

# The period function for Hamiltonian systems with homogeneous nonlinearities\*

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## Abstract

The paper deals with Hamiltonian systems with homogeneous nonlinearities. We prove that such systems have no isochronous centers, that the period annulus of any of its centres is either bounded or the whole plane and that the period function associated to the origin has at most one critical point.

## 1 Introduction and statement of main results.

This paper deals with Hamiltonian systems of the form

$$\begin{cases} \dot{x} = -H_y(x, y), \\ \dot{y} = H_x(x, y), \end{cases} \quad (1)$$

where  $H(x, y) = \frac{x^2+y^2}{2} + H_{n+1}(x, y)$ , and  $H_{n+1}$  is a non zero homogeneous polynomial of degree  $n + 1, n \geq 2$ . The solutions of system (1) are contained in the level curves  $\{H(x, y) = h, h \in \mathbb{R}\}$ . Furthermore, the origin is a centre. For any centre  $p$  of a planar differential system, the largest neighbourhood of  $p$  which is entirely covered by periodic orbits is called the *period annulus* of  $p$ . The function which associates to any closed curve its period is called the

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*period function*. When the period function is constant, the centre is called *isochronous centre*. We are interested in obtaining the global description of the period function  $T(h)$  defined in the origin's period annulus.

It has been proved by several authors that the origin of (1) cannot be an isochronous centre: For  $n = 2$  and  $3$  this fact was observed by Loud [17] and Pleshkan [19] respectively. In the general case, Christopher and Devlin [6] used geometrical and dynamical methods, and Schuman [20] used Birkhoff's normal form. Another natural approach is the computation of the period constants (see [5] for definitions). Using this last approach we obtain the same result (see Corollary 1 of the Appendix). One advantage of this method is that it also provides information about the behaviour in a neighbourhood of the origin of the period function, giving lower bounds for the number of critical points of this function (*critical periods*) associated to the origin's period annulus. Our strategy for the study of  $T(h)$  consists of using the knowledge of the period constants and the knowledge of some properties of the phase portrait of (1).

To enunciate the main result we must introduce the following notation: system (1) can be written in complex coordinates as

$$\dot{z} = iz + F_n(z, \bar{z}), \quad \text{with } z \in \mathbb{C},$$

$$F_n(z, \bar{z}) = \sum_{k+l=n} f_{kl} z^k \bar{z}^l, \text{ and } \operatorname{Re} \left( \frac{\partial F_n(z, \bar{z})}{\partial z} \right) \equiv 0.$$

**Theorem A.** (a) Let  $T(h)$  be the period function associated to the origin's period annulus of system (1).  $T(h)$  satisfies one of the following properties:

- (i) It is monotonous decreasing.
- (ii) It is monotonous increasing and it tends to infinity when the periodic orbit tends to the boundary of the period annulus.
- (iii) It has a unique non degenerate critical period (a minimum) and it tends to infinity when the periodic orbit tends to the boundary of the period annulus.

Furthermore,

- (i) It is monotonous decreasing if and only if  $n$  is odd and  $g(\theta) = H_{n+1}(\cos \theta, \sin \theta) \geq 0$ ,
- (ii) It is monotonous increasing if and only if
  - (I) either  $n$  is even,
  - (II) or  $n$  is odd, and  $\operatorname{Im}(f_{\frac{n+1}{2}, \frac{n-1}{2}}) \leq 0$ .
- (iii) It has a unique non degenerate critical period if and only if  $n$  is odd,  $\operatorname{Im}(f_{\frac{n+1}{2}, \frac{n-1}{2}}) > 0$ , and there exists  $\theta \in [0, 2\pi)$  such that  $g(\theta) < 0$ .

(b) *There are systems of type (1) having a critical point of center type (different from the origin) for which the period function has at least two critical periods.*

Theorem A (a) was obtained for  $n = 2$  by Li Ji-Bin [16], Coppel and Gavrilov [12]. After the present work was finished, we knew about the paper of Collins [9], where it is proved the global monotonicity of the origin's period function in the  $n$  even case (i.e., statement (a.ii.I) of the Theorem A).

Notice that Theorem A (a) cannot be applied to other centres different from the origin because the structure of (1) is broken under translations (except for  $n = 2$ ). Statement b) of the above Theorem shows that the period function is more complicated for these centers.

A similar difference could exist with other problems. The most relevant is that of isochronicity. From Theorem A, it is obvious that systems of type (1) cannot have isochronous centres at the origin. In fact, this result is already known, see [6] and [20]. But, since the structure of (1) is broken under translations, what can be said about the isochronicity of the other centres different from the origin? Are there isochronous centres inside the family of Hamiltonian systems with homogeneous nonlinearities? As far as we know, there was no answer to this question. In this paper, we prove that:

**Theorem B.** *System (1) has no isochronous centres.*

Our proof of Theorems A and B uses some knowledge of the phase portrait of (1). In particular, we need to study the structure of the hyperbolic sectors at *infinity* in the Poincaré's compactification of (1). According to the definitions used in [7], given an infinite critical point  $q$  and a hyperbolic sector  $\mathcal{H}$  associated to  $q$ , we say that  $\mathcal{H}$  is *degenerate* if its two separatrices are contained in the equator of the Poincaré's disk. Otherwise, we say that  $\mathcal{H}$  is *non-degenerate*. The control of this kind of points is important for knowing the type of boundary of the period annulus, and for solving Conti's problem for system (1), see [10]. We prove the following result.

**Theorem C.** *The following statements hold for systems of type (1).*

- (i) *If  $q$  is a infinite critical point in the Poincaré's compactification having a hyperbolic sector at the infinity  $\mathcal{H}$ , then  $\mathcal{H}$  is degenerate.*
- (ii) *The origin of (1) is either a global center or has a bounded period annulus. Furthermore, the origin is a global centre of (1) if and only if  $g(\theta) = H_{n+1}(\cos \theta, \sin \theta) \geq 0$ , and this can only occur when  $n$  is odd.*
- (iii) *A centre  $p$  of (1) different from the origin has a bounded period annulus.*

For  $n = 2$ , statements (ii) and (iii) of the above theorem can be deduced

from [2].

In Section 2 we give the proof of Theorem C and Section 3 is devoted to prove Theorems A and B.

Finally, in the Appendix, we compute the first Liapunov and period constants for the origin of a system with homogeneous nonlinearities (not necessarily Hamiltonian). They play a key role in the proof of Theorem A, but we prefer to show the computations apart, as a technical result. Furthermore, the way of computing these constants and their final expressions help to improve a known result of Conti (see [11]) about the characterization of the centres at the origin of (1) with constant angular speed, see also [18]. While Conti gave an integral characterization of those systems, we provide an explicit expression.

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## 2 Hyperbolic sectors at the infinity and proof of Theorem C.

First of all we need a preliminary result that can be also found in [7]. We include the proof here for the sake of completeness and because it is simpler than that of [7].

Let  $q$  be an infinite critical point of any planar polynomial Hamiltonian vector field in the Poincaré's compactification. We will say that  $\mathcal{H}$  *does not contain straight lines* if given any finite straight line  $l$  which passes through  $q$  (in the Poincaré's compactification) it exists a big enough compact set  $K$  such that  $l \cap (\mathbb{R}^2 \setminus K)$  is not contained in the interior of  $\mathcal{H}$ .

