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Dimension of the orbit of marked subspaces ☆

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Abstract

Given a nilpotent endomorphism, we consider the manifold of invariant subspaces having a fixed Segre characteristic. In [Linear Algebra Appl., 332–334 (2001) 569], the implicit form of a miniversal deformation of an invariant subspace with respect to the usual equivalence relation between subspaces is obtained. Here we obtain the explicit form of this deformation when the invariant subspace is marked, and we use it to calculate the dimension of the orbit and in particular to characterize the stable marked subspaces (those with open orbit). Moreover, we study the rank of the endomorphisms in the quotient space by the subspaces in the miniversal deformation of the giving subspace.

Keywords: Linear map; Invariant subspaces; Marked subspaces; Miniversal deformation

1. Introduction

Let $\mathrm{Inv}(p, q)$ be the manifold of invariant subspaces with Segre characteristic q with respect to a nilpotent endomorphism of Segre characteristic p. In [2] the implicit form of a miniversal deformation of an invariant subspace with respect to the usual equivalence relation between subspaces is obtained (see Theorem 2.9). In [3], an interesting class of invariant subspaces, which are called marked, is introduced (see Definition 3.1). In this note we obtain the explicit form of this deformation when the invariant subspace is marked. The fact that the dimension of a miniversal

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deformation is the codimension of the orbit allows us to obtain the dimension of the orbit. As a consequence, we characterize the stable marked subspaces, that is to say, the subspaces with open orbit. It is very interesting to know what classes of subspaces are in a neighbourhood of the considered marked subspace. Our contribution to this problem is to obtain the quotient rank.

The paper is organized as follows. In Section 2, the concepts and results that we use to obtain our results are summarized. Particularly, in Section 2.1 we present the differentiable structure of the set $\mathrm{Inv}(p, q)$ formed by the invariant subspaces with Segre characteristic $q = (q_1, q_2, \ldots, q_m)$ with respect to a nilpotent endomorphism with Segre characteristic $p = (p_1, p_2, \ldots, p_n)$ studied in [4]. In Section 2.2 the Arnold techniques (see [1]) necessary for our study are briefly summarized, and in Section 2.3 we present the statement of Theorem 2.9 [2], which gives us the implicit form of a miniversal deformation of an invariant subspace in $\mathrm{Inv}(p, q)$ with respect to the usual equivalence relation between subspaces.

In Section 3, we first recall the definition of a marked subspace 3.1, and we define what we understand as its canonical matrix representation. Propositions 3.5 and 3.6 solve the equations of Theorem 2.9 when the invariant subspace is marked, and Theorem 3.7 gives the explicit form of the miniversal deformation of a marked subspace. In Theorem 3.10, we calculate the codimension of a marked subspace orbit, and we use this result to describe the stable marked subspaces in Corollary 3.11.

2. Prerequisites

For the convenience of the reader, and in order to fix the notation we recall the differentiable structure of $\mathrm{Inv}(p, q)$ as an orbit space introduced in [4], which is used together with the Arnold techniques in [2].

2.1. The differentiable structure of $\mathrm{Inv}(p, q)$

Let E be a $\mathbb{C}$-vectorial space of dimension N and $f \in \mathrm{End}(E)$ a nilpotent endomorphism with Segre characteristic $p = (p_1, p_2, \ldots, p_n)$. If M is a space of matrices we denote by M^* the subset of matrices in M having full rank. We denote by $\mathrm{Inv}(p, q)$ the subset of the Grassman manifold $\mathrm{Gr}(d)$ formed by the d-dimensional invariant subspaces with Segre characteristic $q = (q_1, q_2, \ldots, q_m)$. We fix a Jordan basis of E and, if $Y \in M^*_{N,d}(\mathbb{C})$, we denote by $\mathrm{Sp}(Y)$ the subspace spanned by the vectors whose components in this basis are the columns of Y.

Definition 2.1

1. A matrix is called lower diagonal (LD) if it is a lower triangular matrix constant along the diagonals.
2. A partitioned matrix whose blocks are LD matrices will be called block lower diagonal (BLD).

