AN EXPLICIT EXPRESSION OF THE FIRST LIAPUNOV AND PERIOD CONSTANTS WITH APPLICATIONS*

ARMENGOL GASULL[1], ANTONI GUILLAMON[2], VÍCTOR MAÑOSA[3]

[1]Departament de Matemàtiques,
Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain.
[2]Departament de Matemàtica Aplicada I, E.U.P.B,
Universitat Politècnica de Catalunya, 08028 Barcelona, Catalonia, Spain.
[3]Departament de Matemàtica Aplicada III,
Universitat Politècnica de Catalunya, 08222 Terrassa, Catalonia, Spain.

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Abstract: In this paper, we study systems in the plane having a critical point with pure imaginary eigenvalues, and we search for effective conditions to discern whether this critical point is a focus or a center; in the case of it being a center, we look for additional conditions in order to be isochronous. We wish to stress that the essential differences between the techniques used in this work and the more usual ones are basically two: the elimination of the integration constants when we consider primitives of functions (see also Remark 3.2) and the fact that we maintain the complex notation in the whole study. Thanks to these aspects, we reach with relative ease an expression of the first three Liapunov constants, v_3, v_5 *and* v_7, *and of the first two period ones,* p_2 *and* p_4, *for a general system. As far as we know, this is the first time that a general and compact expression of* v_7 *has been given. Moreover, the use of a computer algebra system is only needed in the computation of* v_7 *and* p_4. *These results are applied to give a classification of centers and isochronous centers for certain families of differential equations.*

1. INTRODUCTION.

In the Qualitative Theory of planar differential equations, the problem of determining whether a critical point with pure imaginary eigenvalues is a center or a weak focus is known as the *center-focus* problem.

The solution of the center-focus problem for a particular system involves the knowledge of the sign of the so-called *return map*, $P(\rho)$, in some neighbourhood of the origin. This sign can be studied by computing the terms of the series expansion of $P(\rho)$ which can be obtained recurrently and are generically called the *Liapunov constants*. They are usually denoted by v_{2i+1}, for $i \in \mathbf{N}$, and will be defined accurately afterwards.

There exist several ways to compute the Liapunov constants but all of them run into troubles from some particular v_i on. These troubles are mainly due to the large amount of computations that are involved which break down the capacity of the computers.

The method we present is a development of [10]. Its main advantages are:

(1) Until the last step we do not need to write the constants in terms of the coefficients of the system. Most of the methods to compute such constants use the coefficients from the first step on, thereby increasing the difficulties in handling the formulas.
(2) It maintains the algebraic structure of the constants and helps to detect relations that avoid some further unnecessary computations.
(3) The treatment in complex coordinates leads obviously to the shortening of the expressions. This is not only an aesthetic advantage, but also a practical one because in many particular cases it implies a reduction of the total number of operations.

In the supposition that we have characterized the centers, it is also interesting to know whether the center is isochronous or not. A progressive way to find isochronicity conditions for centers is by computing the terms of the series expansion of the period function, that is to say, the so-called *period constants*. They are usually denoted by p_{2i}, for $i \in \mathbf{N}$, and will also be defined afterwards.

In this paper, we obtain a general shortened expression of v_3, v_5, v_7, p_2 and p_4. The newest results are the expressions of v_7 and p_4 for a general system. The expressions of v_3, v_5 and p_2 have been calculated in several ways by many authors (see [2], [3], [6], [10], [12], [14], [16], [17] among others). The expression of p_4 (using normal forms) is also obtained in [7] and [14].

In Section 2 we give the main statements: the expressions of these Liapunov (Theorem A) and period (Theorem B) constants. Section 3 is devoted to their proofs.

To illustrate the use of Theorems A and B we will enunciate in this introduction four results that are straightforward applications of them. They will be proved in the last section (Section 4).

Our first application is the study of a system derived from the Fitzhugh model for the nerve impulse. There are also other reasons to study this system. In [15], Lloyd proposed new lines to be studied in the field of Liénard equations; one of them, is to give information about the systems $\dot{x} = F(x) - y$, $\dot{y} = g(x)$, with $F(x)$ and $g(x)$ polynomials of degree n and m, respectively. Observe that the next system corresponds to the $n = 3$, $m = 4$ case.

THEOREM 1. *Consider the system*

$$\begin{cases} \dot{x} &= -y + ax^2 + bx^3, \\ \dot{y} &= x + dx^2 + gx^3 + hx^4, \end{cases} \tag{1}$$

where $a, b, d, g, h \in \mathbf{R}$. *Then,*
(a) the origin of (1) is a center if and only if one of the following conditions holds:

(a.1) $a = b = 0$.
(a.2) $a \neq 0,\ b = d = h = 0$.
(a.3) $ad \neq 0,\ b = \frac{2}{3}ad,\ g = h = 0$.

(b) the origin of (1) is an isochronous center if and only if $b = d = h = 0$ *and* $g = \frac{4}{9}a^2$.

As we said above, an interesting particular case of system (1) is the Fitzhugh equation, which is a simplified model of the nerve impulse proposed by Fitzhugh in 1961 [5] and has been studied by many authors. Hsü and Kazarinoff in 1976, see [13], computed the first focal value, and wondered about the possibility of having a second focal value vanishing. Numerical experiments with the Fitzhugh equation were done in 1979 by Göbber and Willamowski, [12]. They predicted the possibility of having only a second order degenerate Hopf bifurcation, that is only the first focal value vanishing. In 1981 Golubitsky and Langford [9] predicted, observing some preliminary calculations not shown, the possibility of having a third order degenerate Hopf bifurcation, that is the second focal value vanishing for some choice of the parameters. Finally, in 1990 Gamero, [7], computed the first and second focal values for the equation, showing that the system has only a second order degenerate Hopf bifurcation. This result also follows from our study:

PROPOSITION 2 (see also [7]). *Consider the Fitzhugh model given by*

$$\begin{cases} \dot{x} &= \lambda + y + x - \frac{1}{3}x^3, \\ \dot{y} &= -\rho x - \rho\beta y \end{cases}$$

where $\beta, \rho \in (0,1)$*, and* $\lambda \geq 0$*. Then, for each value of* λ*, at most two limit cycles can bifurcate from the critical point of the system.*

The next application is a generalization of the Liénard equation studied also in [3]. We give a classification of the centers and of the isochronous centers.

THEOREM 3. *Consider the system*

$$\begin{cases} \dot{x} = & -y + ax^2 + bx^3, \\ \dot{y} = & x + my^2 + ny^3, \end{cases} \tag{2}$$

where a*,* b*,* m*,* $n \in \mathbf{R}$*. Then,*

(a) *the origin is a center if and only if* $b + n = 0$ *and, moreover, either* $a^2 - m^2 = 0$*, or* $b = am = 0$*.*

(b) *the origin of (2) is an isochronous center if and only if* $a = b = m = n = 0$*.*

Finally, in the last application, we study the following predator-prey model obtained from a perturbation of the Lotka-Volterra system:

$$\begin{cases} \dot{x} &= 1 - e^y - c\varphi(\frac{x}{d})e^y, \\ \dot{y} &= d\left(e^{x/d} - 1 + c\varphi(\frac{x}{d})e^{x/d}\right), \end{cases} \tag{3}$$

where $\varphi(x) = ax + bx^2 + (e^{-x} - 1)$ and a, b, $c \in \mathbf{R}$ and $d \in \mathbf{R}^+$.