**Lemma 1** *Let  $q$  be an infinite critical point of a Hamiltonian system with a hyperbolic sector  $\mathcal{H}$ . Then,  $\mathcal{H}$  is either degenerate or it does not contain straight lines. Moreover, in this case, the Hamiltonian takes the same value on both separatrices, which are finite.*

**Proof.** Let  $s_1$  and  $s_2$  be the two separatrices of  $\mathcal{H}$ . First we will prove that if  $s_1$  is not included in the equator of the Poincaré's disk, then  $s_2$  is not contained either. Set  $x \in s_2$  and  $\{p_n\}_n$  a sequence of points in the interior of  $\mathcal{H}$  such that  $\lim_{n \rightarrow +\infty} p_n = x$ . Since  $\mathcal{H}$  is a hyperbolic sector, there exists a sequence  $\{p'_n\}_n$  in the interior of  $\mathcal{H}$  such that  $H(p_n) = H(p'_n)$  and moreover,  $\lim_{n \rightarrow +\infty} p'_n = x' \in s_1$ .

Then,

$$H(x') = \lim_{n \rightarrow +\infty} H(p'_n) = \lim_{n \rightarrow +\infty} H(p_n).$$

Hence, we have that  $\lim_{z \rightarrow x} H(z)$  exists for all  $x \in s_2$  when  $z$  is in the interior of  $\mathcal{H}$ . Since  $H$  is a polynomial, then  $s_2$  cannot lay on the equator of the Poincaré's disk, and we are done.

When  $\mathcal{H}$  is non-degenerate we can assume, then, that  $\mathcal{H}$  has two finite separatrices  $s_1$  and  $s_2$ . From the above equality these separatrices have the same value of the energy ( $h$ ). First we will proof that if  $\Gamma \subset \mathcal{H}$  is any path going to  $q$ , we have that (see Fig. 1)

$$\lim_{p \rightarrow q, p \in \Gamma} H(p) = h.$$

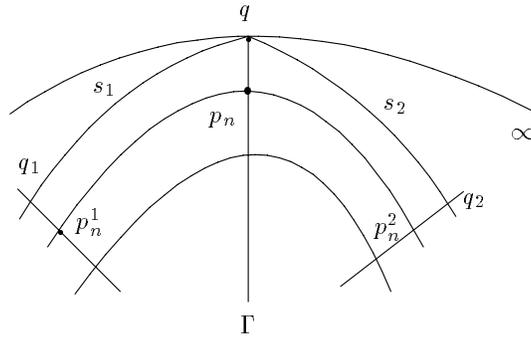


Figure 1: Construction used in the proof of Lemma 1.

Let  $\{p_n\}_n$  be a sequence of points in the interior of  $\mathcal{H}$  satisfying  $\lim_{n \rightarrow +\infty} p_n = q$ . Since  $\mathcal{H}$  is a hyperbolic sector, there exist sequences of points  $\{p_n^i\}_n$ , for  $i = 1, 2$ , such that  $\lim_{n \rightarrow +\infty} p_n^i = q_i \in s_i$  and  $H(p_n^i) = H(p_n)$ , for  $i = 1, 2$ . Then,

$$\lim_{n \rightarrow +\infty} H(p_n^i) = \lim_{n \rightarrow +\infty} H(p_n) = H(q_i) = h,$$

and so,

$$\lim_{p \rightarrow q} H(p) = h.$$

Suppose now that  $\Gamma$  is a straight line. Without loss of generality, we can suppose that this straight line is  $x = 0$ . From the above argument, if we set

$$H(x, y) = H_0(x) + yH_1(x) + y^2H_2(x) + \dots + y^{n+1}H_{n+1}(x),$$

then  $\lim_{y \rightarrow +\infty} H(0, y) = h$ . However, this is only possible if and only if  $H_0(0) = h$ , and  $H_j(0) = 0$  for all  $j = 1, \dots, n + 1$ ; that is,  $H(x, y)|_{x=0} \equiv h$  and so  $x = 0$  is

formed by solutions, which is a contradiction with the fact that  $\Gamma$  is included in  $\mathcal{H}$ .  $\blacksquare$

We will introduce polar coordinates in order to prove Theorem C. The Hamiltonian function now writes as

$$H(r, \theta) = \frac{r^2}{2} + g(\theta) r^{n+1},$$

where  $g(\theta)$  is a trigonometric polynomial of degree  $n+1$ , and system (1) becomes

$$\begin{cases} \dot{r} = -g'(\theta) r^n, \\ \dot{\theta} = 1 + (n+1)g(\theta) r^{n-1}, \end{cases} \quad (2)$$

defined on the cylinder  $C = \{(r, \theta) : r \in \mathbb{R}^+, \theta \in [0, 2\pi]\}$ . Observe that the critical points of (2) are  $(r_*, \theta_*)$  such that  $g(\theta_*) < 0$  and  $g'(\theta_*) = 0$ , and  $r_* = (-1/((n+1)g(\theta_*)))^{1/(n-1)}$ .

**Proof of Theorem C.**

(i) Suppose that  $q$  is an infinite critical point of system (1) having a non-degenerate hyperbolic sector  $\mathcal{H}$ . From Lemma 1, we know that both separatrices must be finite. Without loss of generality, we can suppose that  $q$  is determined by the direction  $x = 0$  and again, by Lemma 1, that the separatrices lay in the right side of  $x = 0$ .

Assume that the separatrices of  $\mathcal{H}$  have energy level  $h$ . Then, the energy equation written in polar coordinates is  $\frac{r^2}{2} + g(\theta) r^{n+1} = h$ , and so we have that

$$g(\theta) = \frac{2h - r^2}{2r^{n+1}}. \quad (3)$$

We set  $F_{h,n}(r) := \frac{2h-r^2}{2r^{n+1}}$ . If the situation described above was possible for any fixed  $\theta \in (\pi/2 - \varepsilon, \pi/2)$ , there would be two arbitrarily large pre-images of  $F_{h,n}(r)$  satisfying (3), but this is a contradiction with the behaviour of  $F_{h,n}(r)$ , for any value of  $h$  (see Figure 2).

(ii-iii) Suppose that  $p$  is a centre whose period annulus,  $N_p$ , is unbounded but not global. Under this assumption, there must exist a hyperbolic sector at infinity with at least one separatrix contained in  $\partial N_p$ . This implies the existence of a non-degenerate hyperbolic sector at the infinity, in contradiction with statement (i).

Therefore,  $\partial N_p$  is either bounded (moreover, by the analyticity of (1),  $\partial N_p$  cannot be a periodic orbit and it contains at least one critical point) or is the empty set. In the latter case,  $p$  is the unique critical point and it is a global centre (in fact,  $p$  is the origin).

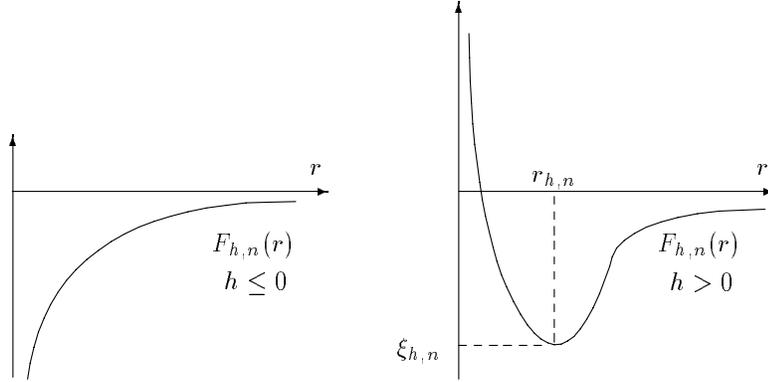


Figure 2: Graph of  $F_{h,n}(r)$  for  $h \leq 0$  (left) and for  $h > 0$  (right).