3. We denote by $\mathrm{BLD}(p, q)$ the BLD matrices with respect to the block partition (p, q), that is to say, its rows are partitioned according to $p = (p_1, p_2, \dots, p_n)$ and its columns are partitioned according to $q = (q_1, q_2, \dots, q_m)$.
4. If $X \in \mathrm{BLD}(p, q)$, we denote by $X_{i,j}$ the block in $M_{p_i,q_j}(\mathbb{C})$ and by $x_{i,j,k}$ the entries in the k-diagonal of this block, where the first diagonal is the element in the left bottom corner.

Example 2.2 $X \in \mathrm{BLD}((3, 2, 2), (2, 1))$ is

$$\left|\begin{array}{cc|c}
0 & 0 & 0 \\
x_{1,1,2} & 0 & 0 \\
x_{1,1,1} & x_{1,1,2} & x_{1,2,1} \\
\hline
x_{2,1,2} & 0 & 0 \\
x_{2,1,1} & x_{2,1,2} & x_{2,2,1} \\
\hline
x_{3,1,2} & 0 & 0 \\
x_{3,1,1} & x_{3,1,2} & x_{3,2,1}
\end{array}\right|$$

Lemma 2.3
1. $\mathrm{BLD}^*(p, p)$ *is the closed subgroup of* $\mathrm{Gl}(N)$ *formed by the non singular matrices which commute with the lower Jordan matrix of* f.
2. *Analogously for* $\mathrm{BLD}^*(q, q)$.

Proposition 2.4
1. *If* $Y \in \mathrm{BLD}^*(p, q)$, *then* $\mathrm{Sp}(Y) \in \mathrm{Inv}(p, q)$, *and the columns of* Y *form a Jordan basis of* $\mathrm{Sp}(Y)$.
2. *Conversely, for each* $F \in \mathrm{Inv}(p, q)$ *there is some* $Y \in \mathrm{BLD}^*(p, q)$ *such that* $F = \mathrm{Sp}(Y)$.
3. $\mathrm{BLD}^*(q, q)$ *acts freely on* $\mathrm{BLD}^*(p, q)$ *by right multiplication*.
4. *Let* $Y, Y' \in \mathrm{BLD}^*(p, q)$. *Then* $\mathrm{Sp}(Y) = \mathrm{Sp}(Y')$ *if and only if there is some* $Q \in \mathrm{BLD}^*(q, q)$ *such that* $Y = Y'Q$.

Corollary 2.5. *We can identify*

$$\mathrm{Inv}(p, q) = \mathrm{BLD}^*(p, q)/\mathrm{BLD}^*(q, q).$$

By means of this identification, a differentiable structure is introduced in $\mathrm{Inv}(p, q)$ as an orbit space:

Theorem 2.6
1. *The quotient space* $\mathrm{BLD}^*(p, q)/\mathrm{BLD}^*(q, q)$ *has a differentiable structure, such that the natural projection* π *is a submersion (that is, the linear map* $(d\pi)_x$ *is surjective for all* $X \in \mathrm{BLD}^*(p, q)$*)*.
2. *With this differentiable structure*, $\mathrm{Inv}(p, q)$ *is a submanifold of the Grassman manifold* $\mathrm{Gr}(d)$.

Corollary 2.7. $\dim \mathrm{Inv}(p, q) = \dim \mathrm{BLD}(p, q) - \dim \mathrm{BLD}(q, q)$.

2.2. Arnold's deformations

Let M be a differentiable manifold and G a Lie group acting on it. We consider the equivalence relation associated to this action, so that the equivalence class of a point $x \in M$ is its orbit $\mathcal{O}(x)$ with respect to the action

$$\mathcal{O}(x) = \{g \cdot x,\ g \in G\}.$$

We will assume that the orbit is a submanifold of M (as it will be in our case).

An l-parameterized (local) *deformation* of $x \in M$ is a differentiable map $\Phi : U \to M$, where U is a neighbourhood of the origin of $\mathbb{C}^l$, and $\Phi(0) = x$. For example, any local parameterizations of a submanifold $N \subset M$ containing x is a deformation of x.