The Lotka-Volterra model written as a Hamiltonian system is the same as system (3) with $c = 0$. The system (3) was considered in [8] to study the number of limit cycles that can appear. To know the cyclicity of the origin, the Liapunov constants are needed. In [8], the last Liapunov constant that can vanish for the origin of (3), which is v_7, is computed in a numerical way. Here, we give the exact expression, as a result of Theorem A, and so we conclude that no more than three limit cycles can be obtained in a neighbourhood of the critical point for systems (3). Summing up, we have that:

THEOREM 4. *The origin of the system (3) is a weak focus of order at most three.*

2. MAIN RESULTS.

We are interested in systems such that the origin is a critical point with pure imaginary eigenvalues. Such systems can be written in complex coordinates as:

$$\dot{z} = iz + \sum_{k\geq 2} F_k(z, \overline{z}), \qquad z \in \mathbf{C}, \tag{4}$$

where $F_k(z,\overline{z})$ are homogeneous polynomials of degree k.

To state the main results, we need to introduce the following notation for system (4):

$$
\begin{aligned}
F_2(z,\overline{z}) &= Az^2 + Bz\overline{z} + C\overline{z}^2;\\
F_3(z,\overline{z}) &= Dz^3 + Ez^2\overline{z} + Fz\overline{z}^2 + G\overline{z}^3;\\
F_4(z,\overline{z}) &= Hz^4 + Iz^3\overline{z} + Jz^2\overline{z}^2 + Kz\overline{z}^3 + L\overline{z}^4;\\
F_5(z,\overline{z}) &= Mz^5 + Nz^4\overline{z} + Oz^3\overline{z}^2 + Pz^2\overline{z}^3 + Qz\overline{z}^4 + R\overline{z}^5;\\
F_6(z,\overline{z}) &= Sz^6 + Tz^5\overline{z} + Uz^4\overline{z}^2 + Vz^3\overline{z}^3 + Wz^2\overline{z}^4 + Xz\overline{z}^5 + Y\overline{z}^6;\\
F_7(z,\overline{z}) &= Zz^4\overline{z}^3 + g_7(z,\overline{z}),
\end{aligned}
\tag{5}
$$

where $g_7(z,\overline{z})$ does not contain the monomial $z^4\overline{z}^3$.

THEOREM A. *With the notation given in (5), the first three Liapunov constants of system (4) are:*

$$
\begin{aligned}
v_3 &= 2\pi\left(\mathrm{Re}(E) - \mathrm{Im}(AB)\right).\\
v_5 &= \frac{\pi}{3}\Big(6\,\mathrm{Re}(O) + \mathrm{Im}(3E^2 - 6DF + 6A\overline{I} - 12BI - 6B\overline{J} - 8CH - 2C\overline{K})\\
&\quad + \mathrm{Re}(-8C\overline{C}E + 4AC\overline{F} + 6A\overline{B}F + 6B\overline{C}F - 12B^2D - 4ACD - 6A\overline{BD} + 10B\overline{CD} + 4A\overline{C}G + 2BC\overline{G})\\
&\quad + \mathrm{Im}(6A\overline{B}^2C + 3A^2B^2 - 4A^2\overline{B}C + 4\overline{B}^3C)\Big).
\end{aligned}
$$

$$
\begin{aligned}
v_7 = \frac{\pi}{216} (& \\
& 432\,\mathrm{Re}(Z) \\
& - 36\,\mathrm{Im}\,(-24A\overline{U} - 12AV + 36BU + 24B\overline{V} + 20CT + 8C\overline{W} - 6D\overline{N} + 6DP + 12EO + 18FN \\
& \quad +6F\overline{P} + 15GM + 3G\overline{Q} + 12HK + 12IJ) \\
& + 9\,\mathrm{Re}\,(-48A^2\overline{N} + 72A\overline{A}O - 216A\overline{BN} + 72ABO + 168AB\overline{O} - 24A\overline{B}P - 120A\overline{CM} + 16ACN \\
& \quad + 48AC\overline{P} + 8A\overline{C}Q + 24A\overline{D}I - 48A\overline{DJ} + 24ADK + 24AE\overline{I} - 72A\overline{EI} + 24AEJ \\
& \quad + 24A\overline{E}J - 96A\overline{FH} + 24AFI + 48AF\overline{J} + 24A\overline{F}K + 24AGH + 36AG\overline{K} + 24A\overline{G}L \\
& \quad - 216B^2N + 264B\overline{B}O - 72B^2\overline{P} - 260BCM + 168B\overline{CN} + 104B\overline{C}P - 36BC\overline{Q} - 24B\overline{D}H \\
& \quad + 48BD\overline{I} - 120BDJ + 96B\overline{DK} - 168BEI + 24B\overline{E}I + 24BE\overline{J} + 120B\overline{EJ} - 216BFH \\
& \quad + 144B\overline{FI} + 72B\overline{F}J + 132B\overline{GH} + 48B\overline{G}K + 12BG\overline{L} + 72C\overline{C}O - 72CDI + 32CD\overline{J} \\
& \quad + 48C\overline{DL} - 104CEH + 24C\overline{E}H + 24CE\overline{K} + 56C\overline{EK} + 56C\overline{FI} + 64C\overline{FJ} + 16CF\overline{L} \\
& \quad + 60C\overline{GI} + 40C\overline{G}J - 18D^2G - 72DEF + 24D\overline{E}F + 48D\overline{F}G - 12E^3 + 12E^2\overline{E} \\
& \quad +48EF\overline{F} + 36EG\overline{G} + 18F^2\overline{G}) \\
& - 3\,\mathrm{Im}\,(-72A^2\overline{AI} - 144A^2B\overline{I} + 72A^2BJ + 144A^2\overline{BH} - 72A^2\overline{B}K + 48A^2CI \\
& \quad - 48A^2\overline{C}L + 36A^2DG - 36A^2\overline{D}E + 36A^2\overline{DE} + 36A^2EF - 36A^2\overline{E}F \\
& \quad - 36A^2\overline{F}G + 216A\overline{A}BI + 144A\overline{A}B\overline{J} + 48A\overline{A}CH + 48A\overline{A}C\overline{K} + 72A\overline{A}DF \\
& \quad + 36A\overline{A}E^2 + 432AB^2\overline{J} - 1008AB\overline{BI} + 216AB\overline{B}J - 144ABCH + 372ABC\overline{K} \\
& \quad - 624AB\overline{CH} + 216AB\overline{C}K + 72ABD\overline{D} - 432AB\overline{DF} + 216ABE\overline{E} - 216AB\overline{E}^2 \\
& \quad + 360ABF\overline{F} + 288ABG\overline{G} + 648A\overline{B}^2\overline{H} - 216A\overline{B}^2K + 192A\overline{B}CI - 384A\overline{B}C\overline{J} \\
& \quad - 336A\overline{BC}L + 36A\overline{B}DG - 216A\overline{BD}E + 432A\overline{BDE} + 144A\overline{B}EF - 360A\overline{BE}F \\
& \quad - 396A\overline{BF}G + 96AC^2\overline{L} - 240AC\overline{CI} + 144AC\overline{C}J - 24ACDE + 72ACD\overline{E} \\
& \quad - 144AC\overline{DG} + 168ACE\overline{F} - 216AC\overline{EF} + 192ACF\overline{G} + 108A\overline{CD}^2 - 192A\overline{CD}F \\
& \quad + 96A\overline{C}EG - 216A\overline{CE}G + 84A\overline{C}F^2 - 648B^3H + 1152B^2\overline{BI} + 576B^2\overline{BJ} \\
& \quad + 156B^2C\overline{L} + 456B^2\overline{CI} + 552B^2\overline{C}J - 756B^2DE + 252B^2D\overline{E} + 396B^2\overline{DG} \\
& \quad + 396B^2E\overline{F} + 396B^2\overline{EF} + 324B^2F\overline{G} + 1596B\overline{B}CH + 408B\overline{B}C\overline{K} + 1224B\overline{B}DF \\
& \quad + 612B\overline{B}E^2 + 804BC\overline{CI} + 384BC\overline{CJ} - 378BCD^2 + 600BCD\overline{F} + 516BCE\overline{G} \\
& \quad + 360BC\overline{EG} + 162BC\overline{F}^2 - 48B\overline{C}^2L + 612B\overline{C}DG + 168B\overline{CD}E - 720B\overline{CDE} \\
& \quad + 720B\overline{C}EF + 216B\overline{CE}F + 84B\overline{CF}G + 288C^2\overline{C}H + 72C^2\overline{CK} + 180C^2D\overline{G} \\
& \quad +60C^2\overline{FG} + 456C\overline{C}DF + 228C\overline{C}E^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,\mathrm{Re}\,\big(54A^3B\overline{D} - 54A^2\overline{ABD} - 54A^2\overline{A}BE + 54A^2\overline{A}B\overline{E} - 54A^3BF \\
&\quad + 54A^2\overline{AB}F + 54A^3\overline{B}G - 36A^2\overline{A}CD - 36A^3CE + 36A^3C\overline{E} + 36A^2\overline{A}C\overline{F} \\
&\quad + 36A^2\overline{AC}G - 108A^2B^2E + 27A^2\overline{B}^2G - 126A^2BCD - 108A\overline{A}B\overline{CD} - 18A^2\overline{B}CE \\
&\quad + 162A^2\overline{B}C\overline{E} - 162A^2BC\overline{F} + 108A\overline{A}B\overline{C}F + 18A^2B\overline{C}G - 108A\overline{A}BC\overline{G} + 72A^2C\overline{CD} \\
&\quad - 72A^2C\overline{C}F - 72A^2C^2\overline{G} + 378AB^3D - 1242AB\overline{B}^2\overline{D} - 1350AB^2\overline{B}E + 1350AB^2\overline{BE} \\
&\quad - 594AB^3\overline{F} + 1458AB\overline{B}^2F - 486A\overline{B}^3G + 756AB^2\overline{CD} - 1638AB\overline{B}CD + 900A\overline{B}^2CE \\
&\quad - 684A\overline{B}^2C\overline{E} - 1044AB^2\overline{C}F + 1926AB\overline{B}C\overline{F} - 1080AB^2C\overline{G} + 1854AB\overline{BC}G - 720A\overline{B}C\overline{CD} \\
&\quad - 1296ABC\overline{C}E + 1008ABC\overline{CE} + 1296A\overline{B}C\overline{C}F + 558A\overline{B}C^2\overline{G} - 396AC^2\overline{C}D + 396AC^2\overline{CF} \\
&\quad + 396AC\overline{C}^2G - 1728B^3\overline{B}D + 1404B^2\overline{B}^2E - 243B^4\overline{G} + 2286B^2\overline{BCD} - 1062B^3\overline{C}E \\
&\quad - 234B^3\overline{CE} + 1350B^2\overline{BC}F + 414B^2\overline{B}C\overline{G} - 2214B^2C\overline{C}D + 2448B\overline{B}C\overline{C}E + 54B^2C\overline{CF} \\
&\quad + 297B^2\overline{C}^2G + 990BC\overline{C}^2\overline{D} + 594BC\overline{C}^2F + 198BC^2\overline{CG} + 272C^2\overline{C}^2E\big)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\,\mathrm{Im}\,\Big(-36A^4BC - 54A^3B^3 + 54A^2\overline{A}B^2\overline{B} + 72A^3B\overline{B}C - 72A^2\overline{AB}^2C + 144A^2\overline{A}BC\overline{C} \\
&\quad - 432A^2B^3\overline{B} + 1017A^2B\overline{B}^2C + 108A\overline{A}B^3\overline{C} - 684A^2B^2C\overline{C} + 540A^2\overline{B}C^2\overline{C} + 1404AB^3\overline{B}^2 \\
&\quad - 648AB^4\overline{C} - 1512AB\overline{B}^3C + 4266AB^2\overline{B}C\overline{C} - 1080A\overline{B}^2C^2\overline{C} + 1064ABC^2\overline{C}^2 + 657B^4\overline{BC} \\
&\quad + 675B^3C\overline{C}^2\Big)\Big)
\end{aligned}
$$