To end the proof we will characterize global centers. From equation (2), we see that any critical point  $(r_*, \theta_*)$  different from the origin must satisfy  $g(\theta_*) < 0$  and  $g'(\theta_*) = 0$ . Then, it is clear that the origin is the unique critical point if and only if  $g(\theta) \geq 0$ , for all  $\theta \in [0, 2\pi)$  and from part (i) this implies that it is a global center. Finally, notice that if  $n$  is even, then  $g(\theta)$  is a trigonometric polynomial of odd degree and so,  $g(\theta) = -g(\theta + \pi)$ . Consequently, the property  $g(\theta) \geq 0$ , for all  $\theta \in [0, 2\pi)$ , can only hold when  $n$  is odd. ■

### 3 Proofs of Theorems A and B.

In order to prove them, we need the following preliminary results.

**Theorem 1** *An analytic function  $f : I = (i^-, i^+) \subset \mathbb{R} \rightarrow \mathbb{R}$  has at most one non-degenerate critical point if and only if there exists an analytic function  $\varphi : I \rightarrow \mathbb{R}$  such that, for all  $x \in I$ ,*

$$f''(x) + \varphi(x)f'(x) \neq 0. \quad (4)$$

**Proof.** Suppose that there exists an analytic function  $\varphi : I \rightarrow \mathbb{R}$  such that equation (4) holds. Let  $\psi$  be a primitive of  $\varphi$ . Consider  $h : J = (j^-, j^+) \rightarrow I$ , a solution of the differential equation  $h' = \exp(\psi(h))$ , defined in its maximal interval of definition. Observe that since  $h' > 0$  and it is defined in its maximal interval of definition, then  $\lim_{x \rightarrow j^\pm} h(x) = i^\pm$ . So,  $h$  is a diffeomorphism.

Since  $h' \neq 0$  and  $h$  is bijective, then  $f$  has at most one non-degenerate critical point if and only if  $f \circ h$  does so. In order to see this last property it

suffices to see that  $(f \circ h)'' \neq 0$ . We prove this as follows:

$$\begin{aligned} (f \circ h)''(x) &= (f'(h(x))h'(x))' = f''(h(x))(h'(x))^2 + f'(h(x))h''(x) = \\ &= f''(h(x))e^{2\psi(h(x))} + f'(h(x))e^{2\psi(h(x))}\psi'(h(x)) = \\ &= e^{2\psi(h(x))} (f''(h(x)) + \varphi(h(x))f'(h(x))) \neq 0. \end{aligned}$$

Let us prove now the converse.

Suppose that  $f$  has no critical points. Then, it suffices to choose  $\varphi(x) = \frac{f'(x) - f''(x)}{f'(x)}$ .

If  $f$  has a non-degenerate critical point, we can assume, without loss of generality, that it is  $x = 0$  and that  $f(0) = f'(0) = 0$  and  $f''(0) = A > 0$ . Hence

$$f(x) = Ax^2 + Bx^k + O(x^{k+1}),$$

with  $k \geq 3$ . We choose

$$\varphi(x) = \frac{(f')^2 - 2f''f}{2ff'}(x).$$

Clearly, since  $f$  is an analytic function for all  $x \neq 0$ ,  $\varphi$  is analytic. We must prove that it is analytic also on  $x = 0$ . An easy computation shows that if  $k = 3$ , we have  $\lim_{x \rightarrow 0} \varphi(x) = -\frac{B}{A}$  and, if  $k > 3$ , we have  $\lim_{x \rightarrow 0} \varphi(x) = 0$ . So,  $\varphi$  is analytic on  $I$ .

Since  $x = 0$  is the unique finite critical point of  $f$ ,  $f(x) \neq 0$  and  $f'(x) \neq 0$  for all  $x \neq 0$ . Hence, we have that, as we wanted to prove,  $(f'' + \varphi f')(x) = \frac{(f')^2}{2f} \neq 0$ , for all  $x \neq 0$ . On the other hand, it is easy to see that  $\lim_{x \rightarrow 0} (f'' + \varphi f')(x) = 2A \neq 0$ . ■

We will apply this last result to prove that the period function associated to the origin's period annulus has at most one critical period. Before proving this fact, we will see that in any Hamiltonian system the set of all periodic orbits,  $\Gamma$ , can be parameterized by the energy in any period annulus  $W$ .

Consider in  $W$  the following total ordering:

Given  $\gamma_1, \gamma_2 \in \Gamma$  we say that  $\gamma_1 < \gamma_2$  if and only if  $\text{Int}\gamma_1 \subset \text{Int}\gamma_2$ , where  $\text{Int}\gamma_i$  denotes the bounded domain of  $\mathbb{R}^2$  surrounded by  $\gamma_i$ .

Now we endow  $\Gamma$  with the order topology. Clearly, the Hamiltonian function  $H$  over  $\Gamma$  is continuous with respect this topology and applies  $\Gamma$  in some interval  $I = (0, b)$  of the real line ( $b \in \mathbb{R} \cup \{+\infty\}$ ). To see that this map is order-preserving it suffices to show that it is one to one. To prove this, suppose that  $H(\gamma_1) = H(\gamma_2)$  with  $\gamma_1 < \gamma_2$  and consider the map  $H$  restricted to the compact

annulus  $K = \overline{\text{Int}\gamma_2} \setminus \overline{\text{Int}\gamma_1}$ . This map attaches a maximum and a minimum in  $K$ . Since  $\partial K = \gamma_1 \cup \gamma_2$  and  $H|_{\partial K}$  is constant, either  $H|_K$  is constant or  $H|_K$  has a local extremum in its interior. In both cases we can ensure the existence of  $x \in K \subset W$  with  $(\nabla H)_x = 0$ , that is a critical point of the Hamiltonian vector field in the interior of  $W$ , which is a contradiction. So the map  $H$  over  $\Gamma$  is order preserving (in fact it is an order-preserving homeomorphism).

Hence, it seems natural to consider the period function over  $I$  instead of the original period function which is defined over the period annulus  $W$ , because we can use standard techniques of differential analysis to study the properties of the period function. Therefore, in the sequel we will talk about the period function  $T(h)$  which gives the period of the closed orbit with energy  $H = h$ .

From equation (2),  $T(h)$  can be computed as

$$T(h) = \int_0^{2\pi} \frac{d\theta}{1 + (n+1)g(\theta)r(\theta, h)^{n-1}}, \quad (5)$$

(for shortness, in the following we denote  $r := r(\theta, h)$ ) while  $\dot{\theta}|_{H=h} = 1 + (n+1)g(\theta)r(\theta, h)^{n-1}$  does not vanish. This condition is verified in a deleted neighbourhood of the origin because  $\lim_{r \rightarrow 0} \dot{\theta} = 1$ . The following lemma asserts that this condition holds in the whole period annulus of the origin  $W$ . This result is well-known, see [3], [8] or [9], but we include here, for the sake of completeness, a different proof.

