in (1) and by $\mathfrak{B}^t$ the particular case of semiindex with binomial coefficients:

$$\mathfrak{B}_i^t(\mathcal{W}) = \sum_{S \in \mathcal{C}_i} t^{s-1}(1-t)^{n-s}. \tag{2}$$

1.2 Simple games with probability distributions over coalitions

A *probability distribution* over coalitions is a function $P : 2^N \to \mathbb{R}$ such that $P(S) \geq 0$ for all $S \subseteq N$ and $\sum_{S \in 2^N} P(S) = 1$. $P(S)$ is interpreted as the probability that voters in S vote "yes" and voters is $N \setminus S$ vote "no".

Let $\mathcal{P}_N$ be the set of probability distributions over coalitions on N. For any $P \in \mathcal{P}_N$, a decisiveness index can be defined on $\mathcal{S}_N$, as pointed out by Dubey and Shapley in [5]. The decisiveness index we are considering captures the idea that voters become crucial for a given coalition by either changing its status from winning to losing or conversely, and it measures the probability of a voter being decisive.

Given $P \in \mathcal{P}_N$ then, for any voter $i \in N$, the *decisiveness index* is defined as:

$$\Phi_i^P(\mathcal{W}) = \sum_{\substack{S\,:\, i \in S \in \mathcal{W} \\ S \setminus \{i\} \notin \mathcal{W}}} P(S) + \sum_{\substack{S\,:\, i \notin S \notin \mathcal{W} \\ S \cup \{i\} \in \mathcal{W}}} P(S) = \sum_{S \in \mathcal{C}_i} \left[P(S) + P(S \setminus \{i\})\right] \tag{3}$$

The Banzhaf index β and the Shapley-Shubik index φ are examples of decisiveness indices for particular probability distributions. Indeed, $\beta = \Phi^{P^*}$ and $\varphi = \Phi^{P^{**}}$, where $P^*(S) = 1/2^n$ and $P^{**}(S) = 1/\left[(n+1)\binom{n}{s}\right]$ for all $S \subseteq N$.

We center now our attention in anonymous probability distributions.

A probability distribution P is:

i) *anonymous* if $P(S) = P(T)$ whenever $|S| = |T|$,

ii) *independent* if every voter's vote is independent of the others. In this case, if t_i is the probability of i voting "yes":

$$P(S) = \prod_{i \in S} t_i \prod_{i \notin S} (1 - t_i). \tag{4}$$

Consequently in an *anonymous and independent* probability distribution every voter casts the vote independently of the other voters with the same probability $t_i = t$ for all $i \in N$ so that

$$P(S) = t^s (1 - t)^{n-s}. \tag{5}$$

Note that the Banzhaf index is obtained from a probability distribution P^* which is both, anonymous and independent, with $t_i = 1/2$ for all $i \in N$. The Shapley-Shubik index is obtained from the probability distribution P^{**} which is anonymous but not independent.

The purpose of this note is to show that the decisiveness indices Φ^P are closely related to semiindices when the probability distribution P is anonymous and this relation is still greater when it is anonymous and independent.

2 Decisiveness indices for anonymous probability distributions

Let $\mathcal{A}_N \subseteq \mathcal{P}_N$ denote the set of anonymous probability distributions. If $P \in \mathcal{A}_N$ denote $P_s = P(S)$ for all $S \subseteq N$. Then the decisiveness index in (3) reduces to

$$\Phi_i^P = \sum_{S \in \mathcal{C}_i} [P_s + P_{s-1}] \tag{6}$$

with $\sum_{s=0}^{n} \binom{n}{s} P_s = 1$, $P_s \geq 0$ for all $s = 0, 1, \ldots, n$.