THEOREM B. *With the notation given in (5), the first two period constants of system (4) are:*

$$
\begin{aligned}
p_2 &= \frac{2\pi}{3}\left(-3\,\mathrm{Re}(AB) + 3B\overline{B} + 2C\overline{C} - 3\,\mathrm{Im}\,E\right) \\
p_4 &= 2\pi\Bigg(-\mathrm{Im}\,O + \mathrm{Re}\left(-A\overline{I} + 3B\overline{J} - 2BI + \frac{5}{3}C\overline{K} - \frac{4}{3}CH - DF + F\overline{F} + \frac{3}{4}G\overline{G}\right) \\
&\quad - \mathrm{Im}\left(-2B^2D - \frac{2}{3}ACD + A\overline{BD} - \frac{11}{3}B\overline{CD} + A\overline{B}F + 5B\overline{C}F + 4B^2\overline{F} + \frac{2}{3}AC\overline{F}\right. \\
&\qquad \left. + \frac{1}{3}A\overline{C}G + \frac{29}{6}BC\overline{G}\right) \\
&\quad + \mathrm{Re}\left(A\overline{A}B\overline{B} + AB^2\overline{B} - 2B^2\overline{B}^2 - \frac{2}{3}A^2\overline{B}C + 3A\overline{B}^2C - \frac{10}{3}B^3\overline{C}\right. \\
&\qquad \left. + \frac{1}{3}A\overline{A}C\overline{C} - \frac{7}{3}ABC\overline{C} + \frac{7}{4}B\overline{B}C\overline{C} - \frac{4}{9}C^2\overline{C}^2\right)\Bigg).
\end{aligned}
$$

From the proof of our results it will be clear that the proposed method allows to compute (with a reasonable effort) v_3, v_5 and p_2 without any computer algebra system.

3. NOTATION AND PROOF OF THE BASIC RESULTS.

First of all, we need to clarify the main notation as well as the definitions of the constants.

3.a. Definitions.

We start making the change of variables given by $r^2 = z\overline{z}$ and $\theta = \arctan \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$, to transform equation (4) into:

$$\frac{dr}{d\theta} = ir\frac{\dot{z}\overline{z} + z\dot{\overline{z}}}{\dot{z}\overline{z} - z\dot{\overline{z}}} = \frac{\frac{F\overline{z}+\overline{F}z}{2r}}{1 + \frac{F\overline{z}-\overline{F}z}{2ir^2}}.$$

Since $F(z, \overline{z}) = \sum_{k\geq 2} r^k F_k(e^{i\theta}, e^{-i\theta})$, the equation can be written as:

$$\frac{dr}{d\theta} = \frac{\sum_{k\geq 2} r^k \,\mathrm{Re}(S_k(\theta))}{1 + \sum_{k\geq 1} r^k \,\mathrm{Im}(S_{k+1}(\theta))} =: \sum_{k=2}^{+\infty} R_k(\theta) r^k, \tag{6}$$

where $S_k(\theta) = e^{-i\theta} F_k(e^{i\theta}, e^{-i\theta})$. Note that (6) is only defined for r small enough.

Following [2] we denote by $r(\theta, \rho)$ the solution of (6) that takes the value ρ when $\theta = 0$. Then, it can be expanded as:

$$r(\theta, \rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \ldots, \text{ with } u_k(0) = 0 \text{ for } k \geq 2.$$

Let $P(\rho) = r(2\pi, \rho)$ be the return map defined on the $\theta = 0$ axis. The values $u_k(2\pi)$, for $k \geq 2$, control the behaviour of the solutions of (6) near the origin. We will say that the system (6) has a center at the origin if and only if $u_k(2\pi) = 0$, for all $k \geq 2$. On the other hand, it has a focus if there exists a k such that $u_k(2\pi) \neq 0$.