**Lemma 2** *The period annulus associated the origin of (1),  $W$ , has no points  $(r, \theta)$  on which  $\dot{\theta} = 1 + (n+1)g(\theta)r^{n-1} = 0$ .*

**Proof.** First we prove that there are no points in  $x \in W$  for which  $\dot{\theta}(x) = \ddot{\theta}(x) = 0$ . Consider  $\dot{\theta}(x) = 1 + (n+1)g(\theta)r^{n-1}$ . Then,  $\ddot{\theta}(x) = (n^2-1)g(\theta)r^{n-2}\dot{r} + (n+1)g'(\theta)r^{n-1}\dot{\theta}$ . Hence,  $\dot{\theta}(x) = \ddot{\theta}(x) = 0$  implies that  $\dot{\theta} = \dot{r} = 0$  and, as a consequence,  $x$  is a critical point different from the origin, which contradicts the fact that  $x \in W$ .

Set  $I = [0, a)$ , the image of  $W$  by  $H$  (remember that  $H$  is a homeomorphism between the set of periodic orbits  $\Gamma$  and  $I$ ). For each  $h \in I$  denote by  $\gamma_h$  the closed curve of  $H = h$  contained in  $W$ . Define the map  $L : I \rightarrow \mathbb{R}$  by

$$L(h) = \min |1 + (n+1)g(\theta)r^{n-1}|_{\gamma_h}.$$

This function is clearly well defined and continuous. If  $L^{-1}(0) = \emptyset$  there is nothing to prove. Suppose that  $L^{-1}(0) \neq \emptyset$ . Then  $L^{-1}(0)$  is a closed set which does not contain 0 because  $L(0) = 1$ . Let  $h_0$  be the infimum of  $L^{-1}(0)$ . Then the orbit  $\gamma_{h_0}$  is the first orbit (in the ordering of  $\Gamma$ ) such that there exists  $x \in \gamma_{h_0}$  with  $\dot{\theta}(x) = 0$ . Set  $\varphi_y(t) = (r_y(t), \theta_y(t))$  be the solution of (2) with

initial condition  $y$ . Since  $\ddot{\theta}(x) \neq 0$ , the function  $\theta_x(t)$  has a local extremum at 0. This implies that, for  $\varepsilon > 0$  small enough, the function  $\theta_y(t)$  has also a local extremum for  $y \in \gamma_{h_0 - \varepsilon}$ . Therefore there exists  $z \in \gamma_{h_0 - \varepsilon}$  with  $\dot{\theta}(z) = 0$  and hence  $L(h_0 - \varepsilon) = 0$ . This last equality is in contradiction with the fact that  $h_0 = \inf L^{-1}(0)$ .  $\blacksquare$

From the above result and the energy equation  $\frac{r^2}{2} + g(\theta)r^{n+1} = h$ , it follows that

$$\frac{dh}{dr} = r(1 + (n+1)g(\theta)r^{n-1}) > 0, \quad (6)$$

in the whole period annulus. Furthermore, any fixed periodic orbit in the origin's period annulus has positive energy.

**Lemma 3** *The period function associated to the origin's period annulus satisfies*

$$T(h) = \frac{d}{dh} \int_0^{2\pi} \frac{r^2}{2} d\theta.$$

**Proof.** Let  $\gamma$  denote a closed orbit of energy  $h$  corresponding to a solution  $r(\theta, h)$  of (2). From the expression (5), using (6) and Green's Theorem we have

$$\begin{aligned} T(h) &= \int_{\gamma} \frac{d\theta}{1 + (n+1)g(\theta)r(\theta, h)^{n-1}} = \\ &= \iint_{\text{Int}(\gamma)} r \frac{dr}{dh} d\theta = \frac{d}{dh} \iint_{\text{Int}(\gamma)} r dr d\theta = \\ &= \frac{d}{dh} \int_{\gamma} \frac{r^2}{2} d\theta. \end{aligned}$$

$\blacksquare$

**Theorem 2** *The period function associated to the period annulus of the origin of (1) has at most one critical period.*

**Proof.** As we have seen in Lemma 3,  $T(h) = \frac{d}{dh} \int_0^{2\pi} \frac{r^2}{2} d\theta$ . So, equation (4) can be written as

$$T''(h) + \varphi(h)T'(h) = \frac{1}{2} \int_0^{2\pi} \frac{d^3}{dh^3}(r^2) + \varphi(h) \frac{d^2}{dh^2}(r^2) d\theta \neq 0 \quad (7)$$

We set  $M(r, \theta) = 1 + (n+1)g(\theta)r^{n-1}$  (we call it  $M$ , for the sake of brevity). Taking into account equation (6), we have that the middle part of expression

(7) can be written as

$$\frac{1}{2} \int_0^{2\pi} \frac{-2(n^2-1)(n-3)g(\theta)r^{n-5}M+6(n^2-1)^2g^2(\theta)r^{2n-6}}{M^5} + \varphi(h) \frac{-2(n^2-1)g(\theta)r^{n-3}}{M^3} d\theta. \quad (8)$$

We choose  $\varphi(h) = -\frac{n-3}{2} \frac{1}{h}$ , defined in  $I = (0, a)$ , for some  $a \in \mathbb{R}^+ \cup \{+\infty\}$  (notice that the fact that the energy in the period annulus takes only positive values plays an important role here). Tedious computations transforms the expression (8) into

$$2 \int_0^{2\pi} \frac{(n+1)n(n-1)^2}{h M^5} g(\theta)^2 r^{2n-4} \left( 1 + \frac{n+3}{4n} (n+1)g(\theta)r^{n-1} \right) d\theta$$

Note that  $0 < \frac{n+3}{4n} < 1$ . Then, since by Lemma 2 in the whole origin's period annulus  $1 + (n+1)g(\theta)r^{n-1} > 0$  holds, we have that

$$1 + \frac{n+3}{4n} (n+1)g(\theta)r^{n-1} > 0,$$

and then,

$$\int_0^{2\pi} \frac{(n+1)n(n-1)^2}{h M^5} g(\theta)^2 r^{2n-4} \left( 1 + \frac{n+3}{4n} (n+1)g(\theta)r^{n-1} \right) d\theta > 0.$$

By Theorem 1, the theorem follows. ■

Now we are able to prove Theorem A.

### Proof of Theorem A.

(a) From equation (5), and taking into account (6), we have that

$$\frac{dT(h)}{dh} = -(n+1)(n-1) \int_0^{2\pi} \frac{g(\theta)r^{n-3} d\theta}{(1 + (n+1)g(\theta)r^{n-1})^3}. \quad (9)$$

To prove statement (i), we recall that  $1 + (n+1)g(\theta)r^{n-1} \neq 0$  in the whole origin's period annulus. Hence, if  $g(\theta) = H_{n+1}(\cos \theta, \sin \theta) \geq 0$ , from (9) we directly obtain that  $\frac{dT}{dh}(h) < 0$ . Conversely, suppose that  $T(h)$  is monotonous decreasing. This implies- using Theorem C (ii)- that the origin is a global centre (otherwise, the boundary of the origin's period annulus would have a critical point and  $T(h)$  would tend increasingly to infinity) and, again by Theorem C (ii), if the origin is a global centre then  $g(\theta) \geq 0$ .

Suppose now that  $g(\theta)$  takes negative values. By Theorem C (ii), we also know that the period annulus of the origin is bounded and contains some critical point. This fact implies that the period function tends to infinity as the closed orbits tend to this boundary.