It is called *versal* if for any other deformation

$$\Psi : V \to M, \quad V \subset \mathbb{C}^k,$$

there is a neighbourhood $V' \subset V$, $0 \in V'$, a differentiable map

$$\alpha : V' \to U, \quad \alpha(0) = 0,$$

and a deformation $\beta : V' \to G$ of the identity element of G such that

$$\Psi(v) = \beta(v) \cdot \Phi(\alpha(v)), \quad v \in V'.$$

Among the versal deformations, those having a minimal number of parameters are called *miniversal*.

Theorem 2.8. *In the above conditions*, (*any parameterization of*) *the submanifold $N \subset M$ is a versal deformation of $x \in N$ if and only if N is transversal to $\mathcal{O}(x)$ at x, that is to say, if*

$$T_x(N) + T_x\mathcal{O}(x) = T_x(M),$$

where $T_x(\cdot)$ means the tangent space to $(\cdot)$ at point x. Furthermore, it is miniversal if and only if it is minitransversal, that is to say, if the sum is direct.

Notice that, if N is a miniversal deformation of $x \in M$, then

$$\dim \mathcal{O}(x) = \dim M - \dim N.$$

Notice also that, if we have a versal deformation of a point in the manifold, we can obtain a versal deformation of another point of its orbit in a natural way.

2.3. Miniversal deformation of a subspace

We consider the manifold $M = \mathrm{Inv}(p, q)$, and the action of the Lie group $G = \mathrm{BLD}^*(p, p)$ by left multiplication as the action, then the orbit of a subspace is the set of its equivalent subspaces. Then we have the next result.

Theorem 2.9 [2]. *Let $F \in \mathrm{Inv}(p, q)$. If $F = \mathrm{Sp}(Y)$ and $Y \in \mathrm{BLD}^*(p, q)$, then a miniversal deformation of F in $\mathrm{Inv}(p, q)$ is formed by $\mathrm{Sp}(Y + \mathscr{X})$, where $\mathscr{X}$ is (a neighbourhood of the origin of) the set of matrices $X \in \mathrm{BLD}(p, q)$ such that*

$$\mathrm{trace}(PYX^*) = \mathrm{trace}(YQX^*) = 0$$

for all $P \in \mathrm{BLD}(p, p)$ and $Q \in \mathrm{BLD}(q, q)$.

3. Miniversal deformation of a marked subspace

Definition 3.1 [3]. An invariant subspace with respect to f is a marked subspace if there is a Jordan basis of the restriction extendible to a Jordan basis of the whole space.

Remark 3.2. When $F \in \mathrm{Inv}(p, q)$ is marked, there is no restriction to suppose that $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_m > q_{m+1} = \cdots = q_n = 0$ and $p = (p_1, p_2, \ldots, p_n)$ is ordered in such a way that the i-Jordan chain of length q_i of F is contained in the i-Jordan chain of length p_i of E and $p_i \geqslant p_{i+1}$ if $q_i = q_{i+1}$.

Lemma 3.3. *Let $F \in \mathrm{Inv}(p, q)$ be a marked subspace. Then, there is $Y \in \mathrm{BLD}^*(p, q)$ such that $\mathrm{Sp}(Y) \in \mathscr{O}(F)$ with $Y = (Y_{i,j})$, $Y_{i,j} \in M_{p_i,q_j}(\mathbb{C})$, where for all $1 \leqslant j \leqslant m$,*

$$Y_{j,j} = \begin{pmatrix} 0 \\ I_{q_j} \end{pmatrix}$$

and $Y_{i,j} = 0$ otherwise.

Proof. By Proposition 2.4 there is $Y' \in \mathrm{BLD}^*(p, q)$ such that $F = \mathrm{Sp}(Y')$. Then $F = \mathrm{Sp}(Y'T)$ for all $T \in \mathrm{BLD}^*(q, q)$ and $F \in \mathscr{O}(\mathrm{Sp}(Y))$ if and only if there are matrices $S \in \mathrm{BLD}^*(p, p)$ and $T \in \mathrm{BLD}^*(q, q)$ such that $SY'T = Y$.