It is well-known that the first non-vanishing $u_k(2\pi)$ has an odd subscript, $k = 2m+1$. We will say that $v_{2m+1} = u_{2m+1}(2\pi)$ is the *m-th Liapunov constant.*

In principle, the purpose is to obtain the functions $u_k(\theta)$ and evaluate them at $\theta = 2\pi$. When substituting the solution $r(\theta, \rho)$ into equation (6), we get some recurrent relations among the $u_k(\theta)$. The effect of this process is displayed in Proposition 3.1.

Assuming that (6) has a center at the origin, the *period function*, $T(\rho)$, at ρ is defined as the time spent by the closed orbit $r(\theta, \rho)$ to turn once around the origin. In polar coordinates, we have that

$$\frac{d\theta}{dt} = 1 + \mathrm{Im}\left(\sum_{k\geq 2} r^{k-1} S_k(\theta)\right) = 1 + \sum_{k\geq 2} r^{k-1}\,\mathrm{Im}(S_k(\theta)).$$

Therefore,

$$T(\rho) = \int_0^{2\pi} \frac{d\theta}{1 + \sum\limits_{k\geq 2} r(\theta,\rho)^{k-1}\,\mathrm{Im}(S_k(\theta))},$$

which can be expanded in series as:

$$\begin{aligned} T(\rho) &= \int_0^{2\pi} 1 + \sum_{k\geq 1} H_k(\theta) r(\theta,\rho)^k \; d\theta = 2\pi + \sum_{k\geq 1} \int_0^{2\pi} H_k(\theta) r(\theta,\rho)^k \; d\theta \\ &:= 2\pi + \sum_{k\geq 1} \int_0^{2\pi} g_k(\theta)\; d\theta \; \rho^k. \end{aligned}$$

If we denote $t_k(\theta) = \int_0^\theta g_k(\varphi)\, d\varphi$, we have that $T(\rho) = 2\pi + \sum\limits_{k\geq 1} t_k(2\pi)\rho^k$.

Then, if $k = 2m$ and $t_j(2\pi) = 0$ for all $j = 1, \ldots, k-1$, we define the *m-th period constant* p_{2m} as $p_{2m} = t_{2m}(2\pi)$. (it is well-known $t_1(2\pi) = \cdots = t_{2j}(2\pi) = 0$ implies that $t_{2j+1}(2\pi) = 0$, for all $j \in \mathbf{N}$, see [4]).

Apart from the above basic definitions, some other notations will be used in the proof of Theorem A:

Given a trigonometric polynomial in the variable θ,

$$p(\theta) = \sum_{k\in K} p_k e^{ik\theta} + p_0,$$

where K is a finite subset of $\mathbf{Z} \setminus \{0\}$, we set:

$$\widetilde{p}(\theta) = \int_0^\theta p(\varphi)\; d\varphi = \sum_{k\in K} \frac{p_k}{ik}(e^{ik\theta} - 1) + p_0\theta, \qquad \textit{and}$$

$\widehat{p}(\theta)$ is the primitive of $p(\theta)$ without independent term on θ, that is to say,

$$\widehat{p}(\theta) = \sum_{k\in K} \frac{p_k}{ik} e^{ik\theta} + p_0\theta \tag{7}$$

Let us now enter into the proof (we only prove Theorem A since Theorem B can be proved in a similar way).

3.b. Proof of Theorem A.

The first step consists of substituting the expression of $r(\theta,\rho)$ into the differential equation (6), obtaining a general relation among the derivatives $u_k'(\theta)$ and the previous $u_j(\theta)$:

$$\sum_{k\geq 2} u_k'(\theta)\rho^k = \sum_{n\geq 2} R_n(\theta)\left(r(\theta,\rho)\right)^n.$$

It is not difficult to prove that

$$(r(\theta,\rho))^n = \left(\sum_{i\geq 1} u_i(\theta)\rho^i\right)^n$$

$$= \sum_{k\geq n}\left(\sum_{\substack{a_1+a_2+\cdots+a_{k-1}=n\\ a_1+2a_2+\cdots+(k-1)a_{k-1}=k}} \frac{n!}{a_1!a_2!\cdots a_{k-1}!} u_2^{a_2}(\theta)u_3^{a_3}(\theta)\cdots u_{k-1}^{a_{k-1}}(\theta)\right)\rho^k.$$

Hence, by comparison of the coefficients of ρ^k, one gets

$$u_k'(\theta) = \sum_{n=2}^{k} R_n(\theta) \sum_{\substack{a_1+a_2+\cdots+a_{k-1}=n\\ a_1+2a_2+\cdots+(k-1)a_{k-1}=k}} \frac{n!}{a_1!a_2!\cdots a_{k-1}!} u_2^{a_2}(\theta)u_3^{a_3}(\theta)\cdots u_{k-1}^{a_{k-1}}(\theta).$$

This formula gives a recurrent way to compute the $u_k(\theta)$ in terms of the $R_j(\theta)$, with $j \leq k$. If we carry out the successive integrations we reach the following proposition, which generalizes the results of [1] and [10].

PROPOSITION 3.1. *The functions $u_i(\theta)$, which define the solution $r(\theta,\rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \ldots$ of (6), can be expressed for $i = 2,3,4,5,6,7$, in terms of the* *functions $R_k(\theta)$ as:*