If instead of parameterize the closed curves of the period annulus  $W$  by the Hamiltonian energy we use the point of intersection of any closed curve of  $W$  with the  $OX^+$ -axis we get another period function called  $t(r)$ . Observe that this can be done in the whole  $W$ , because in this set  $1 + (n + 1)g(0)r^{n-1} > 0$ , and  $t(r) = T(r^2/2 + g(\theta)r^{n+1})$ . Hence

$$T'(h) = \frac{1}{r(1 + (n + 1)g(0)r^{n-1})} t'(r), \quad (10)$$

where  $h = r^2/2 + g(0)r^{n+1}$ .

From the above expression we get that the main preliminary result we have obtained, Theorem 2, is still valid for  $t(r)$ .

To prove statements (ii) and (iii), we use the results of the Appendix. From Proposition 1 of the Appendix, we know that

$$b_1 = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -2\pi \text{Im}(f_{\frac{n+1}{2}, \frac{n-1}{2}}), & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, in the proof of Corollary 1 of the Appendix, we deduce that  $b_2 > 0$ .

We distinguish then two cases, depending on the value of the first period-Abel constant:

$$(ii) \ b_1 \geq 0$$

Depending on whether  $b_1$  vanishes or not, the period function in polar coordinates, writes as (see Corollary 2 of the Appendix)

$$t(r) = 2\pi + b_2 r^{2n-2} + O(r^{2n-1}), \text{ with } b_2 > 0,$$

or

$$t(r) = 2\pi + b_1 r^{n-1} + O(r^n), \text{ with } b_1 > 0.$$

In both cases, in a neighbourhood  $(0, \delta)$ , the period function  $t(r)$  is monotonously increasing. Then, Theorem C (ii) and Theorem 2 ensure that  $t(r)$  is monotonous increasing in its domain and tends to infinity near the boundary of the origin's period annulus, and so does  $T(h)$ .

$$(iii) \ b_1 < 0.$$

Then, the period function in polar coordinates writes as

$$t(r) = 2\pi + b_1 r^{n-1} + O(r^n), \text{ with } b_1 < 0.$$

Therefore, in a neighbourhood  $(0, \delta)$ , the period function  $T(x)$  is monotonously decreasing. As in the statement (ii), we recall Theorem C (ii) and Theorem 2. In the current case, they imply that  $T(h)$  reaches a unique minimum and then it tends to infinity as the closed orbits tend to boundary of the period annulus. ■

(b) Consider the following system

$$\begin{cases} \dot{x} &= -y - ex^4 - 2dx^3y + 3x^2y^2 + y^4, \\ \dot{y} &= x + 5cx^4 + 4ex^3y + 3dx^2y^2 - 2xy^3. \end{cases} \quad (11)$$

It has a centre at the point  $(0, 1)$ . By a translation to the origin and the linear change of time  $\frac{dt}{d\tau} = -1/\sqrt{3}$ , it is transformed in the following quartic system:

$$\begin{cases} \dot{x} = & -y - 3x^2 - 2y^2 + 2\sqrt{3}dx^3 - 6x^2y - \frac{4}{3}y^3 \\ & + 3ex^4 + 2\sqrt{3}dx^3y - 3x^2y^2 - \frac{1}{3}y^4, \\ \dot{y} = & x - 3\sqrt{3}dx^2 + 6xy - 12ex^3 - 6\sqrt{3}dx^2y \\ & + 6xy^2 - 15\sqrt{3}cx^4 - 12ex^3y - 3\sqrt{3}dx^2y^2 + 2xy^3. \end{cases} \quad (12)$$

The first two period constants (we call them  $p_2$  and  $p_4$ ) are known for general systems, see for instance [15]. Straightforward computations give that:

$$\begin{aligned} p_2 &= \frac{129}{4} + \frac{135}{4}d^2 + \frac{9}{2}e. \\ p_4 &= \frac{832883}{2304} + \frac{945}{8}cd - \frac{25095}{128}d^2 - \frac{152685}{256}d^4. \end{aligned}$$

Since  $p_2$  and  $p_4$  are independent and can take any real value, standard arguments imply that there are values of the parameters for which the period function associated to the period annulus of  $(0, 1)$  has at least two critical points in a neighbourhood of the critical point. ■

**Proof of Theorem B.** Let  $p$  be a centre of system (1) and  $N_p$  its period annulus. From Theorem C (ii), we know that either  $N_p$  is bounded and its boundary contains a critical point - and, then, it cannot be an isochronous centre-, or  $p$  is a global centre. The last case is only possible if and only if  $n$  is odd and  $g(\theta) = H_{n+1}(\cos \theta, \sin \theta) \geq 0$ . From Theorem A (i), the period function  $T(h)$  defined in the origin's period annulus is globally monotonous decreasing, and so it can neither be an isochronous centre. ■

## Appendix: Liapunov and period constants

Consider

$$\dot{z} = iz + F_n(z, \bar{z}), \quad \text{with } z \in \mathbb{C}, \quad (13)$$

where  $F_n(z, \bar{z})$  is a homogeneous polynomial of degree  $n$ . We will usually write  $F_n(z, \bar{z}) = \sum_{k+l=n} f_{k,l} z^k \bar{z}^l$ , where  $f_{k,l} \in \mathbb{C}$ . For the sake of simplicity, we define, for a fixed  $n$ :

$$g_l = \begin{cases} \frac{f_{\frac{n+l+1}{2}, \frac{n-l-1}{2}}}{2} & \text{if } l \in \Omega_n, \\ 0 & \text{if } l \notin \Omega_n, \end{cases} \quad (14)$$

where  $\Omega_n = \{l \in \mathbb{Z} : (n+l) \text{ is odd and } -(n+1) \leq l \leq n-1\}$ .

Our interest is mainly focused on computing the so-called Liapunov and period constants for system (13). To this end, we perform the following changes of variables:

If we first introduce the usual polar coordinates by setting  $R^2 = z\bar{z}$  and  $\theta = \arctan \frac{\text{Im}z}{\text{Re}z}$ , and then apply the change  $r = \frac{R^{n-1}}{1 + \text{Im}(S_n(\theta))R^{n-1}}$  (suggested in [4]), system (13) writes as:

$$\begin{cases} \dot{r} &= \frac{A_2(\theta)r^2 + A_3(\theta)r^3}{1 - \text{Im}(S_n(\theta))r}, \\ \dot{\theta} &= \frac{1}{1 - \text{Im}(S_n(\theta))r}, \end{cases} \quad (15)$$

where  $S_n(\theta)$  is a trigonometric polynomial defined by  $S_n(\theta) = e^{-i\theta} F_n(e^{i\theta}, e^{-i\theta})$ ; then,  $A_2(\theta) = \text{Re}((n-1)S_n(\theta) + iS_n'(\theta))$  and  $A_3(\theta) = \frac{n-1}{2} \text{Re}(iS_n^2(\theta))$ . By eliminating the time, we reach the Abel equation:

$$\frac{dr}{d\theta} = A_2(\theta)r^2 + A_3(\theta)r^3, \quad (16)$$

Following [1], for this differential equation, consider the solution  $r(\theta, \rho)$  that takes the value  $\rho$  when  $\theta = 0$ . Therefore,

$$r(\theta, \rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \dots, \text{ with } u_k(0) = 0 \text{ for } k \geq 2. \quad (17)$$

Let  $P(\rho) = r(2\pi, \rho)$  be the return map between  $\mathbb{R} \times \{0\}$  and  $\mathbb{R} \times \{2\pi\}$ . We will say that system (16) has a centre at  $r = 0$  if and only if  $u_k(2\pi) = 0$ , for all  $k \geq 2$ . On the other hand, it has a focus if it exists some  $k$  such that  $u_k(2\pi) \neq 0$ . When for system (13),  $u_j(2\pi) = 0$  for  $j = 1, \dots, m-1$ , we will say that its *Liapunov-Abel constant of order  $m$*  is  $a_m = u_m(2\pi)$ .