We consider a Jordan basis of F extendible to a Jordan basis $\mathscr{B}'$ of E. Let $Y' \in \mathrm{BLD}^*(p, q)$ the matrix formed by the components of the vectors of this basis in the fixed Jordan basis $\mathscr{B}$ of E, that is, $F = \mathrm{Sp}(Y')$.

If $S \in \mathrm{BLD}^*(p, p)$ is the matrix formed by the components of the vectors of $\mathscr{B}$ in the basis $\mathscr{B}'$, then $Y = SY'$ has the desired form. □

Definition 3.4. Let $F \in \mathrm{Inv}(p, q)$ be a marked subspace. We say that the matrix $Y \in \mathrm{BLD}^*(p, q)$ obtained in Lemma 3.3 is its canonical matrix representation or the canonical matrix representation of its orbit in $\mathrm{Inv}(p, q)$ (notice that all the equivalent marked subspaces have the same canonical matrix representation).

Proposition 3.5. *Let $Y \in \mathrm{BLD}^*(p, q)$ be a canonical matrix representation of a marked subspace. The matrix $X \in \mathrm{BLD}^*(p, q)$ is a solution of*

$$\operatorname{trace}(PYX^*) = 0 \quad \textit{for all } P \in \mathrm{BLD}(p, p)$$

if and only if

$$x_{i,k,h} = 0, \quad 1 \leqslant h \leqslant \min(p_i - p_k + q_k, q_k) \ \textit{for all}\ 1 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant m.$$

Proof. The form of Y implies that $(PY)_{i,j} = \sum_{k=1}^{n} P_{i,k}Y_{k,j} = P_{i,j}Y_{j,j}$. Hence, $(PYX^*)_{i,j} = \sum_{k=1}^{m} P_{i,k}Y_{k,k}X^*_{j,k}$. Then,

$$\begin{aligned}\operatorname{trace}(PYX^*) &= \sum_{i=1}^{n} \operatorname{trace}\left(\sum_{k=1}^{m} P_{i,k}Y_{k,k}X^*_{i,k}\right)\\ &= \sum_{i\leqslant n, k\leqslant m} \operatorname{trace}\bigl(P_{i,k}Y_{k,k}X^*_{i,k}\bigr) = \sum_{i\leqslant n, k\leqslant m} \operatorname{trace}\bigl(P^{(q_k)}_{i,k}X^*_{i,k}\bigr),\end{aligned}$$

where $P^{(q_k)}_{i,k}$ is the matrix formed by the last q_k columns of $P_{i,k}$.

Hence, computing these traces we obtain that

$$\operatorname{trace}(PYX^*) = \sum_{i\leqslant n, k\leqslant m} \sum_{1\leqslant h\leqslant \delta_{i,j}} h\, p_{i,k,p_k-q_k+h}\, x^*_{i,k,h} = 0,$$

where $\delta_{i,k} = \min(p_i - p_k + q_k, q_k)$.

Then, considering that the equality holds for all $P \in \mathrm{BLD}(p, q)$, we obtain that

$$x_{i,k,h} = 0, \quad 1 \leqslant h \leqslant \delta_{i,k} \ \text{ for all } 1 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant m$$

and the proposition is proved. $\square$

Notice that if $p_i - p_k + q_k < 0$ there is not any condition for the corresponding block $X_{i,k}$.

Proposition 3.6. *Let $Y \in \mathrm{BLD}^*(p, q)$ be a canonical matrix representation of a marked subspace. The matrix $X \in \mathrm{BLD}^*(p, q)$ is a solution of*

$$\operatorname{trace}(YQX^*) = 0 \quad \textit{for all } Q \in \mathrm{BLD}(q, q)$$

if and only if

$$x_{i,k,h} = 0, \quad 1 \leqslant h \leqslant \min(q_i, q_k) \ \textit{for all}\ 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant m.$$