$$
\begin{aligned}
u_2(\theta) =& \tilde{R}_2, \\
u_3(\theta) =& \tilde{R}_3 + (\tilde{R}_2)^2, \\
u_4(\theta) =& \tilde{R}_4 + \widetilde{\tilde{R}_2 R_3} + 2\tilde{R}_2\tilde{R}_3 + (\tilde{R}_2)^3, \\
u_5(\theta) =& \tilde{R}_5 + 2\widetilde{\tilde{R}_2 R_4} + 2\tilde{R}_2\tilde{R}_4 + \frac{3}{2}(\tilde{R}_3)^2 + 2\tilde{R}_2\widetilde{\tilde{R}_2 R_3} + \\
& \widetilde{(\tilde{R}_2)^2 R_3} + 3(\tilde{R}_2)^2\tilde{R}_3 + (\tilde{R}_2)^4, \\
u_6(\theta) =& \tilde{R}_6 + 3\widetilde{R_5\tilde{R}_2} + 2\tilde{R}_2\tilde{R}_5 + \widetilde{R_4\tilde{R}_3} + \\
& 3\tilde{R}_3\tilde{R}_4 + 4\tilde{R}_2\widetilde{R_4\tilde{R}_2} + 3\tilde{R}_4(\tilde{R}_2)^2 + 3\widetilde{R_4(\tilde{R}_2)^2} + \\
& \widetilde{R_3\tilde{R}_2}\tilde{R}_3 + 3\tilde{R}_3\widetilde{R_3\tilde{R}_2} + 4\tilde{R}_2(\tilde{R}_3)^2 + 2\tilde{R}_2\widetilde{R_3(\tilde{R}_2)^2} + \\
& 4\tilde{R}_3(\tilde{R}_2)^3 + 3(\tilde{R}_2)^2\widetilde{R_3\tilde{R}_2} + \widetilde{R_3(\tilde{R}_2)^3} + (\tilde{R}_2)^5, \\
u_7(\theta) =& \tilde{R}_7 + 2\tilde{R}_6\tilde{R}_2 + 4\widetilde{R_6\tilde{R}_2} + 3\tilde{R}_5\tilde{R}_3 + \\
& 2\widetilde{R_5\tilde{R}_3} + 3\tilde{R}_5(\tilde{R}_2)^2 + 6\tilde{R}_2\widetilde{R_5\tilde{R}_2} + 6\widetilde{R_5(\tilde{R}_2)^2} + \\
& 4\tilde{R}_4\widetilde{R_3\tilde{R}_2} + 2(\tilde{R}_4)^2 + 2\tilde{R}_2\widetilde{R_4\tilde{R}_3} + 6\tilde{R}_3\widetilde{R_4\tilde{R}_2} + \\
& 8\tilde{R}_2\tilde{R}_3\tilde{R}_4 + 4\widetilde{R_4\tilde{R}_2}\tilde{R}_3 + 4\tilde{R}_4(\tilde{R}_2)^3 + 6\tilde{R}_2\widetilde{R_4(\tilde{R}_2)^2} + \\
& 6(\tilde{R}_2)^2\widetilde{R_4\tilde{R}_2} + 4\widetilde{R_4(\tilde{R}_2)^3} + \frac{5}{2}(\tilde{R}_3)^3 + 3\tilde{R}_3\widetilde{R_3(\tilde{R}_2)^2} + \\
& \frac{15}{2}(\tilde{R}_2)^2(\tilde{R}_3)^2 + 2\left(\widetilde{R_3\tilde{R}_2}\right)^2 + 2\tilde{R}_2\widetilde{R_3\tilde{R}_2}\tilde{R}_3 + 8\tilde{R}_2\tilde{R}_3\widetilde{R_3\tilde{R}_2} + \\
& 2R_3\widetilde{\tilde{R}_3(\tilde{R}_2)^2} + 5(\tilde{R}_2)^4\tilde{R}_3 + 3(\tilde{R}_2)^2\widetilde{R_3(\tilde{R}_2)^2} + 4(\tilde{R}_2)^3\widetilde{R_3\tilde{R}_2} + \\
& 2\tilde{R}_2\widetilde{R_3(\tilde{R}_2)^3} + \widetilde{R_3(\tilde{R}_2)^4} + (\tilde{R}_2)^6.
\end{aligned}
$$

Remark 3.2. *In fact, the computation of the Liapunov constants consists of evaluating the latest expressions at $\theta = 2\pi$, supposing that the previous constants vanish. Since $\tilde{R}_i(\theta) = \int_0^\theta R_i(\varphi)\, d\varphi$ has a term independent of θ, that arises from the evaluation of a primitive of $R_i(\theta)$ at $\theta = 0$, the successive integration steps that appear in the statement of Proposition 3.1 could be shortened if this independent term vanished. In the following proposition we prove that the $\tilde{R}_i(\theta)$ can be changed by the $\hat{R}_i(\theta)$, which have already been defined as primitives of $R_i(\theta)$ without independent term, see (7). This fact makes easier the effective computation of the constants, and settles one of the differences between the technique that we use and the usual one.*

PROPOSITION 3.3. *The first three Liapunov constants for system (4) are given by:*

$$
\begin{aligned}
v_3 &= \int_0^{2\pi} R_3(\theta)\ d\theta, \\
v_5 &= \int_0^{2\pi} \left(R_5 + 2R_4\hat{R}_2 + R_3(\hat{R}_2)^2\right)(\theta)\ d\theta, \\
v_7 &= \int_0^{2\pi} \left(R_7 + 4R_6\hat{R}_2 + 2R_5\hat{R}_3 + 6R_5(\hat{R}_2)^2 + 4R_4(\hat{R}_2)^3\right. \\
&\qquad \left. + 4R_4\hat{R}_3\hat{R}_2 + 2R_3\hat{R}_3(\hat{R}_2)^2 + R_3(\hat{R}_2)^4\right)(\theta)\ d\theta.
\end{aligned}
$$

Proof. We will show only the proof of v_7. Of course, the other cases are easier. By (7), we have that for $i \geq 2$, $\tilde{R}_i(\theta) = \hat{R}_i(\theta) + c_i$, with $c_i \in \mathbf{R}$.

Let us see what would it happen if in the computation of $u_7(2\pi)$ we changed the $\tilde{R}_i$ by $\hat{R}_i + c_i$. By Proposition 3.1 and the fact that $u_2(2\pi) = u_3(2\pi) = u_4(2\pi) = 0$ imply that $\tilde{R}_2(2\pi) = \tilde{R}_3(2\pi) = \left(\tilde{R}_4 + \widetilde{R_3\tilde{R}_2}\right)(2\pi) = 0$, we can compute $u_7(2\pi)$ as:

$$
\begin{aligned}
u_7(2\pi) &= \int_0^{2\pi} \left(R_7 + 4R_6\tilde{R}_2 + 2R_5\tilde{R}_3 + 6R_5(\tilde{R}_2)^2 + 4R_4\tilde{R}_2\tilde{R}_3 + 4R_4(\tilde{R}_2)^3\right. \\
&\quad \left. + 2R_3(\tilde{R}_2)^2\tilde{R}_3 + R_3(\tilde{R}_2)^4\right)(\theta)\ d\theta \\
&= \int_0^{2\pi} \left(R_7 + 4R_6(\hat{R}_2 + c_2) + 2R_5(\hat{R}_3 + c_3) + 6R_5(\hat{R}_2 + c_2)^2 + 4R_4(\hat{R}_2 + c_2)^3\right. \\
&\quad \left. + 4R_4(\hat{R}_2 + c_2)(\hat{R}_3 + c_3) + 2R_3(\hat{R}_2 + c_2)^2(\hat{R}_3 + c_3) + R_3(\hat{R}_2 + c_2)^4\right)(\theta)\ d\theta \\
&= \int_0^{2\pi} \left(R_7 + 4R_6\hat{R}_2 + 2R_5\hat{R}_3 + 6R_5(\hat{R}_2)^2 + 4R_4(\hat{R}_2)^3 + 4R_4\hat{R}_2\hat{R}_3\right. \\
&\quad \left. + 2R_3(\hat{R}_2)^2\hat{R}_3 + R_3(\hat{R}_2)^4\right)(\theta)\ d\theta \\
&\quad + 4c_2 \int_0^{2\pi} \left(R_6 + 3R_5\hat{R}_2 + R_4\hat{R}_3 + 3R_4(\hat{R}_2)^2 + R_3\hat{R}_2\hat{R}_3 + R_3(\hat{R}_2)^3\right)(\theta)\ d\theta \\
&\quad + 2(c_3 + 3c_2^2) \int_0^{2\pi} \left(R_5 + 2R_4\hat{R}_2 + R_3(\hat{R}_2)^2\right)(\theta)\ d\theta + 2c_2^2 \int_0^{2\pi} (R_3\hat{R}_3)(\theta)\ d\theta \\
&\quad + 4(c_2c_3 + c_2^3) \int_0^{2\pi} \left(R_4 + R_3\hat{R}_2\right)(\theta)\ d\theta + (c_2^4 + 2c_2^2c_3) \int_0^{2\pi} R_3(\theta)\ d\theta
\end{aligned}
$$

Among the above last six integrals, the first one is the v_7 of the statement, the second, third, fifth and sixth ones vanish under the hypothesis $v_5 = v_3 = 0$, while the fourth one also vanishes since $v_3 = 0$ implies that $\hat{R}_3(2\pi) = \hat{R}_3(0)$. $\square$

At this point, the problem of the computation of the functions $u_j(\theta)$ has been reduced to the computation of the functions $R_j(\theta)$, which are easier to be expressed in terms of the coefficients, but not easy enough.

