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ADDITIONAL CONSTRAINTS IN VARIATIONAL PROCEDURES FOR ARMA SPECTRAL ESTIMATION

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ABSTRACT

A new variational ARMA power spectral estimation procedure is proposed. The resulting is a generalization of the ARMA estimation method proposed by Lagunas et al. |1| which used correlation and cepstral constraints.

The non linear problem of solving the Lagrange multipliers that appear in any variational procedure is translated into an eigenanalysis one since the autocorrelation "causal image" must be a minimum phase sequence in a general case.

The method can be considered a generalization of the up till now known because makes use of second and fourth order statistics in front of the first and second order ones.

1. INTRODUCTION

The variational procedure for spectral analysis consists of minimizing (or maximizing) an objective function which depends on the estimator S(w) and that usually has the form $\int F[S(w)]dw$. This optimization is carried out under a set of constraints which are frequently obtained from the periodogram $P(w) = |X(w)|^2/N$ of the recorded data (where X(w) is the Fourier Transform of the data and N the number of available data samples) and have the form (1).

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(S(w))\exp(jmw)dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(P(w))\exp(jmw)dw \qquad |m|<Q \qquad (1)$$

It must be pointed out that modifications of the periodogram can be used here. The constraints usually used are those coming from considering G(.)=.; i.e. the correlations coming from the biased estimator.

It is important to note that the constraints and their number serve to incorporate the specific knowledge we have about the signal under analysis to the final estimator.
However, the final estimator will depend not only on the constraints and their number but also on the objective function. This means that the election constraints-objective function can't be arbitrary for a given spectral model.

As an example, let's see what results from considering the entropy $\int \log S(w)dw$ as objective function and the constraints (2) as proposed primarily by Lagunas et al. |1|.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(S(w))e^{jnw}dw=c(n); \quad |n|<P,\ n\neq 0 \qquad (2\text{-a})$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(w)e^{jmw}dw=r(m); \qquad |m|<Q \qquad (2\text{-b})$$

The basic method consists of forming the Lagrangian and setting it's derivative to zero. The above leads to an ARMA model (3) for S(w)

$$S(w) = \frac{1+\sum_{\substack{n=-P\\ n\neq 0}}^{P} \lambda_n \exp(-jnw)}{\sum_{n=-Q}^{Q} \beta_n \exp(-jnw)} \qquad (3)$$

In the same way, it is easily shown that if the entropy is the objective, a MA model for S(w) evolves when considering the cepstral (2-a) constraints and an AR model when considering the autocorrelation (2-b) ones.

So, varying objective function and constraints, different spectral models can be obtained. An example of which is the following.

2. PROPOSED METHOD

First, let us define a function $\emptyset(S(w))$ accordingly to (4).

$$\emptyset(S(w)) = S(w)+jH\{S(w)\} \tag{4}$$

where $H\{.\}$ denotes Hilbert transform and j is the imaginary unit.

It must be noted that $|\emptyset(S(w))|$ is the spectral envelope and the inverse Fourier transform of $\emptyset(S(w))$ is two times the autocorrelation "causal image" as defined by Cadzow |2|.

Then, with $\emptyset(.)$ so defined, a variational problem can be formulated dealing with the maximization of (5)

$$\int_{-\pi}^{\pi} \log |\emptyset(w)|^2 dw \tag{5}$$

under the set of constraints (6).

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\emptyset(w)|^2 e^{jnw} dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(P^2(w)+H^2\{P(w)\})e^{jnw}dw=\zeta(n) \quad |n|<P,\ n\neq 0 \tag{6-a}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\emptyset(w)|^2 e^{jmw} dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} (P^2(w)+H^2\{P(w)\})e^{jmw}dw=\emptyset(m) \quad |m|<Q \tag{6-b}$$

Where $P(w)$ is the periodogram and $\zeta(0)$ is not included in the constraints because of being the objective itself.

The above gives rise to (7), i.e. a rational model for $|\emptyset(z)|^2$.

$$|\emptyset(z)|^2 = \frac{1+\sum_{n=-P,\ n\neq 0}^{P} \lambda_n z^{-n}}{\sum_{n=-Q}^{Q} \beta_n z^{-n}} \tag{7}$$

The problem is then, finding the Lagrange multipliers λ_n and β_n such that the constraints are fulfilled.

3. REGULAR PROCESSES

It can be shown that if $\int_{-\pi}^{\pi} |\log S(w)| dw < \infty$ (Paley-Wiener condition for discrete signals) and $S(w)$ is bounded, then $\int_{-\pi}^{\pi} |\log |\emptyset(z)|^2| dw < \infty$, and then $|\emptyset(z)|^2$ can be factored into (8).

$$|\emptyset(z)|^2 = \gamma^2 \frac{N^-(z)}{D^-(z)}\frac{N^+(z)}{D^+(z)} \tag{8}$$

where $\gamma \frac{N^+(z)}{D^+(z)}$ corresponds to two times the autocorrelation "causal image" z-transform and verifies that is a minimum phase function.