Substituting (17) in (16) we easily get the following relations, that suggest a recurrent way to find the Liapunov-Abel constants  $a_j$ :

$$\begin{aligned} u_2' &= A_2, \\ u_3' &= A_3 + 2A_2u_2, \\ u_4' &= A_2u_2^2 + 2A_2u_3 + 3A_3u_2, \dots \end{aligned} \quad (18)$$

Once we know that the origin of (13) is a centre, there is a simple way to give the conditions to be an isochronous centre. We observe that we cannot

use the Abel equation (16), since this equation does not take into account the time variable. The idea we will use is suggested in [13]: if we take the second equation of (15) and we integrate the time, we obtain:

$$\bar{I}(\rho) = \int_0^{2\pi} 1 - \text{Im}(S_n(\theta))r(\theta, \rho)d\theta = 2\pi - \int_0^{2\pi} \text{Im}(S_n(\theta))r(\theta, \rho) d\theta, \quad (19)$$

where  $r(\theta, \rho)$  is given above.

The system (13) has an isochronous centre at the origin if it is a centre and, furthermore,

$$\begin{aligned} \int_0^{2\pi} \text{Im}(S_n(\theta))r(\theta, \rho) d\theta &= \int_0^{2\pi} \text{Im}(S_n(\theta))\left(\sum_{j \geq 1} u_j(\theta)\rho^j\right) d\theta = \\ \sum_{j \geq 1} \left(\int_0^{2\pi} \text{Im}(S_n(\theta))u_j(\theta) d\theta\right) \rho^j &= 0. \end{aligned}$$

Hence, the conditions to have an isochronous centre are:

$$b_j := - \int_0^{2\pi} \text{Im}(S_n(\theta))u_j(\theta) d\theta = 0, \text{ for } j \geq 1. \quad (20)$$

The numbers  $b_j$  will be called *period-Abel constants*.

In the main result we give some of the first Liapunov-Abel and period-Abel constants for all systems of type (13) in terms of the coefficients of the equation and valid for all  $n \in \mathbb{N}$ . The above approach has been already used in [14] to give integral expressions for the Liapunov and period constants. As we will see in the applications, our result allows to establish general properties for systems of type (13) of any degree, see for instance Corollary 1 and Proposition 2 in this Appendix.

**Proposition 1** *The following assertions are true for systems of type (13), with  $F_n(z, \bar{z})$  homogeneous of degree  $n$ :*

(a) *The first three Liapunov-Abel constants are:*

$$\begin{aligned} a_2 &= 2\pi(n-1)\text{Re}(g_0), \\ a_3 &= (1-n)\pi \sum \text{Im}(g_l g_{-l}), \\ a_4 &= \frac{\pi(1-n)}{2}\text{Re} \left( \sum_{l, k, l+k \neq 0} \frac{g_l g_k}{l+k} \left( (n-1+l+k)g_{-(l+k)} + (n-1-l-k)\bar{g}_{l+k} \right) \right). \end{aligned}$$

(b) *The first two period-Abel constants are:*

$$\begin{aligned} b_1 &= -2\pi \text{Im}(g_0), \\ b_2 &= -\pi \left( \sum_{l \neq 0} \frac{n-l-1}{l} g_l \bar{g}_l + 2 \sum_{l > 0} g_l g_{-l} \right). \end{aligned}$$

The statement basically follows by integrating the recurrences given in (18). We set first some useful notation for the integration steps:

Given a trigonometric polynomial  $p(\theta) = \sum_{k \in K} p_k e^{ik\theta} + p_0$  with  $K$  a finite subset of  $\mathbb{Z} \setminus \{0\}$ , we define

$$\begin{aligned}\tilde{p}(\theta) &= \int_0^\theta p(\xi) d\xi = \sum_{k \in K} \left[ \frac{p_k}{ik} e^{ik\xi} + p_0 \theta \right]_0^\theta = \sum_{k \in K} \frac{p_k}{ik} (e^{ik\theta} - 1) + p_0 \theta, \\ \hat{p}(\theta) &= \sum_{k \in K} \frac{p_k}{ik} e^{ik\theta} + p_0 \theta,\end{aligned}$$

and  $\{p\}^\sim = \tilde{p}$ ,  $\{p\}^\wedge = \hat{p}$ . In general, we can write  $\tilde{p}(\theta) = \hat{p}(\theta) - p(0)$ .

The difference between both primitives of  $p(\theta)$  is that  $\tilde{p}$  contains an ‘‘extra’’ constant term, while  $\hat{p}(\theta)$  is the primitive of  $p(\theta)$  which has no constants terms. This fact will be crucial for the fluency of our computations.

Observe also that

$$\begin{aligned}\tilde{p}'(\theta) &= \left\{ \sum_{k \neq 0} ik p_k e^{ik\theta} \right\}^\sim = \sum_{k \neq 0} p_k (e^{ik\theta} - 1) = p(\theta) - p(0), \\ \hat{p}'(\theta) &= \left\{ \sum_{k \neq 0} ik p_k e^{ik\theta} \right\}^\wedge = \sum_{k \neq 0} p_k e^{ik\theta} = p(\theta) - p_0.\end{aligned}\tag{21}$$

The last one, then, becomes a trigonometrical polynomial without constant terms.

### Proof of Proposition 1

(i) To integrate (18) we compute the expressions of  $S_n(\theta)$ ,  $A_2(\theta)$  and  $A_3(\theta)$  in terms of the coefficients given in (14).

$$\begin{aligned}S_n(\theta) &= \sum_l g_l e^{il\theta}, \\ A_2(\theta) &= \operatorname{Re} \sum_l (n-1-l) g_l e^{il\theta}, \\ A_3(\theta) &= -\frac{n-1}{2} \operatorname{Im} \sum_{l,k} g_l g_k e^{i(l+k)\theta}.\end{aligned}\tag{22}$$

By using (18) and the above expressions we have that:

$$u_2'(\theta) = A_2(\theta) = \operatorname{Re} \sum_l (n-1-l) g_l e^{il\theta}.$$

This implies that

$$\begin{aligned}u_2(\theta) &= \widetilde{A}_2(\theta) = \operatorname{Re} \sum_l \int_0^\theta (n-1-l) g_l e^{il\theta} d\theta = \\ &= \operatorname{Re} \left[ (n-1)g_0\theta + \sum_{l \neq 0} \frac{(n-1-l)}{i-l} g_l e^{il\theta} \right]_0^\theta.\end{aligned}$$

Thus,  $a_2 = u_2(2\pi) = 2\pi(n-1)\text{Reg}_0$ .