We define the following polynomials in $e^{i\theta}$, which will act as the link between the $R_j(\theta)$ and the coefficients of the system:

$$
\begin{aligned}
&S_k(\theta) = e^{-i\theta} F_k(e^{i\theta}, e^{-i\theta});\\
&T_k(\theta) = -i\hat{S}_k(\theta);\\
&W_k(\theta) = -i\widehat{S_k^2}(\theta);\\
&s_k(\theta) = S_k(\theta) - \overline{S_k(\theta)};\\
&t_k(\theta) = T_k(\theta) - \overline{T_k(\theta)};\\
&w_k(\theta) = W_k(\theta) + \overline{W_k}(\theta), \qquad \text{for } k \geq 2.
\end{aligned} \tag{8}
$$

Their relation with $R_j(\theta)$ is displayed in the next result:

LEMMA 3.4. *The following assertions hold:*

(a)

$$R_n = \mathrm{Re}(S_n) - \sum_{k=1}^{n-2} \mathrm{Im}(S_{k+1}) R_{n-k}.$$

(b) *Consider the trigonometric polynomials $U_n(\theta)$, for $n \geq 1$, such that $R_n(\theta) = \mathrm{Re}(U_n(\theta))$. Then,*

$$U_n = S_n + \frac{i}{2} \sum_{k=1}^{n-2} U_{n-k}(S_{k+1} - \overline{S_{k+1}}). \tag{9}$$

Moreover, the polynomials $U_n(\theta)$ can be expressed as $U_n(\theta) = X_n(\theta) + Y_n(\theta)$, *where* $\mathrm{Re}(Y_n(\theta)) = 0$ *and*

$$X_2 = S_2,$$

$$X_3 = S_3 + \frac{i}{2}(S_2^2),$$

$$X_4 = S_4 + iS_2S_3 + \frac{1}{4}S_2(\overline{S_2}^2 - S_2^2),$$

$$X_5 = S_5 + i\left(S_2S_4 + \frac{1}{2}S_3^2\right) + \left(\frac{1}{2}S_2\overline{S_2}S_3 - \frac{3}{4}S_2^2S_3 + \frac{1}{4}S_2^2\overline{S_3}\right) + \frac{i}{8}S_2^3(2\overline{S_2} - S_2),$$

$$X_6 = S_6 + i(S_2S_5 + S_3S_4) - \frac{1}{2}\left((S_2S_4 + \frac{1}{2}S_3^2)s_2 + S_2S_3s_3 + \frac{1}{2}S_2^2s_4\right) + \frac{i}{2}S_2^2(s_2S_3 + \frac{1}{2}(S_2\overline{S_3} + \overline{S_2}S_3)) - \frac{1}{8}s_2S_2^3(\overline{S_2} - \frac{1}{2}S_2),$$

$$\begin{aligned} X_7 = {} & S_7 + i\left(S_2S_6 + S_3S_5 + \frac{1}{2}S_4^2\right) \\ & - \frac{1}{2}\left((S_2S_5 + S_3S_4)s_2 + (S_2S_4 + \frac{1}{2}S_3^2)s_3 + S_2S_3s_4 + \frac{1}{2}S_2^2s_5\right) \\ & + \frac{i}{2}\left(-\frac{1}{2}(S_2S_4 + \frac{1}{2}S_3^2)s_2^2 - \frac{1}{2}S_2S_3s_2s_3 - \frac{1}{4}S_2^2s_4s_2\right. \\ & + \left.\left(\frac{1}{2}S_2\overline{S_2}S_3 - \frac{3}{4}S_2^2S_3 + \frac{1}{4}S_2^2\overline{S_3}\right)s_3 + \frac{1}{4}S_2(\overline{S_2}^2 - S_2^2)s_4\right) \\ & - \frac{1}{4}\left(S_2^2s_2(s_2S_3 + \frac{1}{2}(S_2\overline{S_3} + \overline{S_2}S_3)) + \frac{1}{2}S_2^3s_3(\overline{S_2} - \frac{1}{2}S_2)\right) \\ & - \frac{i}{16}(S_2^3s_2^2(\overline{S_2} - \frac{1}{2}S_2)). \end{aligned}$$

Proof of Lemma 3.4. (a) It can be easily obtained applying to (6) known results about the computation of the coefficients of a series expansion of a quotient of series, see for instance [11, p.14].

(b) Applying basic rules of calculus with complex numbers to (a), we have

$$\begin{aligned} R_n = \mathrm{Re}(U_n) &= \mathrm{Re}(S_n) + \sum_{k=1}^{n-2} \mathrm{Re}(iS_{k+1})\,\mathrm{Re}(U_{n-k}) \\ &= \mathrm{Re}\left(S_n + \frac{1}{2}\sum_{k=1}^{n-2} U_{n-k}(iS_{k+1} + (-i)\overline{S_{k+1}})\right), \end{aligned}$$

and so the equality given in the statement.

To get the expanded expressions of $X_k(\theta)$, we use the recurrence (9) to obtain first the $U_k(\theta)$ (recall that $S_0 \equiv S_1 \equiv 0$). For instance, X_3 is computed from:

$$U_3 = S_3 + \frac{i}{2}U_2(S_2 - \overline{S_2}) = S_3 + \frac{i}{2}S_2^2 - \frac{i}{2}S_2\overline{S_2}.$$

It is clear that $\mathrm{Re}(\frac{i}{2}S_2\overline{S_2}) = 0$, and so we define $X_3 = S_3 + \frac{i}{2}S_2^2$ and $Y_3 = -\frac{i}{2}S_2\overline{S_2}$. □

Then, using this lemma together with Proposition 3.3 we will be able to reach the last result, from which Theorem A follows. This result describes the Liapunov constants in terms of the polynomials given in (8).

THEOREM 3.5. *Consider the differential equation (4), with the definitions given in (8). Then, the first three Liapunov constants v_3, v_5 and v_7 can be written as:*

$$v_3 = \mathrm{Re}\int_0^{2\pi} S_3(\theta)\ d\theta - \mathrm{Im}\int_0^{2\pi} \frac{1}{2}S_2^2(\theta)\ d\theta,$$

$$\begin{aligned} v_5 = {} & \mathrm{Re}\int_0^{2\pi} S_5(\theta)\ d\theta - \mathrm{Im}\int_0^{2\pi} (S_2S_4 + \frac{1}{2}S_3^2 + S_4t_2)(\theta)\ d\theta \\ & + \mathrm{Re}\int_0^{2\pi} (\frac{1}{2}S_2\overline{S_2}S_3 - \frac{3}{4}S_2^2S_3 + \frac{1}{4}S_2^2\overline{S_3} - S_2S_3t_2 - \frac{1}{4}S_3t_2^2)(\theta)\ d\theta \\ & - \mathrm{Im}\int_0^{2\pi} (\frac{1}{8}S_2^3(2\overline{S_2} - S_2) + \frac{1}{4}S_2t_2(\overline{S_2}^2 - S_2^2) - \frac{1}{8}S_2^2t_2^2)(\theta)\ d\theta, \end{aligned}$$

$$v_7 = \mathrm{Re}\left(\sum_{j=1}^{3}\int_0^{2\pi} v_7(2j-1)(\theta)\ d\theta\right) - \mathrm{Im}\left(\sum_{j=1}^{3}\int_0^{2\pi} v_7(2j)(\theta)\ d\theta\right),$$

where:

$$\begin{aligned} v_7(1) = {} & S_7, \\ v_7(2) = {} & S_6S_2 + S_3S_5 + \frac{1}{2}S_4^2 + 2S_6t_2 + S_5t_3, \\ v_7(3) = {} & -(S_2S_5 + S_3S_4)(\frac{s_2}{2} + 2t_2) - (S_2S_4 + \frac{1}{2}S_3^2)(\frac{s_3}{2} + t_3) \\ & - S_4(\frac{S_2S_3 - \overline{S_2S_3}}{2} + t_3t_2) - S_5(\frac{S_2^2 - \overline{S_2}^2}{4} + \frac{1}{2}w_2 + \frac{3}{2}t_2^2), \\ v_7(4) = {} & -(S_2S_4 + \frac{1}{2}S_3^2)(\frac{1}{4}s_2^2 + s_2t_2 + \frac{3}{2}t_2^2 + \frac{1}{2}w_2) \\ & + S_2(\frac{1}{2}\overline{S_2}S_3 - \frac{3}{4}S_2S_3 + \frac{1}{4}S_2\overline{S_3})(\frac{1}{2}s_3 + t_3) \\ & - S_3(\frac{1}{4}S_2s_2s_3 + S_2s_3t_2 + S_2t_2t_3 + \frac{1}{4}t_2^2t_3) \\ & - S_4(\frac{1}{2}s_2t_2(S_2 + \overline{S_2}) + \frac{1}{2}w_2t_2 + \frac{1}{2}t_2^3) - \frac{1}{4}S_2^2s_2s_4, \\ v_7(5) = {} & -S_2^2(-s_2S_3 + \frac{1}{2}(S_2\overline{S_3} + \overline{S_2}S_3))(\frac{1}{4}s_2 + t_2) - \\ & \frac{1}{4}S_2^3(\overline{S_2} - \frac{1}{2}S_2)(\frac{1}{2}s_3 + t_3) - S_2(\frac{1}{2}\overline{S_2}S_3 - \frac{3}{4}S_2S_3 + \frac{1}{4}S_2\overline{S_3})(\frac{1}{2}w_2 + \frac{3}{2}t_2^2) + \\ & \frac{1}{2}S_2(\frac{1}{2}S_2s_2t_2t_3 + S_3w_2t_2 + S_3t_2^3 + \frac{1}{4}S_2t_2^2t_3) + \frac{1}{8}S_3t_2^2(w_2 + \frac{1}{2}t_2^2), \\ v_7(6) = {} & -\frac{1}{4}S_2^3(\overline{S_2} - \frac{1}{2}S_2)(\frac{1}{4}s_2^2 + s_2t_2 + \frac{1}{2}w_2 + \frac{3}{2}t_2^2) \\ & + \frac{1}{8}S_2^2(s_2w_2t_2 + s_2t_2^3 + \frac{1}{2}w_2t_2^2 + \frac{1}{4}t_2^4) \end{aligned}$$

Proof. Recall that $R_k = \mathrm{Re}(X_k)$. Here, we wish to write also $\hat{R}_2$ and $\hat{R}_3$ as the real part of some polynomial. We must resort, then, to the definitions of T_k and W_k given in (8). Hence,

$$\hat{R}_2 = \widehat{\mathrm{Re}(X_2)} = \widehat{\mathrm{Re}(S_2)} = \mathrm{Re}\,\hat{S}_2 = \mathrm{Re}(iT_2),$$
$$\hat{R}_3 = \widehat{\mathrm{Re}(X_3)} = \widehat{\mathrm{Re}(S_3 + \frac{i}{2}S_2^2)} = \mathrm{Re}\,\hat{S}_3 + \frac{1}{2}\,\mathrm{Re}(i\widehat{S_2^2}) = \mathrm{Re}(iT_3 - \frac{1}{2}W_2).$$

From Proposition 3.3, if we want to compute v_5, we must do the following expansions:

$$\begin{aligned} R_5 &= \mathrm{Re}\,U_5 = \mathrm{Re}\,X_5, \\ 2R_4\hat{R}_2 &= 2\,\mathrm{Re}\,U_4\,\mathrm{Re}(iT_2) = \mathrm{Re}(X_4(iT_2 - i\overline{T_2})) = \mathrm{Re}(iX_4t_2), \\ R_3(\hat{R}_2)^2 &= -\frac{1}{4}\,\mathrm{Re}(X_3t_2^2). \end{aligned}$$

Analogously, for v_7, we expand:

$$\begin{aligned} R_7 &= \mathrm{Re}\,U_7 = \mathrm{Re}\,X_7, \\ 4R_6\hat{R}_2 &= \mathrm{Re}(U_6)\,\mathrm{Re}(\hat{S}_2) = 4\,\mathrm{Re}(X_6)\,\mathrm{Re}(iT_2) \\ &= 2\,\mathrm{Re}(iX_6(T_2 - \overline{T_2})) = 2\,\mathrm{Re}(iX_6t_2), \\ 2R_5\hat{R}_3 &= 2\,\mathrm{Re}(U_5)\,\mathrm{Re}(iT_3 - \frac{1}{2}W_2) = \mathrm{Re}(iX_5t_3 - \frac{1}{2}X_5w_2), \\ 6R_5(\hat{R}_2)^2 &= 6\,\mathrm{Re}(U_5)(\mathrm{Re}(iT_2))^2 = -\frac{3}{2}\,\mathrm{Re}(X_5t_2^2), \\ 4R_4\hat{R}_3\hat{R}_2 &= 4\,\mathrm{Re}(U_4)\,\mathrm{Re}(iT_3 - \frac{1}{2}W_2)\,\mathrm{Re}(iT_2) = \mathrm{Re}(-X_4t_3t_2 - \frac{i}{2}X_4w_2t_2), \\ 4R_4(\hat{R}_2)^3 &= 4\,\mathrm{Re}(U_4)(\mathrm{Re}(iT_2))^3 = -\frac{1}{2}\,\mathrm{Re}(X_4it_2^3), \\ 2R_3\hat{R}_3(\hat{R}_2)^2 &= 2\,\mathrm{Re}(U_3)\,\mathrm{Re}(iT_3 - \frac{1}{2}W_2)(\mathrm{Re}(iT_2))^2 = -\frac{1}{4}\,\mathrm{Re}(iX_3t_3t_2^2 - \frac{1}{2}X_3w_2t_2^2), \\ R_3(\hat{R}_2)^4 &= \mathrm{Re}(U_3)\,\mathrm{Re}(iT_2)^4 = \frac{1}{16}\,\mathrm{Re}(X_3t_2^4). \end{aligned}$$

We substitute the X_k in the above expressions according to the equalities provided in Lemma 3.4(b) and we distribute the result by degrees. Finally, applying Proposition 3.3 the theorem is obtained. □

From the expressions of Theorem 3.5, the computation of the constants in terms of the coefficients is reduced to substitute the expressions given in (8). It is very important that the computation of the integrals that appear in the statements can be done in a very easy way. Observe that all their integrands are functions of the form $C + f(e^{i\theta}, e^{-i\theta})$, where $f(e^{i\theta}, e^{-i\theta})$ is a sum of non-constant trigonometric polynomials. So, when integrating from 0 to 2π, we obtain $2\pi C$. Then, we can take advantage of the option of some computer algebra systems that allows to isolate the coefficients of some especific degree to avoid the step of integration. For instance, the command *Coefficient List* of *Mathematica* has been useful in our work. The computations of v_3, v_5 and p_2 can be easily done (without computer algebra systems).

Proof of Theorem A. The formulas of v_3 and v_7 arise directly from the operations indicated in Theorem 3.5, without any substitution (in v_3 because it is not necessary, and in v_7 by difficulties beyond our goals). On the contrary, v_5 has been computed and afterwards we have used more explicitly the fact that $v_3 = 0$. Then, of course, the formula we provide for v_5 is a little bit shorter than the one that could be obtained directly from Theorem 3.5. □

4. PROOFS OF THE APPLICATION RESULTS.

Proof of Theorem 1. (a) We consider equation (1) in complex notation. Applying Theorem A we obtain the first three focal values which give the center conditions for (1).