The underlying considerations lead us to the following result holding when the Paley-Wiener condition is accomplished and $S(w)$ is a bounded function:

- $S(z)$ (because of the P-W condition) can be decomposed into the product of a minimum phase rational function times its corresponding maximum phase rational function.

And $S(z)$ can also be decomposed into the sum of the autocorrelation "causal image" z-transform (which is a minimum phase z-polynomial) plus the corresponding to the "anticausal image" one.

4. OBTAINING THE LAGRANGE MULTIPLIERS

The problem of obtaining the Lagrange multipliers that match the constraints, or equivalently, the gain γ^2 and the polynomial coefficients of $N(z)$ and $D(z)$ is highly nonlinear if one attempts to solve it directly from the constraints.

The proposed method for solving λ_n and β_n in (7) is identical to that proposed by Musicus et Kabel |3| in a similar context. It makes use of the fact that if the Paley-Wiener condition is satisfied and $S(w)<\infty$ then $\frac{N^+(z)}{D^+(z)}$ must be a minimum phase rational function.

So, let us assume that we are in the above section situation and suppose that $N^+(z)$ and $D^+(z)$ are polynomials of order P and Q respectively.

Then we can write (9) and (10).

$$\frac{N^+(z)}{D^+(z)} = 1 + \sum_{n=1}^{\infty} g(n) z^{-n} \quad (9)$$

$$N^+(z) = 1 + \sum_{n=1}^{P} a_n z^{-n}$$
$$D^+(z) = 1 + \sum_{n=1}^{Q} b_n z^{-n} \quad (10)$$

Multiplying (9) by $D^+(z)$ and matching equal powers of z, the next matrix equation can be stated (11)

$$\begin{bmatrix} 1 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_P \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ g(1) & 1 & & \\ \cdot & & \ddots & \\ \cdot & & & \\ g(P) & \cdots & 1 & \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_Q \end{bmatrix} \quad (11)$$

or equivalently: $\underline{a} = \underline{\underline{G}}\ \underline{b}$.

Since g(n) must be a minimum phase sequence (the "causal image" sequence normalized respect the value at lag zero), it can be obtained from the causal part of the $\zeta(.)$ constraints by means of the recursion (12) proposed by Oppenheim-Schafer |4|

$$g(0) = 1$$

$$g(n) = \frac{1}{n} \sum_{K=1}^{n} K\,\zeta(K) g(n-K); \quad n=1,2,.. \quad (12)$$

Now, equation (8) is rewrited into the form (13)

$$D^+(z)\,|\phi(z)|^2 = \gamma^2 N^+(z) \frac{N^-(z)}{D^-(z)} \quad (13)$$

Again, matching equal powers of z, (14) will be obtained

$$\underline{\underline{\phi}}\ \underline{b} = \gamma^2\ \underline{\underline{G}}^T\ \underline{a} \quad (14)$$

where $\underline{\underline{\phi}}$ is the following matrix (Toeplitz and symmetric).

$$\underline{\underline{\phi}} = \begin{bmatrix} \phi(0) & \phi(1) & \cdots & \phi(Q) \\ \phi(1) & \phi(0) & \ddots & \\ \cdot & & \ddots & \\ \cdot & & & \phi(1) \\ \phi(Q) & \cdots & \phi(1) & \phi(0) \end{bmatrix}$$

Equations (11) and (14) yiedd

$$\underline{\underline{\phi}}^{-1}\ \underline{\underline{G}}^T\ \underline{\underline{G}}\ \underline{b} = \frac{1}{\gamma^2}\ \underline{b} \quad (15)$$

So the nonlinear initial problem has becomed an eigenanalysis problem. It can be shown (see |3|) that the adequate election for $\frac{1}{\gamma^2}$ is $\frac{1}{\gamma^2} = \lambda_{max}$; where λ_{max} is the maximum eigenvalue of $\underline{\underline{\phi}}^{-1}\ \underline{\underline{G}}^T\ \underline{\underline{G}}$.

Once that γ^2, $\underline{a}$ and $\underline{b}$ have been calculated, the spectrum can be obtained from (16)

$$S(w) = \mathrm{Re}\,|\phi(w)| \quad (16)$$

Since $\frac{\gamma}{2} \frac{N^+(z)}{D^+(z)} = R^+(z)$, where $R^+(z)$ is the z-transform of the "causal image" of the autocorrelation, an easy recursion for r(n) can be found (see |2|).

5. CONCLUSIONS

A new method of variational spectral estimation has been presented. Generally those methods make identifications of first order sequences. The method proposed here can then be considered a generalization of the other ones since second order identifications are carried out.

It is important to note:

- The objective function (log γ^2) is related to the extrapolated correlation at lag zero, (R(0)=); so this method tries to maximize the power of the process.
- The objective function depends upon the spectral envelope.
- The $\zeta(.)$ constraints are translated into an equivalent set ofautocorrelation constraints.
- The spectrum S(w) equals the real part of $\phi(w)$.

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