Proof. The form of Y implies that $(YQ)_{i,j} = \sum_{k=1}^{m} Y_{i,k}Q_{k,j} = Y_{i,i}Q_{i,j}$. Hence, $(YQX^*)_{i,j} = \sum_{k=1}^{m} Y_{i,i}Q_{i,k}X^*_{j,k}$. Then,

$$\begin{aligned}\operatorname{trace}(YQX^*) &= \sum_{i=1}^{m} \operatorname{trace}\left(\sum_{k=1}^{m} Y_{i,i}Q_{i,k}X^*_{i,k}\right)\\ &= \sum_{i,k\leqslant m} \operatorname{trace}(Y_{i,i}Q_{i,k}X^*_{i,k}) = \sum_{i,k\leqslant m} \operatorname{trace}\left(\begin{pmatrix} 0 \\ Q_{i,k}\end{pmatrix} X^*_{i,k}\right).\end{aligned}$$

Hence, computing these traces we obtain that

$$\operatorname{trace}(YQX^*) = \sum_{i,k \leqslant m} \sum_{1 \leqslant h \leqslant \varepsilon_{i,k}} h\, q_{i,k,h}\, x^*_{i,k,h} = 0,$$

where $\varepsilon_{i,k} = \min(q_i, q_k)$.

Then, considering that the equality holds for all $Q \in \mathrm{BLD}(q, q)$, we obtain that

$$x_{i,k,h} = 0, \quad 1 \leqslant h \leqslant \varepsilon_{i,k} \text{ for all } 1 \leqslant i, k \leqslant m.$$

Notice that the equality also holds for $i > m$ since $q_i = 0$, and the proposition is proved. $\square$

Theorem 3.7. *Let $F \in \mathrm{Inv}(p, q)$ be a marked subspace. If $F = \mathrm{Sp}(Y)$ and $Y \in \mathrm{BLD}^*(p, q)$ is its canonical matrix representation, then a miniversal deformation of F in $\mathrm{Inv}(p, q)$ is formed by $\mathrm{Sp}(Y + \mathscr{X})$, where $\mathscr{X}$ is (a neighbourhood of the origin of) the set of matrices $X \in \mathrm{BLD}(p, q)$ such that*

$$\begin{aligned} x_{i,k,h} = 0, \quad & 1 \leqslant h \leqslant \max\big(\min(p_i - p_k + q_k, q_k), \min(q_i, q_k)\big), \\ & 1 \leqslant i \leqslant n, \quad 1 \leqslant k \leqslant m. \end{aligned}$$

Proof. The result is a consequence of Theorem 2.9, Propositions 3.5 and 3.6. $\square$

Corollary 3.8. *The condition of Theorem 3.7 is equivalent to*

1. *If $q_i \geqslant q_k$ or $p_i \geqslant p_k$, then the block $X_{i,k} = 0$.*
2. *If $q_i < q_k$ and $p_i < p_k$, then $x_{i,k,h} = 0$ for $1 \leqslant h \leqslant \max(p_i - p_k + q_k, q_i)$.*

We illustrate the above theorem with the following example.

Example 3.9. Let $q = (4, 2, 1, 0)$ and $p = (6, 3, 4, 2)$. In this case, the matrices $Y + X$ are

$$\left|\begin{array}{cccc|cc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
x_{213} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{213} & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & x_{213} & 0 & 0 & 1 & 0 \\
\hline
x_{314} & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{313} & x_{314} & 0 & 0 & 0 & 0 & 0 \\
0 & x_{313} & x_{314} & 0 & 0 & 0 & 0 \\
0 & 0 & x_{313} & x_{314} & 0 & 0 & 1 \\
\hline
x_{412} & 0 & 0 & 0 & x_{422} & 0 & 0 \\
x_{411} & x_{412} & 0 & 0 & 0 & x_{422} & x_{431}
\end{array}\right|$$

Theorem 3.10. *Let* $F \in \mathrm{Inv}(p, q)$ *be a marked subspace, then*

$$\dim \mathcal{O}(F)^{\perp} = \sum_{1 \leqslant k < i \leqslant n} \max\big(0, \min(p_i - q_i, p_k - q_k, p_k - p_i, q_k - q_i)\big),$$

where q *and* p *are ordered as in Remark* 3.2.