The first focal value is $v_3 = 2\pi\left(\frac{3}{8}b - \frac{1}{4}ad\right)$. When $b = \frac{2}{3}ad$ we obtain:

$$
\begin{aligned}
v_5 &= \frac{\pi a}{12}\left(5dg - 3h\right), \\
v_7 &= \frac{\pi a}{384}\left(-285dg^2 + 210a^2dg + 240d^3g - 200d^2h + 171gh - 126a^2h\right).
\end{aligned}
$$

The constraints $v_3 = v_5 = v_7 = 0$ give the three conditions (a.i), with $i = 1, 2, 3$, stated in Theorem 1. Case (a.1) corresponds to a Hamiltonian system. Following Poincaré's Criterion, in case (a.2), equation (1) has a center because it is invariant by the change of variables $(x, y, t) \longrightarrow (-x, y, -t)$. The last case admits a differentiable integrating factor depending on y and therefore, it has a center at the origin. Then, (a) is proved.

(b) In case (a.1) the system is gradient. From [4] we know that these type of systems cannot exhibit isochronous centers unless they are linear.

By Theorem B, the first period constant for system (1) is

$$p_2 = \frac{a^2}{6} + \frac{5}{12}d^2 - \frac{3}{8}g.$$

In case (a.2), $p_2 = 0$ implies $g = \frac{4}{9}a^2$. Furthermore, in this case $p_4 = 0$. This restriction leads to the system

$$\left\{\begin{array}{ll} \dot{x} & = -y\, +\, ax^2, \\ \dot{y} & = x\, +\, \frac{4}{9}a^2x^3, \end{array}\right. \tag{10}$$

$\dot{\mathbf{x}} = X(\mathbf{x})$, with $\mathbf{x} = (x, y)$, for short. We have proved that taking $\dot{\mathbf{x}} = Y(\mathbf{x})$ as

$$\begin{cases} \dot{x} &= x + \frac{2}{3}axy - \frac{2}{9}a^2x^3 \\ \dot{y} &= y + \frac{a}{3}x^2 + \frac{2}{3}ay^2 - \frac{2}{27}a^3x^4, \end{cases}$$

then $[X(\mathbf{x}), Y(\mathbf{x})] = 0$, where $[\cdot,\cdot]$ denotes the Lie bracket. Then, using the result of [18] we obtain that the origin of (10) is an isochronous center.

For the case (a.3), if we impose $p_2 = 0$, Theorem B gives

$$p_4 = \frac{85}{864}a^2d^2 + \frac{1505}{1728}d^4 \neq 0,$$

and therefore no isochronous centers can appear. $\square$

Proof of Proposition 2. The Fitzhugh model is given by

$$\begin{cases} \dot{x} &= \lambda + y + x - \frac{1}{3}x^3, \\ \dot{y} &= -\rho x - \rho\beta y \end{cases}$$

where $\beta, \rho \in (0,1)$, and $\lambda \geq 0$.

When $\lambda = \sqrt{1-\rho\beta}\,\frac{3-2\beta-\rho\beta^2}{3\beta}$, the system has a unique critical point (x_λ, y_λ) at which the divergence vanishes. We translate this critical point to the origin, and consider the change given by $(X, Y) = (x, -\rho\beta\, x - \omega_0\, y)$, where $\omega_0 = \sqrt{\rho(1-\rho\beta^2)}$. Consider also the rescaling of time given by $t = \tau/\omega_0$ and denote again by a dot the derivative with respect to τ. Then, we obtain

$$\begin{cases} \dot{X} = -Y - \frac{X_0}{\omega_0}X^2 - \frac{1}{3\omega_0}X^3, \\ \dot{Y} = X + \frac{\rho\beta}{\omega_0{}^2}\left(X_0X^2 + \frac{1}{3}X^3\right), \end{cases}$$

where $X_0 = \sqrt{1-\rho\beta}$. This system corresponds to a system of type (1) with $a = -\frac{X_0}{\omega_0}$, $b = -\frac{1}{3\omega_0}$, $d = \frac{\rho\beta X_0}{\omega_0^2}$, $g = \frac{\rho\beta}{3\omega_0^2}$ and $h = 0$.

From Theorem A we get

$$v_3 = -\frac{\pi\rho}{4\omega_0^3}\left(\beta^2\rho - 2\beta + 1\right).$$

Hence, in our range of parameters v_3 only vanishes when $\beta \in (\frac{1}{2}, 1)$ and $\rho = \frac{2\beta-1}{\beta^2}$. For these values $v_5 = \frac{5\pi\,\beta}{48\omega_0\,(\beta-1)} \neq 0$, as we wanted to prove. $\square$

Proof of Theorem 3. System (2) writes in complex coordinates as:

$$\dot{z} = iz + Az^2 + 2\overline{A}z\overline{z} + A\overline{z}^2 + dz^3 + 3ez^2\overline{z} + 3dz\overline{z}^2 + e\overline{z}^3,$$

where $A = \frac{1}{4}(a - mi)$, $d = \frac{1}{8}(b - n) \in \mathbf{R}$ and $e = \frac{1}{8}(b + n) \in \mathbf{R}$.

From Theorem A, we know that $v_3 = 2\pi e = \frac{\pi}{4}(b + n)$. Imposing that $b + n = 0$, we have that $v_5 = \frac{\pi}{24}(a^2 - m^2)(5b - 6am)$ and $v_7 = \frac{\pi}{128}(a^4 - m^4)(55b - 42am)$.

To end the proof of (a) it suffices to observe that the conditions $v_3 = v_5 = v_7 = 0$ lead to systems with centers, by using Poincaré's Criterion, as in the proof of Theorem 1.

To prove (b), it suffices to compute the period constants using Theorem B. Assuming that $e = b + n = 0$, we have that $p_2 = \frac{\pi}{3}(a^2 + m^2)$, and also imposing that $p_2 = 0$, $p_4 = \frac{3\pi}{4}b^2$. With these data, the only possibility of having an isochronous center becomes when $a = b = m = n = 0$. $\square$

The above result has also been proved in [3], but because of the lack of formulas for v_7 and p_4, the way of finding the constants was more industrious.

Proof of Theorem 4. Using Theorem A, we compute first

$$v_3 = c\frac{2 + 2b + c + 4bc + 4b^2 c}{16d^3},$$

and so from $v_3 = 0$ we get the relations $b_{\pm} = \frac{-2c-1\pm\sqrt{1-4c}}{4c}$.

With these values, we get the respective v_5,

$$v_{5,\pm} = 15 - 21c - 20c^2 \mp \sqrt{1-4c}(15 - 31c).$$

While $v_{5,+}$ never vanish, it can be seen that $v_{5,-} = 0$ if and only if c is equal to the only real solution of the equation $300 - 1210c + 1171c^2 + 100c^3 = 0$:

$$c^\star = -\frac{1171}{300} - \frac{1734241}{90000\left(\frac{2283832711}{27000000} - \frac{\sqrt{409051}}{6000\sqrt{3}}\right)^{(1/3)}} - \left(\frac{2283832711}{27000000} - \frac{\sqrt{409051}}{6000\sqrt{3}}\right)^{(1/3)}.$$

Finally, we compute v_7 by substituting the values $b = b_-(c^\star)$ and $c = c^\star$. By reasons of space, we do not reproduce v_7 on these pages. Its numerical approximation (of the final result only) with twenty exact digits, is:

$$v_7 \approx -1.0259286464934122\,\pi/d^7.$$

Then, v_7 does not vanish and we get the conclusion of the theorem. $\square$

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