Proposition 2.1 *Let $P \in \mathcal{A}_N$. Then the decisiveness index Φ^P coincides with the semiindex $\mathfrak{S}^{\mathbf{p}}$ defined in* (1) *with coefficients $p_s = P_s + P_{s-1}$ for all $s = 1, \ldots, n$.*

Proof: We have to prove that the vector $\mathbf{p} = (p_1, \ldots, p_n)$ defined by $p_s = P_s + P_{s-1}$ for all $s = 1, \ldots, n$ determines a semiindex, i.e., $p_s \geq 0$ for all $s = 1, \ldots, n$ and $\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$. Then, from (1) and (6), we deduce that $\Phi^P = \mathfrak{S}^{\mathbf{p}}$.

It is clear that $p_s \geq 0$ for $s = 1, \ldots, n$ because $P_s \geq 0$ for all $s = 0, 1, \ldots, n$.

Using that $\sum_{s=0}^{n} \binom{n}{s} P_s = 1$ we deduce that $\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$ as follows:

$$\sum_{s=0}^{n} \binom{n}{s} P_s = 1 \iff P_0 + \sum_{s=1}^{n-1} \binom{n}{s} P_s + P_n = 1$$

$$\iff P_0 + \sum_{s=1}^{n-1} \left[\binom{n-1}{s-1} + \binom{n-1}{s} \right] P_s + P_n = 1 \iff P_0 + \sum_{s=1}^{n-1} \binom{n-1}{s-1} P_s + \sum_{s=2}^{n} \binom{n-1}{s-1} P_{s-1} + P_n = 1$$

$$\iff (P_0 + P_1) + \sum_{s=2}^{n-1} \binom{n-1}{s-1} P_s + \sum_{s=2}^{n-1} \binom{n-1}{s-1} P_{s-1} + (P_{n-1} + P_n) = 1 \iff \sum_{s=1}^{n} \binom{n-1}{s-1} (P_s + P_{s-1}) = 1$$

$$\iff \sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$$

□

Nevertheless, not every semiindex comes from a decisiveness index. As an example consider for $n = 3$ the semiindex $\mathfrak{S}^{\mathbf{p}}$ defined by $\mathbf{p} = (p_0, p_1, p_2) = (1/8, 3/8, 1/8)$. In fact, if there were $P \in \mathcal{A}_N$ satisfying the three equalities in Proposition 2.1 then it should be $P_1 \leq 1/8$ and $P_2 \leq 1/8$ from the first and third equalities, which would contradict the second equality $P_1 + P_2 = 3/8$. The characterization of the semiindices which come from a decisiveness index is an open problem.

A direct consequence of Proposition 2.1 is the following corollary, which establishes a bijective correspondence between the semiindices with binomial coefficients defined in (2) and the decisiveness indices given by anonymous and independent probability distributions. Let $\mathcal{I}_N \subseteq \mathcal{A}_N$ denote the set of anonymous and independent probability distributions.

Corollary 2.2 *If $P \in \mathcal{I}_N$ then $\Phi^P = \mathfrak{B}^t$ for some $t \in [0, 1]$, and, conversely, any semiindex $\mathfrak{B}^t$ can be obtained in this way.*

Proof: From (5), if $P \in \mathcal{I}_N$ then there exist some $t \in [0, 1]$ such that $P_s = t^s(1-t)^{n-s}$, for $s = 0, 1, \ldots, n$. From Proposition 2.1 we know that $\Phi^P = \mathfrak{S}^{\mathbf{p}}$ with the vector $\mathbf{p} = (p_1, \ldots, p_n)$ defined by $p_s = P_s + P_{s-1}$ for all $s = 1, \ldots, n$. Thus, $p_s = t^s(1-t)^{n-s} + t^{s-1}(1-t)^{n-s+1} = t^{s-1}(1-t)^{n-s}[t + (1-t)] = t^{s-1}(1-t)^{n-s}$, and these coefficients define the semiindex $\mathfrak{B}^t$. As this proof is independent of t we can conclude that *any* semiindex with binomial coeficients is a decisiveness index Φ^P with $P \in \mathcal{I}_N$. □

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