To compute the subsequent  $a_i$ , we will assume that  $a_2 = 0$  and so,  $\text{Reg}_0 = 0$  (this assumption also reads as  $u_2(2\pi) = 0$ ,  $\widetilde{A}_2(2\pi) = 0$  or  $\widehat{A}_2(2\pi) = \widehat{A}_2(0)$ ). Of course, we also must re-consider the functions  $A_2$ ,  $\widetilde{A}_2 = u_2$  and  $\widehat{A}_2$  under this restriction. As a consequence,  $\widehat{A}_2(\theta)$  becomes a trigonometrical polynomial without constant terms.

The second equality of (18) gives that

$$\begin{aligned} u_3(\theta) &= \{A_3 + 2A_2u_2\} \widetilde{(\theta)} = \widetilde{A}_3(\theta) + 2\{A_2u_2\} \widetilde{(\theta)} = \widetilde{A}_3(\theta) + \{(\widetilde{A}_2^2)'\} \widetilde{(\theta)} = \\ &\widetilde{A}_3(\theta) + \widetilde{A}_2^2(\theta) - \widetilde{A}_2^2(0). \end{aligned}$$

Then, imposing that  $a_2 = 0$ ,

$$\begin{aligned} a_3 &= u_3(2\pi) = \widetilde{A}_3(2\pi) = \frac{1-n}{2} \text{Im} \left( \sum_{l+k=0} g_l g_k \xi + \sum_{l+k \neq 0} \frac{g_l g_k}{i(l+k)} e^{i(l+k)\xi} \right)_0^{2\pi} = \\ &\pi(1-n) \text{Im} \sum_{l+k=0} g_l g_k = \pi(1-n) \text{Im} \sum_l g_l g_{-l}. \end{aligned}$$

Again from (18), and using that  $u_2 = \widetilde{A}_2$ , we get that

$$\begin{aligned} u_4(\theta) &= \{A_2u_2^2 + 2A_2u_3 + 3A_3u_2\} \widetilde{(\theta)} = \{A_2\widetilde{A}_2^2 + 2A_2u_3 + 3A_3u_2\} \widetilde{(\theta)} = \\ &\frac{1}{3} \{(\widetilde{A}_2^3)'\} \widetilde{(\theta)} + 2\{(\widetilde{A}_2\widetilde{A}_3)'\} \widetilde{(\theta)} + \{A_3\widetilde{A}_2\} \widetilde{(\theta)}. \end{aligned}$$

To compute  $a_4$  we must assume that  $\widetilde{A}_2(2\pi) = \widetilde{A}_3(2\pi) = 0$ . Thus,

$$a_4 = u_4(2\pi) = \{A_3\widetilde{A}_2\} \widetilde{(2\pi)}.$$

Moreover, there exists some constant  $C$  such that  $\{A_3\widetilde{A}_2\} \widetilde{(2\pi)} = \{A_3\widehat{A}_2\} \widetilde{(2\pi)} + C \widehat{A}_3(2\pi)$  and so,

$$a_4 = \{A_3\widehat{A}_2\} \widetilde{(2\pi)}.$$

This simple trick clarifies the forthcoming computations:

$$\begin{aligned} A_3\widehat{A}_2 &= \left( \frac{1-n}{2} \text{Im} \sum_{l+k \neq 0} g_l g_k e^{i(l+k)\theta} \right) \left( \text{Re} \sum_{j \neq 0} \frac{n-j-1}{ij} g_j e^{ij\theta} \right) = \\ &\frac{1-n}{4} \text{Im} \sum_{\Delta} g_l g_k e^{i(l+k)\theta} \frac{n-j-1}{ij} (g_j e^{ij\theta} - \overline{g_j} e^{-ij\theta}), \end{aligned}$$

where  $\Delta = \{(j, l, k) : l + k \neq 0, j \neq 0\}$ ; and so,

$$\begin{aligned}
a_4 &= \frac{1-n}{4} \text{Im} \sum_{\Delta} \int_0^{2\pi} g_l g_k e^{i(l+k)\theta} \frac{n-j-1}{ij} (g_j e^{ij\theta} - \overline{g_j} e^{-ij\theta}) d\theta, \\
&= \frac{1-n}{4} \text{Im} \left[ \sum_{j+k+l=0} \frac{n-j-1}{ij} g_l g_k g_j \theta - \sum_{-j+k+l=0} \frac{n-j-1}{ij} g_l g_k \overline{g_j} \theta + \sum_{s \neq 0} \psi_s e^{is\theta} \right]_0^{2\pi} = \\
&= \frac{\pi(1-n)}{2} \text{Im} \sum_{l, k, l+k \neq 0} \frac{g_l g_k}{i(l+k)} (-g_{-(l+k)}(n+l+k-1) - \overline{g_{l+k}}(n-l-k-1)) = \\
&= \frac{\pi(1-n)}{2} \text{Re} \sum_{l, k, l+k \neq 0} \frac{g_l g_k}{l+k} (g_{-(l+k)}(n+l+k-1) + \overline{g_{l+k}}(n-l-k-1)),
\end{aligned}$$

as we wanted to prove.

(ii) Referring to the period constants, since  $u_1(\theta) \equiv 1$ , it comes immediately the expression of  $b_1$ :

$$b_1 = - \int_0^{2\pi} \text{Im} S_n(\theta) d\theta = -2\pi \text{Im} g_0.$$

On the other hand, from (20), and assuming that  $a_i = 0$  for all  $i$  and  $b_1 = 0$ , we see that

$$\begin{aligned}
b_2 &= - \int_0^{2\pi} \text{Im} S_n(\theta) \widetilde{A}_2(\theta) = - \int_0^{2\pi} \text{Im} S_n(\theta) \widehat{A}_2(\theta) = \\
&= - \int_0^{2\pi} \left( \text{Im} \sum_{l \neq 0} g_l e^{il\theta} \right) \left( \text{Re} \sum_{j \neq 0} \frac{n-j-1}{ij} g_j e^{ij\theta} \right) = \\
&= - \frac{1}{2} \text{Im} \int_0^{2\pi} \sum_{j, l \neq 0} \frac{n-j-1}{ij} g_j e^{ij\theta} (g_l e^{il\theta} - \overline{g_l} e^{-il\theta}) = \\
&= \frac{1}{2} \text{Re} \int_0^{2\pi} \sum_{j, l \neq 0} \frac{n-j-1}{j} g_j (g_l e^{i(j+l)\theta} - \overline{g_l} e^{-i(j-l)\theta}) = \\
&= -\pi \text{Re} \sum_{l \neq 0} \frac{1}{l} ((n+l-1)g_l g_{-l} + (n-l-1)g_l \overline{g_l}).
\end{aligned}$$

By using that  $\frac{n+l-1}{l} - \frac{n-l-1}{l} = 2$ , and that  $a_3 = 0$  we get that the real part of the above expression can be removed and then

$$b_2 = -\pi \sum_{l \neq 0} \frac{1}{l} ((n+l-1)g_l g_{-l} + (n-l-1)g_l \overline{g_l}) = \quad (23)$$

$$-\pi \left( 2 \sum_{l > 0} g_l g_{-l} + \sum_{l \neq 0} \frac{n-l-1}{l} g_l \overline{g_l} \right),$$

which gives expression for  $b_2$ . ■

As a consequence of Proposition 1, we can state the following results.

**Corollary 1** *Suppose that system (13) is Hamiltonian. Then, the origin cannot be an isochronous centre.*

**Proof.** We will prove that for such systems the second period-Abel constant is always positive, and hence that the origin cannot be an isochronous centre.