Proof. The dimension of $\mathcal{O}(F)^{\perp}$ is the number of parameters in the miniversal deformation. Using Corollary 3.8 we have

$$\dim \mathcal{O}(F)^{\perp} = \sum_{q_i < q_k, p_i < p_k} \min(p_i, q_k) - \max(p_i - p_k + q_k, q_i).$$

But $\max(p_i - p_k + q_k, q_i) = q_k - \min(p_k - p_i, q_k - q_i)$. Hence

$$\min(p_i, q_k) - \max(p_i - p_k + q_k, q_i)$$
$$= \min(p_i - q_k, 0) + \min(p_k - p_i, q_k - q_i).$$

We study the two possible cases separately:

If $p_i \geqslant q_k$, $\min(p_i - q_k, 0) + \min(p_k - p_i, q_k - q_i) = \min(p_k - p_i, q_k - q_i)$.
If $p_i < q_k$, $\min(p_i - q_k, 0) + \min(p_k - p_i, q_k - q_i) = \min(p_k - q_k, p_i - q_i)$.

Because $q_i < q_k$ and $p_i < p_k$, we can summarize the two cases as

$$\min(p_i, q_k) - \max(p_i - p_k + q_k, q_i)$$
$$= \min(p_i - q_i, p_k - q_k, p_k - p_i, q_k - q_i).$$

Hence,

$$\dim \mathcal{O}(F)^{\perp} = \sum_{q_i < q_k, p_i < p_k} \min(p_i - q_i, p_k - q_k, p_k - p_i, q_k - q_i).$$

And finally, we can include the cases $q_i \geqslant q_k$ or $p_i \geqslant p_k$ in the sum considering the maximum between 0 and the minimum above since this minimum is negative or 0. $\square$

Now we are going to study the marked subspaces with open orbit, i.e. the ones with the dimension of the manifold as dimension of their orbit. They are known as stable marked subspaces.

Corollary 3.11. *A marked subspace* $F \in \mathrm{Inv}(p, q)$ *is structurally stable if and only if*

$$q_k \in \,]q_i, p_k[\implies p_i \notin \,]q_i, p_k[\quad \textit{for all } 1 \leqslant k < i \leqslant n.$$

Proof. The result is a consequence of Theorem 3.10 because $F \in \mathrm{Inv}(p, q)$ is structurally stable if and only if $\mathcal{O}(F) = \mathrm{Inv}(p, q)$. $\square$

Finally, we contribute to the study of the possible classes of subspaces that are in a neighbourhood of a marked subspace by giving the possible quotient ranks.

Lemma 3.12. *Let $F \in \mathrm{Inv}(p, q)$ be a marked subspace, $F = \mathrm{Sp}(Y)$ and $Y \in \mathrm{BLD}^*(p, q)$ its canonical matrix representation.*

If we denote by $\{e_{i,1}, e_{i,2}, \dots, e_{i,p_i}\}_{1 \leqslant i \leqslant n}$ the Jordan chains of a basis of the space E adapted to F, that is, $\{e_{i,p_i-q_i+1}, \dots, e_{i,p_i}\}_{1 \leqslant i \leqslant m}$ is a Jordan basis of F, then a basis of a subspace $\mathrm{Sp}(Y + X)$ belonging to the miniversal deformation of F in Theorem 3.7 is

$$\left\{ e_{k,p_k-q_k+l} + \sum_{q_i<q_k,\, p_i<p_k} \sum_{\eta_{ikl}<j\leqslant \min(p_i,q_k)} x_{i,k,j}\, e_{i,p_i-j+l} \right\}_{1\leqslant k\leqslant m,\, 1\leqslant l\leqslant q_k},$$

where $\eta_{ikl} = \max(p_i - p_k + q_k, q_i, l - 1)$ and $e_{i,t} = 0$ for $t > p_i$.