In the case of Hamiltonian systems we have that  $\operatorname{Re} \frac{\partial F}{\partial z} \equiv 0$  and so we get the following characterization:

$$(n+l+1)\overline{g_l} + (n-l+1)g_{-l} = 0$$

By substituting the relation given by (23), we get:

$$\begin{aligned} b_2 &= -\pi \sum_{l \neq 0} \frac{g_l \overline{g_l}}{l} \left( \frac{-(n+l-1)(n+l+1)}{n-l+1} + (n-l-1) \right) \\ &= -\pi \sum_{l \neq 0} \frac{g_l \overline{g_l}}{l} \frac{-4nl}{n-l+1} = \pi \sum_{l \neq 0} \frac{4ng_l \overline{g_l}}{n-l+1} > 0. \end{aligned}$$

■

**Corollary 2** *Assume that system (13) has a center at the origin. For  $r$  small enough let  $t(r)$  denote the period function of the solution of (13) which starts at the point  $z = r+0i$ . Let  $b_1$  and  $b_2$  be given by Proposition 1. Then, the following holds:*

- (i) *If  $b_1 \neq 0$  then  $t(r) = 2\pi + b_1 r^{n-1} + O(r^n)$ ,*
- (ii) *If  $b_1 = 0$  and  $b_2 \neq 0$  then  $t(r) = 2\pi + b_2 r^{2n-2} + O(r^{2n-1})$ .*

**Proof.** Consider  $b_1 \neq 0$ . By the definition of  $b_1$ , see (20), it turns out that

$$\bar{t}(\rho) = 2\pi + b_1 \rho + O(\rho^2),$$

where  $\bar{t}(\rho)$  is given in (19). From the change used to get (15), we have that

$$t(r) = \bar{t} \left( \frac{r^{n-1}}{1 + \operatorname{Im}(S_n(0))r^{n-1}} \right).$$

Hence the proof follows by direct substitution. The case  $b_1 = 0$  and  $b_2 \neq 0$  can be proved in a similar way. ■

The expression of the Liapunov-Abel constants in the way given in Proposition 1 is also a good language to prove and write more explicitly a result of Conti, see [11], which gives necessary conditions for the origin of a system of type (13) satisfying

$$\frac{d\theta}{dt} \equiv 1$$

to be a centre. When this centre exists, it is obvious that it is an isochronous one.

In real variables, these systems admit the general form:

$$\begin{cases} \dot{x} &= -y + x \sum_{k=0}^n c_{n-k,k} x^{n-k} y^k \\ \dot{y} &= x + y \sum_{k=0}^n c_{n-k,k} x^{n-k} y^k. \end{cases} \quad (24)$$

The above system expressed in complex coordinates turns out to be:

$$\dot{z} = iz + F_{n+1}(z, \bar{z}), \quad (25)$$

with

$$F_{n+1}(z, \bar{z}) = \frac{1}{2^n} \sum_{k=0}^n c_{n-k,k} z (z + \bar{z})^{n-k} (z - \bar{z})^k (-i)^k.$$

Expanding the binomials, we finally obtain that:

$$F_{n+1}(z, \bar{z}) = \sum_{l+m=n+1} f_{l,m} z^l \bar{z}^m,$$

where

$$\begin{aligned} f_{l,m} &= \frac{1}{2^n} \sum_{\Delta} (-1)^{j_2} (-i)^k \binom{n-k}{j_1} \binom{k}{j_2} c_{n-k,k}, \\ n &= l + m - 1, \text{ and,} \\ \Delta &= \{(k, j_1, j_2) : 0 \leq k \leq n, 0 \leq j_1 \leq n-k, 0 \leq j_2 \leq k, j_1 + j_2 = m\}. \end{aligned}$$

**Proposition 2** (i) A system of type (24) (which in complex coordinates writes as (25)) has a center at the origin if and only if its first Liapunov-Abel constant  $a_2$  is zero.

(i)

$$a_2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{2\pi n}{2^n} \sum_{\Delta'} (-1)^{j_2} (-i)^k \binom{n-k}{j_1} \binom{k}{j_2} c_{n-k,k} & \text{if } n \text{ is even,} \end{cases} \quad (26)$$

where

$$\Delta' = \{(k, j_1, j_2) : 0 \leq k \leq n, 0 \leq j_1 \leq n-k, 0 \leq j_2 \leq k, j_1 + j_2 = n/2\}.$$

Conditions for several  $n$  obtained applying (26) are

$$n = 2 \quad c_{0,2} + c_{2,0} = 0,$$

$$n = 4 \quad 3c_{0,4} + c_{2,2} + 3c_{4,0} = 0,$$

$$n = 6 \quad 5c_{0,6} + c_{2,4} + c_{4,2} + 5c_{6,0} = 0,$$

$$n = 14 \quad 429c_{0,14} + 33c_{2,12} + 9c_{4,10} + 5c_{6,8} + 5c_{8,6} + 9c_{10,4} + 33c_{12,2} + 429c_{14,0} = 0,$$

$$n = 20 \quad 46189c_{0,20} + 2431c_{2,18} + 429c_{4,16} + 143c_{6,14} + 77c_{8,12} + 63c_{10,10} + 77c_{12,8} + 143c_{14,6} + 429c_{16,4} + 2431c_{18,2} + 46189c_{20,0} = 0.$$

**Proof of Proposition 2** (i) The necessity is obvious. To prove the sufficiency, suppose that  $\text{Reg}_0 = 0$ . By using (22) this last equality is equivalent to

$$\text{Re} \int_0^{2\pi} S_{n+1}(\theta) d\theta = 0.$$

Then, integrating system (25) in polar coordinates, we will obtain that all the orbits are closed, and so that the origin is a centre. This is done in the sequel.

From  $r^2 = z\bar{z}$  and (25), it follows that

$$\begin{aligned} r\dot{r} &= \text{Re}(\bar{z}F_{n+1}(z, \bar{z})) = \text{Re}(re^{-i\theta}F_{n+1}(re^{i\theta}, re^{-i\theta})), \\ \frac{\dot{r}}{r^{n+1}} &= \text{Re}(e^{-i\theta}F_{n+1}(e^{i\theta}, e^{-i\theta})) = \text{Re}(S_{n+1}(\theta)), \text{ and, finally,} \\ \frac{1}{nr^n} \int_0^{2\pi} &= \text{Re} \int_0^{2\pi} S_{n+1}(\theta) d\theta = 0. \end{aligned}$$

Finally we will prove (ii). In our notation, this constant writes as  $a_2 = 2\pi n \text{Reg}_0$  (see Proposition 1), where  $g_l$  are defined as in (14). As we have pointed out before, if there is a center in this sytem it is isochronous. So the first period-Abel constant  $b_1$  is always zero. Therefore (see Proposition 1),  $\text{Reg}_0 = g_0$ .

From (14) we obtain that  $g_0 = 0$  if  $n$  is odd, and that

$$g_0 = f_{\frac{n+2}{2}, \frac{n}{2}} = \frac{1}{2^n} \sum_{\Delta'} (-1)^{j_2} (-i)^k \binom{n-k}{j_1} \binom{k}{j_2} c_{n-k, k},$$

where

$$\Delta' = \{(k, j_1, j_2) : 0 \leq k \leq n, 0 \leq j_1 \leq n-k, 0 \leq j_2 \leq k, j_1 + j_2 = \frac{n}{2}\},$$

if  $n$  is even, as we wanted to prove. ■

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