Proof. The result is a consequence of Theorem 3.7 and Corollary 3.8. □

Proposition 3.13. *Let $F \in \mathrm{Inv}(p, q)$ be a marked subspace with respect to f, $F = \mathrm{Sp}(Y)$, $Y \in \mathrm{BLD}^*(p, q)$ its canonical matrix representation and $\mathrm{Sp}(Y + X)$ a subspace in its miniversal deformation. Then, the quotient map*

$$\tilde{f}_X : \frac{E}{\mathrm{Sp}(Y+X)} \to \frac{E}{\mathrm{Sp}(Y+X)}$$

induced by f is defined by

$$\begin{aligned}
&\tilde{f}_X(\tilde{e}_{k,j}) = \tilde{e}_{k,j+1} \quad \text{for } 1 \leqslant k \leqslant n,\ 1 \leqslant j < p_k - q_k,\\
&\tilde{f}_X(\tilde{e}_{k,p_k-q_k}) = -\sum_{q_i<q_k,\, p_i<p_k} \sum_{\eta_{ik1}<j\leqslant \min(p_i,q_k)} x_{i,k,j}\, \tilde{e}_{i,p_i-j+1} \quad \text{for } 1 \leqslant k \leqslant m,\\
&\tilde{f}_X(\tilde{e}_{k,p_k-q_k}) = 0 \quad \text{for } m < k \leqslant n.
\end{aligned}$$

Proof. It is immediate that the classes of the set $\{e_{k,j}\}_{1\leqslant k\leqslant n, 1\leqslant j\leqslant p_k-q_k}$ form a basis of $\frac{E}{\mathrm{Sp}(Y+X)}$. Hence, the expression of $\tilde{f}_X$ is obvious by applying Lemma 3.12. □

Corollary 3.14. *The matrix of the map $\tilde{f}_X$ in the basis $\{e_{k,j}\}_{1\leqslant k\leqslant n,1\leqslant j\leqslant p_k-q_k}$ is a partitioned lower triangular matrix whose blocks $C_{i,k} \in M_{p_i-q_i,p_k-q_k}(\mathbb{C})$ have the following forms:*

- *$C_{i,i}$ is a nilpotent matrix.*
- *$C_{i,k}$ $(i \neq k)$ has all the entries 0 except the last column which is $(-x_{i,k,p_i}, -x_{i,k,p_{i-1}}, \dots, -x_{i,k,q_i+1})^{\mathrm{t}}$ if $q_i < q_k$ and $p_i < p_k$ (where $x_{i,k,j} = 0$ if $j \leqslant \eta_{i,k,1}$ or $j > \min(p_i, q_k)$).*

Example 3.15. In Example 3.9 the matrix of the quotient map is

$$
\left|\begin{array}{cc|c|ccc|cc|}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -x_{213} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -x_{314} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_{313} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & -x_{412} & -x_{422} & 1 & 0 & 0 & 0 & 0 \\
0 & -x_{411} & 0 & 0 & 0 & -x_{431} & 1 & 0
\end{array}\right.
$$

Theorem 3.16. *Let* $F \in \mathrm{Inv}(p, q)$ *be a marked subspace with respect to* f, $F = \mathrm{Sp}(Y)$, $Y \in \mathrm{BLD}^*(p, q)$ *its canonical matrix representation and* $\mathrm{Sp}(Y + X)$ *a subspace in its miniversal deformation. Then,*

$$\operatorname{rank} \tilde{f}_X = \sum_{i=1}^{n} \max(0, p_i - q_i - 1) + \operatorname{rank} (x_{i,k,p_i})_{i,k}.$$

Proof. The rows with 1 in the $C_{i,i}$ blocks give the first sumand. The matrix in the second sumand is obtained eliminating their rows and columns. □

Notice that the matrix $(x_{i,k,p_i})_{i,k}$ is lower triangular with null diagonal and $\operatorname{rank}(x_{i,k,p_i})_{i,k} \leqslant \min(n-1, m)$ because $x_{i,k,p_i} = 0$ if $p_i \leqslant \max(p_i - p_k + q_k, q_i)$ or $p_i > \min(p_i, q_k)$.

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